International Conference on Game Theory Stony Brook University

Nash Equilibria of the Quantum Prisoner's Dilemma

Marek Szopa

University of Silesia

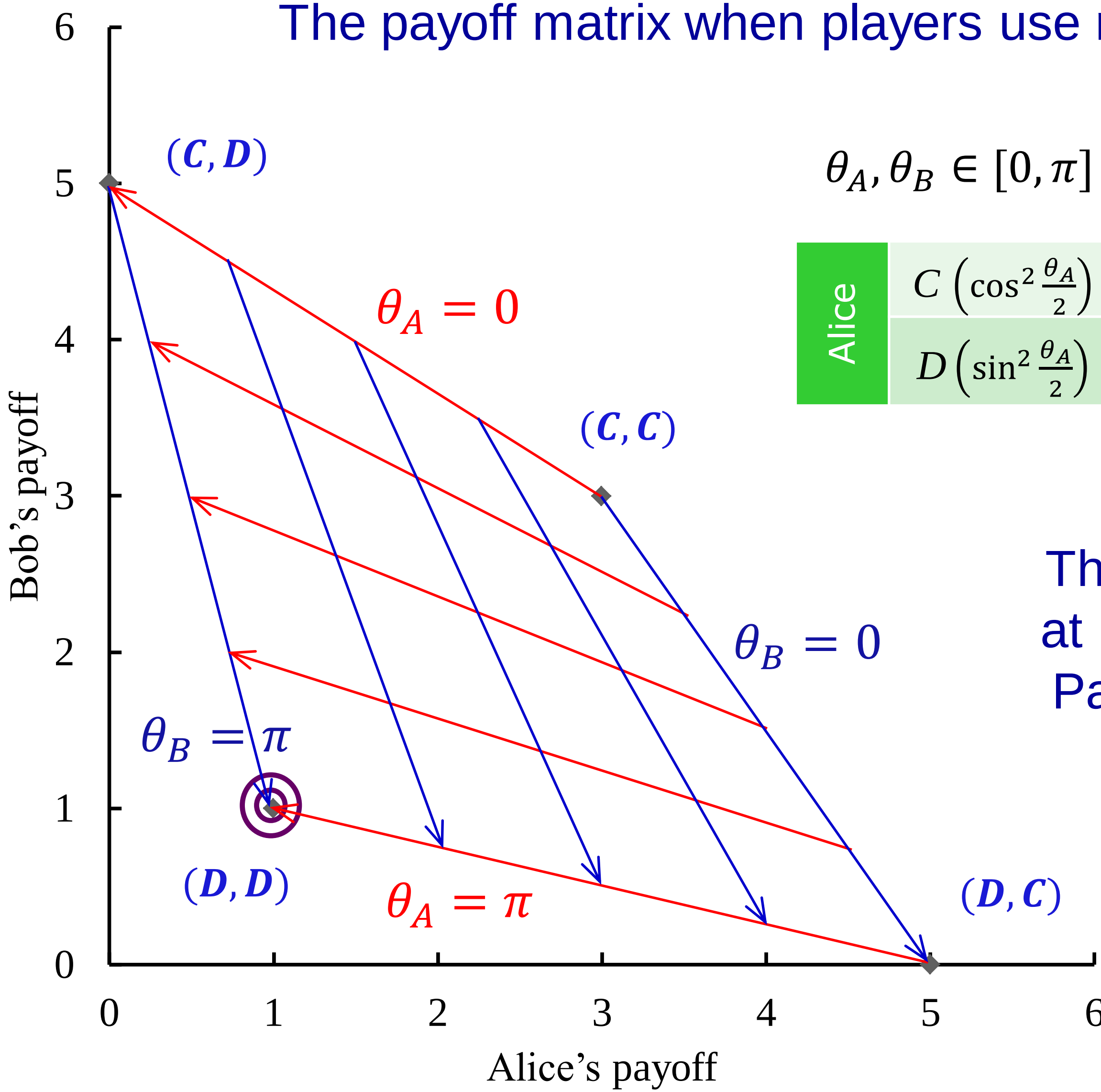
Poland

Classical Prisoner's Dilemma Utility Diagramm

The payoff matrix when players use mixed strategies is:

$\theta_A, \theta_B \in [0, \pi]$

		Bob	
		$C\left(\cos^2 \frac{\theta_B}{2}\right)$	$D\left(\sin^2 \frac{\theta_B}{2}\right)$
Alice	$C\left(\cos^2 \frac{\theta_A}{2}\right)$	(r, r)	(s, t)
	$D\left(\sin^2 \frac{\theta_A}{2}\right)$	(t, s)	(p, p)



The Nash equilibrium at (1,1) is far from the Pareto optimal result

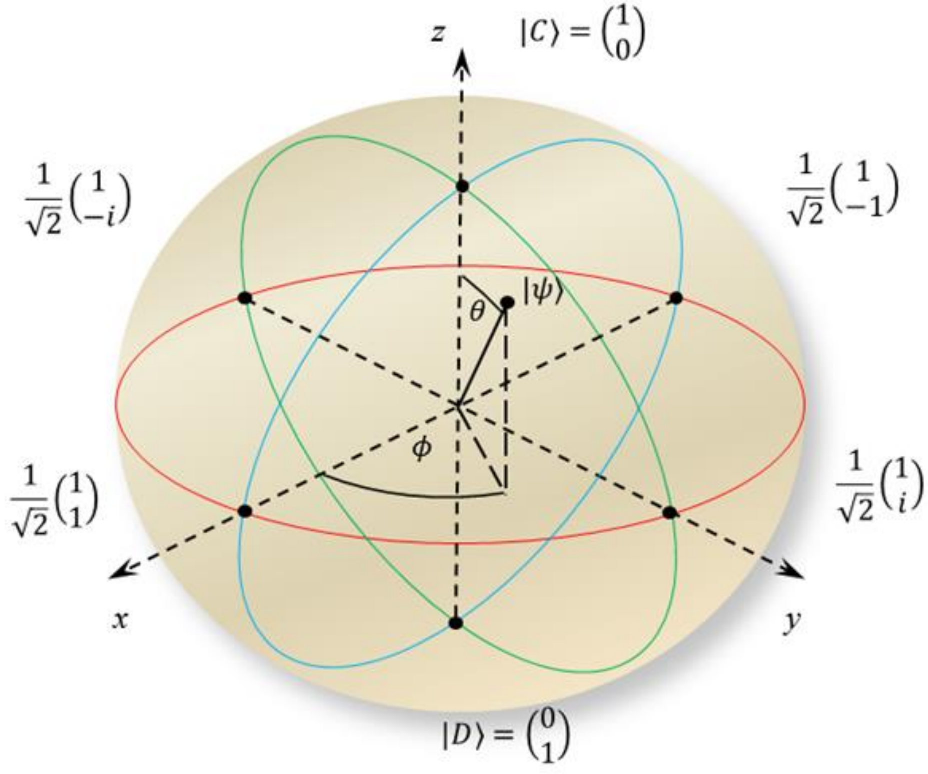
$t = 5, r = 3,$
$p = 1$ and $s = 0$

Quantum game – the Bloch sphere

Mixed strategies resemble superposition of states

$$|\psi\rangle = \cos\frac{\theta}{2}|C\rangle + e^{i\phi}\sin\frac{\theta}{2}|D\rangle, \qquad \theta \in [0,\pi]$$

called qubits $\qquad \phi \in [-\pi,\pi]$



The collapse of the wave function

The qubit is a superposition of two quantum states. It means that unless we take a measurement we cannot tell in which of the two states, the qubit actually is:

$$|\psi\rangle = \cos\frac{\theta}{2}|C\rangle + e^{i\phi}\sin\frac{\theta}{2}|D\rangle \qquad \theta \in [0,\pi] \qquad \phi \in [-\pi,\pi]$$

As a result of measurement, the wave function **collapses** into one of the states:

- $|C\rangle$ can occur with probability of $\cos^2\frac{\theta}{2}$
- $|D\rangle$ can occur with probability of $\sin^2\frac{\theta}{2}$,

Unmeasured possible states simply disappear from sight like losing lottery tickets.

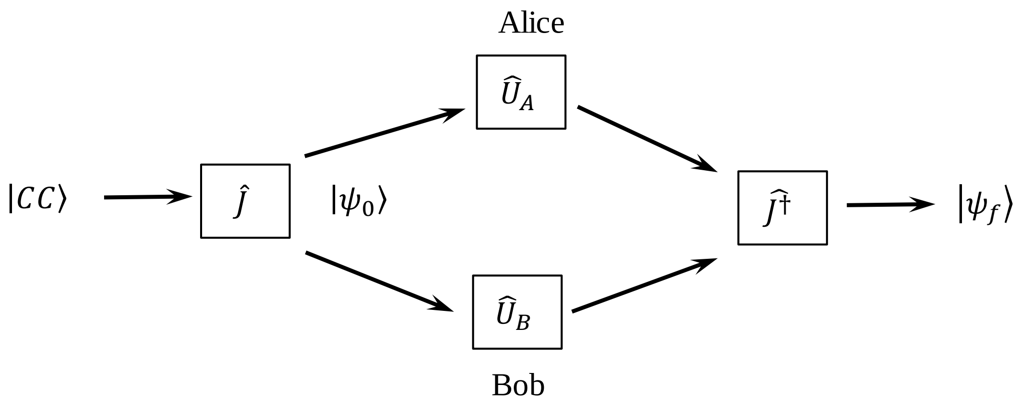
The case of the Schroedinger's cat is the same:

$$\Psi(cat) = \frac{1}{\sqrt{2}}|alive\rangle + \frac{1}{\sqrt{2}}|dead\rangle$$

UNIWERSYTET ŚLĄSKI W KATOWICACH

The quantum game

The quantum game is defined by: $|\psi_f\rangle = \hat{J}^{\dagger}(\hat{U}_A \otimes \hat{U}_B)\hat{J}\,|CC\rangle$



where $|CC\rangle$ is the initial state of two (Alice's and Bob's) qubits,

$\hat{J} = \frac{1}{\sqrt{2}}(\hat{I} + i\sigma_x \otimes \sigma_x)$, is the entangling operator

$\hat{U}_A = \hat{U}(\theta_A, \phi_A, \alpha_A)$ and $\hat{U}_B = \hat{U}(\theta_B, \phi_B, \alpha_B)$, where

$$\hat{U}(\theta, \phi, \alpha) = \begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} & e^{i\alpha}\sin\frac{\theta}{2} \\ -e^{-i\alpha}\sin\frac{\theta}{2} & e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix},$$

are Alice's and Bob's quantum strategies.

$\hat{J}^{\dagger}$ is a disentangling operator and $|\psi_f\rangle$ is the final state defining payoffs.

Quantum game – result

The final state of the game is an entangled state

$$|\psi_f\rangle = p_{CC}\,|CC\rangle + p_{CD}\,|CD\rangle + p_{DC}\,|DC\rangle + p_{DD}\,|DD\rangle$$

where $|p_{CC}|^2, \dots, |p_{DD}|^2$ are probabilities:

$$p_{CC} = \cos\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\cos(\phi_A+\phi_B) - \sin\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\sin(\alpha_A+\alpha_B),$$
$$p_{CD} = \sin\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\cos(\alpha_A-\phi_B) - \cos\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\sin(\phi_A-\alpha_B),$$
$$p_{DC} = \sin\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\sin(\alpha_A-\phi_B) + \cos\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\cos(\phi_A-\alpha_B),$$
$$p_{DD} = \cos\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\sin(\phi_A+\phi_B) + \sin\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\cos(\alpha_A+\alpha_B),$$

that, after the measurement the final state $|\psi_f\rangle$ will collapse to one of four states. The expected values of payoffs are:

$$\$_A = r\,|p_{CC}|^2 + s\,|p_{CD}|^2 + t\,|p_{DC}|^2 + p\,|p_{DD}|^2$$

$$\$_B = r\,|p_{CC}|^2 + t\,|p_{CD}|^2 + s\,|p_{DC}|^2 + p\,|p_{DD}|^2$$

Examples of the game results

If e.g. $\theta_A = \theta_B = 0$ and $\phi_A + \phi_B = {}^{\pi}/_{4}$ then

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}|CC\rangle + \frac{1}{\sqrt{2}}|DD\rangle$$

		Bob	
		C	D
Alice	C	(r,r)	(s,t)
	D	(t,s)	(p,p)

i.e. regardless of the random nature of the final state both payers will play the same way.

If e.g. $\theta_A = {}^{\pi}/_{2}, \theta_B = 0$ and $\alpha_A - \phi_B = {}^{\pi}/_{4}$ then

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}|CD\rangle + \frac{1}{\sqrt{2}}|DC\rangle$$

		Bob	
		C	D
Alice	C	(r,r)	(s,t)
	D	(t,s)	(p,p)

it's obvious that players will play the opposite.

This type of entangled result of the game is characteristic of quantum games and is not present in classical games.

Similar entangled states are in the famous EPR paradox.

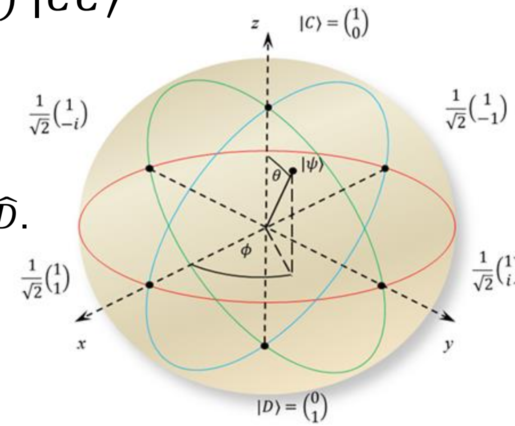
Quantum PD in classical limit

If the strategies of players do not contain complex phases i.e. $\widetilde{U}_A = \widehat{U}(\theta_A, 0, 0\,)$ and $\widetilde{U}_B = \widehat{U}(\theta_B, 0, 0)$, then

$$|\psi_f\rangle = \hat{J}^{\dagger}(\widehat{U}_A \otimes \widehat{U}_B)\hat{J}\,|CC\rangle = (\widetilde{U}_A \otimes \widetilde{U}_B)\,|CC\rangle$$

where

$$\widehat{U}(\theta, 0{,}0) = \begin{pmatrix} e^{-i0}\cos\frac{\theta}{2} & e^{i0}\sin\frac{\theta}{2} \\ -e^{-i0}\sin\frac{\theta}{2} & e^{i0}\cos\frac{\theta}{2} \end{pmatrix} = \widetilde{U}(\theta) = \cos\frac{\theta}{2}\hat{C} + \sin\frac{\theta}{2}\widehat{D}.$$



In this case the Alice's (Bob's) payoff is

$$\$_{A(B)} = r\ \cos^2\frac{\theta_A}{2}\cos^2\frac{\theta_B}{2} + s(t)\cos^2\frac{\theta_A}{2}\sin^2\frac{\theta_B}{2} + t(s)\sin^2\frac{\theta_A}{2}\cos^2\frac{\theta_B}{2} + p\ \sin^2\frac{\theta_A}{2}\sin^2\frac{\theta_B}{2}$$

- identical with the classical.

		Bob	
		$C\left(\cos^2\frac{\theta_B}{2}\right)$	$D\left(\sin^2\frac{\theta_B}{2}\right)$
Alice	$C\left(\cos^2\frac{\theta_A}{2}\right)$	(r,r)	(s,t)
	$D\left(\sin^2\frac{\theta_A}{2}\right)$	(t,s)	(p,p)

$\theta_B = 0$ $\theta_B = \pi$

Quantum entanglement is not possible!

Quantum PD strategies

Let $\hat{A} = \hat{U}(\theta_A, \phi_A, \alpha_A)$ be an arbitrary Alice's quantum strategy and denote by $\widehat{A'} = \hat{U}\left(\theta_A, \phi_A - \frac{\pi}{2}, \alpha_A - \frac{\pi}{2}\right)$ her another strategy.

Let $\hat{B}$ and $\widehat{B'}$ are Bob's strategies (depend also on $\theta_A, \phi_A, \alpha_A$):

$$\hat{B} = \hat{U}\left(\theta_A + \pi, \alpha_A, \phi_A - \frac{\pi}{2}\right), \ \widehat{B'} = \hat{U}\left(\theta_A + \pi, \alpha_A - \frac{\pi}{2}, \phi_A - \pi\right)$$

The quantum game gives the following results:

		Bob	
		$\hat{B}$	$\widehat{B'}$
Alice	$\hat{A}$	(s,t)	(t,s)
	$\widehat{A'}$	(t,s)	(s,t)

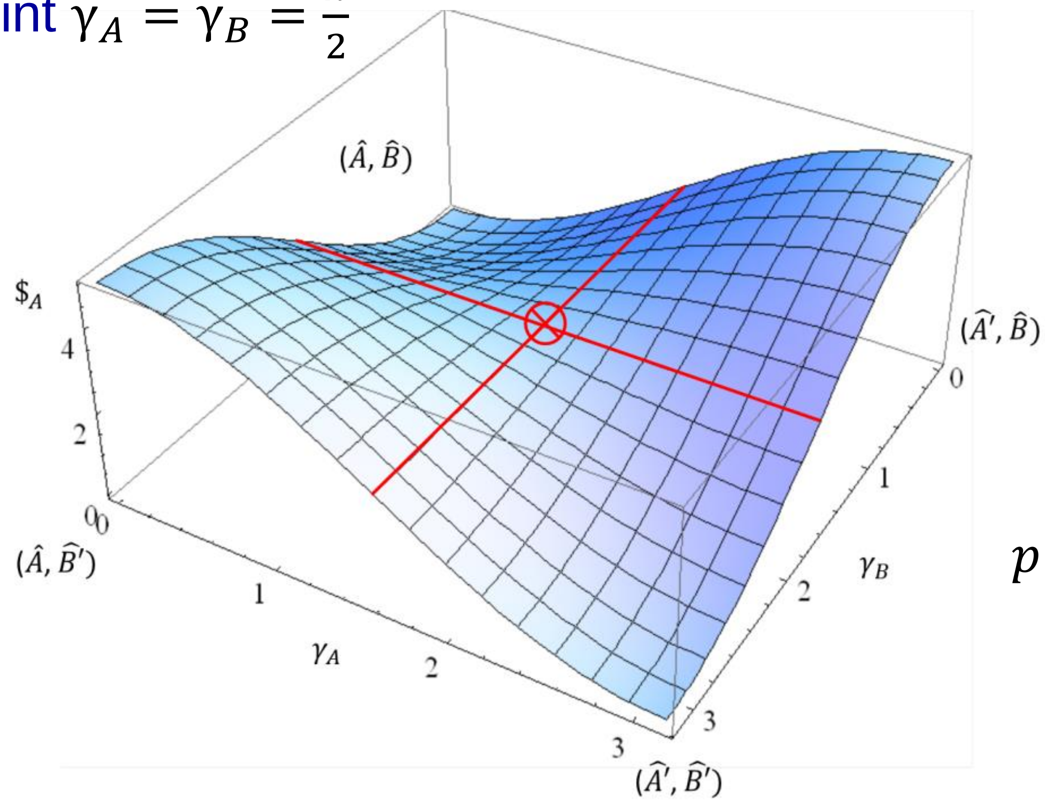
And is equivalent to the zero sum matching pennies game.

Nash equilibria of quantum PD

Now, if the opponents are playing mixed strategies:

$\cos^2\frac{\gamma_A}{2}\hat{A} + \sin^2\frac{\gamma_A}{2}\widehat{A}'$ and $\cos^2\frac{\gamma_B}{2}\hat{B} + \sin^2\frac{\gamma_B}{2}\widehat{B}'$, where $\gamma_A, \gamma_B \in [0, \pi]$

Then this game has the only one (for given $\theta_A, \phi_A, \alpha_A$) NE in the saddle point $\gamma_A = \gamma_B = \frac{\pi}{2}$

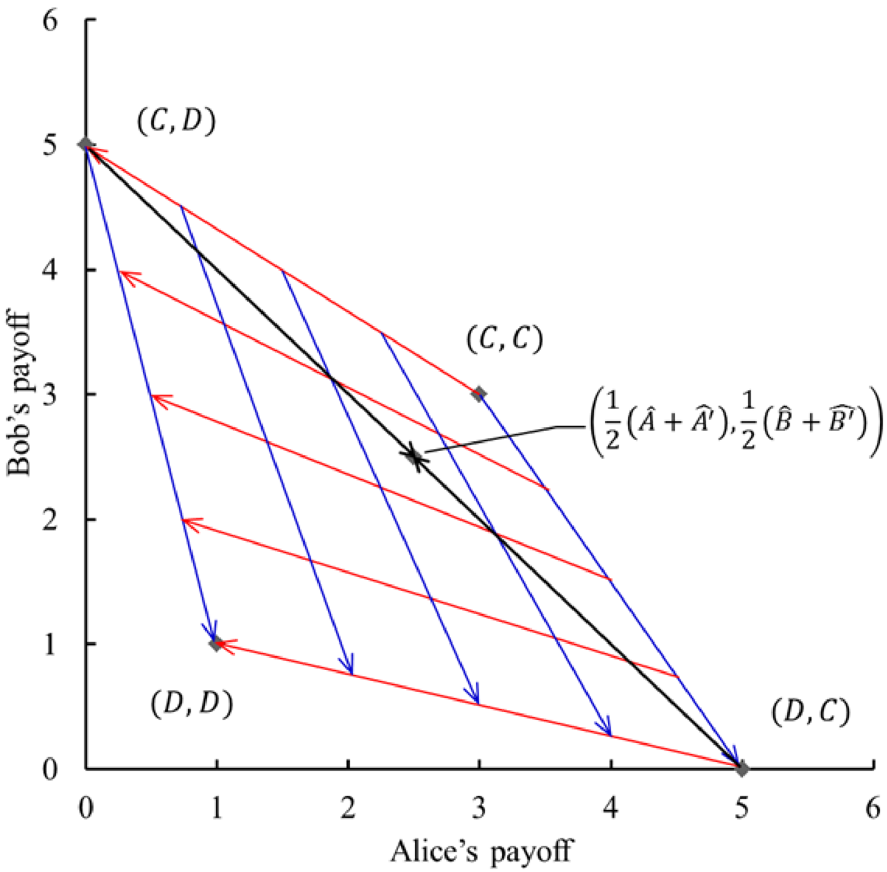


$t = 5, r = 3,$
$p = 1$ and $s = 0$

Utility diagram for the quantum PD

If the players apply their strategies $\gamma_A = \gamma_B = \frac{\pi}{2}$

then the expectation value of their payoffs is $\frac{s+t}{2} = 2{,}5$



$t = 5, r = 3,$
$p = 1$ and $s = 0$

2013-07-03

Summary

1. Nash equilibrium of the quantum PD is more favorable than the NE of the classical PD.
2. Achieving favorable NE for quantum games is made possible by the phenomenon of entanglement of the final states.
3. Quantum games can be played using a quantum computer.
4. Classical simulations of quantum games are possible.
5. Some market processes resemble NE of the quantum PD.

Questions:

1. Is the collapse of the wave function observed in classical games?
2. Does quantum entanglement have its counterpart in the classical games?

UNIWERSYTET ŚLĄSKI W KATOWICACH