COINCIDENCE OF TWO SOLUTIONS TO NASH'S BARGAINING PROBLEM

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Abstract. John Nash in [2] gave an elegant solution to the bargaining problem using his somewhat controversial IIA axiom. Ehud Kalai and Meir Smorodinsky in [1] gave a different solution replacing the IIA condition by their own Monotonicity condition. While the two solutions do not coincide in general they obviously do so when the problem is symmetric. Are there other cases where the two solutions coincide? Indeed there are and we give a complete characterization.

1. Formulation of the Problem

A two-person bargaining situation can be represented as a pair (a, S), where S is a subset of $\mathbb{R}^2$ and $a = (a_1, a_2)$ a point in S called the base point. Every point $x = (x_1, x_2) \in S$ represents the utilities, to agents 1 and 2 respectively, of engaging in a possible bargain with the base point representing the utility of not engaging in any bargain at all. Every pair (a, S) must satisfy the following properties:

(Bargaining Incentive)	$\exists x \in S$ such that $x_i > a_i$ for $i = 1, 2$.
(Convexity)	S is convex.
(Compactness)	S is compact.
(Loss Aversion)	$\forall x \in S, a_i \leqslant x_i,$ for $i = 1, 2$.

Bargaining Incentive reflects the natural assumption that both agents would only engage in a bargaining situation in which they each stand to gain. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are the utilities associated to two potential bargains, any probability combination of the two bargains would itself be a bargain, yielding utilities $px+(1-p)y$ for some $p \in [0, 1]$. Compactness in this context means that S is closed and bounded, which is reasonable from realistic concerns. Loss Aversion reflects the idea that no player would agree to a bargain in which he is worse off than he would be not bargaining at all. Loss Aversion is not assumed by Nash, though all of his results still hold with this included, as it only limits the space of possible bargaining pairs.

Let U be the set of all bargaining pairs (a, S) with the above properties. A solution to the bargaining problem is a function $f : U \to \mathbb{R}^2$ such that $f(a, S) = (f_1(a, S), f_2(a, S)) \in S$. In general, f is meant to give the "fair" agreement for the two agents.

2. Normalized Bargaining Pairs

We will only consider solutions which are invariant under affine transformations of utility, i.e. transformations of the form $A(x_1, x_2) = (c_1x_1 + d_1, c_2x_2 + d_2)$ for $c_1, c_2, d_1, d_2 \in \mathbb{R}$.[1] This means first that regardless the original base point a, we may consider without loss of generality the bargaining pairs $(0, S)$ (by translating the original base point). Further, let

$$b_1(S) = \sup \left\{ x_1 \in \mathbb{R} \;\middle|\; \text{there exists } x_2 \in \mathbb{R}, (x_1, x_2) \in S \right\}$$
$$b_2(S) = \sup \left\{ x_2 \in \mathbb{R} \;\middle|\; \text{there exists } x_1 \in \mathbb{R}, (x_1, x_2) \in S \right\}$$

$$b(S) = (b_1(S), b_2(S))$$

As before, we may consider only bargaining pairs $(0, S)$ such that $b(S) = (1, 1)$ (by scaling, we can still keep the base point $(0, 0)$).

A bargaining pair is called *normalized* if $a = (0, 0)$ and $b(S) = (1, 1)$. In the following sections, we assume (a, S) is normalized unless otherwise stated.

3. Nash Solution η

Nash gives three axioms regarding properties that a solution should satisfy, along with philosophical justifications.

N1 Pareto Optimality: For every bargaining pair (a, S), if $x \in S$ such that $\exists y \in S$ with $y_1 > x_1$ and $y_2 > x_2$, then $x \neq f(a, S)$.

N2 Symmetry: If S is symmetric with respect to the line $x_1 = x_2$, then $f(0, S)$ lies on the line $x_1 = x_2$.

N3 Independence of Irrelevant Alternatives (IIA): If (a, S) and (a, T) are bargaining pairs such that $S \subset T$ and $f(a, T) \in S$, then $f(a, S) = f(a, T)$.

N1 reflects the assumption that each agent is interested in maximizing his own utility. We assume that the players are equally skilled at negotiating, thus N2. N3 has been criticized, but according to Nash, if two rational individuals would agree that $f(T)$ is fair if T were the set of possible bargains, then they should be willing to agree to the same deal with a smaller set of bargains available to them.

Nash proved that there is one and only one function, η, given below, which satisfies these three axioms.

$$\eta(a, S) = (\eta_1, \eta_2) \text{ where } (\eta_1, \eta_2) \in S \text{ and } \eta_1 \cdot \eta_2 \geqslant x_1 \cdot x_2 \text{ for any } x \in S$$

[1] Nash assumes this as a property of the utilities defined for each player. Kalai and Smorodinsky give this as an axiom for a solution. Significant philosophical objections have been brought up regarding this assumption, but for the purposes that follow, we may ignore these issues.

4. Kalai-Smorodinsky Solution μ

Kalai and Smorodinsky give a different set of axioms that a solution must satisfy, motivated by issues raised regarding N3. Before proceeding we introduce some additional notation.

$$\text{Let } g_S(x) = \sup\left\{y \in \mathbb{R} \;\middle|\; x \leqslant x' \text{ and } (x', y) \in S\right\} \text{ (defined for } x \leqslant b_1)$$

Intuitively, $g_S(x)$ is the greatest utility the agent 2 can get if agent 1 gets at least x. A similar function can be defined for agent 1, but with symmetry, this is not necessary.

KS1 Pareto Optimality: For every bargaining pair (a, S), if $x \in S$ such that $\exists y \in S$ with $y_1 > x_1$ and $y_2 > x_2$, then $x \neq f(a, S)$.

KS2 Symmetry: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T((x_1, x_2)) = (x_2, x_1)$. For every bargaining pair (a, S), $f(T(a), T(S)) = T(f(a, S))$.

KS3 Invariance with Respect to Affine Transformations of Utility:
If $A : \mathbb{R}^2 \to \mathbb{R}^2$, $A(x_1, x_2) = (c_1x_1 + d_1, c_2x_2 + d_2)$ for constants c_1, d_1, c_2, and d_2, then $f(A(a), A(s)) = A(f(a, S))$.

KS4 Monotonicity: If (a, S) and (a, T) are bargaining pairs such that $b_1(S) = b_1(T)$ and $g_S \leqslant g_T$, then $f_2(a, S) \leqslant f_2(a, T)$.

The Pareto axiom remains unchanged. The purpose of the Symmetry axiom is the same, though it is formulated differently. This statement implies the Nash's, though both solutions satisfy both versions of the axiom. KS4 reflects the idea that if for any demand agent 1 can make, the maximum possible utility of agent 2 increases, then the utility for agent 2 under the solution should not decrease. KS3 reflects an assumption about the nature of the utility functions which define S, namely that they are determined up to changes in scale.

Kalai and Smorodinsky also give a unique function μ which satisfies this new set of axioms.

$$\mu(a, S) = (\mu_1, \mu_2) \text{ is the maximal point in } S \text{ on the line through } a \text{ and } b(S)$$

5. Coincidence of η and μ

Kalai and Smorodinsky showed by example that η does not satisfy KS4, so in general $\eta \neq \mu$. However, it is clear from the Symmetry axioms for both solutions that the two will always coincide when S is symmetric about the line $x_1 = x_2$. It is easy to build an example for which S is not symmetric, but the two solutions are the same.[2]

Example. *Let* S' *be the convex hull of the points* $(0, 0), (1, 0), (0, 1), (.9, .9), (.5, 1)$ *and* $(1, .5)$. *It is easy to check that* $\eta(0, S') = \mu(0, S') = (.9, .9)$

[2]Acknowledgment is due to Rann Smorodinsky for giving an example of such a bargaining pair, different from the example included.

Let S be the convex hull of $(0,0),(1,0),(0,1),(.9,.9)$ and $(.5,1)$ which is clearly not symmetric. $S \subset S'$ both of which contain $\eta(0,S')$, so $\eta(0,S)=\eta(0,S')$. Since $b(S)=b(S')=(1,1)$ and the line segment between $(0,0)$ and $(.9,.9)$ is contained in S, $\eta(0,S)=\eta(0,S')=\mu(0,S)$.

It turns out that there is a simple geometric characterization of the properties of (a,S) for which $\eta(a,S)=\mu(a,S)$.[3]

Theorem. *Let $(0,S)$ be a normalized bargaining pair.*

$$\eta(0,S)=\mu(0,S) \text{ if and only if for every } (x_1,x_2)\in S, x_1+x_2 \leqslant \mu_1+\mu_2.$$

Proof. Let $\mu(a,S)=(\mu_1,\mu_2), T=\left\{(x_1,x_2\in\mathbb{R}^2\middle|x_1,x_2\in[0,1] \text{ and } x_1+x_2\leqslant\mu_1+\mu_2\right\}$. Suppose $S\subset\left\{(x_1,x_2)\in\mathbb{R}^2\middle|x_1+x_2\leqslant\mu_1+\mu_2\right\}$. Clearly $S\subseteq T$, T is symmetric about the line $x_1=x_2$, and $(0,T)$ is a bargaining pair. By N2, $\eta(0,T)$ lies on the line $x_1=x_2$. Since $x_1\cdot x_2$ is strictly increasing on this line in the first quadrant, $\eta(0,T)$ must be the maximal point in S on the line. Since $(0,T)$ is normalized, this point must be $\mu(0,T)=\mu(0,S)$. By N3, since $\eta(0,T)=\mu(0,S)\in S, \eta(0,S)=\mu(0,S)$.

Suppose there is a point $x\in S$ not in T. Without loss of generality let $x=(\mu_1+c,\mu_2+c-d)$, for some appropriate choice of $c,d>0$.[4] Since $(0,S)$ is normalized, $\mu_1=\mu_2$. Since S is convex, the line segment between μ and x is contained in S. Let $0<\varepsilon<\frac{d}{c}\cdot\mu_1$. Then the point $(\mu_1+\varepsilon,\mu_1-\frac{c-d}{c}\varepsilon)\in S$ for ε sufficiently small, as it is easy to check that this point lies on the line segment between x and μ.

$$\begin{aligned}
(\mu_1+\varepsilon)(\mu_1-\frac{c-d}{c}\varepsilon) &= \mu_1^2-\mu_1\varepsilon+\frac{d}{c}\mu_1\varepsilon+\mu_1\varepsilon-\varepsilon^2+\frac{d}{c}\varepsilon^2\\
&= \mu_1^2+\frac{d}{c}\mu_1\varepsilon-\varepsilon^2+\frac{d}{c}\varepsilon^2\\
\text{(by the choice of }\varepsilon) \quad &> \mu_1^2+\varepsilon^2-\varepsilon^2+\frac{d}{c}\varepsilon^2\\
&= \mu_1^2+\frac{d}{c}\varepsilon^2\\
&> \mu_1^2
\end{aligned}$$

This means that there is a point in S, namely, x, for which $x_1x_2>\mu_1\mu_2$, therefore $\eta(0,S)\neq\mu(0,S)$.

□

Geometrically, the two solutions are equal precisely if S lies below the line tangent to $x_1x_2=\mu_1\mu_2$ at the point (μ_1,μ_2).

[3]This question was brought up by Rohit Parikh of the CUNY Graduate Center, who was instrumental in the completion of this work.

[4]We consider only the case where the point lies to the right of μ_1. Using KS1, we can use this case to handle all other cases.

Corollary. *Let (a, S) be a bargaining pair (not necessarily normalized).*

$$\eta(a, S) = \mu(a, S) \text{ if and only if for every } (x_1, x_2) \in S, \mu_2 x_1 + \mu_1 x_2 \leqslant 2\mu_1\mu_2.$$

This follows from the invariance of the solutions under affine transformations of utility. Geometrically, we interpret the result the same as for the case for normalized pairs. This means that the two solutions are equal precisely if S is contained in the half-plane below the line tangent to $x_1 x_2 = \mu_1 \mu_2$ at μ, which is precisely the line which bounds the half-plane described by the given inequality.

References

[1] Kalai, Ehud; Smorodinsky, Meir, "Other Solutions to Nash's Bargaining Problem," *Econometrica*, Vol. 43, No. 3 (May, 1975), 513-518

[2] Nash, John F., "The Bargaining Problem," *Econometrica*, Vol. 18 No. 2 (April, 1950), 155-162