A non-cooperative analysis of the estate division problem

Hans Peters*, Marc Schröder*, Dries Vermeulen*

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Abstract

This paper considers the estate division problem from a non-cooperative perspective. The integer claim game initiated by O'Neill (1982) and extended by Atlamaz et al. (2011) is generalized by allowing for an arbitrary sharing rule to divide every interval among the claimants. Our main focus is on games that use the constrained equal awards rule, the constrained equal losses rule, or the Talmud rule as sharing rule. A full characterization of the set of Nash equilibria with corresponding payoffs is obtained for these games. It is shown that for a subset of problems, the Talmud rule is the only rule from a particular family of rules that always results in an equilibrium payoff. A variation on the claim game is considered by allowing for arbitrary instead of integer claims.

1 Introduction

The estate division problem, also known as bankruptcy problem or rationing problem, concerns the issue of dividing an estate among a group of claimants who have entitlements to the estate, when the sum of these entitlements exceeds the size of the estate. A seminal paper on this problem is O'Neill (1982). Subsequently, most research has focused on comparing different solution rules by their properties. For an overview of this normative, axiomatic approach, see Thomson (2003).

The estate division problem can also be approach strategically, i.e., by a non-cooperative game. O'Neill (1982) already formulates a non-cooperative game, associated with an estate division problem, in which players can use their entitlements to claim specific parts of the estate. More precisely, think of an estate with size E as an interval $[0, E]$. Each player can partition this interval into finitely many subintervals and on each of those subintervals put a claim such that the total amount claimed is equal to his entitlement. O'Neill (1982) considers the Nash equilibria of this game. Atlamaz et al. (2011) extend this game by allowing for multiple claims on every subinterval. Then every subinterval is divided among the players according to the proportional rule with respect to the claims. We generalize their game by allowing for other sharing rules to divide the subintervals.

We focus on the rules from the TAL-family (Moreno-Ternero and Villar, 2001) as sharing rules. This family of rules contains three of the best-known rules, namely the constrained equal awards rule, the constrained equal losses rule and the Talmud rule. See Herrero and Villar (2001) for a comparative analysis between these three solutions and the proportional rule. The constrained equal awards and the constrained equal losses rules implement the idea of equal division, the

*Address: Department of Quantitative Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands. Email: m.schroeder@maastrichtuniversity.nl, h.peters@maastrichtuniversity.nl, d.vermeulen@maastrichtuniversity.nl

former rule with respect to awards and the latter rule with respect to losses. The Talmud rule (first formulated by Aumann and Maschler, 1985) combines the underlying ideas of these two rules. The Talmud rule behaves like the constrained equal awards rule if the estate is less than half of the total entitlements and like the constrained equal losses rule if the estate is larger than half of the total entitlements.

Although we use the estate division terminology, our model has applications other than the division of a heritage or the leftovers of a bankrupt firm. For instance, think of the interval $[0, E]$ as representing a continuum of uniformly distributed consumers (cf. Hotelling, 1929), and of the claimants as firms who provide services to the consumers, with total services equal to the entitlements. Every claim can be thought of as an investment in a particular consumer segment. Since we allow for multiple claims, this interpretation allows for competitive investments by different firms in the same consumer segment. As we can choose the sharing rule, it is possible to allow for different forms of competition among the firms. Aside from proportional division of the consumers, one could think of a form of competition in which the firms that invest maximally in a given segment, equally share the consumers in that segment – this is achieved by using the constrained equal awards rule in our model. Or one could imagine a competition in which all investing firms equally share a segment – achieved by using the constrained equal losses rule.

It is also possible (but postponed to future work) to extend to non-homogeneous preferences over the estate, like in Pálvölgyi et al. (2010). This extension has several other applications like, for example, land division problems (Berliant et al., 1992). Still other applications are political elections (cf. Merolla et al., 2005) or auctions (cf. Cramton et al., 2003).

Our main focus is on restricted estate division problems, in which individual entitlements do not exceed the size of the estate. In fact, for the sharing rules we consider, the unrestricted problem can be solved conveniently with the help of the restricted problem. For the case in which claims are integer-valued, we characterize all Nash equilibria of the associated claim game and the corresponding payoffs in case we use the constrained equal awards rule, the constrained equal losses rule or the Talmud rule as the sharing rule. Then we compare these equilibria and find that for a subset of estate division problems (those for which the estate is larger than half of the total entitlements), the equilibria and associated payoffs are the same, independent of the sharing rule. We show that the Talmud rule is the only rule from the TAL-family that always results in an equilibrium payoff for these kind of problems. This result can be seen as an equilibrium argument to use the Talmud rule for the problems under consideration.

We also investigate what happens if we relax the assumption of placing integer-valued claims and allow for arbitrary claim heights. Unlike for the proportional case (Atlamaz et al., 2011), the claims profile in which every player has a uniform claim over the estate, is usually not the unique equilibrium claims profile.

The organization of the paper is as follows. Section 2 explains the basic model and the relevant sharing rules. In Section 3, we consider restricted problems and claim games in which players are allowed to place integer-valued claims; in Section 4, players are allowed to place arbitrary claims. Section 5 analyzes the relation between restricted and unrestricted problems and Section 6 concludes.

2 The model

The set of *players* is $N = \{1, \dots, n\}$, where $n \geq 2$. An *estate division problem* is a pair (E, c), where $E \in \mathbb{R}$, $E > 0$, is the *estate* and $c = (c_1, \dots, c_n) \in \mathbb{R}^N$ with $c_i > 0$ for all $i \in N$ and $\sum_{i \in N} c_i \geq E$, is the vector of *entitlements*. A *payoff vector* for (E, c) is a vector $x = (x_1, \dots, x_n) \in \mathbb{R}_+^N$ with $\sum_{i \in N} x_i \leq E$, where x_i is the payoff to player i, and $\mathbb{R}_+ := [0, \infty)$.

The purpose of this paper is to find payoff vectors for estate division problems as equilibrium outcomes of a suitable non-cooperative game. To this end, we first define a *sharing rule* to be a function f that assigns to every $b \in \mathbb{R}_+^N$ a vector $f(b) \in [0, 1]^N$ such that $\sum_{i \in N} f_i(b) \leq 1$. Given a sharing rule f we associate with an estate division problem (E, c) a *claim game*, denoted by (E, c, f). First, a *strategy* of player $i \in N$ in this claim game consist of a finite division of the interval $[0, E]$ into subintervals and on each subinterval a non-negative number of claims, such that the total amount claimed is equal to c_i. It will be without loss of generality to assume that the strategies of all players have the same division of $[0, E]$ in common, since otherwise we can consider the common refinement of the player divisions instead. The following definition therefore introduces so-called claims profiles and, based on these, the game (E, c, f).

Definition 2.1. A *claims profile* for problem (E, c) is a triple (y, β, m), where

(i) $m \in \mathbb{N}$,

(ii) $y = (y_0, \dots, y_m) \in \mathbb{R}^{m+1}$ with $0 = y_0 < y_1 \dots < y_{m-1} < y_m = E$,

(iii) $\beta = (\beta_1, \dots, \beta_n)$ with $\beta_i : \{1, \dots, m\} \to \mathbb{R}_+$ such that

$$\sum_{t=1}^{m} \beta_i(t) \cdot (y_t - y_{t-1}) = c_i \text{ for all } i \in N.$$

With a slight abuse of language, we refer to the interval (y_{t-1}, y_t) as interval t. We write $\beta(t) = (\beta_i(t))_{i \in N}$, where $\beta_i(t)$ is interpreted as the amount that player i claims on interval t; and we write $M = \{1, \dots, m\}$. We now use the sharing rule f to distribute every interval t among the claimants of the interval. Specifically, $f_i(\beta(t))$ is the *share* of player i of interval t, and player i's payoff is determined by the payoff function $u_i^f : (y, \beta, m) \mapsto u_i^f(y, \beta, m) \in \mathbb{R}$ defined by

$$u_i^f(y, \beta, m) = \sum_{t \in M} f_i(\beta(t)) \cdot (y_t - y_{t-1})$$

for every claims profile (y, β, m). We write $u^f = (u_1^f, \dots, u_n^f)$. This concludes the definition of the game (E, c, f).

Atlamaz et al. (2011) analyze this game with the proportional rule as the sharing rule. We generalize their results by considering different sharing rules. More precisely, we focus on sharing rules derived from a particular family of rules, the TAL-family, to which the constrained equal awards rule, the constrained equal losses rule and the Talmud rule belong. For the original definitions of these rules see Remark 2.6

We first describe the *constrained equal awards rule* f^{CEA} as a sharing rule.

Definition 2.2. For every $b \in \mathbb{R}^N_+$ and every $i \in N$,

$$f_i^{CEA}(b) = \begin{cases} b_i & \text{if } \sum_{j \in N} b_j < 1, \\ \min\{b_i, \lambda\} & \text{if } \sum_{j \in N} b_j \geq 1, \end{cases}$$

where λ is the unique solution to the equation $\sum_{j \in N} \min\{b_j, \lambda\} = 1$.

The constrained equal awards rule assigns equal shares to all claimants subject to no one receiving more than his claim. Note that $f^{CEA}(0, \ldots, 0) = (0, \ldots, 0)$. When used in a claim game this implies that if a part (an interval) of the estate is not claimed, then it is not distributed. We write u^{CEA} instead of $u^{f^{CEA}}$.

The *constrained equal losses rule* f^{CEL} (as a sharing rule) is defined as follows.

Definition 2.3. For every $b \in \mathbb{R}^N_+$ and every $i \in N$,

$$f_i^{CEL}(b) = \begin{cases} b_i & \text{if } \sum_{j \in N} b_j < 1, \\ \max\{0, b_i - \mu\} & \text{if } \sum_{j \in N} b_j \geq 1, \end{cases}$$

where μ is the unique solution to the equation $\sum_{j \in N} \max\{0, b_j - \mu\} = 1$.

The constrained equal losses rule focuses on the loss each claimant incurs. The rule divides these losses equally among all claimants subject to no one receiving a negative amount. Again, in the claim game, if a part of the estate is not claimed, then it is not distributed. We write u^{CEL} instead of $u^{f^{CEL}}$.

The sharing rules from the *TAL-family* f^θ, identified by a single parameter $\theta \in [0, 1]$, are defined as follows.

Definition 2.4. Let $\theta \in [0, 1]$. For every $b \in \mathbb{R}^N_+$ and every $i \in N$,

$$f_i^\theta(b) = \begin{cases} b_i & \text{if } \sum_{j \in N} b_j < 1, \\ \max\{\theta b_i, b_i - \mu\} & \text{if } \sum_{j \in N} b_j \geq 1 \text{ and } \theta \sum_{j \in N} b_j < 1, \\ \min\{\theta b_i, \lambda\} & \text{if } \theta \sum_{j \in N} b_j \geq 1, \end{cases}$$

where μ and λ are the unique solutions to the equations $\sum_{j \in N} \max\{\theta b_j, b_j - \mu\} = 1$ and $\sum_{j \in N} \min\{\theta b_j, \lambda\} = 1$, respectively.

Every rule from the TAL-family combines the principles of the constrained equal awards rule and the constrained equal losses rule. Namely, if the estate available does not exceed θ times the aggregate claims, no one receives more than a fraction of θ of his claim. In this case, the constrained equal awards rule is applied with θb as claims. If the estate available exceeds θ times the aggregate claim, everyone receives at least a θ-fraction of his claim and the remainder is divided using the constrained equal losses rule with $(1 - \theta)b$ as claims. This family of rules generalizes the Talmud rule, for which this switch happens exactly halfway, so for $\theta = \frac{1}{2}$. For every $\theta \in [0, 1]$, we write u^θ instead of u^{f^θ}. Note that $f^1 = f^{CEA}$ and $f^0 = f^{CEL}$. We refer to $f^{\frac{1}{2}}$ as the *Talmud sharing rule*, and denote the corresponding payoff function by u^T.

For completeness, we define the *proportional rule* f^P (as a sharing rule), used in Atlamaz et al. (2011).

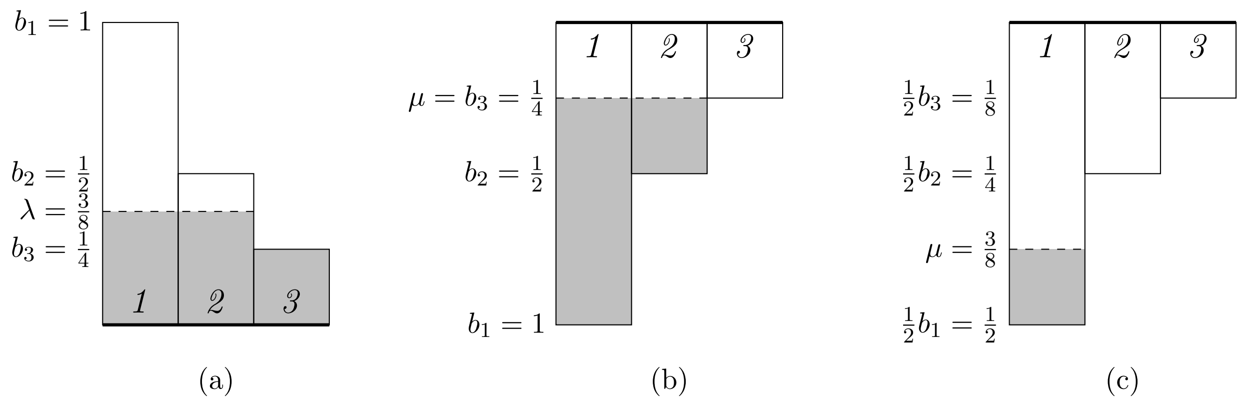

Figure 1: An illustration of the outcome of different sharing rules for $b = (1, \frac{1}{2}, \frac{1}{4})$. The height of every bar represents the claim height and the number in the bar is the name of the player of that claim. (a) The constrained equal awards rule: the height of the shaded area of the bar represents the share of the player. (b) The constrained equal losses rule: the share of each player is represented by the height of the shaded area of the bar. (c) The Talmud rule: every player receives a share of $\frac{1}{2}b_i$ plus the height of the shaded area of the bar.

Definition 2.5. For every $b \in \mathbb{R}^N_+$ and every $i \in N$,

$$f_i^P(b) = \begin{cases} 0 & \text{if } b_i = 0, \\ \frac{b_i}{\sum_{j \in N} b_j} & \text{if } b_i > 0. \end{cases}$$

We denote the corresponding vector of payoff functions by u^P.

Remark 2.6. The original constrained equal awards rule assigns to player i in an estate division problem (E, c) the amount $\min\{c_i, \lambda\}$, where λ solves $\sum_{j \in N} \min\{c_j, \lambda\} = E$. Recall that, by assumption, $\sum_{j \in N} c_j \geq E$. The original constrained equal losses rule assigns to player i the amount $\max\{0, c_i - \lambda\}$, where λ solves $\sum_{j \in N} \max\{0, c_j - \lambda\} = E$. The original TAL-rule with parameter $\theta \in [0, 1]$ assigns to player i the amount $\max\{\theta c_i, c_i - \mu\}$ if $\theta \sum_{j \in N} b_j < E$, and $\min\{\theta c_i, \lambda\}$ if $\theta \sum_{j \in N} c_j \geq E$, where μ and λ are the unique solutions to the equations $\sum_{j \in N} \max\{\theta c_j, c_j - \mu\} = E$ and $\sum_{j \in N} \min\{\theta c_j, \lambda\} = E$, respectively. The original proportional rule assigns the player i the amount $\frac{c_i}{\sum_{j \in N} c_j} E$.

The payoffs assigned by the original rules in Remark 2.6 are also obtained by applying the associated sharing rules to the claims profile in which each player puts a constant claim $\frac{c_i}{E}$ on the entire estate. We call this claims profile, the profile $(y, \beta, 1)$, the *uniform claims profile*.

Example 2.7. Consider the estate division problem (E, c) with $E = 4$ and $c = (4, 2, 1)$. The payoffs assigned to the players by the (original) constrained equal awards rule, the constrained equal losses rule and the Talmud rule are found by considering the uniform claims profile, that is, $\beta(1) = (1, \frac{1}{2}, \frac{1}{4})$. Figure 1 illustrates the three sharing rules. The associated shares are $f^{CEA}(\beta(1)) = (\frac{3}{8}, \frac{3}{8}, \frac{1}{4})$, $f^{CEL}(\beta(1)) = (\frac{3}{4}, \frac{1}{4}, 0)$ and $f^T(\beta(1)) = (\frac{5}{8}, \frac{1}{4}, \frac{1}{8})$. The corresponding payoffs are $u^{CEA} = (1\frac{1}{2}, 1\frac{1}{2}, 1)$, $u^{CEL} = (3, 1, 0)$ and $u^T = (2\frac{1}{2}, 1, \frac{1}{2})$.

Our purpose is to analyze Nash equilibrium outcomes of the claim game under different sharing rules. The following definition is standard.

Definition 2.8. A claims profile (y, β, m) is a *Nash equilibrium profile* (NEP) in (E, c, u^f) if each player maximizes his own payoff, given his opponents' claims.

Hence, in an NEP, no player can increase his payoff by reshuffling his claims. Let the *marginal* gain or loss be defined as the gain or loss per unit interval from increasing or decreasing one's claim on that interval with a smallest claim unit, which is 1 in the case that only integer claims are allowed, and infinitesimal otherwise. Then, in an NEP, the marginal loss of decreasing one's claim on some interval should be at least as large as the marginal gain of increasing one's claim on some other interval. For proportional sharing this condition is also sufficient for a claims profile to be an NEP (Atlamaz et al., 2011), but this does not hold in general. Suppose, as an example, that claims are restricted to be integer-valued – a case that we will study extensively in Section 3 – and that the constrained equal losses rule is used as the sharing rule; and suppose that $\beta_i(t) = 2$ for some player i and interval t, whereas $\beta_j(t) = 0$ for all $j \neq i$. If player $j \neq i$ places a claim of size 1 on t his gain on t is zero, hence his marginal gain is zero. If, however, j places a claim of size 2 on t his gain on t is $\frac{1}{2}$ and, thus, his average gain is $\frac{1}{4}$ (times the length of interval t). In Section 3 we present a characterization of NEP in terms of average – rather than marginal – gains and losses in claim games with f^θ, $0 \leq \theta \leq 1$, as sharing rules.

We are interested in the payoffs associated with NEP's with respect to different sharing rules. For every claim game (E, c, f) we denote the *set of equilibrium payoffs* by

$$U(E, c, f) = \left\{ \left(u_i^f(y, \beta, m) \right)_{i \in N} \mid (y, \beta, m) \text{ is an NEP in } (E, c, f) \right\}.$$

A *restricted problem* is an estate division problem (E, c) with $c_i \leq E$ for all $i \in N$. O'Neill (1982) considers claim games with the proportional sharing rule for restricted problems, in which $\beta_i(t) \in \{0, 1\}$ for each claims profile (y, β, m), each $i \in N$, and each $t \in M$. We generalize to multiple and not per se integer claims and different sharing rules. We start with restricted problems and multiple integer claims in the next section.

Notation. We introduce some convenient notation, related to a claims profile (y, β, m). For all $t \in M$, we denote $P(t) = \{i \in N \mid \beta_i(t) > 0\}$, $\beta_{min}(t) = \min_{i \in P(t)} \beta_i(t)$, $\beta_{max}(t) = \max_{i \in P(t)} \beta_i(t)$, and $\beta_N(t) = \sum_{i \in N} \beta_i(t)$.

3 Restricted problems and integer claims

In this section we consider restricted problems (E, c), i.e., $c_i \leq E$ for all $i \in N$, and integer claims in each associated claim game, i.e., $\beta_i : M \to \{0\} \cup \mathbb{N}$ for every claims profile (y, β, m) and every $i \in N$. This is the setting also considered in O'Neill (1982), with the difference that $\beta_i(t) > 1$ is allowed.

In the following lemma we characterize Nash equilibrium profiles (Definition 2.8) for claim games with sharing rule f^θ, in terms of average gains and losses. Fix $0 \leq \theta \leq 1$ and let (y, β, m) be a claims profile. For $i \in N$ and $t \in M$ with $i \in P(t)$, and $\Delta \in \mathbb{N}$ with $1 \leq \Delta \leq \beta_i(t)$, define

$$AL_i(\Delta, t) = \frac{u_i^\theta(y, \beta, m) - u_i^\theta(y, \beta', m)}{(y_t - y_{t-1})\Delta},$$

where (y, β', m) is a claims profile in the problem (E, c') such that β' is equal to β except that $\beta'_i(t) = \beta_i(t) - \Delta$, and c' is equal to c except that $c'_i = c_i - (y_t - y_{t-1})\Delta$. Similarly, define

$$AG_i(\Delta, t) = \frac{u_i^\theta(y, \beta'', m) - u_i^\theta(y, \beta, m)}{(y_t - y_{t-1})\Delta} ,$$

where (y, β'', m) is a claims profile in the problem (E, c'') such that β'' is equal to β except that $\beta''_i(t) = \beta_i(t) + \Delta$, and c'' is equal to c except that $c''_i = c_i + (y_t - y_{t-1})\Delta$. Hence, $AL_i(\Delta, t)$ is the average loss to player $i \in N$ of removing Δ claims from interval $t \in M$, and $AG_i(\Delta, t)$ is the average gain of adding Δ claims to interval $t \in M$, both measured per unit interval. Observe that for $\Delta = 1$ we obtain the marginal loss and the marginal gain.

Lemma 3.1. *Let $\theta \in [0, 1]$. A claims profile (y, β, m) is an NEP in (E, c, f^θ) if and only if $\beta_N(t) \geq 1$ for all $t \in M$ and for every $i \in N$ we have*

$$\min_{t \in M : i \in P(t)} \min_{\Delta \in \{1, \ldots, \beta_i(t)\}} AL_i(\Delta, t) \geq \max_{t \in M} \max_{\Delta \in \mathbb{N}} AG_i(\Delta, t). \tag{1}$$

Proof. (i) Let claims profile (y, β, m) be an NEP. Suppose, contrary to what we want to show, that there exists a $t \in M$ with $\beta_N(t) = 0$. Since $\sum_{i \in N} c_i \geq E$, there exists a $t' \in M$ such that $\beta_N(t') \geq 2$. Suppose $i \in P(t')$. Player i's gain from putting a claim of size 1 on interval t equals 1, whereas his loss incurred from claiming interval t' one time less is always strictly less than 1, since $\beta_N(t') \geq 2$.[1] Hence, player i can improve, which contradicts that (y, β, m) is an NEP. Thus, $\beta_N(t) \geq 1$ for all $t \in M$.

Next suppose, contrary to (1), that there is a player i for which there exist $t, t' \in M$ with $i \in P(t)$, $\Delta_1 \in \{1, \ldots, \beta_i(t)\}$ and $\Delta_2 \in \mathbb{N}$, such that $AL_i(\Delta_1, t) < AG_i(\Delta_2, t')$. Taking away Δ_1 claims from (a part of) interval t and placing Δ_2 claims on (a sufficiently small part of) interval t' implies an improvement for player i, which contradicts that (y, β, m) is an NEP.

(ii) Now assume that $\beta_N(t) \geq 1$ for all $t \in M$ and (1) is satisfied. We show that (y, β, m) is an NEP.

Consider a claims vector $\bar{\beta}_i \neq \beta_i$ for player $i \in N$ (potentially resulting in a different partition of $[0, E]$, but in that case we consider the common refinement of both partitions). We argue that the payoff from claims vector β_i is at least as large as the payoff from $\bar{\beta}_i$. The difference in payoff between the two claims profiles arises from intervals on which the numbers of claims of player i differ. Let $M_l = \{t \in M \mid \beta_i(t) > \bar{\beta}_i(t)\}$ and $M_h = \{t \in M \mid \beta_i(t) < \bar{\beta}_i(t)\}$, respectively denote the intervals with a lower and a higher amount of claims when going from β_i to $\bar{\beta}_i$. Note that the total difference in claim on intervals from M_l is equal to the total difference in claim on intervals M_h, because player i must use his full entitlement. Moreover, the average loss from intervals from M_l is at least as high as the average gain from intervals from M_h, due to (1). Hence the claims vector β_i is a best response, as it results in a payoff at least as high as the payoff from all other claims vectors. $\square$

In the rest of the paper we will often use (sometimes without explicit mentioning) Lemma 3.1 instead of Definition 2.8 when determining Nash equilibria of claim games based on sharing rules from the TAL-family. The following lemma is useful for computing average gains and losses.

[1]In this argument and in many arguments in the sequel, we mean, implicitly, that a player may shift a claim amount from a small enough subinterval of some interval s to a small enough subinterval of some interval s'.

Lemma 3.2. *Let $\theta \in [0,1]$. Let (y, β, m) be a claims profile for the game (E, c, f^{θ}). Then for all $i \in N$ and $t \in M$,*

$$\min_{\Delta \in \{1,\dots,\beta_i(t)\}} AL_i(\Delta, t) = \begin{cases} AL_i(\beta_i(t), t) & \text{if } \theta\beta_N(t) < 1 \text{ and } i \in P(t), \\ AL_i(1, t) & \text{if } \theta\beta_N(t) \geq 1 \text{ and } i \in P(t), \end{cases} \tag{2}$$

and

$$\max_{\Delta \in \mathbb{N}} AG_i(\Delta, t) = \begin{cases} \max\{AG_i(1,t), AG_i(\beta_{max}(t) - \beta_i(t), t), \\ \quad AG_i(\beta_{max}(t) + 1 - \beta_i(t), t)\} \\ \qquad \text{if } \theta\beta_N(t) < 1 \text{ and } i \notin P_{max}(t) \\ AG_i(1,t) \qquad \text{otherwise}. \end{cases} \tag{3}$$

Proof. We first prove (2). For intervals t with $\theta\beta_N(t) < 1$, $f_i^{\theta}(\beta_i(t)) = \theta\beta_i(t)$ if $i \notin P_{max}(t)$. So the marginal loss for player $i \in P(t)$ equals θ if $i \notin P_{max}(t)$ and is strictly larger than θ if $i \in P_{max}(t)$. Thus, the marginal loss either decreases or remains constant as i removes more claims from t. Hence in all such situations, the minimum average loss is obtained when i reduces his claim to zero, i.e., $\Delta = \beta_i(t)$.

On the other hand, if $\theta\beta_N(t) \geq 1$ then the marginal loss for player $i \in P(t)$ never decreases as i removes claims from t such that $\theta\beta_N(t) \geq 1$. Moreover, the marginal loss is at most θ if $\theta\beta_N(t) \geq 1$ and at least θ if $\theta\beta_N(t) < 1$. These two observations combined imply that the minimum average loss on those intervals is equal to the marginal loss for player i, i.e., $\Delta = 1$.

We next prove (3). If $\theta\beta_N(t) < 1$ and $i \notin P_{max}(t)$ then the maximum average gain on the interval is either the marginal gain (in case $\theta\,(\beta_N(t) + \beta_{max}(t) - \beta_i(t)) \geq 1$), or the average gain of increasing the claim to $\beta_{max}(t)$ (such that i shares the remainder of $1 - \theta\,(\beta_N(t) + \beta_{max}(t) - \beta_i(t))$ with the other players $j \in P_{max}(t)$), or the average gain of increasing his claim to $\beta_{max}(t) + 1$ (such that i is the only player with the largest claim and thus has no incentive to increase his claim any further).

In all other situations, where either $i \in P_{max}(t)$ (no incentive to add more than one claim) or $\theta\beta_N(t) \geq 1$ (marginal gain can only decrease if i adds more claims), the maximum average gain is equal to the marginal gain of player i. □

3.1 Constrained equal awards

In this subsection we analyze the game (E, c, f^{CEA}). Since claims are restricted to be integer valued and, thus, $\beta_i(t) \geq 1$ for all $i \in P(t)$, we obtain for each $t \in M$:

$$f_i^{CEA}(\beta(t)) = \begin{cases} \frac{1}{|P(t)|} & \text{if } i \in P(t), \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma states that in an NEP each interval is claimed at most once by a player.

Lemma 3.3. *Let (y, β, m) be an NEP. Then $\beta_i(t) \in \{0, 1\}$ for every $i \in N$ and $t \in M$.*

Proof. Let (y, β, m) be an NEP. Suppose, contrary to what we wish to prove, that $\beta_i(t) > 1$ for some $i \in N$ and some $t \in M$. Since $c_i \leq E$, there exists an interval $t' \in M$ with $\beta_i(t') = 0$. The loss of removing one claim from t is zero and thus there is a positive gain for player i when shifting one claim from (a part of) t to (a part of) t', contradicting the assumption that (y, β, m) is an NEP. □

Since each interval t is claimed at most once by each player in equilibrium, we end up in the same situation as considered by O'Neill (1982). We obtain the following theorem.

Theorem 3.4. *Let (y, β, m) be a claims profile for the restricted problem (E, c). Equivalent are:*

(i) (y, β, m) *is an NEP in* (E, c, f^{CEA}),

(ii) *there exists a* $k \in \mathbb{N}$ *such that for all* $t \in M$, $|P(t)| \in \{k, k+1\}$ *and* $\beta_i(t) \in \{0, 1\}$ *for every* $i \in N$.

Proof. Suppose (y, β, m) is an NEP in (E, c, f^{CEA}). Since $\sum_{i \in N} c_i \geq E$, every part is claimed at least once. Lemma 3.3 implies that every player puts at most one claim on each interval. Consider interval t with the minimum number of claimants and let $k = |P(t)|$. It is sufficient to show that each interval has either k or $k+1$ claimants. Suppose, contrary to what we wish to prove, that there exists an interval $t' \in M$ which is claimed at least $k+2$ times. If player i with $\beta_i(t') = 1$ claims (a part of) t instead of t', his net gain will be at least $\frac{1}{k+1} - \frac{1}{k+2} = \frac{1}{(k+1)(k+2)} > 0$, which is a contradiction.

For the converse implication, suppose there exists a $k \in \mathbb{N}$ such that for all $t \in M$, $|P(t)| \in \{k, k+1\}$ and $\beta_i(t) \in \{0, 1\}$ for every $i \in N$. The marginal gain for every player of adding a claim to an interval yet unclaimed by that player is at most $\frac{1}{k+1}$, and the marginal gain of placing a second claim is zero. As the marginal loss for every player is at least $\frac{1}{k+1}$, (1) in Lemma 3.1 is satisfied. Hence, this claims profile is an NEP. □

The result that each interval is claimed either k or $k+1$ times is similar to the result found by Atlamaz et al. (2011) for the proportional case. In that case, however, it is possible that a player claims an interval twice, which does not happen in the equilibria in the game (E, c, f^{CEA}). This means that the set of NEPs when using the constrained equal awards rule is a subset of the equilibria found for the proportional rule. Moreover, if $\beta_i(t) \in \{0, 1\}$ for every $i \in N$ then both rules result in the same payoff vectors. Thus, we have the following result.

Corollary 3.5. *Let (E, c) be a restricted problem. Then*

$$U(E, c, f^{CEA}) \subseteq U(E, c, f^P).$$

In order to describe the associated payoff vectors, let $R \in \mathbb{R}$ denote the length of the part that is claimed $k+1$ times and $E - R$ the length of the part that is claimed k times. Then we may assume $0 \leq R < E$ and we can write

$$\sum_{i \in N} c_i = k(E - R) + (k+1)R = kE + R.$$

Notice that k and R are uniquely determined by E and $\sum_{i \in N} c_i$. To find the payoff of each player, let r_i denote the part of player i's claim invested in intervals with $k+1$ claims. Then, $c_i - r_i$ is put on intervals with k claims. Clearly, $0 \leq r_i \leq c_i$, $r_i \leq R$ and the sum of all r_i should equal $(k+1)R$. On the other hand, $c_i - r_i \leq E - R$. Summarizing, each NEP corresponds to a vector $(r_1, \ldots, r_n)$ satisfying:

$$\sum_{i \in N} r_i = (k+1)R \text{ and } \max\{c_i - (E - R), 0\} \leq r_i \leq \min\{c_i, R\} \text{ for every } i \in N.$$

Conversely, each such vector gives rise to an NEP: distribute parts of the entitlement with sizes r_i on the interval $[0, R]$ and distribute the remaining parts of the entitlements $c_i - r_i$ on the interval $[R, E]$ such that every part is claimed at most once by each player, $[0, R]$ is claimed by $k+1$ players, and $[R, E]$ is claimed by k players. Note that although there are many NEPs associated with the same vector $(r_1, \dots, r_n)$, the corresponding payoff vector is the same in all of them, namely $v = (v_i)_{i \in N}$, given by

$$v_i = \frac{r_i}{k+1} + \frac{c_i - r_i}{k} = \frac{c_i}{k} - \frac{r_i}{k(k+1)}$$

for every $i \in N$. This implies that the set of payoff vectors attainable by NEPs is determined by linear inequalities and, in particular, is a polytope.

Example 3.6. Consider the restricted problem (E, c) with $E = 4$, $n = 4$, and $c = (4, 3, 2, 1)$. For this problem, $k = 2$ and $R = 2$, and thus $r_1 = 2$, $1 \leq r_2 \leq 2$, $0 \leq r_3 \leq 2$ and $0 \leq r_4 \leq 1$, while $r_1 + r_2 + r_3 + r_4 = 6$. Hence, in an NEP in (E, c, f^{CEA}) player 1's payoff is $1\frac{2}{3}$, player 2's payoff is in $[1\frac{1}{6}, 1\frac{1}{3}]$, player 3's payoff is in $[\frac{2}{3}, \frac{5}{6}]$ (since, more precisely, $1 \leq r_3 \leq 2$, as $r_3 = 0$ contradicts with $r_1 + r_2 + r_3 + r_4 = 6$), and player 4's payoff is in $[\frac{1}{3}, \frac{1}{2}]$. An example of such a claims profile is represented in Figure 2. The corresponding equilibrium payoffs are given by $(1\frac{2}{3}, 1\frac{1}{6}, \frac{5}{6}, \frac{1}{3})$.

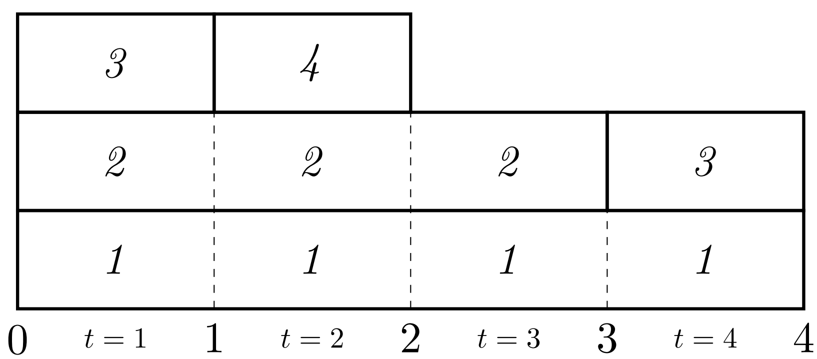

Figure 2: An illustration of a Nash equilibrium claims profile (y, β, m) for problem (E, c) with $E = 4$ and $c = (4, 3, 2, 1)$. Each square corresponds to a claim: the number in the square is the name of the player who puts that claim on the interval. Here $r_1 = 2$, $r_2 = 2$, $r_3 = 1$ and $r_4 = 1$.

3.2 Constrained equal losses

In this subsection we analyze the game (E, c, f^{CEL}). We write $P_{max}(t) = \{i \in N \mid \beta_i(t) = \beta_{max}(t)\}$ for all $t \in M$. Since $\beta_i(t) \in \mathbb{N}$ for every $i \in P(t)$, only those players $i \in P_{max}(t)$ obtain a positive share from interval t. Observe the resemblance with a first-price auction in which the winners have equal probability to win the object. More precisely, for every $i \in N$ and all $t \in M$:

$$f_i^{CEL}(\beta(t)) = \begin{cases} \frac{1}{|P_{max}(t)|} & \text{if } i \in P_{max}(t), \\ 0 & \text{otherwise.} \end{cases}$$

For the next two lemmas, let claims profile (y, β, m) be an NEP of (E, c, f^{CEL}). The first lemma implies that every part is claimed by at most two different players.

Lemma 3.7. *For all $t \in M$, we have $|P_{max}(t)| \in \{1, 2\}$.*

Proof. Since $\beta_N(t) \geq 1$ for all $t \in M$, we suppose that $|P_{max}(t)| \geq 3$ for some $t \in M$ and derive a contradiction. Consider a player i with $\beta_i(t) = \beta_{max}(t)$. The marginal loss on t is equal to $\frac{1}{|P_{max}(t)|}$ and the marginal gain on t is equal to $1 - \frac{1}{|P_{max}(t)|}$. Since $|P_{max}(t)| \geq 3$, we have that the marginal loss is smaller than the marginal gain, which contradicts Lemma 3.1. $\square$

The second lemma states that every interval is claimed at most once by the same player.

Lemma 3.8. *For every $i \in N$ and all $t \in M$, we have $\beta_i(t) \in \{0, 1\}$. Consequently, $|P_{max}(t)| = |P(t)|$ for all $t \in M$.*

Proof. The proof is by contradiction. Suppose there exists an interval $t \in M$ with $\beta_{max}(t) \geq 2$. By Lemma 3.7, $|P_{max}(t)| \in \{1, 2\}$. We derive a contradiction for both cases.

Suppose $|P_{max}(t)| = 1$ and consider player i with $\beta_i(t) = \beta_{max}(t) \geq 2$. If $|P(t)| = 1$, then player i can reduce his claim on t without loss and achieve a positive gain by putting the free claim amount on a part of the estate for which he is not yet the sole winner. If $|P(t)| > 1$ then player $j \in P(t) \setminus P_{max}(t)$ can reduce his claim on t without any loss and put the free claim amount on a part of the estate for which he is not yet the sole winner.

Suppose that $|P_{max}(t)| = 2$ with $\beta_i(t) = \beta_j(t) \geq 2$ for $i, j \in P_{max}(t)$ and $i \neq j$. The minimum average loss of i on t is $\frac{1}{2\beta_i(t)}$, while the marginal gain equals $\frac{1}{2}$. Since $\beta_i(t) \geq 2$, this contradicts Lemma 3.1

Since $\beta_i(t) \in \{0, 1\}$ for all $i \in N$ and $t \in M$, the second statement of the lemma follows immediately. $\square$

The main result of this subsection is the following theorem, which presents a full characterization of the NEPs.

Theorem 3.9. *Let (y, β, m) be a claims profile for the restricted problem (E, c). Equivalent are:*

(i) *(y, β, m) is an NEP in (E, c, f^{CEL}),*

(ii) *for all $t \in M$, $|P(t)| \in \{1, 2\}$ and $\beta_i(t) \in \{0, 1\}$ for every $i \in N$.*

Proof. Suppose (y, β, m) is an NEP in (E, c, u^{CEL}). Lemmas 3.7 and 3.8 imply that $|P(t)| \in \{1, 2\}$ and $\beta_i(t) \in \{0, 1\}$ for every $i \in N$ and all $t \in M$.

Conversely, suppose for all $t \in M$, $|P(t)| \in \{1, 2\}$ and $\beta_i(t) \in \{0, 1\}$ for every $i \in N$. If $|P(t)| = 1$, player $i \in P(t)$ is never able to gain by deviating from t. If $|P(t)| = 2$, player $i \in P(t)$ faces a marginal loss of $\frac{1}{2}$ if he reduces his claim. The marginal gain of placing one claim on either the same or a different interval is at most $\frac{1}{2}$ and the average gain of placing two claims is also at most $\frac{1}{2}$. Hence, Lemma 3.1 implies that (y, β, m) is an NEP. $\square$

Theorem 3.9 shows that the set of NEPs when using the constrained equal losses rule is a subset of the equilibria found for the constrained equal awards rule, which was again a subset of the equilibria found for the proportional rule. Moreover, since $\beta_i(t) \in \{0, 1\}$ for every $i \in N$, all three rules result in the same payoff vectors. We thus have the following result.

Corollary 3.10. *Let (E, c) be a restricted problem. Then*

$$U(E, c, f^{CEL}) \subseteq U(E, c, f^{CEA}) \subseteq U(E, c, f^{P}).$$

Another consequence of the Theorem 3.9 is an existence condition for NEP.

Corollary 3.11. *An NEP exists in the game (E, c, f^{CEL}) if and only if $\sum_{i\in N} c_i \leq 2E$.*

Proof. The only-if part follows from Theorem 3.9. For the if-part, note that if $\sum_{i\in N} c_i \leq 2E$, we can iteratively, from left to right, put the claim of each player on a part of the estate which is not claimed yet and start over again on the left if the complete estate is claimed once. Since we consider restricted problems, we will end up with a claims profile satisfying the conditions of Theorem 3.9. □

The final result in this subsection describes every payoff vector in the set of equilibrium payoff vectors.

Corollary 3.12. *The following two statements are equivalent:*

(i) $v = (v_i)_{i\in N} \in U(E, c, f^{CEL})$,

(ii) *there exists a vector* $r = (r_1, \dots, r_n)$ *such that* $0 \leq r_i \leq \min\{c_i, \sum_{i\in N} c_i - E\}$ *for every* $i \in N$ *and* $\sum_{i\in N} r_i = 2\left(\sum_{i\in N} c_i - E\right)$ *with* $v_i = c_i - \frac{1}{2}r_i$ *for every* $i \in N$.

Proof. Suppose (y, β, m) is an NEP in (E, c, f^{CEL}). In view of Theorem 3.9, we let $R \in \mathbb{R}$ denote the length of the part that is claimed by two different players and $E - R$ the length of the part that is claimed only once. Since $\sum_{i\in N} c_i = 2R + (E - R)$, we have $R = \sum_{i\in N} c_i - E$. To find the payoff of each player, let r_i denote the part of player i's claim invested in intervals with two claims. Then, $c_i - r_i$ is put on intervals with only one claim. Clearly, $0 \leq r_i \leq c_i$, $r_i \leq R$ and the sum of r_i should equal $2R$. These are exactly the conditions of the corollary. The corresponding payoff for each player i in such a claims profile is equal to

$$\frac{1}{2}r_i + c_i - r_i = c_i - \frac{1}{2}r_i.$$

Conversely, suppose there exists a vector r satisfying the above conditions. Each such vector r gives rise to an NEP: distribute parts of the entitlement with sizes r_i on the interval $[0, \sum_{i\in N} c_i - E]$, such that two players claim each part once; and distribute the remaining parts of the entitlements $c_i - r_i$ on the interval $[\sum_{i\in N} c_i - E, E]$, such that each part is claimed once by one player. To see that this distribution is feasible, note that since $\sum_{i\in N}(c_i - r_i) = \sum_{i\in N} c_i - 2\left(\sum_{i\in N} c_i - E\right) = 2E - \sum_{i\in N} c_i$ together with $r_i \leq c_i$, implies that $c_i - r_i \leq 2E - \sum_{i\in N} c_i$. Hence we found a feasible claims profile that satisfies the conditions of Theorem 3.9, that is, we have an NEP. □

3.3 TAL-family

In this subsection we consider the game (E, c, f^θ) with $\theta \in [0, 1]$ and generalize some of the results of the previous two subsections.

First, we assume $\theta \in [\frac{1}{2}, 1]$. Let (y, β, m) be a claims profile. Note that if $|P(t)| \geq 2$ for $t \in M$, then the constrained equal awards rule with $\theta\beta(t)$ as entitlements is used to determine the shares. As $\theta \geq \frac{1}{2}$, this implies that the interval is equally divided among the claimants. Thus, for $\theta \in [\frac{1}{2}, 1]$ and for each $i \in N$ and each $t \in M$:

$$f_i^\theta(\beta(t)) = \begin{cases} \frac{1}{|P(t)|} & \text{if } i \in P(t), \\ 0 & \text{otherwise.} \end{cases}$$

This is exactly the same sharing rule as for the game (E, c, f^{CEA}). Accordingly, the same analysis applies.

If $\theta \in [\frac{1}{3}, \frac{1}{2}]$, then the induced sharing rule is different, but we still obtain the same set of NEPs. (We omit the formal proof, in which the main observation is that if $|P(t)| = 2$ for $t \in M$, then $\beta_i(t) \leq 2$ for all $i \in N$ and otherwise $\beta_i(t) \leq 1$ for all $i \in N$ and all $t \in M$. This is mainly due to the constrained equal awards part of these rules.)

Lemma 3.13. *Any rule in the TAL-family with $\theta \in [\frac{1}{3}, 1]$ results in the same set of NEPs and in the same set of equilibrium payoffs in the game (E, c, f^{θ}), equal to $U(E, c, f^{CEA})$.*

The following theorem characterizes all NEPs for estate division problems with $\sum_{i \in N} c_i \leq 2E$, for any TAL-rule f^{θ} with $0 \leq \theta \leq 1$.

Theorem 3.14. *Let (E, c) be a restricted problem with $\sum_{i \in N} c_i \leq 2E$, let $\theta \in [0, 1]$, and let (y, β, m) be a claims profile. Equivalent are:*

(i) *(y, β, m) is an NEP in (E, c, f^{θ}),*

(ii) *$|P(t)| \in \{1, 2\}$ and $\beta_i(t) \in \{0, 1\}$ for all for all $t \in M$ and $i \in N$.*

Proof. By Theorems 3.4 and 3.9 and Lemma 3.13, we can restrict ourselves to $\theta \in (0, \frac{1}{3})$.

Suppose (y, β, m) is an NEP in (E, c, f^{θ}) with $\theta \in (0, \frac{1}{3})$. Notice that $\beta_N(t) \geq 1$ for all $t \in M$. Since we consider restricted problems, $\beta_N(t) = 1$ if $|P(t)| = 1$. Hence it is sufficient to show that $\beta_N(t) \leq 2$ for all $t \in M$.

Suppose that $\beta_N(t) \geq 3$ for some $t \in M$. We obtain a contradiction by showing that (1) in Lemma 3.1 is not satisfied. More precisely, since $\sum_{i \in N} c_i \leq 2E$ there must exist an interval t' with $|P(t')| = 1$. It is sufficient to prove that at least two players $i \in P(t)$ have a marginal loss strictly less than $\frac{1}{2}$ on t, since this implies that at least one of them has a marginal gain of $\frac{1}{2}$ on t' by putting a claim on t', as $|P(t')| = 1$; which establishes the contradiction. In order to do so, we distinguish two cases: $|P(t)| = 2$ and $|P(t)| \geq 3$.

For the first case, suppose that $\beta_N(t) \geq 3$ with $|P(t)| = 2$. If $\beta_i(t) = \beta_j(t) \geq 2$ for $i, j \in P(t)$ and $i \neq j$, then both i and j's shares are equal to $\frac{1}{2}$. So both i and j have a marginal loss strictly less than $\frac{1}{2}$, as the share on t remains positive after removing one claim.

If, on the other hand, $\beta_i(t) > \beta_j(t) = 1$, then $0 < f_j(\beta(t)) < \frac{1}{2}$ and thus $\frac{1}{2} < f_i(\beta(t)) < 1$. Since j's share is strictly less than $\frac{1}{2}$, his loss of removing one claim is strictly less than $\frac{1}{2}$. If i reduces his claim by one, then $\beta_i(t) \geq \beta_j(t)$ which means that i's share remains at least $\frac{1}{2}$. Hence also his loss in share is strictly less than $\frac{1}{2}$.

For the second case, assume that $P(t) \geq 3$. Since $f_i(\beta(t)) > 0$ for all $i \in P(t)$, there must be at least two players in $P(t)$ with $f_i(\beta(t)) < \frac{1}{2}$, and then these two players face a marginal loss strictly less than $\frac{1}{2}$.

For the converse, assume that $|P(t)| \in \{1, 2\}$ and $\beta_i(t) \in \{0, 1\}$ for all for all $t \in M$ and $i \in N$. If $|P(t)| = 1$, player $i \in P(t)$ is never able to gain by changing his claim on t. If $|P(t)| = 2$, player $i \in P(t)$ faces a marginal loss of $\frac{1}{2}$ if he reduces his claim, whereas the average gain from placing one or two claims is at most $\frac{1}{2}$. Hence this claims profile satisfies (1) in Lemma 3.1, and thus is an NEP. □

The only claim games that we have not discussed yet, are those associated to problems with $\sum_{i \in N} c_i > 2E$ and a sharing rule f^{θ} with $0 < \theta < \frac{1}{3}$. It turns out that in this case an NEP need not always exist. The precise existence conditions are still unclear, but the next lemma provides a sufficient condition, namely a condition under which a claims profile with $\beta_N(t) \in \{k, k+1\}$ and $\beta_i(t) \in \{0,1\}$ for all $i \in N$ constitutes an NEP.

Lemma 3.15. *Let $\sum_{i\in N} c_i > 2E$. Let $\theta \geq \frac{k^2-k-1}{k^3-k}$, where $k = \max\{k' \in \mathbb{N} \mid \sum_{i\in N} c_i > k'E\}$. Then there exists an NEP in (E, c, f^θ).*

Proof. Let (y, β, m) be a claims profile such that for all $t \in M$, $|P(t)| \in \{k, k+1\}$ and $\beta_i(t) \in \{0,1\}$ for all $i \in N$. We show that this profile is an NEP. We consider two cases: $\theta \geq \frac{1}{k+1}$ and $\frac{k^2-k-1}{k^3-k} \leq \theta < \frac{1}{k+1}$.

First, suppose that $\theta \geq \frac{1}{k+1}$. Note that every interval $t \in M$ with $k+1$ claimants uses the constrained equal awards rule as sharing rule, which implies that there is no incentive to put a second claim on those intervals. Moreover, the loss of removing one claim is at least $\frac{1}{k+1}$ (on an interval with $k+1$ claimants). This implies that claiming an unclaimed interval does not result in a positive gain (as this gain is at most $\frac{1}{k+1}$), and that the gain of placing a second claim is at most $\frac{2}{k+1} - \frac{1}{k}$ (on an interval with k claimants and if $\theta = \frac{1}{k+1}$), which is also not profitable. Hence the claims profile is an NEP.

Second, suppose that $\frac{k^2-k-1}{k^3-k} \leq \theta < \frac{1}{k+1}$. The loss of removing one claim is at least $\frac{1}{k+1}$ (on an interval with $k+1$ claimants). The only possible way for a player i to gain is by placing a second claim on an interval t with k claimants and with $i \in P(t)$ (in all other cases the maximum average gain is at most $\frac{1}{k+1}$). As $\theta(k+1) < 1$ and as i becomes the only player with two claims on the interval, the gain of this second claim is equal to $1 - \theta(k-1) - \frac{1}{k}$. Since $\theta \geq \frac{k^2-k-1}{k^3-k}$, we have that $\frac{1}{k+1} \geq 1 - \theta(k-1) - \frac{1}{k}$ and thus no deviation is profitable. □

A consequence of Lemma 3.15 is that if $\theta \geq \frac{5}{24}$ – which is the maximum attained by $\frac{k^2-k-1}{k^3-k}$, namely for $k = 3$ – then there always exists an NEP.

Another unsolved problem is the question of how to characterize NEPs for claim games (E, c, f^θ) with $0 < \theta < \frac{1}{3}$ (in case they exist). The following example shows that such a characterization will be different from the one for the games (E, c, f^{CEA}) or (E, c, f^P). The claims profile in the example is an NEP in $(E, c, f^{\frac{3}{10}})$, although it has a different form and payoffs compared to the equilibria of the other two games.

Example 3.16. Consider the four-player problem (E, c) with $E = 2$ and $c = (2, 2, 1, 1)$. Consider the claims profile represented in Figure 3: let $[0, 1]$ be claimed twice by player 1 and once by player 3, and let $[1, 2]$ be claimed twice by players 2 and once by player 4. This claims profile satisfies the condition in Lemma 3.1 for $\theta = \frac{3}{10}$ and thus is an NEP in $(E, c, f^{\frac{3}{10}})$. The corresponding payoff vector is $(\frac{7}{10}, \frac{7}{10}, \frac{3}{10}, \frac{3}{10})$. Note that since $k = 3$ and $R = 0$, the unique equilibrium payoff vector in the game (E, c, f^{CEA}) or in the game (E, c, f^P) is $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3})$, hence $(\frac{7}{10}, \frac{7}{10}, \frac{3}{10}, \frac{3}{10}) \notin U(E, c, f^P)$.

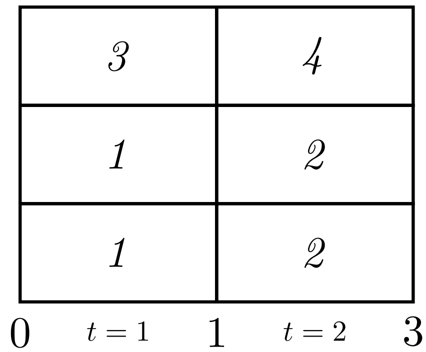

Figure 3: An illustration of a Nash equilibrium claims profile (y, β, m) for $(E, c, u^{\frac{3}{10}})$ with $E = 2$ and $c = (2, 2, 1, 1)$. Each square corresponds to a claim: the number in the square is the name of the player who puts that claim on the interval. Notice that both player 1 and 2 claim an interval twice although $R = 0$.

3.4 Comparison of the results for restricted problems with integer claims

In order to compare the different characterizations of NEPs in claim games with integer claims, associated with restricted problems, we assume that $\sum_{i \in N} c_i \leq 2E$. For these problems, Theorem 3.14 shows that existence of NEPs is assured for all rules from the TAL-family. Surprisingly, if we compare the results of Atlamaz et al. (2011) with Theorem 3.14 for these estate division problems, then we see that all considered rules: the proportional rule and all TAL-family rules, result in the same set of NEPs. In these NEPs every player claims each interval at most once and each interval is claimed by either one or two players. Moreover, for these claims profiles all considered rules yield the same payoff vector. We obtain the following result.

Corollary 3.17. *Let $\sum_{i \in N} c_i \leq 2E$. Then the proportional rule and any rule from the TAL-family results in the same set of NEPs, and in the same set of equilibrium payoff vectors, as described in Corollary 3.12.*

Remark 3.18. If there are two players ($n = 2$) then it is easy to see that in an NEP the claims should have minimal overlap. So the only choice for (r_1, r_2) is $r_1 = r_2 = c_1 + c_2 - E$, resulting in the unique equilibrium payoffs $(\frac{E+c_1-c_2}{2}, \frac{E+c_2-c_1}{2})$. These payoffs coincide with the payoffs assigned by concede-and-divide (Thomson, 2003).

The following example presents an estate division problem for which the Talmud rule is the only rule from the TAL-family that assigns an equilibrium payoff vector to the problem. Also the proportional rule does not result in an equilibrium payoff vector for this problem. (Atlamaz et al., 2011, give a different example to show that the proportional rule need not be obtained in equilibrium.)

Example 3.19. Consider the restricted problem (E, c) with $E = 4$ and $c = (4, 2, 1)$, cf. Example 2.7. Since $r_1 = 3$, $r_2 = 2$ and $r_1 = 1$, there is a unique equilibrium payoff vector (see Figure 4). The payoffs in this equilibrium are equal to $(2\frac{1}{2}, 1, \frac{1}{2})$. The constrained equal awards rule assigns the payoff vector $(1\frac{1}{2}, 1\frac{1}{2}, 1)$, the constrained equal losses rule assigns the payoff vector $(3, 1, 0)$, and the Talmud rule assigns the payoff vector $(2\frac{1}{2}, 1, \frac{1}{2})$ to the problem (E, c), see Example 2.7. Moreover, it can be shown that every rule from the TAL-family with $\theta \in [0, 1]$ assigns a payoff of exactly

θ to player 3. Hence the only payoff vector attainable in equilibrium is assigned by the Talmud rule. The payoff vector of the proportional rule is equal to $(2\frac{2}{7}, 1\frac{1}{7}, \frac{4}{7})$, which is not in the set of equilibrium payoff vectors either.

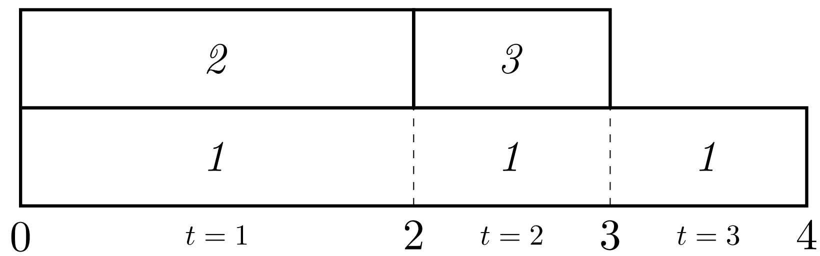

Figure 4: An illustration of a Nash equilibrium claims profile (y, β, m) for problem (E, c) with $E = 4$ and $c = (4, 2, 1)$. Here $r_1 = 3$, $r_2 = 2$ and $r_1 = 1$.

Example 3.19 shows that that the rules of the TAL-family except the Talmud rule, and also the proportional rule, need not result in an equilibrium payoff vector in the claim game with these rules as sharing rules. The next result shows that the Talmud rule, applied to an estate division problem with $\sum_{i \in N} c_i \leq 2E$, *always* leads to an equilibrium vector in the associated claim game based on this rule. This result may be seen as an equilibrium argument in favor of the Talmud rule, at least within the family of TAL-rules and the proportional rule.

Theorem 3.20. *Let $\sum_{i \in N} c_i \leq 2E$. Then the Talmud rule applied to (E, c) results in an equilibrium payoff vector in (E, c, f^T).*

Proof. We show that there exists a vector $(r_1, \ldots, r_n)$, satisfying the conditions of Corollary 3.12, with the same payoffs as the Talmud rule.

Note that for the Talmud rule, we have two sets of players. Let J denote the set of players j who receive strictly more than $\frac{1}{2}c_j$, then players $i \in N \setminus J$ receive exactly $\frac{1}{2}c_i$. The Talmud rule solves the following equation:

$$\sum_{j \in J} (c_j - \mu) + \sum_{i \in N \setminus J} \frac{1}{2} c_i = E,$$

which implies that $\mu = \frac{1}{|J|} \left(\sum_{j \in J} c_j + \sum_{i \in N \setminus J} \frac{1}{2} c_i - E \right)$ if $J \neq \emptyset$. Thus, the Talmud rule prescribes a payoff of $c_j - \mu$ to player $j \in J$ and $\frac{1}{2}c_i$ to player $i \in N \setminus J$.

Consider the vector $(r_1, \ldots, r_n)$ with $r_j = 2\mu$ for $j \in J$ and $r_i = c_i$ for $i \in N \setminus J$. Note the following properties:

(i) $2\mu \geq 0$,

(ii) $2\mu < c_j$ for all $j \in J$, since $c_j - \mu > \frac{1}{2}c_j$,

(iii) if $J \neq \emptyset$, then $c_i \leq 2\mu \leq \sum_{i \in N} c_i - E$ for all $i \in N \setminus J$, since $c_j \leq E$ for $j \in J$ if $|J| = 1$ and $(|J| - 1) \sum_{i \in N} c_i - \sum_{j \in J} c_j \geq (|J| - 2) \sum_{i \in N} c_i \geq (|J| - 2)\, E$ if $|J| \geq 2$,

(iv) $\sum_{i \in N} r_i = \sum_{j \in J} 2\mu + \sum_{i \in N \setminus J} c_i = 2 \left(\sum_{i \in N} c_i - E \right)$.

The vector $(r_1, \ldots, r_n)$ satisfies the conditions of Corollary 3.12, which means that the corresponding claims profile (in which r_i is distributed on intervals that are claimed twice and $c_i - r_i$ is distributed on intervals that are claimed once) is an NEP. Moreover, the payoff to player $j \in J$ is $c_j - \mu$, and to player $i \in N \setminus J$ it is $\frac{1}{2}c_i$. $\square$

For problems with $\sum_{i \in N} c_i > 2E$, different sharing rules in general result in different sets of equilibrium payoffs. The following example shows that the payoff vector from the Talmud rule does not need to be an equilibrium payoff vector in games (E, c, f^θ) with $\theta \in [0, 1]$ or in (E, c, u^P).

Example 3.21. Consider the restricted problem (E, c) with $n = 3$, $E = 4$, and $c = (4, 4, 1)$. The payoff vector of the Talmud rule is $(1\frac{1}{2}, 1\frac{1}{2}, 1)$. Since $k = 2$ and $R = 1$, the unique NEP payoff vector (see Figure 5) in the game (E, c, f^{CEA}) or in the game (E, c, f^P), is equal to $(1\frac{5}{6}, 1\frac{5}{6}, \frac{1}{3})$. Moreover, note that for sharing rules f^θ with $\theta \in (0, \frac{1}{3})$ it is impossible to have an interval t' with $|P(t')| = 1$, as there always exists an interval t with $\beta_N(t) \geq 3$. Thus, player 3's payoff is strictly less than 1 in equilibria in the associated games. Hence, the payoff vector assigned by the Talmud rule is not attainable in any equilibrium, in any of these games.

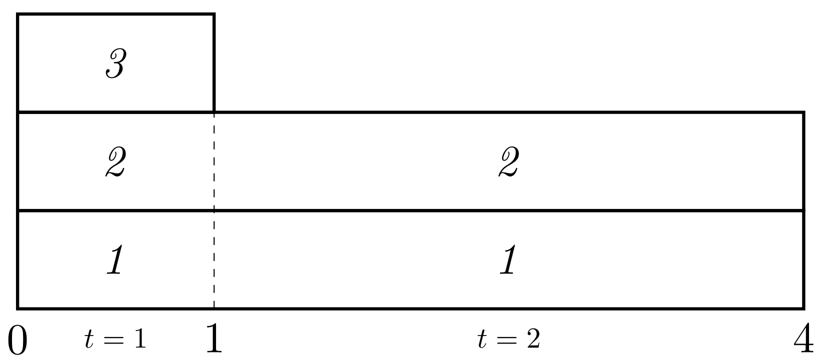

Figure 5: An illustration of a Nash equilibrium claims profile (y, β, m) for problem (E, c) with $E = 4$ and $c = (4, 4, 1)$. Here $r_1 = 1$, $r_2 = 1$ and $r_1 = 1$.

4 Restricted problems and arbitrary claims

Consider the set of restricted problems, and let (y, β, m) be a claims profile in (E, c). We now assume that $\beta_i : M \to \mathbb{R}_+$ for all $i \in N$. In particular, Lemma 3.1 no longer applies.

Recall that the uniform claims profile is the claims profile in which each player puts a claim of size $\frac{c_i}{E}$ on the complete interval $[0, E]$. Also recall that the payoffs assigned by any rule from the TAL-family or by the proportional rule to the estate division problem (E, c) are equal to the payoffs in the claim game associated with that rule under the uniform claims profile.[2]

4.1 Constrained equal awards

In this subsection we analyze the game (E, c, f^{CEA}). Recall that Definition 2.2 gives the corresponding sharing rule. The first lemma shows that the uniform claims profile is an NEP.

[2]The crucial property here is scale invariance of a rule, see Thomson (2003).

Lemma 4.1. *The uniform claims profile $(y, \beta, 1)$ is an NEP in (E, c, f^{CEA}).*

Proof. Consider player $i \in N$, then either $u_i^{CEA}(y, \beta, 1) = c_i$ or $u_i^{CEA}(y, \beta, 1) < c_i$. In the former case, it is impossible for i to improve. In the latter case, note that because of the uniform claims profile, $\beta_i(t) > \lambda$ for all $t \in M$ (with λ as in Definition 2.2). This implies that player i cannot gain by deviating. So $(y, \beta, 1)$ is an NEP. $\square$

The following example shows that the uniform claims profile is not necessarily the unique NEP.

Example 4.2. Consider the two-player problem (E, c) with $E = 2$ and $c = (1\frac{3}{4}, \frac{3}{4})$. Suppose player 1 puts a claim of $\frac{3}{4}$ on $[0, 1]$ and a claim of 1 on $[1, 2]$; and player 2 puts a claim of $\frac{1}{2}$ on $[0, 1]$ and a claim of $\frac{1}{4}$ on $[1, 2]$. This claims profile is an NEP, although it is not the uniform claims profile. Observe that $\lambda = \frac{1}{2}$ on $[0, 1]$ and $\lambda = \frac{3}{4}$ on $[1, 2]$, so that the associated payoffs are $(1\frac{1}{4}, \frac{3}{4})$. These are the same payoffs the players obtain if the constrained equal awards rule is applied to (E, c).

In this example the equilibrium payoffs are equal to the payoffs assigned by the constrained equal awards rule. The following theorem shows that this is true in general.

Theorem 4.3. *All NEPs result in the same payoffs, equal to the payoffs assigned by the constrained equal awards rule.*

Proof. Suppose (y, β, m) is an NEP in (E, c, f^{CEA}). Suppose there exists a player $i \in N$ and $t, t' \in M$ with $\beta_i(t) > \lambda$ and $\beta_i(t') < \lambda$. We derive a contradiction. If i decreases his claim on t to λ, by definition he will not incur any loss. However, since $\beta_i(t') < \lambda$, increasing his claim on t' leads to a positive gain, which is in a contradiction with the NEP assumption.

So in an NEP, we have two sets of players: let J denote the set of players with $\beta_j(t) \geq \lambda$ for all $t \in M$, where at least one of the inequalities is strict, and let $N \setminus J$ denote the set of players with $\beta_i(t) \leq \lambda$ for all $t \in M$.

Note that for all $t \in M$, $f_j^{CEA}(\beta(t)) = \lambda$ for all $j \in J$, and $f_i^{CEA}(\beta(t)) = \beta_i(t)$ for all $i \in N \setminus J$. In other words, all players $i \in N \setminus J$ receive exactly their claim. All players $j \in J$ receive the same payoff, which is at least as much as the players $i \in N \setminus J$, but strictly less than their claim. This is precisely the payoff each player obtains from the constrained equal awards rule applied to the estate division problem (E, c). $\square$

4.2 Constrained equal losses

In this subsection we analyze the game (E, c, f^{CEL}). Recall that Definition 2.3 defines the corresponding sharing rule. The following example shows that the uniform claims profile is not always an NEP.

Example 4.4. Consider the three-player problem (E, c) with $E = 4$ and $c = (4, 2, 1)$. The payoffs in the uniform claims profile are $(3, 1, 0)$. If player 3 puts one claim on $[0, 1]$ instead, he receives payoff $\frac{1}{2}$. Thus, the uniform claims profile is not an NEP in (E, c, f^{CEL}).

The following lemma provides a necessary condition for an NEP in the game (E, c, f^{CEL}).

Lemma 4.5. *Let (y, β, m) be an NEP. Then $\beta_N(t) - \beta_{min}(t) \leq 1$ for all $t \in M$.*

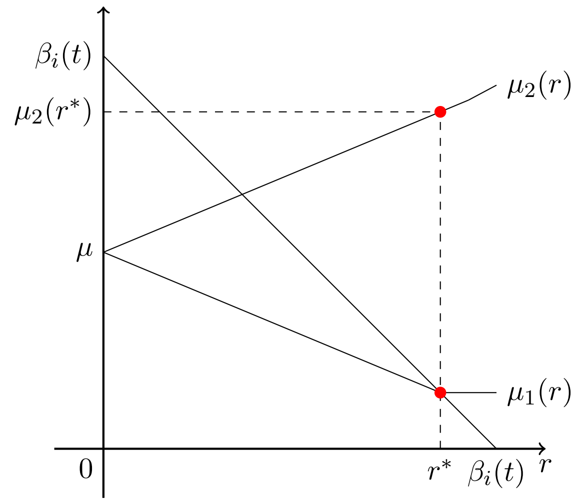

Figure 6: An illustration of the level of μ on the first and second half of t. The value of r represents the amount of claim transferred from the first to the second half of t. The line $\mu_1(r)$ denotes the level of μ on the first half of t and the line $\mu_2(r)$ denotes the level of μ on the second half of t. The line $\beta_i(t) - r$ represents the amount of claim of player i left on the first half of the interval. Notice that the slope of $\mu_1(r)$ is $\frac{-1}{|P(t)|}$ until r^* and the slope of $\mu_2(r)$ is $\frac{1}{|P(t)|}$ until r^*.

Proof. Let $t \in M$. The statement trivially holds in case $|P(t)| = 1$. For the other cases, we argue by contradiction.

Suppose that $\beta_N(t) - \beta_{min}(t) > 1$, which implies that $|P(t)| \geq 2$. We will show that a player i with $\beta_i(t) = \beta_{min}(t)$ can gain by deviating. Observe that in an NEP: $f_j(\beta(t)) > 0$ for all $j \in P(t)$. Otherwise, a player j with $f_j(\beta(t)) = 0$ could put his claim on a sufficiently small subinterval of t in order to gain a positive amount. Let player i divide t into two equally large intervals. We will now show that player i can transfer an amount of r from his claim on the first half to the second half, such that his share on the first half equals zero but without changing anyone's total share of the interval. The remaining claim on the first half can then be used to increase his total share of t, since the marginal loss on the first half after the transfer is zero.

Since the losses are equally distributed among all claimants, the decrease in claim on the first half, leads to a decrease in μ (with μ as in Definition 2.3) at a constant rate of $\frac{1}{|P(t)|}$. On the second half, the increase in claim increases μ at a constant rate of $\frac{1}{|P(t)|}$, until some claimant's share drops to zero. We will show that during this procedure all claimants' shares remain positive on the second half, which implies that total share of every player on t stays the same. To this end, see Figure 6.

At the point $r = r^* = \frac{|P(t)|(\beta_i(t)-\mu)}{|P(t)|-1}$, we see that $\mu_1(r)$ – i.e., the new value of μ on the first half after a transfer of r – intersects with the line $\beta_i(t) - r$. This means that we are at the point at which the share of player i dropped to zero on the first half. In order to see what happens on the second half, note that

$$\mu_2(r^*) = \frac{1}{|P(t)|}r^* + \mu = \frac{\beta_i(t)}{|P(t)| - 1} + \frac{(|P(t)| - 2)\mu}{|P(t)| - 1}.$$

Because of the right-hand side of the expression, we will first treat the case $|P(t)| = 2$ separately.

If $|P(t)| = 2$ with $\beta_i(t) = \beta_j(t) > 1$ for $i, j \in P(t)$, then $\mu_2(r^*) = \beta_i(t) = \beta_j(t)$ – where $\mu_2(r)$ is, analogously, the new value of μ on the second half. This implies that the share of player j dropped to zero on the second half after the transfer, which means that i has a share of one on the second half and j has a share of one on the first half. Notice that the remaining claim of $\mu_1(r^*) = \beta_i(t) - 1$ on the first half can be placed on a sufficiently small subinterval of the first half such that i gains a positive amount on this first half as well. This, however, means that i is able to gain by deviating, which is a contradiction.

On the other hand, if $|P(t)| = 2$ with $\beta_i(t) < \beta_j(t)$ for $i, j \in P(t)$ or if $|P(t)| \geq 3$ then $\mu_2(r^*) < \beta_j(t)$ for all $j \in P(t) \setminus i$. This means that the share of every player $j \in P(t) \setminus i$ remains positive on the second half after the transfer of i. This conclusion is obvious for $|P(t)| = 2$ with $\beta_i(t) < \beta_j(t)$, since then $\mu_2(r^*) = \beta_i(t) < \beta_j(t)$. For $|P(t)| \geq 3$, notice that it is sufficient to show that $\mu_2(r^*) < \beta_i(t)$ (as $\beta_i(t) \leq \beta_j(t)$ for all $j \in P(t) \setminus i$), which is equivalent to showing $\mu < \beta_i(t)$. This is true since $f_j(\beta(t)) > 0$ for all $j \in P(t)$, thus in particular for player j himself.

In order to show that i can actually gain by deviating, observe that since $\mu_1(r^*) = \frac{\beta_N(t) - \beta_i(t) - 1}{n-1} > 0$ by assumption, player i could decrease his claim by an additional positive, but sufficiently small, amount of r on the first half without any loss, while having a marginal gain of $1 - \frac{1}{|P(t)|}$ on the second half. This contradicts that (y, β, m) is an NEP. □

As a corollary to the previous lemma, we obtain a necessary and sufficient condition for the uniform claims profile to be an NEP.

Corollary 4.6. *The uniform claims profile is an NEP in (E, c, f^{CEL}) if and only if $\sum_{i \in N} c_i - \min_{i \in N} c_i \leq E$.*

Proof. If the uniform claims profile is an NEP, then Lemma 4.5 implies $\sum_{i \in N} \frac{c_i}{E} - \min_{i \in N} \frac{c_i}{E} \leq 1$, which is the only-if statement. For the if-part, suppose that $\sum_{i \in N} c_i - \min_{i \in N} c_i \leq E$ and consider the uniform claims profile. The average gain of any increase on an interval is at most $\frac{n-1}{n}$, as the increased loss is equally divided among all players as long as the shares remain positive. The average loss of a decrease is at least $\frac{n-1}{n}$ by similar arguments. Hence, no player has a profitable deviation. □

The following theorem gives a full description of every possible NEP. We restrict our attention to those estate division problems with $\sum_{i \in N} c_i > E$. The reason for this is that if $\sum_{i \in N} c_i = E$ then in equilibrium every interval $t \in M$ satisfies $\beta_N(t) = 1$, which means that everyone receives his claim.

Theorem 4.7. *Let (y, β, m) be a claims profile for the restricted problem (E, c) with $\sum_{i \in N} c_i > E$. Then the following statements are equivalent:*

(i) *(y, β, m) is an NEP in (E, c, f^{CEL}).*

(ii) *Let $k = \max\{\, |P(t)| \mid t \in M\}$. Then $k \geq 2$ and the following three conditions are satisfied:*

(a) *For all $t \in M$, if $|P(t)| < k$, then $\beta_N(t) = 1$.*

(b) *For all $t \in M$, if $|P(t)| = k$ and $P(t) = P(t')$ for all $t' \in M$ with $|P(t')| = k$, then $1 - \beta_{min}(t) \leq \beta_N(t) - \beta_{min}(t) \leq 1$.*

(c) *For all $t \in M$, if $|P(t)| = k$ and $P(t) \neq P(t')$ for some $t' \in M$ with $|P(t')| = k$, then $\beta_N(t) - \beta_{min}(t) = 1$.*

Proof. For the implication $(i) \Rightarrow (ii)$, let (y, β, m) be an NEP in (E, c, f^{CEL}) and define k as in (ii). Observe that since a player can not be the sole winner of every interval, $\beta_N(t) = 1$ if $|P(t)| = 1$. This implies that $k \geq 2$ and that we only need to consider intervals $t \in M$ with $|P(t)| \geq 2$. By Lemma 4.5 and since $\beta_N(t) \geq 1$ for all $t \in M$, we have that all $t \in M$ with $|P(t)| \geq 2$ satisfy

$$1 - \beta_{min}(t) \leq \beta_N(t) - \beta_{min}(t) \leq 1. \tag{4}$$

This proves (b).

Lemma 4.5 only considers deviations within a specific interval. The following claim, which is used to prove (a) and (c), considers deviations between two different intervals. For the remainder of this proof, we denote $\beta_{(1)}(t) \geq \beta_{(2)}(t) \geq \ldots \geq \beta_{(|P(t)|)}(t) > 0$ for all $t \in M$.

Claim. *Let there exists a player $i \in P(t')$, $i \notin P(t)$, where $t, t' \in M$ with $2 \leq |P(t')| \leq |P(t)|$ satisfy (4) with $\beta_N(t') > 1$. Then $\sum_{i=1}^{|P(t')|-1} \beta_{(i)}(t) \geq 1$.*

Proof of claim. In a NEP, player i is not able to gain by putting some of his claim of interval t' on interval t. The average loss of a sufficiently small decrease in claim on t' equals $1 - \frac{1}{|P(t')|}$.

The best player i can do on t is to place a claim such that $\mu = \beta_{(|P(t')|)}(t)$, meaning that at most $|P(t')| - 1$ other players have a positive share left. Player i can not do better, since a further increase would lead to a marginal gain of at most $1 - \frac{1}{|P(t')|}$, whereas placing a lower claim would mean that the opportunity of a marginal gain of at least $\frac{|P(t')|}{|P(t')|+1}$ will be ignored. More precisely, let player i put a claim of size $\beta_i(t) = |P(t')|\beta_{(|P(t')|)}(t) - \sum_{j=1}^{|P(t')|-1} \beta_{(j)}(t) + 1$ on a δ-fraction of t, where $0 < \delta \leq 1$ is chosen such that the average loss on t' does not exceed $\frac{|P(t')|-1}{|P(t')|}$. His loss in payoff from t' is then equal to the total amount of claim needed times the average loss:

$$\beta_i(t)\delta(y_t - y_{t-1}) \left(1 - \frac{1}{|P(t')|}\right) =$$

$$\left(|P(t')|\beta_{(|P(t')|)}(t) - \sum_{j=1}^{|P(t')|-1} \beta_{(j)}(t) + 1\right) \delta(y_t - y_{t-1}) \frac{|P(t')| - 1}{|P(t')|}.$$

Since $\mu = \beta_{(|P(t')|)}(t)$, the gain in payoff from t is equal to

$$(\beta_i(t) - \mu)\, \delta(y_t - y_{t-1}) = \left((|P(t')| - 1)\beta_{(|P(t')|)}(t) - \sum_{j=1}^{|P(t')|-1} \beta_{(j)}(t) + 1\right) \delta(y_t - y_{t-1}).$$

Since the loss in payoff must be at least as large as the gain in payoff, we get after tedious rewriting:

$$\sum_{i=1}^{|P(t')|-1} \beta_{(i)}(t) \geq 1.$$

This completes the proof of the Claim.

For (a), suppose that $\beta_N(t') > 1$ for $t' \in M$ with $|P(t')| < k$. We will derive a contradiction. Consider an interval t with $|P(t)| = k$. Observe that there cannot be a player $i \in P(t') \cap P(t)$ (since then his marginal loss on t' would be smaller than his marginal gain on t). So there exists a player $i \in P(t')$, $i \notin P(t)$. From the claim, $\sum_{j=1}^{|P(t')|-1} \beta_{(j)}(t) \geq 1$. This, however, contradicts with (4), since

$$\beta_N(t) - \beta_{min}(t) = \sum_{j=1}^{k-1} \beta_{(j)}(t) = \sum_{j=1}^{|P(t')|-1} \beta_{(j)}(t) + \sum_{j=|P(t')|}^{k-1} \beta_{(j)}(t) \geq 1 + \sum_{j=|P(t')|}^{k-1} \beta_{(j)}(t) > 1.$$

Hence we have shown (a).

For (c), note that all $t \in M$ with $|P(t)| < k$ satisfy $\beta_N(t) = 1$. This combined with $\sum_{i \in N} c_i > E$, implies that there exists an interval $t' \in M$ with $|P(t')| = k$ and $\sum_{i \in P(t')} \beta_i(t') > 1$. Consider such interval t' together with an interval $t \in M$ with $|P(t)| = k$ and $P(t') \neq P(t)$. Then there is a player i such that $i \in P(t')$ and $i \notin P(t)$. Combining the result of the claim with (4) implies that

$$\beta_N(t) - \beta_{min}(t) = \sum_{j=1}^{k-1} \beta_{(j)}(t) = 1,$$

which shows (c).

For the implication $(ii) \Rightarrow (i)$, suppose that the claims profile satisfies the conditions of (ii). It is sufficient to check that players have no incentive to deviate from intervals $t \in M$ with $|P(t)| = k$ (since for all other intervals the marginal loss equals one, as $\beta_N(t) = 1$). Due to Lemma 4.5 it is not profitable to deviate within the same interval, as the loss of removing Δ is at least $\frac{k-1}{k}\Delta$ (since the losses are equally divided) and the gain of placing it elsewhere is at most $\frac{k-1}{k}\Delta$. By similar arguments, it follows that there is also no incentive do deviate to an interval with a smaller or equal number of claimants.

So the only interesting case arises if there is a player $i \in P(t)$, $i \notin P(t')$ where $t' \in M$ with $|P(t')| = k$. As the minimum average loss on t equals $\frac{k-1}{k}$, the most profitable claim to put on t' is the claim which assures that each player $j \in P(t')$ with $\beta_j(t') = \beta_{min}(t')$ gets a share of zero. An additional increase in claim results in a marginal gain of at most $\frac{k-1}{k}$ and is thus not profitable. A lower claim does not take the marginal gain of $\frac{k}{k+1}$ into account, which means that the claim is not the most profitable one. More precisely, the claim needs to have a size of $k\beta_{min}(t')$, as then $\mu = \beta_{min}(t)$. The total gain on t' then equals $(k-1)\beta_{min}(t')$, whereas the total loss is at least $k\frac{k-1}{k}\beta_{min}(t')$. This proves that no profitable deviation between intervals is possible. □

The following corollary presents the existence condition for an NEP.

Corollary 4.8. *The game (E, c, f^{CEL}) has an NEP if and only if $\sum_{i \in N} c_i \leq 2E$.*

Proof. Let (y, β, m) be an NEP. If $|P(t)| = 1$, then $\beta_N(t) = 1$. If $|P(t)| \geq 2$, we have by Lemma 4.5 that $\beta_N(t) - \beta_{min}(t) \leq 1$ for all $t \in M$. So $|P(t)|\beta_{min}(t) - \beta_{min}(t) \leq \beta_N(t) - \beta_{min}(t) \leq 1$, which implies that $\beta_{min}(t) \leq \frac{1}{|P(t)|-1}$ and thus $\beta_N(t) \leq 1 + \frac{1}{|P(t)|-1} \leq 2$. Together this implies that $\sum_{i \in N} c_i = \sum_{t \in M} \beta_N(t) \leq 2E$.

Suppose $\sum_{i \in N} c_i \leq 2E$. We construct an NEP in the following way: every player claims every interval at most once and every interval is claimed either once or twice. If we distribute the entitlements iteratively from left to right and start again on the left for the second claims, then one can check that the claims profile satisfies the conditions of Theorem 4.7. □

Remark 4.9. We can also use Lemma 4.5 to derive an upper bound on the number of claimants on an interval in an equilibrium as in Corollary 4.8. If $\sum_{i \in N} c_i = E$, then $\beta_N(t) = 1$ for all $t \in M$ in equilibrium, and only n is an upper bound. Now let $\ell \in \mathbb{R}$, $\ell > 1$ be such that $\left(1 + \frac{1}{\ell}\right) E < \sum_{i \in N} c_i \leq 2E$. Then ℓ is an upper bound, which can be seen as follows. If $|P(t)| = 1$, then clearly $|P(t)| < \ell$. If $|P(t)| \geq 2$, we have by Lemma 4.5 that $\beta_N(t) \leq 1 + \beta_{min}(t) \leq 1 + \frac{1}{|P(t)|-1}$ for all $t \in M$. This implies that

$$\left(1 + \frac{1}{\ell}\right) E < \sum_{i \in N} c_i = \sum_{t \in M} \beta_N(t) \leq \left(1 + \frac{1}{|P(t)| - 1}\right) E,$$

which in turn implies $|P(t)| \leq \ell$.

In order to be able to describe the equilibrium payoffs, we show that in equilibrium any interval can be redistributed such that the payoffs for the players remain unchanged and only intervals with one or two claimants are used.

Lemma 4.10. *For every NEP in* (E, c, f^{CEL}) *there exists a payoff-equivalent NEP such that* $|P(t)| \leq 2$ *and* $\beta_i(t) = 1$ *for all* $t \in M$ *and* $i \in P(t)$*.*

Proof. Let (y, β, m) be an NEP. We will show that we can redistribute the claims on every interval $t \in M$ such that $|P(t')| \leq 2$ and $\beta_i(t') = 1$ for all $i \in P(t')$, for every subinterval t' of t, but without changing a player's total share of the interval t. This generates a new finer claims profile which is still an NEP.

We only need to consider intervals $t \in M$ with $|P(t)| \geq 2$, since $\beta_i(t) = 1$ for $i \in P(t)$ if $|P(t)| = 1$. By Lemma 4.5, every interval t with $|P(t)| \geq 2$ satisfies the following inequality: $1 - \beta_{min}(t) \leq \beta_N(t) - \beta_{min}(t) \leq 1$. The share of each player $i \in P(t)$ for such an interval is

$$f_i(\beta(t)) = \beta_i(t) - \frac{\beta_N(t) - 1}{|P(t)|}.$$

We show that for some α with $y_{t-1} < \alpha < y_t$, we can reshuffle all the claims on (y_{t-1}, y_t) such that each subinterval of (y_{t-1}, α) is claimed once by two players, each subinterval of (α, y_t) is claimed once by one player, and all shares of players $i \in P(t)$ remain unchanged. The procedure we use here is similar to the way we describe the payoffs in Corollary 3.12, using the vector r, only now applied to the specific interval t. Since $\beta_N(t)\,(y_t - y_{t-1}) = 2\,(\alpha - y_{t-1}) + (y_t - \alpha)$, we have $\alpha = y_{t-1} + (\beta_N(t) - 1)\,(y_t - y_{t-1})$.

Let $x_i = \frac{2(\beta_N(t)-1)}{|P(t)|}$ for $i \in P(t)$ denote the part of the claim of player i distributed on (y_{t-1}, α) such that every part is claimed once by two players, and let $\beta_i(t) - x_i$ be distributed on (α, y_t) such that every part is claimed by one player.

In order to see that we have a feasible redistribution, note the following properties:

(i) $x_i = \frac{2(\beta_N(t)-1)}{|P(t)|} \geq 0$.

(ii) $x_i = \frac{2(\beta_N(t)-1)}{|P(t)|} \leq \beta_N(t) - 1 \leq \beta_{min}(t) \leq \beta_i(t)$, where the first inequality follows since $|P(t)| \geq 2$ and the second inequality follows from Lemma 4.5.

(iii) $\sum_{i \in P(t)} x_i = 2(\beta_N(t) - 1)$.

Moreover, the share of player $i \in P(t)$ is

$$\frac{1}{2}x_i + \beta_i(t) - x_i = \beta_i(t) - \frac{1}{2}x_i = \beta_i(t) - \frac{\beta_N(t)-1}{|P(t)|} = f_i(\beta(t)).$$

If we reshuffle every interval in the above way, we end up with an equilibrium claims profile without changing the shares of the players. □

Lemma 4.10 makes it possible to describe the payoffs associated with NEPs: all these payoffs can be found by only using intervals with either one or two claimants. Hence the set of payoff vectors is equal to the set found in Corollary 3.12.

4.3 Talmud

In this subsection we analyze the game (E, c, f^T): see Definition 2.4 for the definition of $f^T = f^{\frac{1}{2}}$. We consider two cases: estate division problems with $\sum_{i\in N} c_i > 2E$ and those with $\sum_{i\in N} c_i \le 2E$. We start with the former case.

Theorem 4.11. *Let $\sum_{i\in N} c_i > 2E$. Then all NEPs result in the same payoffs, equal to the payoffs assigned by the Talmud rule to (E, c).*

Proof. Suppose (y, β, m) is an NEP. Recall that $\beta_N(t) \ge 1$ for all $t \in M$. Let J denote the set of players with $\frac{1}{2}\beta_j(t) > \lambda$ for some $t \in M$ with $\beta_N(t) \ge 2$. Since $\beta_N(t) > 2$ for some $t \in M$, there exists a player $j \in P(t)$ with $\frac{1}{2}\beta_j(t) > \lambda$, which shows that $J \ne \emptyset$. We first prove the statement for $|J| = 1$ and afterwards for $|J| \ge 2$.

Suppose that $|J| = 1$. Since $j \in J$ can reduce his claim on t to λ without any loss (since the constrained equal awards rule is applied on this interval), it should not be possible for him to gain a positive amount from an increase in claim on a different interval $t' \ne t$.

If $\beta_N(t') \ge 2$, this implies that $\frac{1}{2}\beta_j(t') \ge \lambda$ for $j \in J$, and that $\frac{1}{2}\beta_i(t') \le \lambda$ for all $i \in N \setminus \{j\}$ by definition of J.

If $\beta_N(t') < 2$, this implies by similar arguments that $\frac{1}{2}\beta_j(t') \ge \mu$ for $j \in J$, while for all other players $i \in N \setminus \{j\}$, we have that $\frac{1}{2}\beta_j(t') \le \mu$, as otherwise $j \in J$ could increase his claim on t' with a positive gain.

Hence in equilibrium every player $i \in N \setminus \{j\}$ receives half of his claim, whereas the remainder, which is at least as much as the payoff of every $i \in N \setminus \{j\}$, is for player j. These are the same payoffs the players obtain if the Talmud rule is applied to the original estate division problem.

Suppose that $|J| \ge 2$. First, we will argue that $\beta_N(t) \ge 2$ for all $t \in M$. By definition of the set J, for every player $j \in J$ there exists an interval $t \in M$ with $\beta_N(t) \ge 2$ and $\frac{1}{2}\beta_j(t) > \lambda$. On this interval t, j can reduce his claim to λ without any loss (again because of the constrained equal awards rule). Since we consider an NEP, this means that it must not be possible for j to gain from an increase on any other interval.

Suppose there exists a $t' \in M$ with $\beta_N(t') < 2$. If there is at most one player $j \in J$ with $\frac{1}{2}\beta_j(t') \ge \mu$, then one of the other players in J gains a positive amount by increasing his claim on t'. If there are at least two players from J with $\frac{1}{2}\beta_j(t') \ge \mu$, then then either of these players gains a positive amount by increasing his claim on t'. Since we consider an NEP, these intervals can not exist in equilibrium.

Next we show that $\frac{1}{2}\beta_j(t) \geq \lambda$ for all $j \in J$ and for all $t \in M$. Suppose, contrary to what we wish to prove, that $\frac{1}{2}\beta_j(t') < \lambda$ for some $t' \in M$. Since j can reduce his claim on some $t \neq t'$ without any loss, he can make a positive gain by increasing his claim on t'. This is a contradiction.

Hence in equilibrium, for all $t \in M$, $\frac{1}{2}\beta_j(t) \geq \lambda$ for all $j \in J$ and $\frac{1}{2}\beta_i(t) \leq \lambda$ for all $i \in N \setminus J$ by definition of J. This means that all players $i \in N \setminus J$ receive half of their claim on every interval, while all players $j \in J$ receive an equal amount which is at least as much as what the players in $N \setminus J$ receive. Again, these payoffs are equal to the payoffs assigned by the Talmud rule to the original estate division problem. □

The following lemma is convenient for finding the equilibrium payoffs in case $\sum_{i \in N} c_i \leq 2E$. It is the analogue of Lemma 4.10.

Lemma 4.12. *Let $\sum_{i \in N} c_i \leq 2E$. For every NEP in (E, c, f^T) there exists a payoff-equivalent NEP such that $|P(t)| \leq 2$ and $\beta_i(t) = 1$ for all $t \in M$ and $i \in P(t)$.*

Proof. Let (y, β, m) be an NEP. We will show that we can redistribute the claims on every interval $t \in M$ such that $|P(t')| \leq 2$ and $\beta_i(t') = 1$ for all $i \in P(t')$, for every subinterval t' of t, but without changing any player's total share of the interval t. This generates a new finer claims profile which is still an NEP.

Observe that $\beta_N(t) \geq 1$ for all $t \in M$, and that $\beta_i(t) = 1$ for $i \in P(t)$ if $|P(t)| = 1$. Thus for this lemma, we only consider intervals with two or more claimants.

Suppose there is some $t \in M$ with $\beta_N(t) > 2$. There can only be one player j with $\frac{1}{2}\beta_j(t) > \lambda$ (otherwise one of such players can gain by increasing his claim on $t' \neq t$ with $\beta_N(t') < 2$). Because j can reduce his claim to λ without any loss and since we consider an NEP, j can increase his claim on any $t' \neq t$ without any gain. As $\sum_{i \in N} c_i \leq 2$, it is thus possible to redistribute j's claims such that all $t \in M$ satisfy $\beta_N(t) \leq 2$ but without changing the shares of the players.

Thus, w.l.o.g., we can assume that all $t \in M$ satisfy $1 \leq \beta_N(t) \leq 2$. For all $t \in M$, divide $P(t)$ into two different groups. Let $J(t)$ denote the set of players j for who $\beta_j(t) - \mu > \frac{1}{2}\beta_j(t)$, then $P(t) \setminus J(t)$ is the set of players i for who $\beta_i(t) - \mu \leq \frac{1}{2}\beta_i(t)$. This means that the share of player $j \in J(t)$ is equal to $\beta_j(t) - \mu$ and the share of player $i \in P(t) \setminus J(t)$ is equal to $\frac{1}{2}\beta_i(t)$, where

$$\mu = \frac{1}{|J(t)|}\left(\sum_{j \in J(t)} \beta_j(t) + \sum_{i \in P(t) \setminus J(t)} \frac{1}{2}\beta_i(t) - 1\right) \text{ if } J(t) \neq \emptyset.$$

We show that for some $y_{t-1} < \alpha < y_t$, we can reshuffle all the claims on (y_{t-1}, y_t) such that every subinterval of (y_{t-1}, α) is claimed once by two players, every subinterval of (α, y_t) is claimed once by one player, and all shares of players $i \in P(t)$ remain unchanged. The procedure we use here is similar to the way we describe the payoffs in Corollary 3.12, using the vector r, only now applied to the specific interval t. Since $\beta_N(t)\,(y_t - y_{t-1}) = 2\,(\alpha - y_{t-1}) + (y_t - \alpha)$, we have $\alpha = y_{t-1} + (\beta_N(t) - 1)\,(y_t - y_{t-1})$.

Let $x_j = 2\mu$ for $j \in J(t)$ and $x_i = \beta_i(t)$ for $i \in P(t) \setminus J(t)$ denote the part of the claim distributed on (y_{t-1}, α) such that every part is claimed once by two players, and let $\beta_j(t) - x_j$ for all $j \in J(t)$ be distributed on (α, y_t) such that every part is claimed by one player.

In order to see that we have a feasible redistribution, note the following properties:

(i) $2\mu \geq 0$.

(ii) $2\mu < \beta_j(t)$ for all $j \in J(t)$, since $\beta_j(t) - \mu > \frac{1}{2}\beta_j(t)$.

(iii) If $|J(t)| \geq 1$, then $\beta_i(t) \leq 2\mu \leq \beta_N(t) - 1$ for all $i \in P(t) \setminus J(t)$. Since $\beta_j(t) \leq 1$ for $j \in J(t)$ if $|J(t)| = 1$ and $(|J(t)| - 1)\,\beta_N(t) - \sum_{j \in J(t)} \beta_j(t) \geq (|J(t)| - 2)\,\beta_N(t) \geq (|J(t)| - 2)$ if $|J(t)| \geq 2$.

(iv) $\sum_{i \in P(t)} x_i = \sum_{j \in J} 2\mu + \sum_{i \in P(t) \setminus J} \beta_i(t) = 2\,(\beta_N(t) - 1)$.

Moreover, the share of player $j \in J(t)$ is

$$\frac{1}{2}x_j + \beta_j(t) - x_j = \beta_j(t) - \mu,$$

and for player $i \in P(t) \setminus J(t)$ it is

$$\frac{1}{2}x_i = \frac{1}{2}\beta_i(t).$$

If we reshuffle every interval in the above way, we end up with an equilibrium claims profile without changing the shares of the players. □

We are now able to describe the payoffs associated with every NEP. If $\sum_{i \in N} c_i > 2E$, then Theorem 4.11 applies, which means that the equilibrium payoffs are equal to the payoffs from the Talmud rule. If $\sum_{i \in N} c_i \leq 2E$, by Lemma 4.12 all NEP payoffs can be found by only using intervals with either one or two claimants. Hence the set of payoff vectors is equal to the set in Corollary 3.12.

5 Unrestricted problems

In this section we discuss how NEPs for unrestricted problems. That is, players $i \in N$ may have entitlements with $c_i > E$.

5.1 Constrained equal awards

If the constrained equal awards rule is used as sharing rule and we assume integer claim heights, then we obtain the following lemma, which says that in equilibrium every player with an entitlement of at least the estate, claims the complete estate once.

Lemma 5.1. *Let (y, β, m) be an NEP. If $c_i \geq E$ for player i, then $\beta_i(t) \geq 1$ for all $t \in M$.*

Proof. Let (y, β, m) be an NEP. Suppose, to the contrary, that player i does not claim interval $t \in M$. Since $c_i \geq E$, there exists an interval $t' \in M$ with $\beta_i(t') \geq 2$. The net gain of removing one claim from t' and putting it on (a part of) t is positive, which contradicts that (y, β, m) is an NEP. □

Hence, under integer claims, every player i with $c_i \geq E$ claims the estate once and is indifferent where to put his remaining claims, as this remainder does not affect his nor his opponents' payoffs. So in fact, we are allowed to ignore the part of the entitlement that is above the amount of the estate without changing the equilibrium outcome. Therefore, we can solve the unrestricted problem as a restricted problem where the entitlement of every player i with $c_i \geq E$ is equal to E. For the analysis of these problems see Subsection 3.1. We obtain the following theorem.

Theorem 5.2. *Under integer claims, $U(E, c, f^{CEA}) = U(E, c', f^{CEA})$, where $c'_i = \min\{c_i, E\}$ for all $i \in N$.*

Also if arbitrary claims are allowed, both the restricted and unrestricted problems result in the same analysis: all NEPs result in the same payoffs, equal to the payoffs assigned by the constrained equal awards rule. Hence, Theorem 4.3 still applies.

5.2 Constrained equal losses

If the constrained equal losses rule is used as sharing rule, the main observation is that no NEP exists if there is some player i with $c_i > E$, independent of whether we consider integer or arbitrary claims.

Theorem 5.3. *If there is a player i with $c_i > E$, then the game (E, c, f^{CEL}) with integer or with arbitrary claims has no NEP.*

Proof. We first prove the statement for integer claims and afterwards for arbitrary claims.

Let player i have $c_i > E$, and suppose that (y, β, m) is an NEP in the game with integer claims. We derive a contradiction. It can be checked that Lemmas 3.7 and 3.8 also apply for unrestricted problems, which implies that in equilibrium $\beta_j(t) \leq 1$ for all $j \in N$ and all $t \in M$. However, since $c_i > E$ for player i, there must exist some interval $t \in M$ with $\beta_i(t) > 1$. This is a contradiction.

Next, let again player i have $c_i > E$, and suppose that (y, β, m) is an NEP in the game with arbitrary claims. We again derive a contradiction. Note that $\beta_N(t) = 1$ if $|P(t)| = 1$, as a free claim can always be used to gain a positive amount somewhere else. Lemma 4.5 (which can be seen to hold also for unrestricted problems) implies that $\beta_j(t) \leq 1$ for all $j \in N$ and all $t \in M$ with $|P(t)| \geq 2$. However, since $c_i > E$ for player i, there must exist some interval $t \in M$ with $\beta_i(t) > 1$. This is a contradiction. □

5.3 Talmud rule

If the Talmud rule is used as sharing rule in the game with integer claims, then recall that this rule is equal to the constrained equal awards sharing. Thus, the same analysis applies.

In case of arbitrary claims, we obtain the following theorem.

Theorem 5.4. *If there is a player i with $c_i > E$, then all NEPs result in the same payoffs, equal to the payoffs assigned by the Talmud rule.*

Proof. Note that if $\sum_{i \in N} c_i > 2E$, the presence or absence of a player i with $c_i > E$ does not make a difference for the analysis in Theorem 4.11. Hence this result still holds.

If $\sum_{i \in N} c_i \leq 2E$, then a player i with $c_i > E$ claims the complete estate once, so that the share of each player $j \in N \setminus \{i\}$ is at most $\frac{1}{2}\beta_j(t)$ for every $t \in M$. Since $\sum_{j \in N} c_j \leq 2E$, every player $j \in N \setminus \{i\}$ is able to assure a share of $\frac{1}{2}\beta_j(t)$ on all $t \in M$, so that in equilibrium every player $j \in N \setminus \{i\}$ receives $\frac{1}{2}c_j$ and i receives $E - \sum_{i \in N \setminus \{i\}} \frac{1}{2}c_i$. This is precisely what every player obtains if the Talmud rule is applied to the estate division problem. □

6 Summary and conclusion

We have analyzed the estate division problem as a non-cooperative game, in which every player uses his entitlement to claim specific parts of the estate. Every part is then distributed based on these integer valued or arbitrary claims, according to a sharing rule. We have investigated the payoffs associated with the equilibrium outcomes of this game. The sharing rules we consider are all rules from the TAL-family, among which are the constrained equal awards rule, the constrained equal losses rule and the Talmud rule.

	Restricted and integer claims	Restricted, arbitrary claims and $\sum_{i\in N} c_i \leq 2E$	Restricted, arbitrary claims and $\sum_{i\in N} c_i > 2E$	Unrestricted
f^{CEA}	$v_i = \frac{c_i}{k} - \frac{r_i}{k(k+1)}$ (Subsection 3.1)	CEA-payoff (Theorem 4.3)	CEA-payoff (Theorem 4.3)	Solve (E, c', f^{CEA}), where $c'_i = \min\{c_i, E\}$ (Theorem 5.2)
f^{CEL}	$v_i = c_i - \frac{r_i}{2}$ (Corollary 3.12)	$v_i = c_i - \frac{r_i}{2}$ (Lemma 4.10)	No NEPs (Corollary 4.8)	No NEPs (Corollary 5.3)
f^T	$v_i = \frac{c_i}{k} - \frac{r_i}{k(k+1)}$ (Lemma 3.13)	$v_i = c_i - \frac{r_i}{2}$ (Lemma 4.12)	Talmud-payoff (Theorem 4.11)	Talmud-payoff (Theorem 5.4)

Table 1: A summary of the equilibrium payoffs for the three main sharing rules for different problems. For clarification, we refer to the corresponding references.

The main results of the paper are summarized in Table 1. The first (most left) column describes the possible NEP payoffs in integer claim games associated with restricted problems. The payoff from the CEL sharing rule arises from taking $k = 1$: NEPs exist if and only if $\sum_{i\in N} c_i \leq 2E$. Under the latter condition, all sets of NEPs coincide, and the same is true for all associated payoffs. For all restricted problems, we have the following relation between the different sets of equilibrium payoffs:

$$U(E, c, f^{CEL}) \subseteq U(E, c, f^{CEA}) = U(E, c, f^T) \subseteq U(E, c, f^P).$$

The second column maintains the condition $\sum_{i\in N} c_i \leq 2E$, but allows for arbitrary claims: this does not essentially effect the payoffs under CEL or Talmud: but for CEA, all payoffs coincide with the payoffs assigned by the CEA rule to the original estate division problem. The third column concerns arbitrary claims under the condition $\sum_{i\in N} c_i > 2E$. The fourth column collects some results for unrestricted problems.

Another interesting result is that for integer claim games associated with restricted problems for which $\sum_{i\in N} c_i \leq 2E$, the Talmud rule is the only sharing rule in the TAL-family for which there is always an NEP with payoffs equal to the payoffs assigned by the Talmud rule to the estate division problem. This raises the question whether there are other rules outside the TAL-family for which there is always an NEP in the associated claim game resulting in the same payoffs as those assigned by the rule directly. The proportional rule is not a candidate for this, as already established in Atlamaz et al. (2011).

Another interesting, general question is whether one can find axiomatic, normative justifications for the various sets of NEP payoffs, possibly related to the axiomatic justifications of the underlying rules.

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