Local coordination and global congestion in random networks☆

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Abstract

This paper analyzes the impact of local and global interactions on individuals' action choices. Players are located in a network and interact with each other with perfect knowledge of their neighborhood and probabilistic knowledge of the complete network topology. Each player chooses an action, from some finite set, which imposes an externality on their neighbors as well as an externality on the complete network. Players deal with two opposing forces: they obtain utility from sharing their choices with their neighbors (positive local externality) but suffer disutility from sharing the same choice with all members of the network (negative global externality). Economic and social phenomena exhibiting these features are: the adoption of cost-reducing innovations, clusters of firms, time schedule choices, the adoption of subcultures and fads, among others.

We find the conditions for the existence of all Bayesian Nash equilibria and translate them to a characterization in terms of the main properties of the network topology. The balance between local satisfaction and global dissatisfaction partially explains the equilibrium outcome. The players who finally decide the type of equilibria are those that are either highly connected (hubs) or poorly connected (peripherals) to the others. On the one hand, hubs try to coordinate their action choices, which will depend on the perceived congestion. On the other, peripherals are only worried about congestion and play the less selected actions of the network. Some examples illustrate our main results as well as the failure of equilibirum existence for some congestion costs.

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1. Introduction

Many social and/or economic activities exhibit local externalities but, at the same time, suffer from the inability of agents to coordinate their actions which, in turn, is a source of congestion in societies. For instance, individuals following the same schedule have the opportunity to share common time with others which is highly valuable for them: colleagues in a firm can meet in their coffee break, friends can see each other after work to speak and have a drink, relatives can get together in the evening on working days or at the weekend. But these individuals access public services such as public transport, highways, sport facilities, supermarkets, cinemas, airports, etc., which are usually congested at rush hours because of the regularity of society's schedules. As a result, individuals suffer the inconvenience of sharing public services with many other people (global negative externality or congestion) in order to share common time with colleagues, friends and relatives (positive local externality). This effect has been called the *tragedy of the commons* in the analysis of air traffic congestion (Mayer and Sinai [15]), but can be extended to many other contexts where there are multiple agents who do not take into account the externality they create for others.

For example, firms often benefit when their business partners (suppliers, firms of complement goods) adopt cost-reducing innovations; this set of partners is a small subset of the total set of potential adopters of these innovations. Firms would like that their business partners adopt the same innovation but too many adopters (substitute firms) may give rise to a negative externality. Similarly, firms' choice of location may suffer from the same coordination problem: firms may like to locate in clusters with other firms in order to obtain increasing returns from sharing local indivisible facilities (or a common local public good), but too many firms in the same cluster may create a congestion problem and reduce the initial advantage of being together.

The adoption of *subcultures*, social groups with particular behaviors or beliefs, by youth can also be explained by our model. A young will adopt a certain subculture if the proportion of their friends following it is big enough. However, young people like belonging to something unique and exclusive, thus the attractiveness of a subculture decreases with the proportion of people in society following it. In like fashion, the adoption of fads where the exclusivity is part of their attractiveness can also be approach by our model.

Coordination failure, or agents' uncertainty about the action of other agents, may be an important source of congestion in large decentralized societies. In the *El Farol* or Santa Fe bar problem, Arthur [2] provides a simple paradigm for congestion and coordination problems that may arise in societies. *El Farol* is a bar in Santa Fe. The bar is popular, but becomes overcrowded when more than sixty people over one hundred attend on any given evening. Everyone enjoys themselves when fewer than sixty people go, but no one has a good time when the bar is overcrowded. The *El Farol* problem emphasizes the difficulty of coordinating the actions of independent agents without a centralized mechanism. Unlike the standard public good framework, in this scenario fully informed optimizing agents will not increase consumption of a publicly available resource until it experiences an inefficient level of congestion. If agents could predict the behavior of other agents perfectly the bar would never be crowded and all the patrons would have a good time. The only source of congestion, at least in a deterministic framework, is the inability of agents to coordinate their actions. Although the *El Farol* problem initially explored the collective dynamics of boundedly rational agents, it is also interesting as a simple model of congestion and coordination behavior that occurs with shared resources like Internet bandwidth. Arthur [2] believed that any solution to the *El Farol* problem would require heterogeneous agents, that is, agents who pursue different strategies.

We analyze the above insights in a static model where individuals enjoy being at the bar with their friends or relatives but suffer from the congestion created by all the agents in society. We will assume that congestion is an increasing function on the number of individuals choosing the same action, unlike the *El Farol* problem where there is only congestion when the proportion of individuals in the bar is above certain threshold. Thus, we consider that the congestion cost is not an *all or nothing* concept but a non-linear continuous function on the proportion of individuals making the same choice. Namely, when the number of players choosing the same is small, then the addition of a new player with the same choice will not increment the congestion cost substantially, while if the number of players following the same action is large, then the new player will cause a large increment in the congestion.

Let us consider that the relationships between individuals can be modeled as a network, where each node is an individual and any (undirected) link between two nodes represents some kind of relationship between them such as friendship, family ties, firms of complement goods, etc. Each individual only has a local knowledge of the network: they know their neighbors (to whom they are linked) but they do not know who their neighbors' neighbors are. This lack of information about the network's topology is modeled by considering it as an instance of a random network where individuals know the degree probability distribution over the nodes of the network.

Individuals simultaneously choose their actions from a finite set, which imposes an externality on their neighbors as well as an externality on the complete network, and then obtain an utility. The optimal (Bayesian Nash) decision taken by each individual depends on: the spread of their connections in the network (their degree), the degree probability distribution, and the balance between positive local externality and negative global externality which impact on their utility.

We assume that each individual's utility depends mostly on two factors: positively on the proportion of neighbors choosing the same action (positive local externality) and negatively on the proportion of the network members doing the same, because it creates congestion (negative global externality).

Our contribution is twofold. We find first the conditions for the existence of all Bayesian Nash equilibria, which at the same time explain the non-existence results. Next we translate the condition on equilibrium existence to an equilibrium characterization in terms of the main properties of the network topology.

To motivate our analysis notice that a rough calculation would give us two possible (Bayesian Nash) equilibrium solutions: the one where all individuals choose the same action (homogeneous pure profile), and the one where each individual chooses their actions randomly giving all options the same probability (uniformly mixed profile). Intuitively, homogeneous pure profiles could be equilibrium outcomes when congestion costs are low, whereas the uniformly mixed profile would be an equilibrium solution if congestion costs were high enough.

However, common intuition needs to be polished since both local and global network properties play an important role in equilibrium choices. This comes from the observation that the network topology defines two important features such as *hub players* (highly connected nodes) and *peripheral players* (poorly connected nodes). Although each individual's value function will depend on both the average action profile followed by the network and the average action profile of their neighbors, their relative weight will depend on the individual's number of connections. Thus, the network average action profile is particularly important for peripherals because, by definition, their number of neighbors is very small and therefore their choice will mostly be driven by the network global topology. A peripheral player is only worried about congestion and to reduce it as much as possible she choose the least frequent action. Thus, when peripherals are frequent homogeneous pure equilibrium profiles are impossible to sustain. On the contrary, the hubs or highly connected players' choices will mainly depend on the average profile of their neighbors' actions, i.e. on the network local properties. It's very likely that hubs will be linked to other hubs, thus they try to coordinate their choices to play the same action and maximize their utility. However, if the proportion of hubs is too high, then congestion disutility may prevail and it makes difficult to guarantee equilibrium profiles. Therefore Bayesian Nash Equilibria are expressed in terms of the proportion of hubs and peripherals which, in turn, is given by both the asymmetry of the degree probability distribution (its skewness) and the weight of its tails

To the best of our knowledge this is the first time that both local and global effects are analyzed from the network perspective. We finish this section with a review of related literature.

1.1. Related literature

Our paper is a contribution to network games, an active area of research over the last few years. A complete review of this literature exceeds the intention of this section, so we refer readers to the extensive overview in Goyal [12] and more recently Jackson [14]. We assume that the network of relationships between individuals is fixed. When an individual chooses their action they obtain utility from sharing their choice with their neighbors but suffer the disutility of sharing the same choice with all members of the network. Thus, an individual's net utility depends both on their action as well as on other individuals' actions, forcing them to play in a strategic way.

The focus is on large networks, where a change in the behavior of one individual could drastically modify their utility while having a marginal impact on other individuals' utilities. Thus, we will consider that there are an infinite number of individuals in the network as in Galeotti and Vega-Redondo [10] and Morris and Shin [16]. This assumption makes the computations easier and does not have any effect on the main results.

We consider a random network, where individuals do not know the complete topology of the network but rather the degree distribution (which is fixed). Random networks were first used in Newman [17]. An overview of this literature can be found in Newman *et al.* [18] and some applications of these models to economic problems in Ioannides [13]. More recently Galeotti and Vega-Redondo [10] use random networks to study how local externalities shape agents' strategic behavior when the underlying network is volatile and complex. Galeotti *et al.* [9] and Sundararajan [20] analyze local networks, where each individual's utility depends on their own action as well as on their neighbors' actions. The first paper provides a framework with random networks to characterize the behavior and payoffs of individuals according to different factors in the model. The second paper presents a model where individuals value the adoption of a product by a heterogeneous subset of other individuals in their neighborhood, and have incomplete information about the structure and strength of adoption complementarities among all other individuals.

Recently several authors have become interested in Local Networks, where a player's payoff depends only on their own actions and those of their neighbors. Galeotti *et al.* [9] provide a general model and analyze how a given individual's behavior is affected by their position within the network and the nature of the game (strategic substitutes versus complements and positive versus negative externalities, and the level of information.) Sundararajan [20] presents a model of local effect for the analysis of an adoption game. Among other results, these articles show the existence of equilibria in pure strategies and give some properties that they verify. Our model differs in that we consider a payoff function which has both local and global externalities, but we use specific functional forms close to the ones found in Galeotti and Vega-Redondo [10] and Ballester *et al.* [3]. Those papers, however, analyze local games and a continuum of players' space of actions.

This paper is organized into seven sections. Section 2 provides the general framework. Section 3 presents the main results on equilibrium existence and Section 4 characterizes equilibria in terms of the network topology. Some comparative statics for two-type players' examples are offered in Section 5. Section 6 illustrates the results of Sections 3 and 4 in Scale free and Poisson random networks. Finally, Section 7 concludes the paper.

2. The model

There is a countable infinity of agents (players) N, and A is a finite set of actions for them. We assume that there are only two possible actions[1], i.e. $A = \{m, e\}$. For each player $i \in N$, we denote their action by $a_i \in A$. What is relevant in the analysis in that if only one individual changes their decision, then the other individuals' payoffs do not change (or change only marginally). Thus, the analysis could be carried out by considering a large number of individuals and the main results will not be affected.

Non-directed graphs are used to model network relationships between individuals. In such graphs the nodes correspond to the agents and the links correspond to the bilateral relationships between them. Let g be such a network.

[1] This is a simplification. The model with a finite number of actions extends easily.

Each individual $i \in N$ has a number of relationships with other agents in g that defines their set of neighbors, N_i, and their degree, k_i. Each agent knows their degree but does not know the degree of the other nodes in the network. We assume that players know the degree distribution that is fixed and characterized by the probability distribution

$$\mathbf{p} = \{p_k\}_{k \in K} \tag{1}$$

where K, its support, is a subset (no necessarily finite) of the positive integer numbers, $K \subseteq \mathbb{N}^*$, and p_k denotes the probability of finding a player in the network who has k neighbors. We assume that the first moment of the random variable defined by $\mathbf{p}$ is finite, thus let d be the average degree, i.e. $d = \sum_{l \in K} l p_l$. Notice that isolated players are not allowed since we assume that each individual in society maintains at least one relationship with another one.

Players interact with each other as determined by g. The network is equiprobable chosen from among all the possible networks that have a given degree distribution $\mathbf{p}$. Thus, we are assuming that no player knows their neighbors' degree but all of them know the overall degree distribution and the random network is fully characterized by it.

The influence of a player in the network is measured by their centrality. The simplest measure of centrality is a player's degree, which only uses local information and is invariant with respect to the rest of the graph.[2] A individual with high degree is a central player with respect to a local portion of the graph and becomes in a *hub* of the network, while another individual with low degree is a *peripheral* in a local portion of the graph. Notice that both hubs and peripherals are relative concepts: they depend on the degree probability distribution of each network. Given a network, an individual's degree should be considered high or low with respect to the average degree of the network she belongs to. Thus, a relative measure of centrality may be helpful to classify a node as either a hub or a peripheral.

Definition 1. *[Relative degree] Given a network g and its degree distribution $\mathbf{p} = \{p_k\}_{k \in K}$, we define the relative degree of a node as the ratio between its degree and the average degree, i.e. given node $i \in N$, its relative degree is k_i/d.*

The relative degree play a central role in the characterization of the equilibria. Nodes with high relative degree will be considered as hubs and nodes with low relative degree will be peripherals.

If a player chooses a neighbor randomly, then they will not know their degree but will know that neighbor's degree distribution. Assuming independence across neighbors' degrees, then the probability of arriving at a node is proportional to its degree, and we can compute that

$$\widetilde{p}_k = \frac{k p_k}{\sum_{l \in K} l p_l} \tag{2}$$

[2] Other concepts of centrality, such as Closeness, Betweenness and Bonacich' measure, need to know the complete topology of the network to be calculated. A description of these measures can be found in Jackson [14]

is the probability of a node having degree k when it is selected randomly from among a player's neighbors. Let this distribution be denoted by $\widetilde{\mathbf{p}} = \{\widetilde{p}_k\}_{k \in K}$.

Mixed strategies are allowed, so that the decision of any player is an element in $\Delta(A)$, which is the set of all probability distributions over A. Given that A has two elements, we can identify the 1-dimensional simplex $\Delta(A)$ with the interval $[0, 1]$. Therefore, player i's action is $x_i \in [0, 1]$, where x_i is the probability of choosing action m, and then $1 - x_i$ is the probability of choosing action e.

Prior to interaction each player i has to decide their action $x_i \in [0, 1]$ individually and independently of the other players. This decision can only depend on their own degree and the degree distribution on the other player degrees. Let $\{x_i, \mathbf{x_{-i}}\}$ be the profile of actions, where x_i is the action chosen by i and $\mathbf{x_{-i}}$ those of the other players. Let $(x_j)_{j \in N_i}$ be the profile of actions of i's neighbors. We assume that the net payoff function of player i has two components, the gross payoff function, f, which measures local externalities, and the congestion function, h, which measures global externalities:

$$u_i[x_i, \mathbf{x_{-i}}] = f[x_i, (x_j)_{j \in N_i}] - h[x_i, \mathbf{x_{-i}}].$$

Assuming ex-ante symmetry across players, player i's gross payoff depends on their action and the actions of their neighbors,

$$f : \Delta(A) \times \Delta(A)^{N_i} \to \mathbb{R}_+,$$

while the congestion depends on the actions chosen by all the players in the network,

$$h : \Delta(A)^N \to \mathbb{R}_+.$$

2.1. The Bayesian Game

The strategic situation is modeled as a classical Bayesian game, where each player's type is identified with their degree and all the players' types are drawn independently according to the prevailing degree distribution $\mathbf{p}$. Therefore, the type space for every player is K and their beliefs on the other types is the degree distribution $\mathbf{p}$. Each player's strategy is a mapping from their type to set $[0, 1]$. In other words, as in Galeotti and Vega-Redondo [10], we posit that each player chooses an action induced only by their own degree, the degree distribution $\mathbf{p}$ and their prediction of the other players' actions $\mathbf{x} = \{x_k\}_{k \in N}$ which specifies how every other player anticipates choosing their action, depending on their degree. Thus, we analyze symmetric Bayesian strategies, i.e., all players with the same degree choose the same strategy.

Denote by $v_k[x, \mathbf{x}]$ the expected payoff function of a k-degree player who chooses action x and expects the degree contingent strategy $\mathbf{x} = \{x_l\}_{l \in K}$,

$$v_k[x, \mathbf{x}] = E_{\widetilde{\mathbf{p}}}[f[x, (x_{k_j})_{j \in N_i}]] - E_{\mathbf{p}}[h[x, (x_{k_j})_{j \in N}]].$$

We have defined an incomplete information game where the player's degree defines their type. The main objective of this paper is to study the strategy profiles (indexed by the degree) that are symmetric Bayesian-Nash Equilibrium (BNE).

Definition 2. *A strategy profile* $\mathbf{x}^* = \{x_k^*\}_{k\in K}$ *is a symmetric* Bayesian-Nash Equilibrium *(BNE) if it satisfies:*

$$x_k^* \in argmax_{x\in[0,1]} v_k[x, \mathbf{x}^*] \tag{3}$$

for all $k \in K$.

A strategy profile is a BNE if no player can deviate unilaterally and benefit from that deviation.

To provide a precise specification of function $v_k[x_k, \mathbf{x}]$ and characterized the BNE we have to consider in detail the two functions that define the expected net payoff function.

2.2. Local and global externalities

The expected gross payoff function

The gross payoff function, f, measures the utility that a player, say i, obtains by the interaction with their neighbors. We define the *gross payoff function* of a player *as the expected proportion of their neighbors choosing their same action.* Therefore, the local interaction component exhibits positive externalities since the player i's gross payoff increases with the proportion of neighbors choosing the same action than i.

The player i's gross payoff function depends on two random variables, the proportion of their neighbors choosing their same action, and their own action. The first random variable has a distribution governed by $\widetilde{\mathbf{p}}$, and the second follows a Bernoulli distribution with probability x_{k_i}. However, by the law of total probability, the expected proportion of their neighbors choosing their same action is given by the expected proportion of their neighbors choosing action m times the probability of player i's action to be m plus the expected proportion of their neighbors choosing action e times the probability of player i's action to be e:

$$\begin{aligned} E_{\widetilde{\mathbf{p}}} f[x_{k_i}, (x_{k_j})_{j\in N_i}] &= E_{\widetilde{\mathbf{p}}} f[m, (x_{k_j})_{j\in N_i}] Prob(a_i = m) + E_{\widetilde{\mathbf{p}}} f[e, (x_{k_j})_{j\in N_i}] Prob(a_i = e) \\ &= E_{\widetilde{\mathbf{p}}} f[m, (x_{k_j})_{j\in N_i}] x_{k_i} + E_{\widetilde{\mathbf{p}}} f[e, (x_{k_j})_{j\in N_i}](1 - x_{k_i}). \end{aligned}$$

The expression $E_{\widetilde{\mathbf{p}}} f[m, (x_{k_j})_{j\in N_i}]$ is the expected value of a random variable, the proportion of neighbors choosing action m. One way to obtain an explicit specification of this expected value is to make explicit the support of the random variable, compute the probability mass of each element in the support and calculate the expectation. This is the way followed in Galeotti and Vega-Redondo [10] to compute a close expression.[3] An alternative

[3]The support of the random variable is the set of all possible distributions of the degrees of player i's neighbors, i.e., the set of all k_i dimensional vectors of integer components such that the sum of the components is equal to k_i. The random variable follows a multinomial distribution

way is to realize that the expected value of $f[m,(x_{k_j})_{j\in N_i}]$ is equal to the probability that a randomly chosen player i's neighbor has played action m, and compute this probability using, again, the law of total probability:

$$\begin{aligned}
E_{\widetilde{\mathbf{p}}} f[m,(x_{k_j})_{j\in N_i}] &= Prob(\text{Randomly chosen neighbor (r.c.n.) plays } m) \\
&= \sum_{l\in K} Prob(\text{r.c.n. plays } m|\text{r.c.n. has degree } l)Prob(\text{r.c.n. has degree } l]) \\
&= \sum_{l\in K} x_l \widetilde{p}_l
\end{aligned}$$

Notice that the last term does not depend on the identity of the player but depends on the player's action which is degree contingent. Let us define the average proportion of neighbors following action m in terms of the distribution of the neighbors' degree, $\widetilde{x}$, as

$$\widetilde{x} = \sum_{l\in K} x_l \widetilde{p}_l. \tag{4}$$

Similarly, we obtain that $E_{\widetilde{\mathbf{p}}} f[e,(x_j)_{j\in N_i}] = 1-\widetilde{x}$. Thus, the expected gross payoff function of a player with degree k can be written as,

$$E_{\widetilde{\mathbf{p}}} f[x_k,\mathbf{x}] = x_k\widetilde{x} + (1-x_k)(1-\widetilde{x}) \tag{5}$$

that is independent of the player's identity. Therefore, the expected gross payoff of k-type player depends on its own action and on the average proportion of neighbors choosing the same action in terms of the distribution of the neighbors' degree.

The expected congestion function

The congestion function, h, measures a player's dissatisfaction from, for example, the use of a public service simultaneously with individuals in the network that have chosen their same action. Thus, this is a global interaction and exhibits negative externalities, as player's payoff will be decreasing on the number of players choosing the same action than theirs.

We propose a *congestion function that is quadratic on the expected proportion of players choosing the same action as that player.* This relationship reflects what is commonly seen in real life. When the number of players choosing the same action is small, then the addition of a new player with the same action will not increment congestion substantially, while if the number of players following the same action is large, then the new player will cause a large increment in congestion. This fact is reflected through the quadratic dependence of the congestion on the number of subjects choosing the same action as theirs.

Let us consider a player i with degree k. Similar reasoning as the above for the gross payoff function allows us to write the expected congestion function, $E_{\mathbf{p}}[h[x,(x_{k_j})_{j\in N}]]$, as,

$$
\begin{aligned}
E_{\mathbf{p}}h[x_{k_i},(x_{k_j})_{j\in N}] &= \left(E_{\mathbf{p}}h[m,(x_{k_j})_{j\in N}]\right)^2 Prob(a_i=m)+\left(E_{\mathbf{p}}h[e,(x_{k_j})_{j\in N}]\right)^2 Prob(a_i=e)\\
&= \left(E_{\mathbf{p}}h[m,(x_{k_j})_{j\in N}]\right)^2 x_{k_i}+\left(E_{\mathbf{p}}h[e,(x_{k_j})_{j\in N}]\right)^2(1-x_{k_i}).
\end{aligned}
$$

However, the expected proportion of players choosing action m, $E_{\mathbf{p}}h[m,(x_{k_j})_{j\in N}]$, is equal to the probability that a randomly chosen player has played action m. This probability is straightforwardly computed as,

$$
\begin{aligned}
E_{\mathbf{p}}h[m,(x_{k_j})_{j\in N}] &= Prob(\text{Randomly chosen player (r.c.p.) plays } m)\\
&= \sum_{l\in K} Prob(\text{r.c.p. plays } m|\text{r.c.p. has degree } l)Prob(\text{r.c.p. has degree } l])\\
&= \sum_{l\in K} x_l p_l
\end{aligned}
$$

Let $\overline{x}$ be the average proportion of the choices of all the network types,

$$\overline{x}=\sum_{l\in K} x_l p_l, \tag{6}$$

then the expected congestion function of a player of degree k, $E_{\mathbf{p}}[h[x_k,\mathbf{x}]]$, can be written as,

$$E_{\mathbf{p}}[h[x_k,\mathbf{x}]]=\frac{c}{2}\left[x_k\overline{x}^2+(1-x_k)(1-\overline{x})^2\right], \tag{7}$$

where c is a parameter bigger than zero.
As the expected gross payoff function, the expected congestion cost is independent of the player's identity and depends on the player's own action, which is degree contingent, and on the expected choice of all the network players.

The expected net payoff function

Combining the two components of the value function, the expected net payoff function can be written as:

$$v_k[x_k,\mathbf{x}]=x_k\widetilde{x}+(1-x_k)(1-\widetilde{x})-\frac{c}{2}\left[x_k\overline{x}^2+(1-x_k)(1-\overline{x})^2\right]. \tag{8}$$

The structure of the expected net payoff function, where both gross payoffs and congestion are quadratic, can be found in other studies which analyze the effect of *local* externalities on players' decisions (see e.g. Galeotti and Vega-Redondo [10] or Ballester *et al.* [3]). Here, in contrast, we take into account both local and global externalities.

Then $v_k[x_k,\mathbf{x}]$ can be expressed as a quadratic function of x_k,

$$v_k[x_k,\mathbf{x}]=\alpha_{kk}x_k^2+\sum_{l\in K\setminus k}\alpha_{kl}x_kx_l+\beta_kx_k+\gamma_k(\{x_l\}_{l\neq k}) \tag{9}$$

where,

$$\alpha_{kk} = 2\widetilde{p}_k - cp_k(\frac{1}{2}p_k + 1), \tag{10}$$
$$\alpha_{kl} = 2\widetilde{p}_l - cp_l(p_k + 1), \text{ for all } l \neq k, \tag{11}$$
$$\beta_k = cp_k - 1 + \frac{c}{2} - \widetilde{p}_k, \text{ and} \tag{12}$$
$$\gamma_k(\{x_l\}_{l\neq k}) = (1 - \frac{c}{2}) - \sum_{l\neq k} x_l\widetilde{p}_l + \frac{c}{2}\big[\sum_{l\neq k} x_l p_l(2 - \sum_{l\neq k} x_l p_l)\big] \tag{13}$$

Function $v_k[x_k, \mathbf{x}]$ depends on x_k and on the weighted aggregation of the expected choices of all other player types in the network, i. e. $\sum_{l\neq k} \alpha_{kl}x_l$. Each weight α_{kl} measures the joint expected contribution of an l-degree player to the marginal payoffs of a k-degree player both as a neighbor and as a member of the whole network.

Notice that α_{kk}, the coefficient of the quadratic term, depending on the congestion cost can be either positive or negative and, therefore, the net payoff function can be either convex or concave. It can be checked that the coefficients of the net payoff function verifies the following property,

$$\alpha_{kk} + \beta_k + \frac{1}{2}\sum_{l\neq k} \alpha_{kl} = 0. \tag{14}$$

3. Existence of Bayesian Nash equilibria

As it is well known a Bayesian Nash equilibrium always exists, possibly in mixed strategies, whenever functions $v_k[x_k, \mathbf{x}]$ are concave in x_k ($\alpha_{kk} < 0$), for all $k \in K$, and each strategy space is compact (as is in our case where $x_k \in [0, 1]$). However, by (10) above, concavity of all v_k's functions only results when c is sufficiently high.

Thus, when the congestion cost parameter c is small enough, then functions $v_k[x_k, \mathbf{x}]$ are convex in x_k ($\alpha_{kk} > 0$) in the interval $[0, 1]$, best replies need not be continuous and have a jump. This is so even for intermediate values of c, when some functions $v_k[x_k, \mathbf{x}]$ may still remain convex while others have already turned to be concave.[4] We need then to borrow results from supermodular games and monotone best responses (Vives [21]).

Function $v_k[x_k, \mathbf{x}]$ depends on x_k and on the choices of all other player types in the network, $\mathbf{x}_{-k}$. Theorem 4.2 in Vives (Vives [21]) states that a Bayesian Nash equilibrium exists if, for all $k \in K$, the set of strategies is a lattice compact, v_k is supermodular on $[0, 1]$ and/or has increasing differences in $(x_k, \mathbf{x}_{-k})$. Moreover, v_k is supermodular if and only if $\partial^2 v_k/\partial x_k\partial x_l \geq 0$ for all $l \neq k$. Given that $\partial^2 v_k/\partial x_k\partial x_l = \alpha_{kl}$, supermodularity is guarantee if and only if $\alpha_{kl} \geq 0$ for all $k, l \in K$, $l \neq k$. On the other hand, v_k has increasing differences

[4]Since for $\alpha_{kk} = 0$, the value function is linear and we are only interested in quadratic value functions, then we disregard the linear case.

in $(x_k, \mathbf{x}_{-k})$ if $v_k[x_k, \mathbf{x}_{-k}] - v_k[x_k, \mathbf{x}'_{-k}]$ is increasing in x_k, for all $\mathbf{x}_{-k} \geq \mathbf{x}'_{-k}$ $(\mathbf{x}_{-k} \neq \mathbf{x}'_{-k})$. By (9) above,

$$v_k[x_k, \mathbf{x}_{-k}] - v_k[x_k, \mathbf{x}'_{-k}] = x_k \sum_{l \neq k} \alpha_{kl}(x_l - x'_l) + (\gamma_k(\mathbf{x}_{-k}) - \gamma_k(\mathbf{x}'_{-k})).$$

If $\mathbf{x}_{-k} \geq \mathbf{x}'_{-k}$, then v_k has increasing differences if and only if $\alpha_{kl} \geq 0$ for all $k, l \in K$, $l \neq k$, i.e., the same conditions guarantee both supermodularity and increasing differences of v_k.

Let $\Psi_k(\mathbf{x}_{-k})$ be the best response of type k to $\mathbf{x}_{-k}$, if v_k is either supermodular or has increasing differences and the strategy sets are lattices (sets [0,1]), then $\Psi_k(\mathbf{x}_{-k})$ is increasing in $\mathbf{x}_{-k}$, the composite best response is also increasing and (by Topkins' Theorem) a Bayesian Nash equilibrium exists (Vives [21]). Thus, *if each player considers each of the other players' action as strategic complement* ($\alpha_{kl} \geq 0$ for all $l, k \in K$, $l \neq k$), *then Bayesian Nash equilibria will exis*t.

Therefore we have two general sufficient conditions under which a Bayesian Nash equilibrium exists:

a) Either all the functions $v_k[x_k, \mathbf{x}]$ are concave in x_k in the interval $[0, 1]$.

b) Functions v_k are either all convex, all concave or some convex and others concave in the interval $[0, 1]$, and players are strategic complements ($\alpha_{kl} \geq 0$, for all $l \neq k$).

Define *homogeneous pure* strategy BNE as the profiles where all types play the same pure action either m or e, i.e., the profiles $\mathbf{x_0} = (0, 0, \ldots, 0)$ and $\mathbf{x_1} = (1, 1, \ldots, 1)$; *heterogeneous pure* strategy as those profiles where all types play a pure action, not necessarily the same, i.e., those profiles $\mathbf{x}$ such that $x_k \in \{0, 1\}$ for all $k \in K$; *mixed* strategy BNE as the profiles where all types play a mixed action ($x_k \in (0, 1)$ for all $k \in K$); and finally *hybrid* BNE as those profiles where some players choose a pure strategy and others a mixed strategy. Condition a) above is mainly concerned with the existence of mixed Bayesian Nash equilibria but condition b) is a monotonicity condition which applies mainly to pure strategy and hybrid Bayesian Nash equilibrium. In the following we relax condition b) and give conditions for the existence of the above different equilibrium profiles, which depend on both the concavity/convexity of the payoff function and the strategic complementarity/substitution relationship between players' actions.

Homogeneous pure strategy Bayesian Nash equilibria

Suppose firstly that all the v_k functions are convex in $[0, 1]$, then each $\Psi_k(\mathbf{x}_{-k}) \in \{0, 1\}$ and there exist two Bayesian Nash equilibria in homogeneous pure strategies, $\mathbf{x_0}$ and $\mathbf{x_1}$. In fact, from (9) above it is easily seen that $v_k[\mathbf{x_1}] = v_k[\mathbf{x_0}]$ and that for $\mathbf{x}_{-k} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \ldots \frac{1}{2})$, then both $x_k = 1$ and $x_k = 0$ belong to $\Psi_k(\mathbf{x}_{-k})$, so that best responses have at most a jump upwards. In this setting, we can notice by (9) and (14) that, for all k,

$$v_k[1, \mathbf{x}_{-k}] \geq v_k[0, \mathbf{x}_{-k}] \iff \sum_{l \neq k} (x_l - \frac{1}{2})\alpha_{kl} \geq 0. \tag{15}$$

Thus, if $\mathbf{x}_{-k}$ is either a vector of 1's or of 0's, then whenever $\sum_{l \neq k} \alpha_{kl} \geq 0$, the best response of each player of type k, for all $k \in K$, is a non-decreasing function of the aggregate of the other players' strategies, being $x_k = 1$ as a best reply to a profile of 1's and a $x_k = 0$ to a profile of 0's. Hence, the equilibrium profiles are homogeneous sequences of either all 1's or all 0's. Thus, when functions v_k are all convex in $[0,1]$, all we need to guarantee an homogeneous pure strategy Bayesian Nash equilibrium is that for any player, the other players' actions are strategic complements in the aggregate, i.e., $\sum_{l \neq k} \alpha_{kl} \geq 0$.

Similarly, suppose now that all the functions v_k where concave then,

$$\Psi_k(\mathbf{x}_{-k}) = \frac{\beta_k + \sum_{l \neq k} \alpha_{kl} x_l}{-2\alpha_{kk}}. \tag{16}$$

By concavity $\alpha_{kk} < 0$, then homogeneous pure strategy equilibria would exist whenever $\beta_k \leq 0$, because by (14) that implies that $\sum_{l \neq k} \alpha_{kl} \geq 0$ and hence each player's best response function is non-decreasing.[5]

These results can be extended to any mix of convex and concave payoff functions provided that for any player, the other players' actions are strategic complements in the aggregate, since in this case each player's best reply is non-decreasing. Then,

Proposition 1. *The two homogeneous pure strategy BNE, $\mathbf{x_0}$ and $\mathbf{x_1}$, will exist if for any player, the other players' actions are strategic complements in the aggregate: for all $k \in K$, $\sum_{l \neq k} \alpha_{kl} \geq 0$. (If v_k is concave, then $\beta_k \leq 0$ will be sufficient to guarantee that condition).*

Notice that the above Proposition gives weakness conditions for the existence of homogeneous pure strategy BNE than supermodularity or increasing difference of functions v_k, where strategic complementarity has to be satisfied for each individual pair of players.

Also notice that under the above conditions no *heterogeneous* pure strategy BNE, sequences with a mix of zeros and ones, will exist. However, these strategy equilibria may also exist under different conditions, as the following example shows:

Example 1. *Let $K = \{15, 16, 17\}$ with $p_k = 1/3$ for all $k \in K$. Then, $\widetilde{p}_{15} = 15/48$, $\widetilde{p}_{16} = 16/48$ and $\widetilde{p}_{17} = 17/48$.*

Here, functions v_k specify to (the terms not depending on x_k are not included):

$$\begin{aligned}
&v_{15}[x_{15}, (x_{16}, x_{17})] = \alpha_{15,15}\, x_{15}^2 + \alpha_{15,16}\, x_{15}x_{16} + \alpha_{15,17}\, x_{15}x_{17} + \beta_{15}\, x_{15} \\
&\ = [5/8 - (7/18)c]x_{15}^2 + [2/3 - (4/9)c]x_{15}x_{16} + [17/24 - (4/9)c]x_{15}x_{17} + [(5/6)c - 21/16]x_{15} \\
&v_{16}[x_{16}, (x_{15}, x_{17})] = \alpha_{16,16}\, x_{16}^2 + \alpha_{16,15}\, x_{16}x_{15} + \alpha_{16,17}\, x_{16}x_{17} + \beta_{16}\, x_{16} \\
&\ = [2/3 - (7/18)c]x_{16}^2 + [5/8 - (4/9)c]x_{16}x_{15} + [17/24 - (4/9)c]x_{16}x_{17} + [(5/6)c - 4/3]x_{16} \\
&v_{17}[x_{17}, (x_{15}, x_{16})] = \alpha_{17,17}\, x_{17}^2 + \alpha_{17,15}\, x_{17}x_{15} + \alpha_{17,16}\, x_{17}x_{16} + \beta_{17}\, x_{17} \\
&\ = [17/24 - (7/18)c]x_{17}^2 + [5/8 - (4/9)c]x_{17}x_{15} + [2/3 - (4/9)c]x_{17}x_{16} + [(5/6)c - 65/48]x_{17}
\end{aligned}$$

[5]Alternativelly, by (14) and (16) the condition for $\Psi_k(\mathbf{x}_{-k}) \geq 1$ is that $\sum_{l \neq k} \alpha_{kl}(x_l - 1) \geq \beta_k$ and then if $\mathbf{x}_{-k}$ is a vector of 1's all we need is that $\beta_k \leq 0$. Similarly as above, the condition for $\Psi_k(\mathbf{x}_{-k}) \leq 0$ is that $\sum_{l \neq k} \alpha_{kl} x_l \leq -\beta_k$, and then if $\mathbf{x}_{-k}$ is a vector of 0's, the result follows.

Whenever $c \leq 93/64 \approx 1.45$, all v_k functions are convex and $\sum_{l \neq k} \alpha_{kl} > 0$, for all $k \in \{15, 16, 17\}$. Then by Proposition 1, the unique BNE's are the two homogeneous pure strategy profiles $\mathbf{x}_0$ and $\mathbf{x}_1$.

However, for $99/64 \leq c < 45/28$, ($99/64 \approx 1.55$ and $45/28 \approx 1.61$), all v_k functions are convex but $\sum_{l \neq k} \alpha_{kl} < 0$, with each $\alpha_{kl} < 0$, for all $k \in \{15, 16, 17\}$, and all $l \neq k$. Since $\alpha_{16,17} \geq \alpha_{16,15}$ and $\alpha_{17,16} \geq \alpha_{17,15}$, the unique BNE's are the two heterogeneous pure strategy profiles $\mathbf{x} = (0, 1, 1)$ and $\mathbf{x}' = (1, 0, 0)$.

Notice that, interestingly enough, no equilibrium will exist in the interval $93/64 < c < 99/64$. For instance, suppose that $c = 95/64$, then $\alpha_{kk} > 0$ for all $k \in \{15, 16, 17\}$, with $\sum_{l \neq k} \alpha_{15,l} > 0$, $\sum_{l \neq k} \alpha_{16,l} > 0$ and $\sum_{l \neq k} \alpha_{17,l} < 0$. Therefore, by Proposition 1, homogeneous pure strategy equilibria do not exist. In addition, heterogeneous pure strategy equilibria neither exist as well. To see that, consider, as an example, the profile $\mathbf{x} = (0, 1, 1)$ as the proposed equilibrium. Type $k = 15$ will deviate and play $x_{15} = 1$ because $\sum_{l \neq k} \alpha_{15,l} > 0$ and then $v_{15}(1, 1, 1) > v_{15}(0, 1, 1)$. But for $(1, 1, 1)$ type $k = 17$ will deviate and play $x_{17} = 0$ because $\sum_{l \neq k} \alpha_{17,l} < 0$ and then $v_{17}(1, 1, 0) > v_{17}(1, 1, 1)$ and so on. Similar arguments apply for other heterogeneous profiles.

Heterogeneous pure strategy Bayesian Nash equilibria

To investigate heterogeneous pure strategy equilibrium, let us consider two type of players, $K = \{1, 2\}$, with payoff functions v_k, $k \in K$ and a proposed equilibrium $(1, 0)$ where type 1 chooses a 1, and type 2 chooses a 0. If both functions are convex, then by (15) the condition for type 1 is that $\alpha_{12}(x_2 - \frac{1}{2}) = \alpha_{12}(0 - \frac{1}{2}) \geq 0$, or $\alpha_{12} \leq 0$. Similarly, for type 2 this condition is that $\alpha_{21} \leq 0$. Therefore, if $\alpha_{ij} \leq 0$, $i, j = 1, 2$, $i \neq j$, an heterogeneous pure strategy equilibrium exists because each best reply is decreasing, but the composite best reply is increasing. Now, suppose that the payoff functions are concave, then since each player best response is given by (16), then for the equilibrium $(1, 0)$ it is needed that $\Psi_1(0) \geq 1$ and $\Psi_2(1) \leq 0$, or $-\alpha_{12} \geq \beta_1$ and $\beta_2 \leq -\alpha_{21}$. Furthermore, since by concavity of v_1, $\alpha_{11} = -[\beta_1 + \frac{1}{2}\alpha_{12}] \leq 0$, and then $\beta_1 \geq -\frac{1}{2}\alpha_{12}$. Combining this expression with the one above, $-\alpha_{12} \geq \beta_1 \geq -\frac{1}{2}\alpha_{12}$ that implies $\alpha_{12} \leq 0$ and $\beta_1 \geq 0$. Similarly, the concavity of v_2 implies that $\alpha_{21} \leq 0$ and $\beta_2 \geq 0$. Therefore, since the composite best response is increasing, the conditions $\alpha_{ij} \leq -\beta_i < 0$, $i, j = 1, 2$, $i \neq j$ guarantee the heterogeneous pure strategy equilibrium $(1, 0)$. Finally, for the two type of players, when one function is convex, say v_1, and the other concave, the conditions are as above, $\alpha_{12} \leq 0$ for the convex function and $\alpha_{21} \leq -\beta_2 < 0$ for the concave one.

The extension of the above conditions for more than two players is as follows. Let K_0 the set of types such that $x_k = 0$ and let K_1 the set of types with $x_k = 1$, with $K_0 \cup K_1 = K$, and assume that all functions v_k are convex. Then, by (15), $\sum_{l \in K_1,\, l \neq k} \alpha_{kl} \geq \sum_{l \in K_0} \alpha_{kl}$, for all $k \in K_1$ and $\sum_{l \in K_1} \alpha_{kl} \leq \sum_{l \in K_0,\, l \neq k} \alpha_{kl}$, for all $k \in K_0$. These conditions characterize heterogeneous pure strategy equilibria. A $k \in K_1$ player's best response will be increasing in the aggregate of all $l \neq k$ and likewise for any $k \in K_0$ player's best response. A sufficient condition for existence is that that, for all $k \in K_1$, $\sum_{l \in K_1,\, l \neq k} \alpha_{kl} \geq 0 \geq \sum_{l \in K_0} \alpha_{kl}$, and for $l \in K_0$, $\sum_{k \in K_0,\, k \neq l} \alpha_{lk} \geq 0 \geq \sum_{k \in K_1} \alpha_{lk}$. Notice that for say, $k \in K_1$, the actions of the other players in K_1 are strategic complements in the aggregate while those of the players in K_0 are strategic substitutes in the aggregate and viceversa.

Similarly, assume now that all functions v_k are concave, then by the best reply function (see 16) and property (14), for all $k \in K_1$

$$\Psi_k(\mathbf{x}_{-k}) = \frac{\beta_k + \sum_{l \neq k} \alpha_{kl} x_l}{-2\alpha_{kk}} = \frac{\beta_k + \sum_{l \in K_1, l \neq k} \alpha_{kl}}{2\beta_k + \sum_{l \neq k} \alpha_{kl}}.$$

Thus, $\Psi_k(\mathbf{x}_{-k}) \geq 1$ translates to condition $\sum_{l \in K_0} \alpha_{kl} \leq -\beta_k$. In addition, $-[\beta_k + \frac{1}{2}\sum_{l \neq k} \alpha_{kl}] < 0$, or $-\beta_k < \frac{1}{2}\sum_{l \neq k} \alpha_{kl}$, by concavity. Then, combining these inequalities, $\sum_{l \in K_0} \alpha_{kl} \leq -\beta_k < \frac{1}{2}[\sum_{l \in K_0} \alpha_{kl} + \sum_{l \in K_1, l \neq k} \alpha_{kl}]$, and hence $\sum_{l \in K_0} \alpha_{kl} < \sum_{l \in K_1, l \neq k} \alpha_{kl}$. Similarly, for all $k \in K_0$ since $\Psi_k(\mathbf{x}_{-k}) \leq 0$, then $\sum_{l \in K_1} \alpha_{kl} \leq -\beta_k$ and by concavity $\sum_{l \in K_1} \alpha_{kl} \leq -\beta_k < \frac{1}{2}[\sum_{l \in K_1} \alpha_{kl} + \sum_{l \in K_0, l \neq k} \alpha_{kl}]$, that implies $\sum_{l \in K_1} \alpha_{kl} < \sum_{l \in K_0, l \neq k} \alpha_{kl}$. If this conditions are satified, by an argument similar to the above, an heterogeneous pure strategy equilibrium exists.

Proposition 2. *Let K_0 the set of types such that $x_k = 0$ and let K_1 the set of types with $x_k = 1$, with $K_0 \cup K_1 = K$, then heterogeneous pure strategy BNE exists whenever,*

1. Functions v_k are all convex, $\sum_{l \in K_1, l \neq k} \alpha_{kl} \geq \sum_{l \in K_0} \alpha_{kl}$ for all $k \in K_1$, and $\sum_{l \in K_1} \alpha_{kl} \leq \sum_{l \in K_0, l \neq k} \alpha_{kl}$ for all $k \in K_0$.

2. Functions $v_k[x_k, x]$ are all concave, $\sum_{l \in K_0} \alpha_{kl} \leq -\beta_k$ for all $k \in K_1$, and $\sum_{l \in K_1} \alpha_{kl} \leq -\beta_k$ for all $k \in K_0$.

We do not further proceed with the conditions for heterogeneous pure strategy equilibria when some functions are concave and some other ones convex.

Mixed strategy Bayesian Nash equilibria

As already mentioned, when functions$v_k[x_k, \mathbf{x}]$ are concave in x_k ($\alpha_{kk} < 0$), for all $k \in K$, and since each strategy space is compact a Bayesian Nash Equilibrium in mixed strategies exits. Alternatively, define the uniformly mixed strategy profile as the one where all players choose randomly between actions, giving each action the same probability of being chosen, i.e., the profile $\mathbf{x}$ such that $x_k = 1/2$ for all $k \in K$. If all payoff functions are concave, then the the uniformly mixed strategy profile is a BNE. By (14) we have that $-\alpha_{kk} = \beta k + \frac{1}{2}\sum_{l \neq k} \alpha_{kl}$, then by (16), $\Psi_k(\{1/2\}_{l \neq k}) = 1/2$.

Proposition 3. *A mixed strategy BNE exists whenever all payoff functions are concave.*

Hybrid pure strategy Bayesian Nash equilibria

It remains to show the existence of *hybrid* BNE equilibria where some players choose a pure strategy and others a mixed one. Given the difficulty of dealing with heterogeneous pure strategies, we restrict the analysis to those hybrid equilibria whose pure strategies are homogeneous. Also, these equilibria can only take place when some v_k functions are convex and others are concave. Consider first that all v_k functions are convex but one, this one, v_n, being concave. The profiles $\mathbf{x} = (x_n, \mathbf{x}_{-n})$ with either $x_n \in [\frac{1}{2}, 1)$ and $x_k = 1$ for all $k \neq n$ or $x_n \in (0, \frac{1}{2}]$ and $x_k = 0$ are BNE if i) each player with a convex payoff function considers the actions of the remaining players as strategic complements in the aggregate, i.e., $\sum_{l \in K \setminus k} \alpha_{kl} \geq 0$ for all $k \neq l$; ii) the player with the concave payoff function considers the other players (with convex functions) as strategic complements in the aggregate ($\sum_{l \neq n} \alpha_{nl} \geq 0$); and iii) $\beta_n > 0$, that guarantees that the n-type player chooses a mixed action $x_n \in (0, 1)$. (It follows from (15) and (16) and similar reasoning that the ones already used.) Again the strategic complementarity in the aggregate between the players' choices guarantees the existence of equilibria. This result is extended to any number of convex and concave functions provided that all the convex functions satisfy that $\sum_{l \neq k} \alpha_{kl} \geq 0$, $l \neq k$ and the concave function that $\sum_{l \neq n} \alpha_{nl} \geq 0$, $l \neq n$. Each player with a concave function will choose $\Psi_n(\mathbf{x}_{-n}) \geq 1/2$ whenever those with convex functions choose 1's and will choose $\Psi_n(\mathbf{x}_{-n}) \leq 1/2$, whenever the players with convex functions choose 0's. Since all the best replies are non-decreasing, the composite best response is non-decreasing and (hybrid) equilibria exist.

Now suppose that all the v_n functions are concave but one, say v_k, this one being convex, and that each player with a concave payoff function considers as strategic substitutes both the actions of the remaining players with concave functions as well as the one of the player with a convex function, k, i.e., $\alpha_{ns} \leq 0$, $\alpha_{nk} \leq 0$ for all $k, s \in K \setminus n$. Furthermore, assume that the player with the convex functions also consider as strategic substitutes (pairwise) the actions of all the other players, i.e.,$\alpha_{kn} \leq 0$ for all $n \in K \setminus k$. Then an equilibrium exists, possibly hybrid. The idea of the proof relies on the fact that the system with concave functions has two solutions, parameterized by $x_k \in \{0, 1\}$ (since the v_n are concave and the strategy space is compact), which are non-increasing in x_k because $\alpha_{nk} \leq 0$ for all n. Let $\mathbf{x}^*_\mathbf{n}(1) = \{x^*_n(1)\}_{n \neq k}$ and $\mathbf{x}^*_\mathbf{n}(0) = \{x^*_n(0)\}_{n \neq k}$, be such solutions with $x^*_n(1) \leq x^*_n(0)$ for all $n \in K$ and $n \neq k$. By the strategic substitution between players' actions, in the Appendix it is shown that $x^*_n(1) \leq 1/2 \leq x^*_n(0)$ for all $n \neq k$. Now, player k will maximize v_k, given the other players' choice. Then, since by (15), $v_k[1, \mathbf{x}^*_\mathbf{n}(1)] \geq v_k[0, \mathbf{x}^*_\mathbf{n}(1)]$ if and only if $\sum_{n \neq k} \{x^*_n(1) - 1/2\} \alpha_{kn} \geq 0$, a solution exists provided that $\alpha_{kn} \leq 0$, for all $n \neq k$. The strategic substitution of actions between the players with concave functions and the one with a convex function imply that the composite best response in non-decreasing and an hybrid equilibrium exist.

Proposition 4. *Hybrid BNE equilibria (with homogeneous pure strategies) exist whenever,*

1. Some functions v_k are convex and some others convex and players' actions are strategic complements in the aggregate, i.e. $\sum_{l \neq k} \alpha_{kl} \geq 0$ for all convex functions, and $\sum_{l \neq n} \alpha_{nl} \geq 0$ with $\beta_n > 0$, for the concave functions.

2. All functions v_n are concave but one being convex, v_k, and for all $n, s \in K \setminus k$, $\alpha_{ns} \leq 0, \alpha_{nk} \leq 0$ and $\alpha_{kn} \leq 0$ (strategic substitutes).

The equilibrium may fail to exist.

The above Propositions give us conditions which are sufficient to guarantee the existence of different kind of BNE. However, it still remains the question of whether the equilibrium

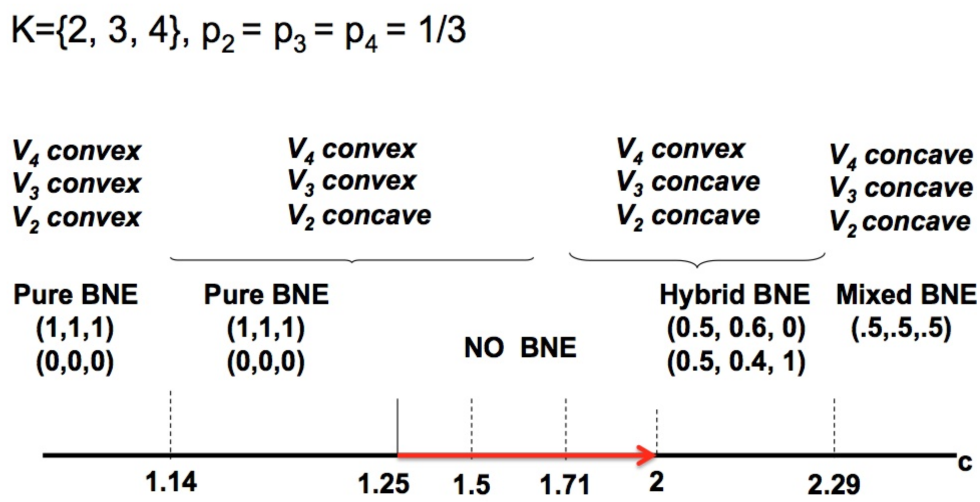

Figura 1: Example 2. Different kind of BNE and as a function of the congestion parameter, concavity/convexity of the payoff functions and strategic complementarity and substitution.

may fail to exist. This situation may happens when some best response function are increasing while some other ones decreasing, thus possibly missing the equilibrium. Notice that all of our results are under the assumption on independence between neighbors's degree and individual degree, and we are not sure about what affiliation between players' degrees may add to the analysis.

The following example with three type of players illustrates the non-existence problem as well as the above Propositions.

Example 2. *Let g be a network where individuals are unformly distributed in degrees 2,3 and 4. Then $K = \{2, 3, 4\}$ and $p_k = 1/3$, for all $k = 2, 3, 4$.*

Here, functions v_k specify to (the terms not depending on x_k are not included):

$v_2[x_2, (x_3, x_4)] = [4/9-(7/18)c]x_2^2+[2/3-(4/9)c]x_2x_3+[8/9-(4/9)c]x_2x_4+[(5/6)c-11/9]x_2$

$v_3[x_3, (x_2, x_4)] = [2/3-(7/18)c]x_3^2+[4/9-(4/9)c]x_3x_2+[8/9-(4/9)c]x_3x_4+[(5/6)c-4/3]x_3$

$v_4[x_4, (x_2, x_3)] = [8/9-(7/18)c]x_4^2+[4/9-(4/9)c]x_4x_2+[2/3-(4/9)c]x_4x_3+[(5/6)c-13/9]x_4$

Notice that v_2 is convex for $c \leq 8/7 \approx 1.143$, v_3 is convex for $c \leq 12/7 \approx 1.714$ and v_4 is convex for $c \leq 16/7 \approx 2.286$.[6]

The equilibrium configuration is displayed in Figure 3

For $c < 1.143$, we find two BNE in homogeneous pure strategies, where conditions of Proposition 1 are fulfilled, i.e., all the v_k functions are convex with $\sum_{j\neq k} \alpha_{kj}>0$, for $k \in K$, i.e., actions of the players are strategic complements in the aggregate.

[6]Also notice that for $c < 1$, $\alpha_{kj}> 0$, for $k, j = 2, 3, 4$ and $k \neq j$. For $1 \leq c \leq 1.25$, $\sum_{j\neq 4} \alpha_{4j}\geq 0$ (although $\alpha_{42}<0$), for $1 \leq c \leq 1.5$, $\sum_{j\neq 3} \alpha_{3j}\geq 0$ (although $\alpha_{32}< 0$) and for $1 \leq c \leq 1.75$, $\sum_{j\neq 2} \alpha_{2j}\geq 0$ (although $\alpha_{23}< 0$ for $c \geq 1.5$).

For $c > 1.143$ function v_2 turns to be concave while v_3 and v_4 still remain convex. Then in the interval $1.143 < c \leq 1.25$, $\sum_{j \neq k} \alpha_{kj} \geq 0$, for $k \in K$ (with $\beta_2 < 0$), the conditions of Proposition 1 still are satisfied and we find that the unique equilibria are the two equilibria in homogeneous pure strategies.

As above, in the interval $1.25 < c < 1.714$, both functions v_3 and v_4 are still convex with v_2 concave but the BNE's fail to exist. In particular, in the interval $1.25 < c < 1.50$, where $\sum_{j \neq 2} \alpha_{2j} > 0$, $\sum_{j \neq 3} \alpha_{3j} > 0$, but $\sum_{j \neq 4} \alpha_{4j} < 0$. The reason is the failure of type $k = 4$ to consider the other players' actions as strategic complements; while in the interval $1.50 < c < 1.71$, $\sum_{j \neq 2} \alpha_{2j} > 0$ but $\sum_{j \neq 3} \alpha_{3j} < 0$ and $\sum_{j \neq 4} \alpha_{4j} < 0$, thus players' actions are neither strategic complement nor strategic substitutes in the aggregate. The same argument applies to the interval $1.71 < c < 2$, where both v_2 and v_3 are now concave while v_4 remains convex, with $\sum_{j \neq 2} \alpha_{2j} > 0$ but $\sum_{j \neq 3} \alpha_{3j} < 0$ and $\sum_{j \neq 4} \alpha_{4j} < 0$.

For $c \geq 2$, the equilibrium is restored because in the interval $2 \leq c < 2.286$, v_4 is convex and v_2 and v_3 are concave with $\alpha_{kj} < 0$ for all $k, j \in K$, $k \neq j$, so that actions are all strategic substitutes. For instance, for $c = 2.2$, the two hybrid BNE are $\{x_2^* = 0.51, x_3^* = 0.60, x_4^* = 0\}$ and $\{x_2^* = 0.49, x_3^* = 0.40, x_4^* = 1\}$. Notice, that in this case the conditions of Proposition 4(2) are satisfied.

Finally, for $c > 2.286$, all the v_k functions, are concave. The unique BNE is the uniformly mixed strategy profile $\{x_2^* = 0.5, x_3^* = 0.5, x_4^* = 0.5\}$ (see Proposition 3).

4. Network characterization of Bayesian Nash Equilibria

As we have seen BNE profiles depend on both the degree distribution of the network and the congestion parameter. Intuition suggests that if congestion is high enough, the unique BNE profile is the one in which players' choice of actions are as heterogeneous as possible. Only for low congestion will the players choose the same action.

However, intuition has to be poolished since *the network global topology plays an important role in the equilibrium characterization*. To see that notice that the degree distribution defines two important network features such as *hub* and *peripheral players*. Although each individual's value function depends on both the average action profile followed by all the individuals of the network and the average profile of their neighbors, their relative weight will depend on the individual's number of connections. Thus, the network average action profile is particularly important for peripherals because, by definition, their number of neighbors is very small and therefore their choices will mostly be driven by the network global topology. On the contrary, the hubs choices will mainly depend on their neighbors average action profile, i.e. on the network's local properties. Therefore both local and global properties determine the equilibrium choices. The proportion of hubs and peripherals depends on the weight of the tails of the degree distribution. As a consequence the equilibrium characterization is driven by the proportion of hubs and peripherals which, in turn, is given by both the asymmetry of the degree distribution (its skewness) and the weight of its tails (its kurtosis).

The next results characterize the BNE profiles of Section 3: *mixed strategy profiles, homogeneous pure strategy profiles and hybrid equilibrium profiles*, in terms of the network topology. First, Proposition 3 can be expressed as,

Proposition 5. *Let g be a network with a degree distribution of $\mathbf{p} = \{p_k\}_{k \in K}$. The unique mixed strategy BNE is the uniformly mixed strategy profile. Moreover, the uniformly mixed strategy will be a BNE if and only if the network relative degree $\frac{k}{d}$ is bounded from above: $\frac{k}{d} < \frac{c}{4}(2 + p_k)$ for all $k \in K$.*

Proof. See the Appendix.

The first statement in Proposition 5 says that if $\mathbf{x}$ is a mixed strategy BNE, i.e. $x_k \in (0, 1)$ for all $k \in K$, then it cannot be otherwise unless $x_k = 1/2$. The second one gives a necessary and sufficient condition, concavity of the v_k functions, for all $k \in K$, in order that the uniformly mixed strategy profile is a BNE. Given a congestion function parameter, concavity imposes an upper bound on the maximum relative degree of the considered network. This means that the degree distribution can be left skewed but not right skewed and thus, the right tail of the degree distribution tells us whether a uniformly mixed strategy BNE exists. Therefore, *uniformly mixed BNE profiles are very difficult to achieve in networks with players with high relative degree (hubs) unless the congestion cost parameter is very high. This is so even when there is only one such a player. In fact, when the maximum degree k is not bounded, then mixed strategy equilibrium will not exist.* (Some examples are given in Section 6.)

Let us give some intuition. Hubs always have an incentive to coordinate their actions and select the same pure strategy. Suppose that all players with different degrees from k choose the uniformly mixed strategy. If k is big enough, then $k\widetilde{p}_k$ will be also big enough, which means that the k-degree players will have many k-degree neighbors. Thus, if these players chose a pure strategy, say action e, their increase in the gross payoff would be high and would compensate for the increase in their congestion cost. More precisely, if all players chose the uniformly mixed strategy their utility function, according to (8), would be $\frac{1}{2} - \frac{c}{8}$. Now, if the k-degree players changed their strategy and all of them choose action e, their value function would be $\frac{1}{2}(1 + \widetilde{p}_k) - \frac{c}{8}(1 + p_k)^2$. Notice that now both the gross payoff and the congestion cost are higher than before. This deviation is not profitable as long as $\frac{1}{2} - \frac{c}{8} > \frac{1}{2}(1 + \widetilde{p}_k) - \frac{c}{8}(1 + p_k)^2$, which implies that $\frac{c}{4}p_k + \frac{c}{8}p_k^2 > \frac{1}{2}\widetilde{p}_k$. Recalling that $\widetilde{p}_k = kp_k/d$, this inequality is equivalent to the condition of the above Proposition.

An alternative interpretation would arise if the inequality of Proposition 5 was re-written in terms of a threshold on the congestion cost. Thus, the uniformly mixed strategy BNE will exist if and only if $c > max_{k \in K} \frac{4k}{d(2+p_k)}$. In other words, if K was unbounded, then the uniformly mixed strategy would not be a BNE. Hence, uniformly mixed profiles will appear whenever there are no hubs in the network or congestion is very high for them.

Next we characterize the existence conditions of homogeneous pure strategy BNE's. The following Proposition translates Proposition 1 to conditions on the network degree distribution.

Proposition 6. *Let g be a network with a degree distribution of $\mathbf{p} = \{p_k\}_{k\in K}$. An homogeneous pure strategy will be a BNE if one of the following conditions is satisfied, either*

$$\frac{c}{4}(2+p_k) \leq \frac{k}{d} \leq \frac{1}{p_k}(1-\frac{c}{2}+\frac{c}{2}p_k^2) \tag{17}$$

for all value of k, or (17) is satisfied for some values of k and

$$\frac{1}{p_k}(cp_k - 1 + \frac{c}{2}) \leq \frac{k}{d} < \frac{c}{4}(2+p_k) \tag{18}$$

for the other values of k.

The left hand side inequality in condition (17) implies that v_k is convex, and the the right hand side inequality implies that $\sum_{l\neq k}\alpha_{kl} \geq 0$. With respect to condition (18), the left hand side inequality implies that v_k is concave and the right hand side that $\beta_k \leq 0$ (and $\sum_{l\neq k}\alpha_{kl} \geq 0$). Let us interpret the above results in terms of hubs, peripherals and the congestion cost parameter. The convexity of the v_k functions is trivially satisfied for low values of c and the right hand side inequality of (17) to constraints on the probability distribution: namely, this inequality is satisfied whenever $p_k \leq \frac{1-c/2}{k/d}$ for all $k > d$.

Therefore, given a low congestion cost parameter, the existence of an homogeneous pure strategy BNE imposes an upper bound on the weight of the right-tail of the degree distribution $\mathbf{p}$, i.e. in the accumulative probability of hubs. Notice that there is now no upper bound in the maximum relative degree (as was necessary for the uniformly mixed strategy BNE, see Proposition 5), but instead *hubs have to be quite unlikely.* The reason is the following: let us assume that all players, except the k-degree ones, choose action m. If the expected number of k- degree neighbors of a k-degree players, $k\widetilde{p}_k$, is high enough, then it will be very likely that these players will be linked to other k-degree players. If k-degree players choose the other pure action, e, then their reduction on the gross payoff will be low and may be offset by the reduction in their congestion cost. To avoid this deviation, $k\widetilde{p}_k$ must be low enough, and therefore p_k also must be low enough.

When c takes intermediate values, relatively high values of k/d have to satisfy condition (17) and relatively low ones condition (18). The left hand side inequality of (18) can be expressed as $p_k \leq \frac{1-c/2}{c-k/d}$, a bound on the left-tail of the degree distribution $\mathbf{p}$. Thus, the proportion of peripherals has to be low in order an homogeneous pure strategy BNE to exist. The reason is the following: peripherals only suffer congestion costs and receive hardly any gross payoffs. Thus, if all players choose action m, then peripherals will have incentives to switch to action e because their gross payoff will not change but their congestion cost will be drastically reduced. Therefore, for moderate values of the congestion cost, condition (17) implies that hubs have to be unlikely and condition (18) says that peripherals have to be so as well.

Under high values of the congestion parameter homogeneous pure strategy profiles cannot be BNE. If c is high enough, then neither the left hand side inequality in (17) nor the left hand side inequality in (18) will be satisfied by any value of the relative degree.

To sum up, homogeneous pure strategy BNE will exist if hubs are quite unlikely whenever the congestion parameter is low enough; if the congestion parameter takes intermediate values, then the existence of homogeneous pure strategy equilibrium profiles will be ensured as long as both hubs and peripherals remain unlikely; finally, there will not be an homogeneous pure strategy BNE if the congestion parameter is high. Notice that since the choice of peripherals is mainly driven by the global network topology, while that of hubs is determined instead by the local network topology, both local and global externalities play a role in the existence of homogeneous pure strategy equilibrium choices

Finally, notice that the conditions for hybrid equilibria of Proposition 4(1) are as those of Proposition 1 but for concave function with interior solutions, and those of Proposition 4(2) refers to the players' action being pairwise strategic substitutes. Therefore Proposition 4 can be expressed as,

Proposition 7. *Let g be a network with a degree distribution of $\mathbf{p} = \{p_k\}_{k \in K}$. Hybrid BNE will exist if one of the following conditions is satisfied, either*

$$\frac{c}{4}(2 + p_k) \leq \frac{k}{d} \leq \frac{1}{p_k}(1 - \frac{c}{2} + \frac{c}{2}p_k^2) \tag{19}$$

for all degree k with a convex v_k, and (20) is satisfied for those k with a concave v_k

$$\frac{k}{d} \leq Min\{\frac{1}{p_k}(1 - \frac{c}{2} + \frac{c}{2}p_k^2),\ \frac{c}{4}(2 + p_k)\} \tag{20}$$

or

$$\frac{c}{4}(2 + p_k) \leq \frac{k}{d} \leq c \ \ and \ \ \frac{l}{d} \leq \frac{c}{4}(2 + p_k) \tag{21}$$

for the k degree with a convex v_k, and for all the remaining l degrees with concave v_l.

Here again, as in Proposition 6, condition (19) imposes an upper bound on the weight of the right-tail of the degree distribution $\mathbf{p}$: for low congestion cost, hubs have to be quite unlikely. When c takes intermediate values, relatively high values of k/d have to verify condition (19) and relatively low ones condition (20). The latter says that either k is bounded or hubs are very unlikely. Notice that theres is not bound on peripherals. Under the second set of conditions (21), both peripheral and hubs have to be bounded in order a hybrid equilibrium to exist. Finally, for high values of c, there is not any hybrid equilibria.

5. Comparative statics for the two-type player case

Propositions 5-7 relate the different kind of the Bayesian Nash equilibria with the proportion of hubs in the network and with the type probability distribution, for given congestion costs. A general comparative static analysis is very complex to be undertaken. However, for a population with two-types of players something more definite can be said. We display here the graphical simulations for two examples with the corresponding calculations in the Appendix.

Consider first how the equilibrium configuration changes under the presence of hubs and peripheral, keeping the type probability distribution constant. Suppose that $K = \{3,\, 4\}$ with $p_3 = p_4 = 0.5$ as opposed to $K = \{3,\, 40\}$ with $p_3 = p_{40} = 0.5$. Thus, the first example illustrates the situation where there are neither peripherals nor hubs in the population, while in the second example the average degree is equal to 21.5, a half of the population (the 3-degree players) consists of peripherals and the other half (the 40-degree players) is composed by hubs. Figure 5 displays the BNE's as a function of the congestion parameter. The left hand side of figure 5) corresponds to the first example while the right hand side displays the second one. Graphs are scaled to enable comparison of the range of existence of the different BNE's.[7]

When $K = \{3,\, 4\}$ -nether peripherals no hubs-, the two homogeneous pure strategy BNE (either $(0, 0)$ or $(1, 1)$) exist up to a congestion cost of 1.14. Then, there is a congestion parameter interval -from $c = 1.14$ to $c = 1.52$- for which there is not any equilibrium. For values of the parameter from 1.52 to 1.83 we find two hybrid BNE with the 4-degree player choosing a pure strategy (either 1 or 0) and the 3-degree player the corresponding mixed one. Finally, for high congestion cost, the uniformly mixed BNE is the unique equilibria.

When $K = \{3,\, 40\}$, both peripherals and hubs are present. This implies with respect to the previous case: i) a decrease on the range of existence of homogeneous pure strategy equilibria (the upper bound of c is now 0.19); ii) a decrease on the existence of uniformly mixed BNE (the lower bound of c is now 2.98); iii) an increment in the range of non-existence, being now from $c = 0.19$ to $c = 2.48$; iv) and an approach to 1/2 of the mixed strategy of the 3-degree players in the hybrid equilibria. Hence, the presence of both peripherals and hubs reduces the range of congestion cost where there exists either pure or mixed equilibrium profiles (see the Appendix).

However, as above mentioned, the degree probability distribution also plays a key role on both the equilibrium configuration and its existence. To illustrate this point, consider again the two type player networks with either $K = \{3, 4\}$ or $K = \{3, 40\}$ and the three probability distributions, $p_3 = 0.25$, $p_3 = 0.50$ and $p_3 = 0.75$. Figure 5 displays the change in the BNE's configurations when both the probability distribution and the ratio between degrees change.

[7] The reader can check that when $K = \{3, 4\}$, the players' expected payoff functions are

$$\begin{aligned} v_3(x_3, x_4) &= (c - 10/7)x_3 + (6/7 - 5c/8)x_3^2 + (8/7 - 3c/4)x_3x_4 + \gamma_3 \\ v_4(x_4, x_3) &= (c - 11/7)x_4 + (8/7 - 5c/8)x_4^2 + (6/7 - 3c/4)x_4x_3 + \gamma_4 \end{aligned}$$

while these function are when $K = \{3, 40\}$,

$$\begin{aligned} v_3(x_3, x_{40}) &= (c - 46/43)x_3 + (6/43 - 5c/8)x_3^2 + (80/43 - 3c/4)x_3x_{40} + \gamma_3 \\ v_{40}(x_{40}, x_3) &= (c - 83/43)x_{40} + (80/43 - 5c/8)x_{40}^2 + (6/43 - 3c/4)x_{40}x_3 + \gamma_{40} \end{aligned}$$

$K = \{3, 4\}$, $p_3 = p_4 = 0.5$ $\qquad\qquad$ $K = \{3, 40\}$, $p_3 = p_{40} = 0.5$

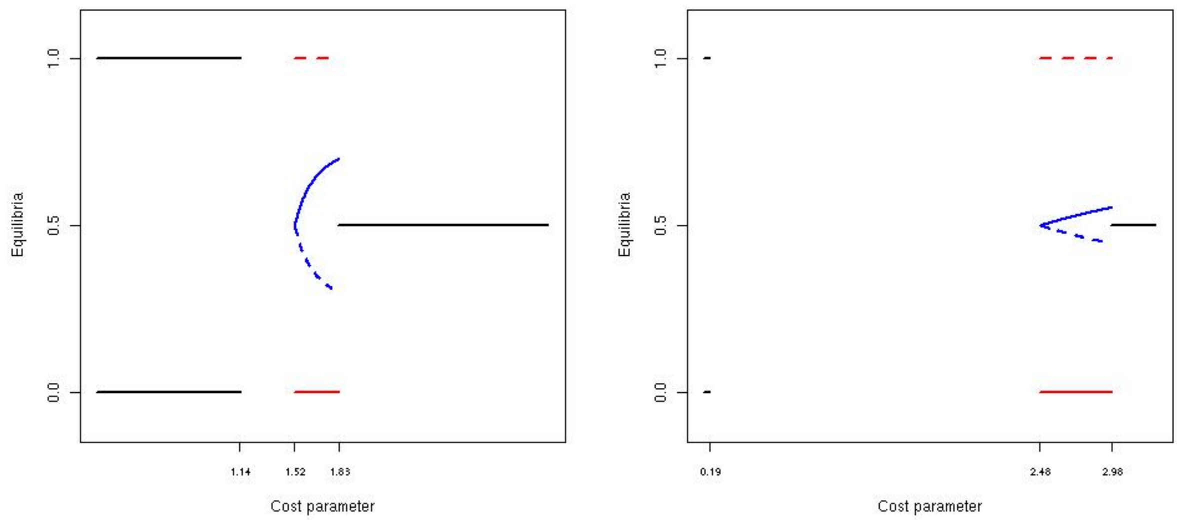

Figura 2: BNEs for two-type players as a function of the congestion parameter, under degree configurations with hubs and without hubs. The solid lines indicate that both players play the same strategy, either homogeneous pure profiles or uniformly mixed profiles; dashed lines together with solid lines mean that each player plays a different strategy (hybrid equilibria). In this case, the dashed lines are one of the possible hybrid equilibrium and the solid lines are the other one.

In the top of the Figure the two type of players have similar degree, $K = \{3, 4\}$ and in the bottom there are peripherals and hubs, $K = \{3, 40\}$. In addition, the left hand side of the two graphs assumes that $p_3 = 0.25$, the middle hand side that $p_3 = 0.50$ and finally, the right hand side that $p_3 = 0.75$, in other words, the probability of hubs decreases as we move to the right.[8]

[8] For $K = \{3, 4\}$, if $p_3 = 0.25$ and $p_4 = 0.75$, the players' expected payoffs are

$$
\begin{aligned}
v_3(x_3, x_4) &= (3c/4 - 6/5)x_3 + (2/5 - 9c/32)x_3^2 + (8/5 - 15c/16)x_3x_4 + \gamma_3 \\
v_4(x_4, x_3) &= (5c/4 - 9/5)x_4 + (8/5 - 33c/32)x_4^2 + (2/5 - 7c/16)x_4x_3 + \gamma_4.
\end{aligned}
$$

If $p_3 = 0.5$ and $p_4 = 0.5$, then see footnote 7.
If $p_3 = 0.75$ and $p_4 = 0.25$, then

$$
\begin{aligned}
v_3(x_3, x_4) &= (5c/4 - 22/13)x_3 + (18/13 - 33c/32)x_3^2 + (8/13 - 7c/16)x_3x_4 + \gamma_3 \\
v_4(x_4, x_3) &= (3c/4 - 17/13)x_4 + (8/13 - 9c/32)x_4^2 + (18/13 - 15c/16)x_4x_3 + \gamma_4.
\end{aligned}
$$

For $K = \{3, 40\}$, if $p_3 = 0.25$ and $p_4 = 0.75$, the players' expected payoffs are

$$
\begin{aligned}
v_3(x_3, x_{40}) &= (3c/4 - 42/41)x_3 + (2/41 - 9c/32)x_3^2 + (80/41 - 15c/16)x_3x_{40} + \gamma_3 \\
v_{40}(x_{40}, x_3) &= (5c/4 - 81/41)x_{40} + (80/41 - 33c/32)x_{40}^2 + (2/41 - 7c/16)x_{40}x_3 + \gamma_{40}.
\end{aligned}
$$

If $p_3 = 0.5$ and $p_4 = 0.5$, then see footnote 7.

Inspection of Figure 3 reveals some facts related with the degree probability distribution. When the probability of hubs decreases three general facts are observed. First, the range of congestion costs where there exists homogeneous pure BNE increases (see Proposition 6). In the three top graphs, this range moves from c in (0, 0.91) to c in (0, 1.34); in the bottom graphs the change is less relevant, being now from c in (0, 0.11) to c in (0, 0.36). Second, the range of congestion costs where there exists mixed BNE decreases (see Proposition 5). Again, in the three top graphs, the lower bound of the congestion cost for which there exists mixed BNE moves from 1.55 to 2.19, and in the bottom graphs moves from 1.89 to 5.80. And third, the range of congestion costs for which there exists hybrid BNE increases: for instance, in the left hand side of the bottom graph there is not any hybrid BNE, in the middle hand side, hybrid BNE exist in the interval of c in (2.48, 2.98), and in the right hand side they exist for c in (3.73, 5.80)(see the Appendix).

However, there is always an interval of the congestion parameter where the BNEs fail to exist. The length of this interval depends on both the degree probability distribution and the ratio between degrees, showing however a not linear behavior. To see this observe that when the probability of the maximum degree players decreases, if players have similar degree, then the length of this interval will also decrease. This is the case displayed in the three top graphs, where the length of the interval of $c's$ precluding the equilibrium existence is reduce from a measure of 0.64 to one of 0.08; while, if players' degree are far apart, then this length will increase as the probability of the maximum degree players decreases. In the three bottom graphs, the length of the interval of $c's$ where the BNEs fail to exists increases from a measure of 1.78 to one of 3.37. Therefore, there is not a monotone linear behavior as the following figure 4 shows.

6. Equilibrium analysis in some common social network distribution

In this section we illustrate the above results on existence of Bayesian Nash equilibria in two common degree probability distributions of social networks.

Empirical analysis of social networks and theoretical models about the dynamic of the networks formation concludes that the most common random networks are the Poisson network and the Scale-free network, where the former has a Poisson degree distribution and the latter a Scale-free degree distribution, also called the power-law degree distribution (Newman [17], Albert and Barabasi [1] and Jackson [14]). Scale-free distributions have *fat right tails*, that is the proportion of nodes with large degrees are higher than it could be expected if the links were formed completely independently as it occurs in Poisson random

---

If $p_3 = 0.75$ and $p_4 = 0.25$, then

$$
\begin{aligned}
v_3(x_3, x_{40}) &= (5c/4 - 58/49)x_3 + (18/49 - 33c/32)x_3^2 + (80/49 - 7c/16)x_3x_{40} + \gamma_3 \\
v_{40}(x_{40}, x_3) &= (3c/4 - 89/49)x_{40} + (80/49 - 9c/32)x_{40}^2 + (18/49 - 15c/16)x_{40}x_3 + \gamma_{40}.
\end{aligned}
$$

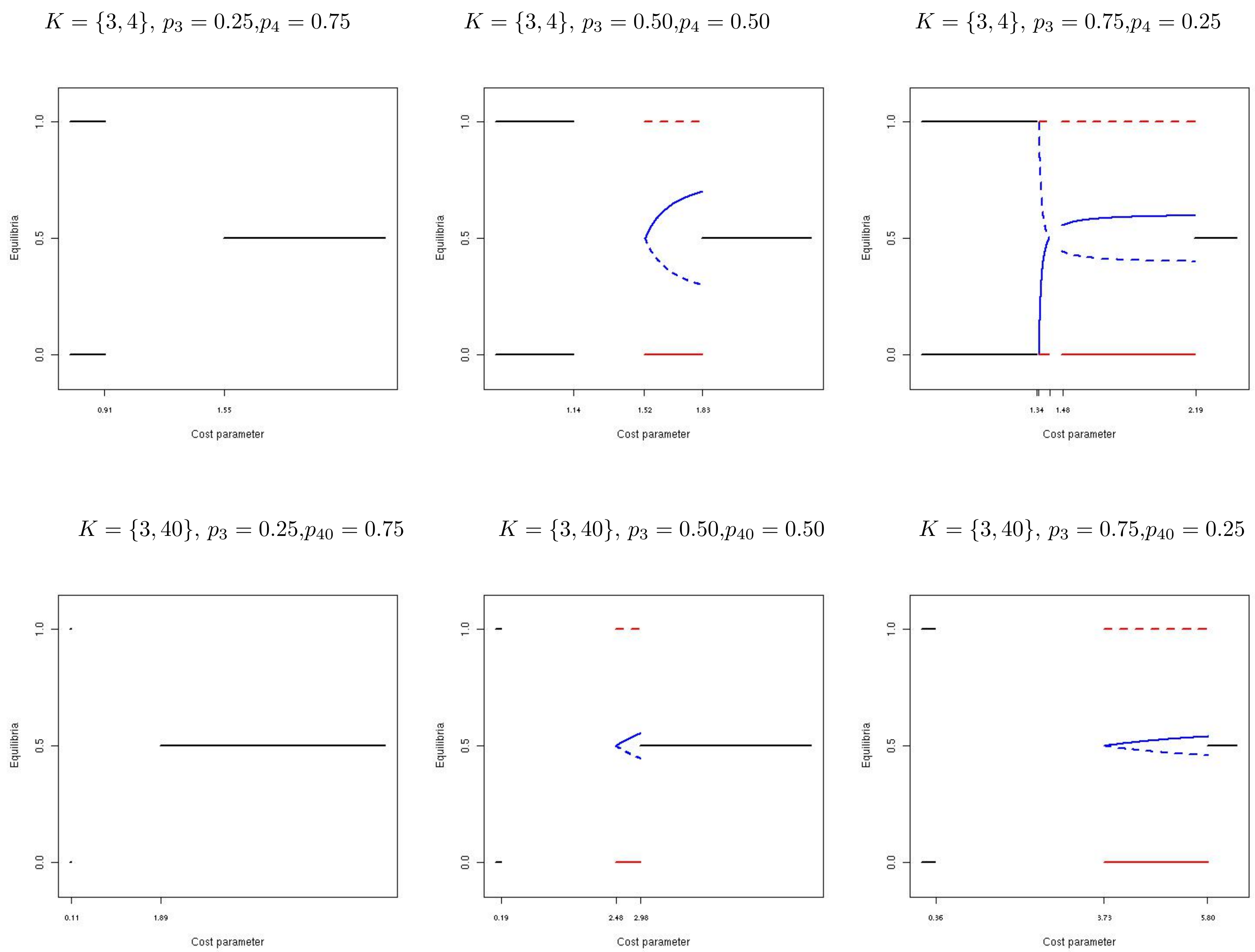

Figura 3: BNEs for two-type players as a function of the congestion parameter, under degree configurations with hubs and without hubs and with different probability distributions.

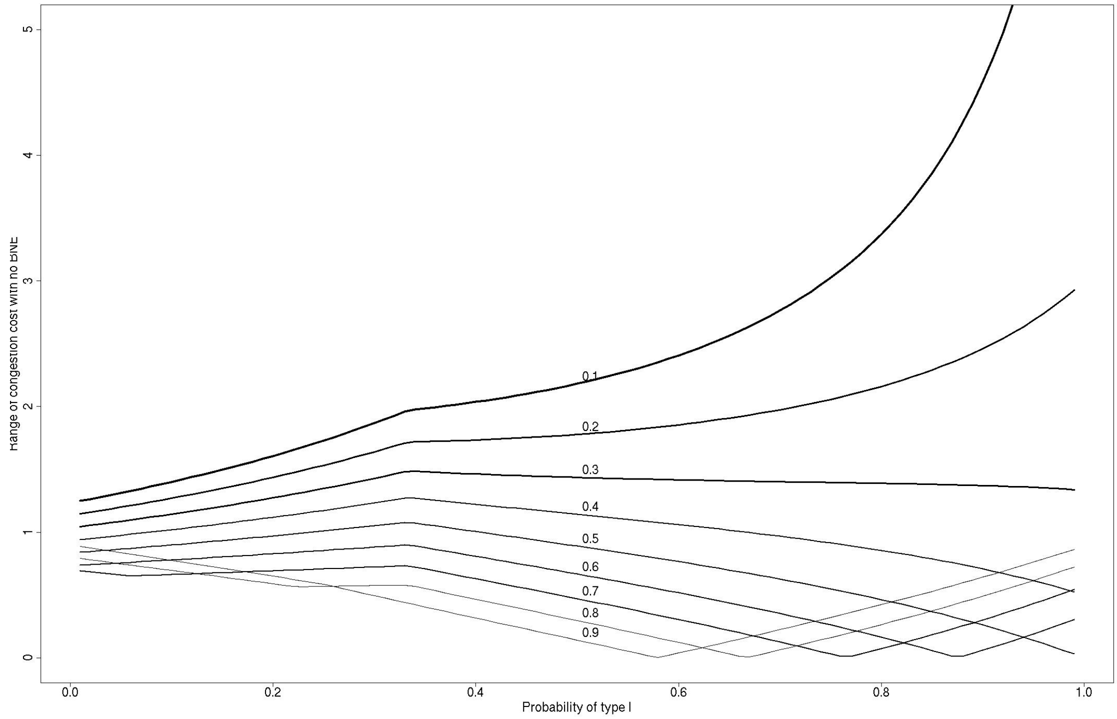

Figura 4: Two type case. The OX exes is the probability of the lower degree type; the OY exes is the range of congestion costs with non-existence of BNE; each line corresponds to a different ratio between the lower degree type and the higher degree type: from top to botton 0.1, 0.2,..., 0.9.

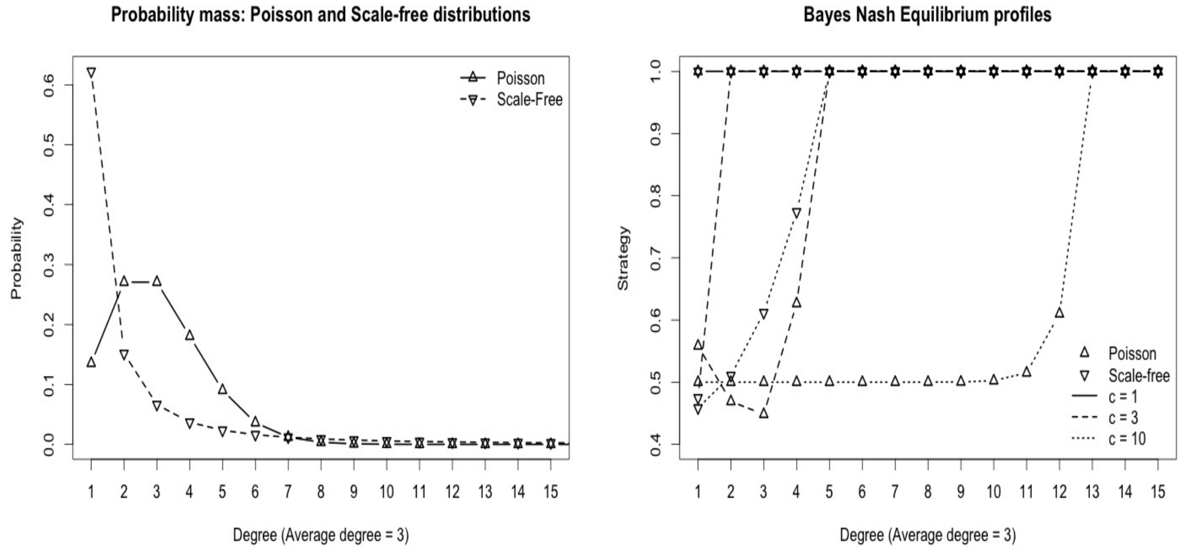

Figura 5: Left: Probability mass of a Poisson distribution (solid lines) and a Scale Free distribution (dotted line) both with an average degree of 3.0. Right: Equilibrium strategies in both distributions as a function of the players' degree and three congestion parameters.

networks (see the left hand side graph of Figure 6). In our terminology, hubs are unlikely in Poisson random networks in comparison with Scale-free random networks, where hubs are very frequent.

The relationship between the congestion parameter and the different kind of BNE under the Poisson network exhibits a complex behavior. For instance, consider a Poisson degree distribution with a given average degree, say3. When the congestion cost is very low, then the conditions in proposition 6 will be satisfied and homogeneous pure strategy profiles will be BNE; there will be a range of values of the congestion parameter (around 2) such that the equilibrium will fail to exist; for higher values of the congestion parameter we will find hybrid BNE: the tail players (either peripherals or hubs) will play the same pure strategy and the players around the average degree of the network will choose a mixed action; for very high congestion parameters a more complex equilibrium profiles arise: peripherals will choose an uniformly mixed action, hubs will select the same pure action and the other players will play any mixed action. Notice that the conditions in Proposition 5 will never be satisfied because there is not an upper bound on the degree of a player and then uniformly mixed strategy profiles will not belong to the set of BNE. This last remark also will hold for the Scale-free distribution and, in general, for any degree distribution over an unbounded domain.

The precise values of the congestion parameter where an equilibrium configuration changes to another one depend on the specific parameter of the Poisson distribution. The right hand side of Figure 6 displays the equilibrium strategies for a Poisson degree distribution with average degree equal to 3 (lines with a triangle pointing-up) and for three congestion parameters, namely 1, 3 and 10 (solid, dashed and dotted lines respectively). For the lowest congestion parameter, $c = 1$, all type of players choose the same pure strategy ($x_k = 1$ for all $k \in K$ in the Figure); when the congestion parameter is higher, $c = 3$, the players with a degree lower than 5 play a mixed strategy and the remaining players choose the same pure strategy ($x_1 = 0.56, x_2 = 0.47, x_3 = 0.45, x_4 = 0.63, x_k = 1$ for $k \geq 5$); for the highest congestion parameter, $c = 10$, the player with a degree lower than 6 play the uniformly mixed strategy, those with a degree between 6 and 12 play a mixed strategy, monotonically increasing on the degree, and finally, the players with a bigger degree play the pure strategy $x_k = 1$, for $k \geq 12$).

Scale-free networks show a simpler relationship between the congestion parameter and the kind of BNE than the Poisson network does. There always exists Bayesian Nash equilibria and these are only of two types. For congestion parameters under a fixed threshold the conditions of proposition 6 are always satisfied and homogeneous pure strategy profiles are BNE; if the congestion parameter is above the threshold only hybrid equilibria exist, with peripherals playing a mixed strategy and the remaining players choosing the same pure strategy.

The right hand side of Figure 6 displays the equilibrium strategies of a Scale-Free degree distribution with average degree equal to 3 (lines with a triangle pointing-down) and for the same congestion parameters than before. For a congestion parameter equal to 1, all type of players choose the same pure strategy; when the congestion parameter is equal to 3, the players with degree 1 play the mixed strategy $x_1 = 0.47$ and the remaining players choose the pure strategy $x_k = 1,\ k > 1$; for $c = 10$, the players with a degree lower than 5 play the a mixed strategy, and those with a bigger degree play the pure strategy $x_k = 1,\ k \geq 5$.

7. Conclusions

This paper analyzes the impact of local and global interaction on individuals' choices. Players are located in a network and interact with each other with perfect knowledge of their own neighborhood and probabilistic knowledge of the complete network topology.

Individuals simultaneously choose their actions from a finite set, which imposes an externality on their neighbors as well as an externality on the complete network, and then obtain an utility. Namely, players obtain utility from sharing their choices with their neighbors (positive local externality) but suffer disutility from sharing that choice with all the members of the network (negative global externality). A variety of economic and social phenomena exhibit these features such as the adoption of cost-reducing innovations, clusters of firms, the choice of time-schedules, etc. The optimal (Bayesian Nash) decision taken by each individual depends on three factors: the spread of their connections in the network (their degree), their knowledge about the network's topology, and the exact nature of the externalities which impact on their utility.

Our main contribution is to show that both local and global network properties play an important role in equilibrium choices. This is so because the network topology defines two important features such as *hubs* (highly connected nodes) and *peripherals* (poorly connected nodes). Although each individual's value function will depend on both the average action profile followed by the network and the average action profile of their neighbors, their relative weight will depend on the individual's number of connections. Thus, the network average action profile is particularly important for peripherals because, by definition, their number of neighbors is very small and therefore their choice will mostly be driven by the network global topology. On the contrary, the hubs or highly connected players' action choices will mainly depend on the average profile of their neighbors' actions, i.e. on the network local properties. Therefore our Bayesian Nash Equilibrium is expressed in terms of the ratio between hubs and peripherals which, in turn, comes from both the asymmetry of the degree probability distribution (its skewness) and the weight of its tails

Bibliography

[1] R. Albert and A. Barabasi, Statistical Mechanics of Complex Networks, American Physics Society's New Journal, 74 (1) (2002) 47-97.

[2] B. Arthur, Inductive Reasoning and Bounded Rationality, American Economic Review, 84 (2) (1994) 406-411.

[3] C. Ballester, A. Calvo-Armengol and Y. Zenou, Who's Who in Networks. Wanted: the Key Player, Econometrica, 74 (5) (2006) 1403-1417.

[4] H. Bernard, P. Killworth and L. Sailer. Informant accuracy in social network data IV, Social Networks, 2 (1980) 191-218.

[5] H. Bernard, P. Killworth and L. Sailer. Informant accuracy in social network data V. Social Science Research, 11 (1982) 30-66.

[6] M. Boguña, R. Pastor-Satorras, A. Díaz-Guilera and A. Arenas, Models of social networks based on social distance attachment, Physical Review E, 70 (2004) 056122.

[7] P. Erdös and A. Rényi, On random graphs, I, Publicationes Mathematicae Debrecen, 6 (1959) 290-297.

[8] P. Erdös and A. Rényi, On the evolution of random graphs, Publ. Math. Inst. Hung. Acad. Sci., 5 (1960) 17-61.

[9] A. Galeotti, S. Goyal, M. O. Jackson, F. Vega-Redondo and L. Yariv, Network Games, Review of Economic Studies, 77 (1) (2010) 218-244.

[10] A. Galeotti and F. Vega-Redondo, Complex Networks and Local Externalities: a strategic approach, International Journal of Economic Theory, 7 (1) (2011) 77-92.

[11] R. Guimerá, L. Danon, A. Díaz-Guilera, F. Giralt and A. Arenas, Self-similar community structure in a network of human interactions, Physical Review E, 68 (2003) 065103.

[12] S. Goyal, Connections: An Introducction to the Economics of Networks, Princeton University Press, New Jersey, 2007.

[13] Y. M. Ioannides, Random Graphs and Social Networks: An Economics Perspective, in preparation, 2006

[14] M. O. Jackson, Social and Economic Networks, Princeton University Press, New Jersey, 2008.

[15] C. Mayer and T. Sinai, Network Effects, Congestion Externalities, and Air Traffic Delays: Or Why not all dalays are evil, The American Economic Review, 93 (4) (2003) 1194-1215.

[16] S. Morris and H. Shin, Global Games: Theory and Applications, Advances in Economics and Econometrics (Proceedings of the Eighth World Congress of the Econometric Society), Cambridge University Press, Cambridge, (2003) 56-114.

[17] M. Newman, The structure and Function of Complex Networks, SIAM Review, 45 2001 167-256.

[18] M. Newman, S. Strogatz and D. Watts, Random Graphs with Arbitrary Degree Distribution and their Applications, Physical Review E, 64 2003 026118.

[19] M. E. J. Newman, Physical Review E, 74 036104 (2006).

[20] A. Sundararajan, Local Network Effects and Complex Network Structure, The B.E. Journal of Theoretical Economics 7 (1) (2008) Article 46.

[21] X. Vives, Nash equilibrium with strategic complementarities, Journal of Mathematical Economics 19 (3) (1990) 305-321.

Appendix

Proof of Proposition 5. Notice that the k-degree players' value function (see Expression 8) is a polynomial of degree 2 on x_k. It can be written as $v_k[x_k, \mathbf{x}] = x_k v_k^m[x_k, \mathbf{x}] + (1 - x_k)v_k^e[x_k, \mathbf{x}]$, where $v_k^m[x_k, \mathbf{x}] = \widetilde{x} - \frac{c}{2}\overline{x}^2$ and $v_k^e[x_k, \mathbf{x}] = 1 - \widetilde{x} - \frac{c}{2}\big(1 - \overline{x}\big)^2$.

Let $\mathbf{x}^*$ be a strategic profile such that $x_k^* \in (0, 1)$ for all $k \in K$. Then $\mathbf{x}^*$ is a BNE profile if and only if both $|\partial v_k[x_k, \mathbf{x}^*]/\partial x_k|_{x_k=x_k^*} = 0$ and $|\partial^2 v_k[x_k, \mathbf{x}^*]/\partial x_k^2|_{x_k=x_k^*} < 0$ are satisfied for all $k \in K$. Computing these partial differentials we have (recall that $\widetilde{p} = kp_k/d$),

$$\begin{aligned}\frac{\partial v_k}{\partial x_k} &= v_k^m - v_k^e + (2x_k - 1)\widetilde{p}_k + cp_k(1 - \overline{x} - x_k),\\ \frac{\partial^2 v_k}{\partial x_k^2} &= 4\widetilde{p}_k - 2cp - cp^2.\end{aligned}$$

The first expression is only equal to zero if $x_k^* = 1/2$ for all $k \in K$, thus as Proposition 5 states the unique mixed BNE profile is the uniformly mixed strategy. The second expression is therefore lower than zero if $k < cd(2 + p_k)/4$. This implies that the uniformly mixed strategy is a BNE if and only if the above inequality holds, which completes the proof. □

Proof of Proposition 4.(2)

Suppose that all the $v_n[x_n, \mathbf{x}]$ functions are concave but one, $v_k[x_k, \mathbf{x}]$, which is convex. The idea of the proof relies on the fact that the system with concave functions has two solutions, parameterized by $x_k \in \{0, 1\}$ (since the $v_n[x_n, \mathbf{x}]$ are concave and the strategy space is compact), which are non-increasing in x_k because $\alpha_{nk} \leq 0$ for all n. Let $\mathbf{x_n^*}(1) = \{x_n^*(1)\}_{n\neq k}$ and $\mathbf{x_n^*}(0) = \{x_n^*(0)\}_{n\neq k}$, be such solutions with $x_n^*(1) \leq x_n^*(0)$ for all $n \in K$ and $n \neq k$. Now, $x_k \in \{0, 1\}$ will maximize v_k, given the other players' choice. Then, $v_k[1, \mathbf{x_n^*}(1)] \geq v_k[0, \mathbf{x_n^*}(1)]$ and $v_k[0, \mathbf{x_n^*}(0)] \geq v_k[1, \mathbf{x_n^*}(0)]$, whenever, $x_n^*(1) \leq 1/2 \leq x_n^*(0)$ for all $n \neq k$ and $\alpha_{kl} \leq 0$, for all $l \neq k$, since k has to satisfy

$$\sum_{l\neq k}\{x_l^*(1) - 1/2\}\alpha_{kl} \geq 0 \tag{22}$$

and

$$-\sum_{l\neq k}\{x_l^*(0) - 1/2\}\alpha_{kl} \geq 0 \tag{23}$$

Therefore, it suffices to show that $x_n^*(1) \leq 1/2 \leq x_n^*(0)$ for all $n \neq k$. Suppose on the contrary that $x_k = 0$ and that $\{x_n^*(0)\}_{n\neq k} < 1/2$, then each n has to satisfy that

$$\sum_{l\neq n, l\neq k}\{x_l^*(0) - 1/2\}\alpha_{nl} \leq \frac{1}{2}\alpha_{nk.}$$

Since $\alpha_{nl} \leq 0$ for all $l \neq n$ and $\alpha_{nk} \leq 0$ for $k \neq n$, and each $x_l^*(0) < 1/2$, the right hand side of the above expression is non-positive but the left hand side is non-negative, which is a contradiction. Hence, when $x_k = 0$, then $\{x_n^*(0)\}_{n\neq k} \geq 1/2$. Similarly, suppose on the contrary that $x_k = 1$ and that $\{x_n^*(1)\}_{n\neq k} > 1/2$, then each n has to satisfy that

$$\sum_{l\neq n, l\neq k} \{x_l^*(1) - 1/2\}\alpha_{nl} \geq -\frac{1}{2}\alpha_{nk.}$$

As above, since $\alpha_{nl} \leq 0$ for all $l \neq n$ and $\alpha_{nk} \leq 0$ for $k \neq n$, and each $x_l^*(1) \geq 1/2$, the right hand side of the above expression is non-negative but the left hand side is non-positive, which is a contradiction. Hence, when $x_k = 1$, then $\{x_n^*(1)\}_{n\neq k} \leq 1/2$.

Calculations and analysis of some relevant bounds for the examples of sections 5.

Let us show here some calculations for the two-type player case. Denote by k and l the two $players'$ degrees with $k > l$, and by $R = \frac{l}{k}$, the ratio between degrees. Since $p_k + p_l = 1$, let $p = p_l$ and recalling the definition of α_{kk} and by some calculations:

$$\alpha_{kk} = 2\widetilde{p}_k - cp_k(\frac{1}{2}p_k + 1) = \frac{2p_k}{p_k + p_l R} - \frac{c}{2}p_k(2 + p_k) = \frac{2(1-p)}{(1-p(1-R))} - \frac{c}{2}(3-p)(1-p).$$

Therefore $v_k[x_k, x]$ is convex whenever:

$$\frac{2(1-p)}{(1-p(1-R))} - \frac{c}{2}(3-p)(1-p) \geq 0 \to \frac{4}{(1-p(1-R))(3-p)} \geq c.$$

Let

$$C_k^+(p, R) = \frac{4}{(1-p(1-R))(3-p)}$$

be the upper bound of c up to which $v_k[x_k, x]$ is convex. Similarly, let

$$C_l^+(p, R) = \frac{4R}{(1-p) + pR)(2+p)}$$

be the corresponding upper bound of c up to which $v_l[x_l, x]$ is convex. In general, unless p is very close to 0 (the condition is $R < \frac{2+p}{3-p}$),

$$C_l^+(p, R) < C_k^+(p, R).$$

With a little algebra it is not difficult to show that:

$$\frac{\partial C_k^+(p, R)}{\partial R} < 0, \ \frac{\partial C_l^+(p, R)}{\partial R} > 0.$$

In words, as k increases (R decreases), the upper bound C_k^+ also increases, making $v[x_k, x]$ convex for more values of c, while C_l^+decreases, and hence $v_l[x_l, x]$ is convex for less values of c. Furthermore, efficient profiles always exist, but we have only found sufficient conditions for the pure homogeneous outcomes to be efficient. Simple examples show us that the efficient profiles exhibit a more complex structure than that of equilibrium outcomes. Hence, it could be interesting to analyze the characterization of efficient profiles more deeply and, in particular, the conditions making mixed equilibrium profiles efficient.

Repeating the algreba for changes of p,

$$\frac{\partial C_k^+(p,R)}{\partial p} > 0, \; \frac{\partial C_l^+(p,R)}{\partial p} \gtreqless 0.$$

Notice that $\frac{\partial C_l^+(p,R)}{\partial p} > 0$, whenever $R < \frac{1+2p}{2+2p}$. Therefore, for any two degrees such that $k > 2l$ as the probability of hubs decreases both upper bounds $C_k^+(p,R)$ and $C_l^+(p,R)$ increase, this meaning that both functions $v_k[x_k, x]$ and $v_l[x_l, x]$ are convex for more values of c. However, when $k < 2l$ (no hubs), $C_l^+(p,R)$ will decrease, making $v_l[x_l, x]$ convex for less values of c.

Changes in the strategic complementarity and substitution: Recalling now the definitions of α_{kl} and $\alpha_{lk}$and by a little algebra, for all $l \neq k$,

$$\alpha_{kl} = 2\widetilde{p}_l - cp_l(p_k+1) = \frac{2pR}{(1-p)+pR} - cp(2-p)$$

therefore, type k of player considers the action of type l as a strategic complement whenever,

$$\frac{2pR}{(1-p)+pR} - cp(2-p) > 0 \rightarrow \frac{2R}{((1-p)+pR)(2-p)} > c.$$

Define

$$C_{kl}^+(p,R) = \frac{2R}{((1-p)+pR)(2-p)},$$

as the the upper bound of c up to which type $l's$ action is a strategic complement of type $k's$ action. Similarly, since

$$\alpha_{lk} = 2\widetilde{p}_k - cp_k(p_l+1) = \frac{2(1-p)}{(1-p)+pR} - c(1-p^2)$$

then define

$$C_{lk}^+(p,R) = \frac{2}{((1-p)+pR)(1+p)},$$

as the the upper bound of c up to which type $k's$ action is a strategic complement of type $l's$ action. Then,

$$\frac{\partial C_{kl}^+(p,R)}{\partial R} > 0, \; \frac{\partial C_{lk}^+(p,R)}{\partial R} < 0.$$

This means that as k increases (R decreases), the upper bound $C_{kl}^+(p,R)$ decreases as well, thus reducing the values of c for which type k of player considers the action of type l as a strategic complement. The opposite results takes place for $C^+_{lk}(p, R)$. Combining this last result on the bounds of $C^+_{kl}(p, R)$ and $C^+_{lk}(p, R)$ as R changes whith the corresponding ones on $C^+_k(p, R)$ and $C^+_l(p, R)$, it can be said that, with p fixed, as k increases the range of values of c for which an uniforms pure strategy equilibrium exist decreases. This is so, because $v[x_k, x]$ is convex for more values of c, but $v_l[x_l, x]$ is convex for less values of c, the range of values of c for which type k of player considers the action of type l as a strategic complement is smaller, but the range of values of c for which type l of player considers the action of type k as a strategic complement is bigger. These opposite effects translate to a lack of players' coordination and thus to a bigger interval of non-existence results (see figure 2).

Next, we analyze the changes in $C^+_{kl}(p, R)$ and $C^+_{lk}(p, R)$ when p changes. Some little algebra shows that

$$\frac{\partial C^+_{kl}(p, R)}{\partial p} > 0, \ \frac{\partial C^+_{lk}(p, R)}{\partial p} \gtrless 0.$$

As above, notice that $\frac{\partial C^+_{lk}(p,R)}{\partial p} > 0$, whenever $R < \frac{2p}{1+2p}$. Therefore, for any two type degrees k and l such that $k > \frac{l(1+2p)}{2p}$ (the case with hubs) as the probability of hubs decreases both upper bounds $C^+_{kl}(p, R)$ and $C^+_{lk}(p, R)$ increase, this meaning that the upper bound on c for which type k (type l) of player considers the action of type l (k) as a strategic complement increases as well. This explains the increase of homogeneous pure strategy equilibrium profiles in the bottom of figure 3 as p increases. When $k < \frac{l(1+2p)}{2p}$ (no hubs), as p increases, $C^+_l(p, R)$ will decrease and type l of player will turn to consider the type $k's$ action as a strategic substitute for lower values of c. However, since $\frac{2p}{1+2p}$ is decreasing in p this effect is higher for k than for l, thus resulting in an increase of homogeneous pure equilibrium and hybrid equilibrium profiles for more values of c as p increases, as shown in the top of figure 3.