Multimarket Contact under Demand Fluctuations: A Limit Result*

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Abstract

The present paper studies repeated oligopoly where the firms compete with price in multiple markets. The markets are subject to independent, stochastic fluctuations in demands. The literature points out that while the demand fluctuations generally hinder collusion, the multimarket contact sometimes facilitates it. We show that on an intermediate range of discount factors where only partial collusion is possible under a single market, the difference between the profit under full collusion and the maximum equilibrium profit converges to zero, if the number of markets goes to infinity. Thus the collusion-deterrence effects of fluctuated demands completely vanish in the limit.

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1 Introduction

The present paper studies a model of repeated Bertrand oligopoly, highlighting two typical features colluding firms often confront. First, they simultaneously interact in two or more markets (*multimarket contact*). Second, the demands in those markets are subject to random shocks (*demand fluctuations*). For instance, large nationwide firms often compete in many local markets simultaneously, and each local market is affected with idiosyncratic demand shocks. Another example is conglomerates competing over several industries, each of which has its own demand shocks.

In a framework of infinitely repeated games, existing results have clarified whether each of the two features facilitates or hinders formation of cartels. First, Bernheim and Whinston [1] point out that the multimarket contact never hinders collusion and sometimes facilitates it. In contrast, demand fluctuations generally hinder collusion, as Rotemberg and Saloner [4] show. The main purpose of this paper is to examine how these two conflicting factors interact and affect the firms' ability to collude.

In this paper, we set up a model of infinitely repeated games, which represents oligopoly with identical firms simultaneously engaging in Bertrand price competition over M *ex ante* identical markets every period. A key assumption is that the payoffs of each Bertrand game are stochastic, being i.i.d. across the markets and over time. The payoffs of each period get known only at the beginning of that period. Thus the firms only know the current stage payoffs, without knowing the stage payoffs of any future period. This formulation of payoff fluctuations follows Rotemberg and Saloner [4]. In this model, we examine the symmetric subgame perfect equilibrium which is *second-best* in the sense that no other symmetric subgame perfect equilibrium attains a greater profit.

Our main results are summarized as follows. Fix the probability distribution of payoffs in the stage game. Then two threshold discount factors exist, $\overline{\delta}$ and $\underline{\delta}$ with $\overline{\delta} > \underline{\delta}$, such that (i) if $\delta < \underline{\delta}$, the second-best equilibrium is repeated play of a static equilibrium regardless of the number of markets, and (ii) if $\delta \geq \overline{\delta}$, the second-best equilibrium attains full collusion under any M. Next, fix $\delta \in (\underline{\delta}, \overline{\delta})$. Then the difference between the profit under full collusion and the second-best equilibrium profit converges to zero, if the number of markets goes to infinity. Hence on this intermediate range of discount factors, the collusion-deterrence effects of fluctuated demands completely vanish in the limit.[1] Another interpretation is that if the firms compete in a large number of markets, their ability to maintain collusion is not much affected by demand fluctuations.

We are not the first to study the effect of multimarket contact under demand fluctuations. Bernheim and Whinston [1] have already studied the case of two markets, and show that the multimarket contact in general increases the per-market profit in comparison with the case $M = 1$. In contrast, we consider an arbitrary number of markets and verify that any possible profit loss due to demand fluctuations goes to zero when the number of markets goes to infinity.

One important assumption in our setup is perfect monitoring; the players can directly observe their past actions. This assumption considerably simplifies analysis, enabling us to explicitly derive the second-best equilibrium payoff for any set of parameters. A more

[1] The result does not entirely deny the collusion-deterrence effects of demand fluctuations in the following two senses. First, if $\underline{\delta} \leq \delta < \overline{\delta}$, full collusion cannot be sustained under any number of markets. Second, the limit result does not hold at $\underline{\delta}$.

realistic assumption would be that the players only receive a noisy signal of their actions (imperfect monitoring). The effect of multimarket contact under imperfect monitoring, especially its role in promoting cooperation and/or collusion, has been investigated by Matsushima [3] and Kobayashi and Ohta [2].[2] [2] derives the second-best equilibrium payoff for any number of markets, assuming that the players are sufficiently patient. We rather consider the case where the players are not so patient. [3] deals with the case of heavy discounting but examines the profit *per market* in the limit case. We instead consider any number of markets, and verify that the difference between the *total* equilibrium profit and the profit under full collusion converges to zero if M goes to infinity.

The effects of multimarket contact when the number of markets is not that large are also examined. We show that for any $\delta \in (\underline{\delta}, \overline{\delta})$, the second-best equilibrium *per-market* profit under M markets is no greater than that under $M+1$ markets, and they are equal only in rare cases to be explained below. That is, on this range of intermediate discount factors, adding one more market almost always improves the per-market profit under the most collusive equilibrium. The case where the per-market profit does not improve is rare in the sense that it never occurs either if $M = 1$ or if two or more markets are added.

Our methodology is worth mentioning. While our main objective is the oligopoly with multimarket contact under demand fluctuations, we rather formulate an abstract model which includes the above oligopoly as a special case. Then we characterize the second-best equilibrium payoff. Since the model is applicable to other contexts such as first-price auctions and moral hazard in teams, which do not necessarily involve simultaneous play of multiple games, we believe the characterization result is interesting in itself.

The rest of this paper is organized as follows. Section 2 introduces the model. Section 3 characterizes the second-best equilibrium and its payoff. Section 4 applies the result in the previous section and studies effects of multimarket contact under demand fluctuations.

2 Model

Two players play a given normal-form game every period.[3] Each player has an identical set of stage actions, denoted by X. Their stage payoff depends not only on the action pair selected in the period, denoted by $(x_1, x_2) \in X \times X$, but also on the *state* of that period. The set of possible states has $M+1$ elements, where M is a natural number. We call them state 0, state 1, ..., state M. $u_i(x_1, x_2, k)$ denotes player i's stage payoff of the action pair (x_1, x_2) under state k. We assume symmetry, so that for any x_1, x_2 and k, $u_1(x_1, x_2, k) = u_2(x_2, x_1, k)$. For $x \in X$ and k, we define $U(x, k) = u_i(x, x, k)$.[4]

We make the following assumptions, which capture some features of Bertrand competition. The stage game may as well be called games with *proportional temptations*.

Assumption 1 *(i) The state of each period follows a common probability distribution, and it is independent over time. For any given period, the state of that period is k with probability $p_k \in (0, 1)$.*

(ii) For any k, $\Delta_k \equiv \max_{x \in X} U(x, k)$ exists and satisfies $\Delta_k > 0$.

[2] In [2] and [3], the stage game is a prisoners' dilemma. Since our stage game is quite similar to their games, the main difference among those models is attributed to the players' monitoring ability.

[3] An extension to the case of three or more players is straightforward.

[4] Due to symmetry, $U(x, k)$ does not depend on choice of i.

(iii) There exists $K > 1$ such that for any k and any $u \in [0, \Delta_k]$, $x \in X$ exists such that

$$U(x,k) = u, \quad \sup_y u_1(y,x,k) = Ku. \tag{1}$$

Moreover, for any k and any $u \in [0, \Delta_k]$, if $x' \in X$ satisfies $U(x',k) = u$, then $\sup_y u_1(y,x',k) \geq Ku$.

(iv) For each k, $\min_{x\in X} \sup_{y\in X} u_1(y,x,k)$ exists and equals zero. Moreover, the normal form game whose payoff function is given by $u_i(x_1,x_2,k)$ for each player i has a unique Nash equilibrium payoff pair.

(v) We have $\Delta_0 \leq \Delta_1 \leq \cdots \leq \Delta_M$, and $\Delta_0 < \Delta_M$.

Assumption 1(i) states that the states are i.i.d. over time. Assumption 1(ii) guarantees existence of a maximum symmetric action pair payoff under any state, which is the value of full collusion under that state. Together with Assumption 1(iv), the maximum is greater than each player's minmax value given the state.[5]

Assumption 1(iii) is the assumption of proportional temptations. It first states that given a state k, any payoff between the maximum symmetric action pair payoff and the minmax value is attained by some symmetric action pair (x,x). It also states that for each player, x is not optimal against x unless $U(x,k) = 0$, and each player can obtain either exactly or approximately K times of $U(x,k)$.[6] Note that the coefficient K is independent of k. For k and $u \in [0,\Delta_k]$, let $x_k(u)$ be an element of X satisfying (1). Assumption 1(iii) also states that any other symmetric action pair (x',x') with a payoff $u \in [0,\Delta_k]$ under state k gives each player no smaller temptation than $x_k(u)$; we have $\sup_y u_1(y,x',k) \geq Ku$.

Evaluating (1) at $u = 0$, we see that $\big(x_k(0), x_k(0)\big)$ is a Nash equilibrium of the game whose payoff function is $u_i(x_1,x_2,k)$ for each player i, and the equilibrium payoff pair is $(0,0)$. Assumption 1(iv) states that it is the only Nash equilibrium payoff pair of that game, and that it minmaxes both players.[7] Since we will exclude randomized actions, the minmax value is defined by pure actions. Finally, Assumption 1(v) states that the states are ordered so that the values of full collusion are nondecreasing, and that the values are not constant.

Let us denote this stage game by G. G has a unique Nash equilibrium payoff pair $(0,0)$, and it is attained by the following simple equilibrium; if the state is k, the players play $\big(x_k(0), x_k(0)\big)$.

We provide three examples of games satisfying Assumption 1(ii)–(v). Thus if we additionally assume that the probabilities of the states satisfy Assumption 1(i), those stage games satisfy Assumption 1.

Example 1 (Bertrand price competition) Let $X = [0,\overline{p}]$ with $\overline{p} > 0$. Fix k, and let

$$u_1(x_1,x_2,k) = \begin{cases} x_1 D(x_1;k) & \text{if } x_1 < x_2, \\ \frac{1}{2} x_1 D(x_1;k) & \text{if } x_1 = x_2, \\ 0 & \text{if } x_1 > x_2, \end{cases}$$

[5] Precisely speaking, the value is the minimum of suprema, but we abuse terminology and call it the minmax value.

[6] (1) is stated in terms of player 1, but the counterpart for player 2 also holds by symmetry.

[7] This is consistent with multiple Nash equilibria, because more than one x may satisfy (1) at $u = 0$.

where $D(p;k)$ is continuous and nonincreasing in p. We also assume $D(0;k) > D(\overline{p};k) = 0$. u_2 is derived from u_1 by symmetry. This is a standard model of Bertrand duopoly, where the costs are assumed to be zero for simplicity.

We have

$$U(x,k) = \frac{1}{2}xD(x;k).$$

By assumption, $\Delta_k = \max_{x \in X} U(x,k)$ exists, and $\Delta_k > 0$. Assumption 1(ii) thus holds.

For $u \in [0, \Delta_k]$, let $x_k(u)$ be the smallest $x \in X$ such that $U(x,k) = u$. Since $U(x,k)$ is continuous in x, $x_k(u)$ indeed exists. By the definition of $x_k(u)$, we have $xD(x;k) < 2u$ for any $x < x_k(u)$. Hence by continuity,

$$\sup_{y \in X} u_1\big(y, x_k(u), k\big) = 2u. \tag{2}$$

Furthermore, for any x such that $U(x,k) = u$, we have $x \geq x_k(u)$. Since $u_1(x_1, x_2, k)$ is nondecreasing in x_2, it holds that

$$\sup_{y \in X} u_1(y, x, k) \geq \sup_{y \in X} u_1\big(y, x_k(u), k\big) = 2u. \tag{3}$$

Since $u \in [0, \Delta_k]$ is arbitrary, (2) and (3) imply that Assumption 1(iii) holds for $K = 2$.

If this game is played only once under state k, its Bertrand structure immediately means that each firm's equilibrium profit is zero, and this equals its minmax value. Thus Assumption 1(iv) is satisfied.

Finally, Assumption 1(v) holds if we assume $D(p;k) > D(p;k-1)$ for any $k \geq 1$ and any $p < \overline{p}$. However, it holds under much weaker assumptions. While it does not hold if we just assume $D(p;k) \geq D(p;k-1)$ for any $k \geq 1$ and any p (then it is possible that $\Delta_0 = \Delta_1 = \cdots = \Delta_M$), it will hold under suitable strengthening of it.

Example 2 (first-price auctions) Let $X = [0, \infty)$. Fix k, and let

$$u_1(x_1, x_2, k) = \begin{cases} v_k - x_1 & \text{if } x_1 > x_2 \text{ and } x_1 \geq r_k, \\ \frac{1}{2}(v_k - x_1) & \text{if } x_1 = x_2 \geq r_k, \\ 0 & \text{if } x_1 < \min\{r_k, x_2\}, \end{cases}$$

where $v_k > r_k \geq 0$. This is interpreted as a first-price auction where under state k, two buyers have a common valuation v_k and the seller sets a reserve price r_k.

We have

$$U(x,k) = \begin{cases} \frac{1}{2}(v_k - x) & \text{if } x \geq r_k, \\ 0 & \text{if } x < r_k. \end{cases}$$

It is easy to see that $\Delta_k = (v_k - r_k)/2 > 0$. Assumption 1(ii) is therefore satisfied.

For any k and any $u \in [0, \Delta_k]$, set $x_k(u) = v_k - 2u$. Then we have $U\big(x_k(u), k\big) = u$. If $u > 0$, no $x \neq x_k(u)$ satisfies $U(x,k) = u$. Moreover, we have $x_k(u) < v_k$ and therefore

$$\sup_{y \in X} u_1\big(y, x_k(u), k\big) = v_k - x_k(u) = 2u. \tag{4}$$

If $u = 0$, we have $x_k(u) = v_k$ and therefore $u_1\big(y, x_k(u), k\big) \leq 0 = 2u$ for any $y \in X$. Hence (4) holds for $u = 0$, too. Since $U(x,k) = 0$ implies $\sup_{y \in X} u_1(y, x, k) \geq 0$, Assumption 1(iii) holds for $K = 2$.

For each k, the normal form game with a payoff function $u_i(x_1, x_2, k)$ has Bertrand structure, so that any equilibrium payoff is zero, and it is also each buyer's minmax value. Hence Assumption 1(iv) is satisfied. Finally, Assumption 1(v) is satisfied if we assume $v_k - r_k$ is nondecreasing and $v_0 - r_0 < v_M - r_M$.

Example 3 (linear payoffs and multiplicative shocks) Let $X = [0,1]$. For each k, let

$$u_i(x_i, x_j, k) = \theta_k(\alpha x_j - x_i),$$

where $\theta_k > 0$ and $\alpha > 1$. Therefore we have $U(x,k) = \theta_k(\alpha - 1)x$, and $\Delta_k = \theta_k(\alpha - 1) > 0$. Hence Assumption 1(ii) is satisfied.

For any $u \in [0, \Delta_k]$, the unique solution of $U(x,k) = u$ is $x = u/\Delta_k$. It is easily seen that

$$\sup_y u_1\Big(y, \frac{u}{\Delta_k}, k\Big) = u_1\Big(0, \frac{u}{\Delta_k}, k\Big) = \frac{\alpha u}{\alpha - 1}.$$

Hence (1) holds for $K = \alpha/(\alpha - 1) > 1$. Assumption 1(iii) is therefore satisfied.

In the normal-form game with a payoff function $u_i(x_i, x_j, k)$, 0 is a dominant action and minmaxes the other player. Hence 0 is indeed a minmax value, and it is also a Nash equilibrium payoff. This guarantees Assumption 1(iv). Finally, Assumption 1(v) holds if we assume $\theta_0 \leq \theta_1 \leq \cdots \leq \theta_M$ with $\theta_0 < \theta_M$.

This is a game with very simple structure, but it includes moral hazard in teams and public goods provision with linear technology as examples.

Note that in all these examples, Δ_k is the maximum of the stage payoff sum, even if we take asymmetric action pairs into account under state k.

The players play G in periods $0, 1, 2, \ldots$. Each player knows the state of each period at the beginning of that period, but does not know the state of any future period until that period arrives. We also assume perfect monitoring. Namely, the players can observe what the other players have selected, together with a sequence of past states. In the present paper, we limit attention to pure strategies. Thus each player i's strategy of this repeated game is a function which maps a history at each period t, consisting of $\big(x_1(\tau), x_2(\tau)\big)_{\tau=0}^{t-1}$ and $\big(k(\tau)\big)_{\tau=0}^{t}$, where $x_i(\tau)$ is the action player i played in period τ and $k(\tau)$ is the realized state of period τ, to an element of X. Note that a history at period t includes the state of period t. Given a strategy pair, player i's payoff of the repeated game is:

$$(1-\delta)E\Bigg[\sum_{t=0}^{\infty} \delta^t u_i\big(x_1(t), x_2(t), k(t)\big)\Bigg],$$

where $\delta \in (0,1)$ is a common discount factor, and the expectation is taken with respect to the states of the entire periods.

Let us denote this infinitely repeated game by $G(\delta)$. Our solution concept for $G(\delta)$ is symmetric subgame perfect equilibrium. A strategy pair is symmetric if at no history the players' actions are different.

3 Second-Best Equilibrium

In what follows, we examine the symmetric subgame perfect equilibrium that is *second-best* in the sense that no other symmetric subgame perfect equilibrium has a greater payoff for each player. We call such an equilibrium the *second-best equilibrium.* In this section, we derive the second-best equilibrium and its payoff for any given $G(\delta)$.

This task is closely related to the following constrained maximization problem.

$$\max_{(B_k)_{k=0}^M} \sum_{k=0}^M p_k B_k \tag{5}$$

$$\text{subject to } (1-\delta)(K-1)B_k \leq \delta \sum_{l=0}^M p_l B_l \quad \forall k \tag{6}$$

$$0 \leq B_k \leq \Delta_k \quad \forall k \tag{7}$$

For any $(B_k)_{k=0}^M$ satisfying the constraints (6) and (7), we have a corresponding trigger strategy pair, defined as follows.

(i) In period 0 with state $k(0)$, the players play $\big(x_{k(0)}(B_{k(0)}), x_{k(0)}(B_{k(0)})\big)$.

(ii) In period t ($t \geq 1$), given the corresponding history $\big(a(\tau)\big)_{\tau=0}^{t-1}$ and $\big(k(\tau)\big)_{\tau=0}^{t}$,

  (a) if $x_1(\tau) = x_2(\tau) = x_{k(\tau)}(B_{k(\tau)})$ for any $\tau \leq t-1$, then the players play $\big(x_{k(t)}(B_{k(t)}), x_{k(t)}(B_{k(t)})\big)$, and

  (b) otherwise, they play $\big(x_{k(t)}(0), x_{k(t)}(0)\big)$.

(7) ensures that this strategy pair is well-defined.

For this strategy pair, (6) states that any one-shot deviation on the path at a period with state k does not improve each player's payoff; its left-hand-side is a supremum of the additional stage payoff from a deviation, and its right-hand-side is the difference in the continuation payoffs. Since this strategy pair has Nash reversion, subgame perfection at any history off the path is satisfied. Consequently, if $(B_k)_{k=0}^M$ is feasible in the above maximization problem, $\sum_{k=0}^M p_k B_k$ is a symmetric subgame perfect equilibrium payoff.

Moreover, no symmetric subgame perfect equilibrium of $G(\delta)$ has a payoff greater than the value of this problem.[8] Thus we can concentrate on the task of solving it. Our first observation is that the solution is simple if δ is sufficiently small. Let us define

$$\underline{\delta} = \frac{K-1}{K}. \tag{8}$$

Proposition 1 *Let $\delta < \underline{\delta}$. Then a unique solution of the maximization problem* (5)–(7) *is $B_k = 0$ for any k, and the value is zero.*

[8] An informal proof is as follows. Fix a symmetric subgame perfect equilibrium of $G(\delta)$. For each k, let $\mathcal{X}_k$ be the set of all actions prescribed at some history whose current state is k. Define $B_k = \sup_{x \in \mathcal{X}_k} U(x, k)$ for each k. Then the vector $(B_k)_{k=0}^M$ satisfies (7). Since the sum $\sum_{k=0}^M p_k B_k$ is no less than the continuation payoff at any history, $(B_k)_{k=0}^M$ also satisfies (6). Thus $(B_k)_{k=0}^M$ is feasible, and $\sum_{k=0}^M p_k B_k$ is no less than the equilibrium payoff, which establishes the claim.

Proof. Fix $(B_k)_{k=0}^M$ satisfying the constraints. Take a weighted sum of (6) over k, where the weight for the k-th inequality is p_k;

$$(1-\delta)(K-1)\sum_{k=0}^{M} p_k B_k \leq \delta \sum_{k=0}^{M} p_k B_k.$$

Since $\delta < \underline{\delta}$, we must have $\sum_{k=0}^{M} p_k B_k \leq 0$. Then from (7), we must have $B_k = 0$ for any k. Since this is the only vector satisfying the constraints, it is trivially a unique solution of the problem. The value is obviously zero. Q.E.D.

Note that $B_k = 0$ for any k corresponds to a symmetric subgame perfect equilibrium where a static equilibrium is played at any history. Proposition 1 reveals that it is a second-best equilibrium if δ is small.[9] Hence no cooperation is possible at all.

If $\delta \geq \underline{\delta}$, the following inequality is important.

$$b \leq \sum_{k=0}^{M} p_k \min\left\{\Delta_k, \frac{\delta b}{(1-\delta)(K-1)}\right\}. \tag{9}$$

It is easy to see that (9) holds at $b = \Delta_0$ (this is due to $\delta \geq \underline{\delta}$), and (9) does not hold if b is sufficiently large. By continuity, the largest b such that (9) holds exists, which we denote by b^*. Note that $b^* \geq \Delta_0$, and that (9) holds with equality at b^*. Note also that the right-hand-side of (9) is a concave function of b. If $\delta > \underline{\delta}$, (9) holds with strict inequality at $b = \Delta_0$, and due to concavity, no $b' \in (\Delta_0, b^*)$ satisfies (9) with equality.

Proposition 2 *If $\delta \geq \underline{\delta}$, the value of the maximization problem* (5)–(7) *is b^*.*

Proof. Fix $(B_k)_{k=0}^M$ satisfying (6) and (7), and define $\overline{B} = \max_k B_k$. From (6) and (7), we have

$$(1-\delta)(K-1)\overline{B} \leq \delta \sum_{k=0}^{M} p_k B_k \leq \delta \sum_{k=0}^{M} p_k \min\left\{\Delta_k . \overline{B}\right\}.$$

This implies that

$$\frac{1-\delta}{\delta}(K-1)\overline{B} \leq b^*,$$

from which we obtain

$$\sum_{k=0}^{M} p_k B_k \leq \sum_{k=0}^{M} p_k \min\left\{\Delta_k, \overline{B}\right\} \leq \sum_{k=0}^{M} p_k \min\left\{\Delta_k, \frac{\delta b^*}{(1-\delta)(K-1)}\right\} = b^*. \tag{10}$$

(10) thus implies that the value of the maximization problem (5)–(7) is at most b^*.

Next, define B^* and $(B_k^*)_{k=0}^M$ as

$$B^* = \frac{\delta b^*}{(1-\delta)(K-1)}, \quad B_k^* = \min\left\{\Delta_k, B^*\right\}.$$

[9] The result does not imply uniqueness of the second-best equilibrium, because Assumption 1(iv) is consistent with multiple stage-game equilibria.

$(B_k^*)_{k=0}^M$ clearly satisfies (7). Since (9) holds with equality at b^*, we have

$$(1-\delta)(K-1)B_k^* \le (1-\delta)(K-1)B^* = \delta \sum_{k=0}^{M} p_k \min\{\Delta_k, B^*\} = \delta \sum_{k=0}^{M} p_k B_k^*$$

for any k. Therefore, (6) also holds. Since

$$\sum_{k=0}^{M} p_k B_k^* = \frac{(1-\delta)(K-1)}{\delta} B^* = b^*,$$

(10) implies that the value of the problem is b^*. Q.E.D.

A special case of Proposition 2 is the one where full collusion can be sustained. That is, $(B_k)_{k=0}^M = (\Delta_k)_{k=0}^M$ is a solution of the maximization problem, and $b^* = \sum_{k=0}^M p_k \Delta_k$. This happens if and only if

$$\Delta_M \le \frac{\delta}{(1-\delta)(K-1)} \sum_{k=0}^{M} p_k \Delta_k, \quad \therefore \delta \ge \overline{\delta} \equiv \frac{(K-1)\Delta_M}{(K-1)\Delta_M + \sum_{k=0}^M p_k \Delta_k}. \tag{11}$$

If $\delta < \overline{\delta}$, the second-best equilibrium only attains partial collusion.

Another special case is when $\delta = \underline{\delta}$. In this case, (9) reduces to

$$b \le \sum_{k=0}^{M} p_k \min\{\Delta_k, b\},$$

and it is easy to see that $b^* = \Delta_0$. Thus the second-best equilibrium payoff equals the payoff of full collusion under the lowest state.

It is also easy to see from (9) that b^* is increasing in δ on $[\underline{\delta}, \overline{\delta}]$. Hence on this range, more patience allows to achieve more collusive outcomes.

The proof of Proposition 2 reveals that the second-best equilibrium is a strategy pair with a "payoff target" B. Namely, in its cooperative phase, the players play a symmetric action pair whose payoff is $\min\{\Delta_k, B\}$ under state k. Unless full collusion is sustainable (if this is the case, the target can be set at Δ_M), the players must give up full collusion under higher states. The payoff target strategy pair stipulates that the players sustain the same payoff level under those states. This is indeed an effective way to collude, because that duplicates the incentive conditions (6) under higher states.

Finally we point out that (9) is important even when $\delta < \underline{\delta}$. In this case, the largest b satisfying (9) is zero, which equals the second-best equilibrium payoff of this case (Proposition 1). Therefore, the largest solution of (9) completely characterizes the second-best equilibrium payoff.

4 Multimarket Contact

This section presents an application of the results in the previous section, which is a main motivation of this paper. Namely, we investigate effects of multimarket contact in Bertrand price competition with demand fluctuations. We first describe the environment and then discuss how it can be formulated as a game satisfying Assumption 1.

There are M ex ante identical markets, and in each market two identical firms compete in price. Each market is subject to demand fluctuations, depending on which it is either in high demand or low demand.[10] The demands are independent across the markets and over time. The probability that a given market is in high demand is $\mu \in (0, 1)$. We assume that each market is associated with the Bertrand price competition game we described in Example 1, except that we now have only two states; ones corresponding to high demands and low demands, respectively. Let π_H (π_L, respectively) be each firm's profit under full collusion (the value corresponding to Δ_k), when the demand is high (low). We assume $\pi_H > \pi_L > 0$. Define $\overline{\pi} = \mu\pi_H + (1 - \mu)\pi_L$, which is the expected value of full collusion per market.

At the beginning of each period, the firms learn which of the M markets are in high demands in that period. This amounts to observing a subset of the set $\{1, 2, \ldots, M\}$, which is the set of markets in high demand. Given that, they decide prices in all markets. Formally, the number of states is 2^M, and the set of actions is $X = [0, \overline{p}]^M$. Since each market satisfies Assumption 1, it is easy to see that this environment also satisfies Assumption 1. Let $\mathcal{M}$ be the set of all subsets of $\{1, 2, \ldots, M\}$. If $M' \in \mathcal{M}$ is the current set of high-demand markets, the maximum symmetric action pair profit is:

$$\Delta_{M'} = M\pi_L + |M'|(\pi_H - \pi_L) > 0.$$

Let us denote the repeated game with this stage game by $G(\delta, M)$, where δ is the discount factor.

As in the argument in Section 3, the second-best equilibrium profit is the value of the following maximization problem (note that this environment satisfies Assumption 1(iii) for $K = 2$, because each market satisfies it for $K = 2$).

$$\begin{aligned} &\max_{(B_{M'})_{M' \in \mathcal{M}}} \sum_{M' \in \mathcal{M}} \mu^{|M'|}(1 - \mu)^{M - |M'|} B_{M'} \\ &\text{subject to } (1 - \delta) B_{M'} \le \delta \sum_{M'' \in \mathcal{M}} \mu^{|M''|}(1 - \mu)^{M - |M''|} B_{M''} \quad \forall M' \in \mathcal{M} \\ &\qquad 0 \le B_{M'} \le \Delta_{M'} \quad \forall M' \in \mathcal{M} \end{aligned}$$

Recall that $\Delta_{M'}$ depends only on $|M'|$. Therefore the solution of this problem must be such that $B_{M'} = B_{M''}$ whenever $|M'| = |M''|$. This observation implies that we have an alternative problem that also specifies the second-best equilibrium profit. For $k \in \{0, 1, \ldots, M\}$, let p_k be the probability that exactly k markets are in high demand. Also, let $\Delta_k = \Delta_{M'}$, where M' is such that $|M'| = k$. Then the maximization problem that exactly coincides with the one (5)–(7) also specifies the second-best equilibrium profit. Thus we will hereafter work with it.

Fix δ, and let b^*_M be the second-best equilibrium profit of $G(\delta, M)$. Define $\beta^*_M \equiv b^*_M / M$, which is the second-best equilibrium *per-market* profit of $G(\delta, M)$. Our objective is to examine how b^*_M and β^*_M depend on M.

Since $K = 2$ in this setup, $\underline{\delta}$ and $\overline{\delta}$ defined by (8) and (11) are

$$\underline{\delta} = \frac{1}{2}, \quad \overline{\delta} = \frac{\pi_H}{\pi_H + \overline{\pi}},$$

[10]The analysis, in principle, extends to the case of three or more demand states.

respectively. Note that they are independent of M. Based on the results in the previous section, three observations are in order.

(I) If $\delta < \underline{\delta} = 1/2$, $b_M^* = \beta_M^* = 0$ for any M. Therefore any collusion is impossible, regardless of the number of markets. In this case, multimarket contact does not help at all.

(II) If $\delta = 1/2$, $b_M^* = \Delta_0$ and $\beta_M^* = \pi_L$ for any M. That is, independent of the number of markets, the second-best equilibrium profit equals the value of full collusion when all markets are in low demand. This is another case where multimarket contact does not help.

(III) If $\delta \geq \overline{\delta}$, $b_M^* = M\overline{\pi}$ and $\beta_M^* = \overline{\pi}$ for any M. Namely, full collusion is sustainable regardless of the number of markets. In this case, multimarket contact does not matter.

The remaining case is $\underline{\delta} < \delta < \overline{\delta}$. Our first result shows how β_M^* varies with M.

Proposition 3 *Let $\delta \in (\underline{\delta}, \overline{\delta})$. For any $M \geq 1$, we have $\beta_M^* \leq \beta_{M+1}^*$, and the equality holds if and only if there exists $\hat{k} \in \{0, 1, \dots, M-1\}$ such that*

$$\frac{\delta}{1-\delta}\beta_M^* = \pi_L + \frac{\hat{k}}{M}(\pi_H - \pi_L). \tag{12}$$

Proof. Fix $\delta \in (\underline{\delta}, \overline{\delta})$ and $M \geq 1$, and consider $G(\delta, M)$. For a later purpose, we explicitly write down p_l, the probability that l out of M markets are in high demand. That is,

$$p_l = \frac{M!}{l!(M-l)!}\mu^l(1-\mu)^{M-l}. \tag{13}$$

Similarly, let q_l ($l \in \{0, 1, \dots, M+1\}$) be the probability that l out of $M+1$ markets are in high demand;

$$q_l = \frac{(M+1)!}{l!(M+1-l)!}\mu^l(1-\mu)^{M+1-l}. \tag{14}$$

From (13) and (14), we have the following equations;

$$p_0(1-\mu) = q_0, \tag{15}$$

$$\begin{aligned} &p_{l-1}\mu\Big\{\pi_L + \frac{l-1}{M}(\pi_H - \pi_L)\Big\} + p_l(1-\mu)\Big\{\pi_L + \frac{l}{M}(\pi_H - \pi_L)\Big\} \\ &= q_l\Big\{\pi_L + \frac{l}{M+1}(\pi_H - \pi_L)\Big\}, \quad \forall l \in \{1, 2, \dots, M-1\} \end{aligned} \tag{16}$$

$$p_k\mu + \sum_{l=k+1}^{M} p_l = \sum_{l=k+1}^{M+1} q_l \quad \forall k \leq M \tag{17}$$

In (17), read $\sum_{l=k+1}^{M} p_l = 0$ in case of $k = M$.

Let $\hat{k}$ be the greatest integer k such that

$$\pi_L + \frac{k}{M}(\pi_H - \pi_L) \leq \frac{\delta}{1-\delta}\beta_M^*. \tag{18}$$

Since $\underline{\delta} < \delta < \overline{\delta}$, we have $\pi_L < \beta_M^* < \overline{\pi}$. Thus (18) holds at $k = 0$, but not at $k = M$. Hence $0 \leq \hat{k} \leq M - 1$.

Since (9) for $G(\delta, M)$ holds with equality at b_M^*, we obtain

$$\beta_M^* = \sum_{l=0}^{\hat{k}} p_l \Big\{ \pi_L + \frac{l}{M}(\pi_H - \pi_L) \Big\} + \sum_{l=\hat{k}+1}^{M} p_l \cdot \frac{\delta}{1-\delta} \beta_M^*. \tag{19}$$

Rearranging (19) yields

$$\begin{aligned} \beta_M^* = &\sum_{l=1}^{\hat{k}} \Big[p_l(1-\mu)\Big\{ \pi_L + \frac{l}{M}(\pi_H - \pi_L) \Big\} + p_{l-1}\mu \Big\{ \pi_L + \frac{l-1}{M}(\pi_H - \pi_L) \Big\} \Big] \\ &+ p_0(1-\mu)\pi_L + p_{\hat{k}}\mu \Big\{ \pi_L + \frac{\hat{k}}{M}(\pi_H - \pi_L) \Big\} + \sum_{l=\hat{k}+1}^{M} p_l \cdot \frac{\delta}{1-\delta}\beta_M^* \\ = &\sum_{l=0}^{\hat{k}} q_l \Big\{ \pi_L + \frac{l}{M+1}(\pi_H - \pi_L) \Big\} + \sum_{l=\hat{k}+2}^{M+1} q_l \cdot \frac{\delta}{1-\delta}\beta_M^* \\ &+ p_{\hat{k}}\mu \Big\{ \pi_L + \frac{\hat{k}}{M}(\pi_H - \pi_L) \Big\} + p_{\hat{k}+1}(1-\mu) \cdot \frac{\delta}{1-\delta}\beta_M^*, \end{aligned} \tag{20}$$

where (20) holds because of (15)–(17). Due to the definition of $\hat{k}$ and (16), it follows that

$$\begin{aligned} &p_{\hat{k}}\mu \Big\{ \pi_L + \frac{\hat{k}}{M}(\pi_H - \pi_L) \Big\} + p_{\hat{k}+1}(1-\mu) \cdot \frac{\delta}{1-\delta}\beta_M^* \\ &\leq q_{\hat{k}+1} \min \Big\{ \Big[\pi_L + \frac{\hat{k}+1}{M+1}(\pi_H - \pi_L) \Big], \frac{\delta}{1-\delta}\beta_M^* \Big\}, \end{aligned} \tag{21}$$

where the equality holds if and only if (12) holds with equality.

Note that the definition of $\hat{k}$ also implies

$$\pi_L + \frac{\hat{k}}{M+1}(\pi_H - \pi_L) < \frac{\delta}{1-\delta}\beta_M^* < \pi_L + \frac{\hat{k}+2}{M+1}(\pi_H - \pi_L). \tag{22}$$

Hence, substituting (21) into (20) and then using (22), we obtain

$$\beta_M^* \leq \sum_{l=0}^{M+1} q_l \min \Big\{ \pi_L + \frac{l}{M+1}(\pi_H - \pi_L), \frac{\delta}{1-\delta}\beta_M^* \Big\},$$

where the equality holds if and only if (12) holds with equality. Therefore, $\beta_M^* \leq \beta_{M+1}^*$, and $\beta_M^* = \beta_{M+1}^*$ holds if and only if (12) holds with equality. Q.E.D.

Proposition 3 implies that if the firms are relatively patient and thus can only attain partial collusion, adding one more market never reduces the second-best equilibrium per-market profit, and increases it in most cases. However, in rare cases where (12) holds, adding one more market does not change the second-best equilibrium per-market profit. In other words, the irrelevance result by Bernheim and Whinston [1] is not entirely denied in this case.

Nevertheless, we may as well claim that the irrelevance result generally fails. First, if $M = 1$, no k satisfies (12). Thus $\beta_1^* < \beta_2^*$. Second, the irrelevance result fails if two or more markets are added. To see this, note that if (12) holds and therefore we have $\beta_M^* = \beta_{M+1}^*$, it follows that

$$\pi_L + \frac{\hat{k}}{M+1}(\pi_H - \pi_L) < \beta_{M+1}^* < \pi_L + \frac{\hat{k}+1}{M+1}(\pi_H - \pi_L).$$

Hence we have $\beta_{M+2}^* > \beta_M^*$. Those observations are summarized by the following corollary.

Corollary 1 *Let $\delta \in (\underline{\delta}, \overline{\delta})$. Then we have (i) $\beta_1^* < \beta_2^*$, and (ii) for any $M \geq 1$ and any $M' \geq M+2$, $\beta_M^* < \beta_{M'}^*$.*

The next result is about the total profit in the second-best equilibrium.

Proposition 4 *Let $\delta \in (\underline{\delta}, \overline{\delta})$, and fix $\varepsilon > 0$ arbitrarily. Then there exists $\underline{M}$ such that for any $M \geq \underline{M}$, $b_M^* > \overline{\pi}M - \varepsilon$. In other words, $\lim_{M\to\infty}(\overline{\pi}M - b_M^*) = 0$.*

Proof. Fix $\delta \in (\underline{\delta}, \overline{\delta})$ and $\varepsilon > 0$. Since $\delta > \underline{\delta}$, a rational number $\hat{\mu} \in (\mu, 1)$ exists such that

$$\frac{\delta}{1-\delta}\overline{\pi} = \frac{\delta}{1-\delta}\big\{\pi_L + \mu(\pi_H - \pi_L)\big\} > \pi_L + \hat{\mu}(\pi_H - \pi_L).$$

It then follows that

$$\frac{\delta}{1-\delta}(\overline{\pi}M - \varepsilon) > M\big\{\pi_L + \hat{\mu}(\pi_H - \pi_L)\big\}$$

for all large M. Therefore,

$$\begin{aligned}\sum_{l=0}^{M} p_l \min\left\{\Delta_l, \frac{\delta}{1-\delta}\big(\overline{\pi}M - \varepsilon\big)\right\} &\geq \sum_{l=0}^{M} p_l \min\Big[\Delta_l, M\big\{\pi_L + \hat{\mu}(\pi_H - \pi_L)\big\}\Big]\\ &= \sum_{l=0}^{M} p_l \Big[\Delta_l + \min\big\{0, (M\hat{\mu} - l)(\pi_H - \pi_L)\big\}\Big]\\ &= \overline{\pi}M - (\pi_H - \pi_L)\sum_{l=0}^{M} p_l \max\big\{0, l - M\hat{\mu}\big\}\end{aligned}$$

for all large M. In Appendix A, we prove that

$$\lim_{M\to\infty}\sum_{l=0}^{M} p_l \max\big\{0, l - M\hat{\mu}\big\} = 0. \tag{23}$$

We thus obtain

$$\overline{\pi}M - \varepsilon < \sum_{l=0}^{M} p_l \min\left\{\Delta_l, \frac{\delta}{1-\delta}\big(\overline{\pi}M - \varepsilon\big)\right\}$$

for all large M. In view of (9), this implies that $b_M^* > \overline{\pi}M - \varepsilon$ for all large M, which completes the proof. Q.E.D.

Proposition 4 shows that given relatively patient firms, the difference between the profit under full collusion and the second-best equilibrium total profit converges to zero if the number of markets goes to infinity. Namely, the collusion-deterrence effects of demand fluctuations completely vanish in the limit.

From the proof of Proposition 4, we see that the second-best equilibrium when M is large enough sets a payoff target greater than $M\{\pi_L + \hat{\mu}(\pi_H - \pi_L)\}$, where $\hat{\mu} > \mu$. That is, the per-market target is greater than the average full collusion profit. By the law of large numbers, the probability that the fraction of high-demand markets is greater than $\hat{\mu}$ converges to zero, if the number of markets goes to infinity. In fact, the convergence is so fast that it converges to zero even if it is multiplied by M. This implies that the firms can fully collude except upon an event with a negligible probability, and that the expected efficiency loss due to demand fluctuations is also negligible. Consequently, full collusion is approximately attained.

A Appendix: Proof of (23)

The left-hand-side of (23) can be further calculated as

$$\sum_{l=0}^{M} p_l \max\{0, l - M\hat{\mu}\} = \sum_{l=\lceil M\hat{\mu}\rceil}^{M} p_l(l - M\hat{\mu}) < (1-\hat{\mu})M \sum_{l=\lceil M\hat{\mu}\rceil}^{M} p_l,$$

where $\lceil k \rceil$ is the smallest integer not less than k. Note that

$$\frac{p_{l+1}}{p_l} = \frac{(M-l)\mu}{(l+1)(1-\mu)} < \frac{(1-\hat{\mu})\mu}{\hat{\mu}(1-\mu)} \equiv \kappa$$

for any $l \geq \lceil M\hat{\mu}\rceil$. Since $\kappa < 1$ from $\hat{\mu} > \mu$, we have

$$(1-\hat{\mu})M \sum_{l=\lceil M\hat{\mu}\rceil}^{M} p_l < \frac{1-\hat{\mu}}{1-\kappa} M p_{\lceil M\hat{\mu}\rceil}.$$

Therefore, if we define

$$\alpha_M \equiv M p_{\lceil M\hat{\mu}\rceil} = M \frac{M!}{\lceil M\hat{\mu}\rceil!\big(M - \lceil M\hat{\mu}\rceil\big)!} \mu^{\lceil M\hat{\mu}\rceil}(1-\mu)^{M-\lceil M\hat{\mu}\rceil},$$

it suffices to prove that $\lim_{M\to\infty} \alpha_M = 0$.

Since $\hat{\mu}$ is rational, we can choose two natural numbers y and Y such that $\hat{\mu} = y/Y$. Since

$$\mu^y(1-\mu)^{Y-y} < \hat{\mu}^y(1-\hat{\mu})^{Y-y},$$

there exists $\eta > 0$ such that

$$\left(\frac{\mu}{\hat{\mu}}\right)^y \left(\frac{1-\mu}{1-\hat{\mu}}\right)^{Y-y} < 1 - \eta. \tag{24}$$

We also have

$$\frac{\alpha_{M+Y}}{\alpha_M} = \frac{M+Y}{M} \cdot \frac{\left[\prod_{k=1}^{Y}(M+k)\right]\mu^y(1-\mu)^{Y-y}}{\left[\prod_{k=1}^{y}\left(\lceil M\hat{\mu}\rceil + k\right)\right]\left[\prod_{k=1}^{Y-y}\left(M - \lceil M\hat{\mu}\rceil + k\right)\right]} \tag{25}$$

for each M, because $\lceil (M+Y)\hat{\mu}\rceil = \lceil M\hat{\mu}\rceil + y$.

Since $M\hat{\mu} \leq \lceil M\hat{\mu}\rceil < M\hat{\mu} + 1$ for any M, it follows that

$$\lim_{M\to\infty} \frac{\lceil M\hat{\mu}\rceil}{M} = \hat{\mu}.$$

Applying this to (25) and using (24), we have

$$\lim_{M\to\infty} \frac{\alpha_{M+Y}}{\alpha_M} = \left(\frac{\mu}{\hat{\mu}}\right)^y \left(\frac{1-\mu}{1-\hat{\mu}}\right)^{Y-y} < 1-\eta.$$

Therefore, for any $z = 1, 2, \ldots, Y$, the sequence $\left(\alpha_{KY+z}\right)_{K=0}^{\infty}$ converges to zero. This proves that $\lim_{M\to\infty} \alpha_M = 0$.

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