A TORTOISE AND A HARE RACE, PART I: FINITE HORIZON

JUSTIN L. JOFFRION[1] AND SÉRGIO O. PARREIRAS[2]

ABSTRACT. We study a dynamic, all-pay auction under complete information. At each round, two contestants (who may have distinct valuations for the prize) simultaneously pick effort levels. The winner is the player with the highest accumulated effort at the last round. The cost of effort is separable across rounds and convex within each round. For two or more rounds, multiple subgame perfect equilibria with varying degrees of rent dissipation may exist. Decreasing the elasticity of inter-temporal effort substitution breaks down the equilibria with low rent dissipation, whereas increasing valuation asymmetries breaks down the equilibria with high rent dissipation.

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1. U.S. Air Force Academy. The views expressed are those of the author and do not reflect the official policy or position of the United States Air Force, the Department of Defense, or the U.S. Government.

2. UNC at Chapel Hill.

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CONTENTS

1. INTRODUCTION

All-pay auctions are used to study: R&D races, lobbying, labor tournaments, sports, and other types of contests. Our goal in this paper, as contests often evolve over time, is to investigate the *substitution* and *dead-line* effects. The first captures the idea that players prefer to smooth out effort; whereas the second says players have common-knowledge of the contest's terminal date.

The substitution effect appears naturally in labor tournaments (*e.g.*, the partnership track in law or consulting firms). It is plausible, nevertheless, in other contexts as well: It may be less costly for a lobbyist to raise a given amount of contributions over time rather than in a single period. Likewise, it maybe easier for an R&D division to secure steady research budgets rather than sudden funding spikes.

Although not prevalent in all applications (*e.g.*, patent races and tennis matches that are better suited to the tug-of-war or other endogenous termination rule), the dead-line effect emerges in many economic environments: In lobbying, one may think of the legislation's calendar or referendum date; as for labor tournaments, consider firms with partnership tracks or, alternatively, firms where an announcement of senior management's retirement date triggers the contest.

In R&D procurement, the dead-line effect also matters. A historical example was the Nexflix Prize (Netflix, 2006) whose aim was to improve the accuracy prediction of Cinematch, Netflix's movie recommendation algorithm. With a grand prize of one million dollars (also progress prizes of fifty thousand dollars), the tournament started in October, 2006. It was set to last until October, 2011 or thirty days after one of the participants achieved a root means square error (RMSE) below the qualifying RMSE, which turned out to be the case as the contest ended by September, 2009. Since then procurement via crowd-sourcing contests platforms such as Topcoder (2013) – used by Alcatel Lucent, AMD, DARPA, and NASA – or Freelancer (2013), are widespread.

The model is a dynamic, all-pay auction under complete information. At each round, two contestants (with possibly distinct valuations for the contest prize) *simultaneously* choose effort levels. At the last round, the winner is the player with the highest accumulated effort. The cost of effort is separable across rounds and the convex *within* each round.

As it is well known (Baye et al., 1996; Hillman and Riley, 1989; Siegel, 2013), one-shot contests have a unique (mixed strategy) equilibrium where the highest valuation player – henceforth, player 1 or, the *favorite*, gets positive payoff while player 2 or, the *underdog*, gets 0 payoff. For symmetric valuations (and no head-starts), rent dissipation is complete: both players receive 0 payoff.

Although in dynamic all-pay auctions (Harris and Vickers, 1985; Konrad, 2011; Konrad and Kovenock, 2009; Leininger, 1991), the definition of *favorite* must be adjusted to account for head-starts and/or first mover's advantage[1], again as in the static models: the favorite

[1] A major contribution of Harris and Vickers (1985) is to characterize the favorite, who is not necessarily the highest valuation player. For example in the R&D race of Leininger (1991), the entrant earns positive payoff provided she moves before the incumbent and her research budget is higher than the incumbent's and thus, one might argue here that the entrant is the favorite.

obtains positive payoffs whereas the underdog gets 0 payoff. Furthermore, the favorite' remains active till she clinches the prize. As for the underdog, he remains inactive with possibly the exception of the first round (Harris and Vickers, 1985; Leininger, 1991).

An alternative to the all-pay auction, the contest success function approach, assumes winning probabilities are 'proportional' to the players' efforts. [2] In this setting and with an endogenous order of moves, Leininger (1993) obtains an unique equilibrium where the underdog leads. A crucial assumption, however, is that each player moves only once. With two periods and simultaneous moves, there is a *continuum* of equilibria, yet the Stackelberg strategy profile where the underdog leads is never an equilibrium (see Yildirim, 2005).

Our first set of main results proves the existence of multiple equilibria. In particular, we characterize equilibria where the underdog leads the race and earns positive payoffs while the favorite slacks and earns 0 payoff. We refer to these equilibria as *tortoise-hare equilibria*. Key ingredients for tortoise-hare equilibria are: 1) multiple periods; 2) simultaneous moves; 3) increasing marginal cost of effort; 4) players' valuations are not 'too far apart'; and 5) the elasticity of inter-temporal effort substitution is 'moderate'.

The intuition for why multiple periods and increasing marginal cost are required is simple. As player 2 builds a lead then it is costly for player 1 to close the gap; and moreover, when player 1 does not exert effort at the initial rounds, player 2 has incentive to spread the effort. Of course, were the valuation of player 1 'too high' relatively

[2] A large literature following Dixit (1987), applies this approach to dynamic settings; see Konrad (2011) for references.

to player 2's valuation, the size of the lead require to deter player 1 from exerting effort would be so high hence as to make unprofitable any player 2 attempt to create a lead. Hence, the intuition for valuations not 'too far apart' is straightforward.

Obviously when effort substitution is perfectly elastic between rounds, it is not that costly for player 1 to try to catch up with player 2. As a matter of fact, the linear cost case is not interesting as any equilibrium is outcome equivalent to the equilibrium of the static model.

The intuition behind the requirement of a 'too low' inter-temporal elasticity of effort is, however, slightly more subtle. Why does the tortoise-hare equilibrium break down in the inelastic case? Now player 2 will build an advantage that is too small (even as he expects player 1 to not exert effort) because the marginal cost of effort is higher in the inelastic case. But then, player 1 will have incentives to catch up.

Our second set of main results have several empirical implications:

First, we classify the equilibria set regarding the degree of rent dissipation. In one pole, we may have equilibria where one of the players is active at $T-1$ and the other slacks. We refer to these equilibria as *collusive* equilibria since rent dissipation is minimal. In particular, *tortoise-hare equilibria* are collusive equilibria where the underdog leads. On the other pole, we may have an equilibrium where both players use symmetric continuous[3] mixed strategies at $T-1$. We refer to it as the *competitive* equilibrium since rent dissipation is maximal – both players get 0 payoff. In between, the collusive and the

[3]That is, continuous for effort levels in the interior of the support since we may have atoms at 0.

competitive equilibrium, we show that there are asymmetric, mixed strategy equilibria where players randomize over finite effort levels. Rent dissipation is linked to the size of the support: the larger the support, the more rent is dissipated.

Second, we show that if valuation asymmetries are introduced, the competitive equilibrium breaks down. As valuation asymmetries increase, near-competitive equilibria also start to break down. And not surprisingly if asymmetries keep increasing, collusive equilibria where the underdog leads (tortoise-hare equilibria) also cease to exist. For sufficiently high asymmetries, the only equilibrium is the equilibrium where players always choose 0 effort before the last round. In this case, the outcome is identical to the one-shot game.

Third, we show that as the inter-temporal elasticity of effort substitution decreases (provided valuation asymmetries are 'small'), collusive equilibria break down.

In sum, valuation asymmetries and the inter-temporal elasticity of effort substitution allows us to predict the range of rent dissipation.

2. The Model

For T rounds, players 1 and 2 simultaneously choose effort levels, e_t^1 and $e_t^2 \in [0, +\infty)$ at every round. After the last round effort levels are cast, the prize is awarded to the player with highest accumulated effort, $\sum_{t=1}^{T} e_t^i$. Ties are randomly broken. Player 1 values the prize at v_1 while 2 values it at v_2 with $v_1 \geq v_2 \geq 0$, when $v_1 > v_2$ we may refer to player 1 player 2 as respectively the *favorite* and the *underdog*.

The cost of effort is separable across rounds and convex. The winner's and loser's, say player i and j, respective payoffs are: $v_i - \sum_{t=1}^{T} c(e_t^i)$ and $-\sum_{t=1}^{T} c(e_t^j)$ where $c(\cdot)$ is the cost of effort function.

We assume c is identical for both players and that it satisfies: $c(0) = 0$, $c'(x) > 0$, and $c''(x) \geq 0$ for all $x > 0$.

Effort choices are perfectly observable, and the description of the game is common-knowledge.

The state variable $\Delta_t \stackrel{\text{def}}{=} \sum_{k=1}^{t-1} e_t^1 - e_t^2$ summarizes the favorite's *advantage* over the underdog at the start of period t. Evidently the favorite is at an disadvantage for $\Delta_t < 0$ thus $-\Delta_t$ is the underdog's advantage.

The equilibrium concept used thru this paper is Markov perfect equilibria; the state is (Δ_t, t). For $T = 1$ or $T = 2$ this is without loss of generality because the set of Markov perfect equilibria coincides with the set of subgame perfect equilibria. However for $T \geq 3$ as there is a *continuum* of equilibria, it is hard to establish whether Markov perfect equilibria imposes additional restrictions.

Despite valuation asymmetries, most proofs of statements regarding player 1 have a *mirror* version regarding player 2: one produces the mirror proof by, in the original proof, mechanically swapping the variables in the following manner: player 1 for player 2, v_1 for v_2, and Δ_t for $-\Delta_t$.

3. The Last Round

The last round is a particular case of Siegel (2013), nonetheless for completeness and reader convenience, we describe its unique (subgame perfect) equilibrium in facts 1, 2 and 3.

For moderate values the absolute value of Δ_T, the contest remains competitive as neither player has a decisive advantage and so contestants play a mixed strategy profile. In contrast, for extreme values of Δ_T as the winner is already determined, both players exert 0 effort.

Let G_1 and G_2 denote the respective cumulative distribution functions of the mixed strategies played at the last round in some subgame-perfect Nash equilibrium. The following lemmata characterize G_1 and G_2.

Fact 1. *If either $\Delta_T \leq -c^{-1}(v_1)$ or $c^{-1}(v_2) \leq \Delta_T$ both players choose 0 effort. In the first case, player 1's stage-T payoffs is 0 and player 2's is v_2 while in the second case, stage-T payoffs are respectively v_1 and 0.*

Fact 2. *If $-c^{-1}(v_1) \leq \Delta_T \leq c^{-1}(v_2) - c^{-1}(v_1)$ players use mixed strategies:*

$$G_1(e) = \begin{cases} \frac{v_2 - c(c^{-1}(v_1)+\Delta_T)}{v_2} & \textit{for } e \in [0, -\Delta_T], \\ \frac{v_2 - c(c^{-1}(v_1)+\Delta_T)+c(e+\Delta_T)}{v_2} & \textit{for } e \in \left(-\Delta_T, c^{-1}(v_1)\right], \quad \textit{and} \\ 1 & \textit{for } e \geq c^{-1}(v_1). \end{cases}$$

$$G_2(e) = \begin{cases} \frac{c(e-\Delta_T)}{v_1} & \textit{for } e \in \left[0, c^{-1}(v_1) + \Delta_T\right], \\ 1 & \textit{for } e \geq c^{-1}(v_1) + \Delta_T. \end{cases}$$

And stage$-T$ payoffs are $u_T^1 = 0$ and $u_T^2 = v_2 - c(c^{-1}(v_1) + \Delta_T)$.

Fact 2 says that when player 2 has a moderate, advantage player 1 still competes in the last round. More exactly player 1 uses a discontinuous strategy, she chooses 0 effort with positive probability and high effort levels ($e > -\Delta_T$) with positive density but intermediate effort levels $0 < e \leq -\Delta_T$ are not chosen. Player 2 also chooses 0 effort with positive probability but unlike player 1 the range of his effort choices is continuous.

Fact 3. *If $c^{-1}(v_2) - c^{-1}(v_1) \leq \Delta_T \leq c^{-1}(v_2)$ equilibrium CDFs are:*

$$G_1(e) = \begin{cases} \frac{c(e+\Delta_T)}{v_2} & \text{if } e \in \left[\underline{e}_1, c^{-1}(v_2) - \Delta_T + \underline{e}_1\right], \\ 1 & \text{otherwise.} \end{cases}$$

$$G_2(e) = \begin{cases} \frac{v_1 - c(c^{-1}(v_2)-\Delta_T) + c(\underline{e}_1)}{v_1} & \text{if } e \leq \underline{e}_2, \\ \frac{v_1 - c(c^{-1}(v_2)-\Delta_T) + c(e-\Delta_T)}{v_1} & \text{if } e \in \left(\underline{e}_2, c^{-1}(v_2)\right], \\ 1 & \text{otherwise.} \end{cases}$$

where $\underline{e}_1 = \max(-\Delta_T, 0)$, $\underline{e}_2 = \max(\Delta_T, 0)$. Stage payoffs are:

$$u_T^1 = v_1 - c(c^{-1}(v_2) - \Delta_T) \text{ and } u_T^2 = 0.$$

4. Continuation Payoffs at $T-1$

Since all our main results require than one round and two rounds suffice to illustrate all our results, we shallalso analyze the $T-1$ case apart. For simplicity, we omit time subscripts and refer to a $T-1$ subgame as the game $\Gamma(\Delta, v_1, v_2)$ with payoffs given by:

$$u_1(e_1, e_2 | \Delta, v_1, v_2) = \begin{cases} -c(e_1) \text{ if } e_1 - e_2 + \Delta \leq c^{-1}(v_2) - c^{-1}(v_1); \\ \\ v_1 - c(c^{-1}(v_2) - e_1 + e_2 - \Delta) - c(e_1) \text{ if} \\ c^{-1}(v_2) - c^{-1}(v_1) \leq e_1 - e_2 + \Delta \leq c^{-1}(v_2); \\ \\ v_1 - c(e_1) \text{ if } c^{-1}(v_2) \leq e_1 - e_2 + \Delta. \end{cases} \quad \text{and}$$

$$u_2(e_1, e_2 | \Delta, v_1, v_2) = u_1(e_2, e_1 | -\Delta, v_2, v_1).$$

With respect to e_1, the payoff is continuous and although not globally concave. It is concave within the two regions where e_1 is either below or above $c^{-1}(v_2) - c^{-1}(v_1) + e_2 - \Delta$.

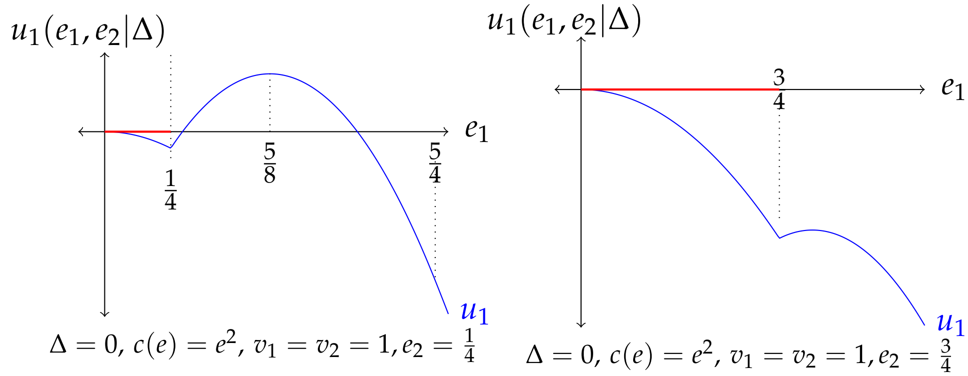

FIG. 1. Continuation payoffs of player 1 given player 2 chooses low effort (left) or high effort (right).

Definition 1. Given player 2's choice, player 1's *dead-zone* is the region of player 1's low positive effort levels, $(0, c^{-1}(v_2) - c^{-1}(v_1) + e_2 - \Delta)$; the *active zone* is the region of high effort levels, $(c^{-1}(v_2) - c^{-1}(v_1) + e_2 - \Delta, c^{-1}(v_2) + e_2 - \Delta)$; and the *safe* zone is the region $(c^{-1}(v_2) + e_2 - \Delta, +\infty)$.

Notice that the marginal payoff is discontinuous only at the frontier of the dead and active zones but continuou between the active and safe zones.

The best response of the favorite is:

$$BR_1(e_2, \Delta) = \begin{cases} \dfrac{c^{-1}(v_2) + e_2 - \Delta}{2} & \text{if } 2c^{-1}(v_1/2) \geq c^{-1}(v_2) + e_2 - \Delta \geq 0 \\ 0 & \text{otherwise.} \end{cases};$$

Proof. See section A.1 in the Appendix. ∎

For the underdog, the dead zone and best-response are similarly defined.

5. Pure Strategy Play: Collusive Equilibria.

We refer to the pure strategy equilibria of the game at $T-1$ as collusive since at least one player remains inactive and the other players (possibly active) player achieves his or her highest subgame perfect equilibrium payoff.

Lemma 1. *In any pure strategy equilibrium of* $\Gamma(\Delta, v_1, v_2)$ *at least one player exerts* 0 *effort.*

Proof. Let (e_1^*, e_2^*) be a pure-strategy Nash eq. At least one of the players, say player j payoff is $-c(e_j^*)$. But for e_j^* to be a best-response, it ought be the the case that $e_j^* = 0$ otherwise $e_j^* > 0$ implies
$-c(e_j^*) < -c(0) = 0$. ■

Proposition 1. *(1) If either* $\Delta \geq c^{-1}(v_2)$ *or* $\Delta \leq -c^{-1}(v_1)$*, the* 0 *effort profile* $(0,0)$ *is the only Nash equilibrium of* $\Gamma(\Delta, v_1, v_2)$*.*

Parts (2) & (3) below assume the complementary case where:
$-c^{-1}(v_1) < \Delta < c^{-1}(v_2)$.

(2) Define: $IC_2 = 4c^{-1}(\frac{v_2}{2}) - 2c^{-1}(v_1) - c^{-1}(v_2)$,
$IR_1 = c^{-1}(v_2) - 2c^{-1}(\frac{v_1}{2})$, $IC_1 = 2c^{-1}(v_2) + c^{-1}(v_1) - 4c^{-1}(\frac{v_1}{2})$,
and $IR_2 = 2c^{-1}(\frac{v_2}{2}) - c^{-1}(v_1)$.
For any $\Delta^* \in [\max(IR_1, IC_2), \min(IR_2, IC_1)]$, *the profile:*

$$e_1(\Delta) = \begin{cases} \frac{c^{-1}(v_2)-\Delta}{2} & \text{if } \Delta^* \leq \Delta < 2c^{-1}(\frac{v_2}{2}) - c^{-1}(v_1) \\ 0 & \text{otherwise;} \end{cases} \quad \text{and}$$

$$e_2(\Delta) = \begin{cases} \frac{c^{-1}(v_1)+\Delta}{2} & \text{if } c^{-1}(v_2) - 2c^{-1}(v_1/2) \leq \Delta < \Delta^*, \\ 0 & \text{otherwise.} \end{cases}$$

is a Nash equilibrium of $\Gamma(\Delta, v_1, v_2)$.

(3) *When* $\min(IR_2, IC_1) < \max(IR_1, IC_2)$ *part (2) above is vacuous, and moreover:*

(a) $\Gamma(\Delta, v_1, v_2)$ *has no pure strategy equilibrium if* $\min(IR_2, IC_1) < \Delta < \max(IR_1, IC_2)$.

(b) $\left(\frac{c^{-1}(v_2)-\Delta}{2}, 0\right)$ *is the only pure strategy eq. for* $\Delta \geq \max(IR_1, IC_2)$.

(c) $\left(0, \frac{c^{-1}(v_1)+\Delta}{2}\right)$ *is the only pure strategy eq. for* $\Delta \leq \min(IR_2, IC_1)$.

Proof. See Appendix A.1. ■

Part (1) of Proposition 1 says that at the last but one stage both players will choose 0 effort if the advantage of any given contestant is too large.

Part (3) of the proposition says that if the advantage is large (but not 'too' large), the player with an advantage builds a lead while the other player chooses 0 effort; it also states when the advantage is 'moderate' but marginal cost of effort increases too fast, there are no collusive equilibria, for further discussion see section 7.

Finally, part (2) states that if the advantage is 'moderate' and the marginal cost of effort does not increase too fast nor too low then it is possible to support either player leading while the other slacks as an equilibrium: The condition $\Delta > IR_1$ implies it is profitable for player 1 to build an advantage when she expects 2 to exert 0 effort; while the other condition, $\Delta > IC_2$, implies it does not payoff for player 2 to try to catch up with player 1 when 2 expects 1 to

FIG. 2. One player takes 'small' effort levels at non-terminal rounds, the other choses no effort.[a]

FIG. 3. At the last round, the laggard either gives up or tries to catch-up and surpass the leader; who tries to keep the advantage.[b]

build an advantage. The combination of the two conditions, $\Delta \geq \max(IR_1, IC_2)$, means that we can support as (part of) a equilibrium, the action profile where player 1 builds an advantage while 2 places no effort at $T-1$. Similarly, we can support player 2 building an advantage while 1 choses no effort when $\Delta \leq \min(IR_2, IC_1)$.

6. MIXED STRATEGY PLAY

6.1. **Competitive Equilibria.** We now study equilibria of $\Gamma(\Delta, v_1, v_2)$ where players choose positive effort levels accordingly to continuous distributions.

a. Fig. 2.: illustration by Harrison Weir (1824-1906). Available online at http://www.gutenberg.org/files/18732/18732-h/18732-h.htm.

b. Fig. 3: illustration by Milo Winter (1886-1956). Available online at http://www.gutenberg.org/files/19994/19994-h/19994-h.htm.

Let $G_i(\cdot|\Delta, v_1, v_2)$ denote the cumulative probability distribution function of player i's effort at Δ_{T-1}. Hereafter for simplicity we write $G_i(\cdot)$ for the CDF and, $g_i(\cdot)$ for the corresponding PDF whenever it is well defined.

Proposition 2. *Assume power cost functions, $c(e) = e^{\theta}$ with $\theta \in \mathbb{Z}_+$, identical valuations, and 0 head-starts. There is a unique, symmetric, mixed strategy, Nash equilibrium. Furthermore, in this equilibrium: players' payoff is 0, g_i is increasing, and G_i is given by,*

$$G_i(x) = (-1)^{\theta-1} + \sum_{k=1}^{\theta-1} C_k \cdot \exp\left(c_k \cdot x\right).$$

The complex-valued constants in the expression for G_i are characterized in the proposition's proof, see appendix B.1.

Example 1. For $\theta = 2$, the symmetric equilibrium is given by:

$$G_i(x) = \begin{cases} \exp\left(\frac{x}{\sqrt{v}}\right) - 1 & \text{for } \ 0 \leq x \leq \ln(2)\sqrt{v} \\ 1 & \text{for } \ x \geq \ln(2)\sqrt{v} \end{cases}$$

Example 2. For $\theta = 3$, the symmetric equilibrium is,

$$G_i(x) = \begin{cases} 1 + \left(\sin\left(\frac{x}{v^{\frac{1}{3}}}\right) - \cos\left(\frac{x}{v^{\frac{1}{3}}}\right)\right)\exp\left(\frac{x}{v^{\frac{1}{3}}}\right) & \text{for } \ 0 \leq x \leq \frac{\pi}{4}v^{\frac{1}{3}} \\ 1 & \text{for } \ x \geq \frac{\pi}{4}v^{\frac{1}{3}} \end{cases}$$

6.2. **Ranked Equilibria.** Besides competitive and collusive equilibria, we may have asymmetric, mixed strategy equilibria. To describe these equilibria, we need some additional notation.

First, consider the set of mixed strategy equilibria of the symmetric game $\Gamma(v, v, 0)$ that have finite support and where player 2's payoff is 0, $\mathcal{E}$. Second, define the set of equilibria in $\mathcal{E}$ where players 1 and

2 use respectively m and n actions:

$$\sigma(m,n) = \{\sigma \in \mathcal{E} : \#\text{supp}\,(\sigma_1) = m \;\text{ and }\; \#\text{supp}\,(\sigma_2) = n\}\,.$$

When $\sigma(m,n)$ is a singleton, we abuse notation and identify it with its element.

Proposition 3. *For the symmetric game with quadratic costs and m and n positive integers less or equal than* 5*:*

(a) $\#\sigma(m,n) = 1$ *if* $n = m$ *or* $n = m+1$*,*
(b) $\#\sigma(m,n) = 0$ *otherwise*
(c) $\frac{v}{2} = u_1(\sigma(1,1)) > u_1(\sigma(1,2)) > u_1(\sigma(2,2)) > u_1(\sigma(2,3)) >$
$> u_1(\sigma(2,3)) > u_1(\sigma(3,4)) > u_1(\sigma(4,4)) > u_1(\sigma(4,5)) >$
$> u_1(\sigma(5,5)) > 0.$

Proof. See Appendix B.2. ∎

7. Comparative Statics

7.1. **Elasticity of Effort Substitution.** For collusive equilibria to exist, the marginal cost of effort can not increase 'too fast'. In addition, for collusive equilibria where the underdog leads (*i.e.* tortoise-hare equilibria) to exist, the marginal cost of effort can not increase 'too slow'. Or in another words, the set of possible threshold levels, $(\max(IR_1, IC_2), \min(IR_2, IC_1))$ in part (a) of proposition 1 that determines which player leads is empty when the inter-temporal elasticity of substitution of effort (which can also be interpreted as risk-aversion) is either 'too high' or 'too low'.

To illustrate consider the power cost case (where the elasticity of effort substitution is given by $\frac{1}{\theta-1}$).

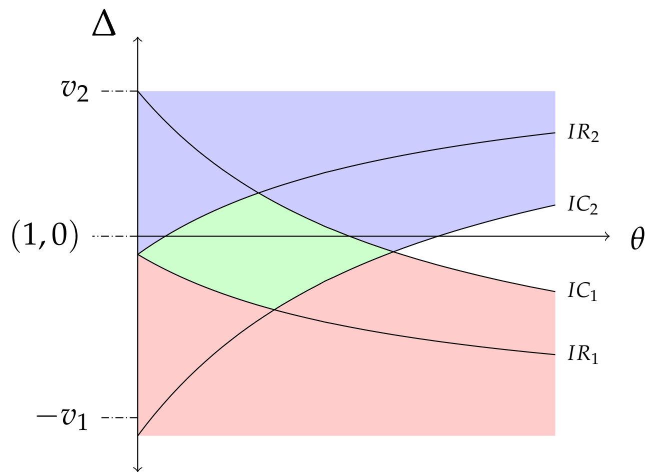

FIG. 4. Collusive equilibria with power costs, Example 3.

Example 3. Let $c(e) = e^{\theta}$ and $v_1 = 1 \geq v_2 = 0.9$. In figure 4, the green region depicts the parameter space where two collusive equilibria exist; the red region indicates where only collusive equilibria with player 2 inactive exist; the blue region is where only collusive equilibria with player 1 inactive exist; and finally, the white region indicates where collusive equilibria do not exist.

For the symmetric case, it is easier to identify the role of the elasticity of effort substitution.

Proposition 4. *In the symmetric case $v_1 = v_2$, there are two pure strategy Nash equilibria if and only if $IC \leq 0$. In particular for power cost functions, $c(e) = e^{\theta}$, pure strategy equilibria exist if and only if $\theta \leq \frac{\log(4/3)}{\log(2)}$.*

Proof. In the symmetric case $\max(IR_1, IC_2) = -\min(IR_2, IC_1)$ thus by proposition 1 we have two pure strategy Nash equilibria if and only if $\max(IR_1, IC_2) < 0$. As in the symmetric case $IR_1 < 0$ by concavity of c^{-1}, the necessary and sufficient condition for the existence of a pure strategy eq. simplifies to $IC \leq 0$. ∎

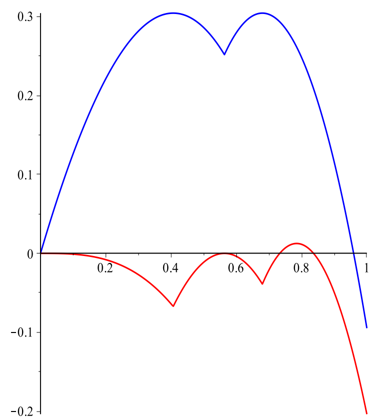

FIG. 5. Cubic costs and symmetric set-up, Example 4: player 2 wants to deviate.

When the cost function is too ‘convex’, equilibria where the size of supports is ’small’ may fail to exist. We already proved that for cubic costs, the symmetric game has no pure-strategy equilibria. In this case also there is no equilibria where supports have size two.

Example 4. In the symmetric case and cubic costs, there is no equilibria where supports have size two. The necessary first-order and indifference conditions for an equilibrium with supports of size two have a unique solution, however, this solution is not an equilibrium as player 2 has incentives to deviate and choose higher higher effort levels as Figure 5 illustrates:

However, despite the fact there is no collusive equilibria nor equilibria where strategies have supports with size two, there is an equilibrium where supports have size three:

Example 5. Let $c(e) = e^3$, $v_1 = v_2 = 1$, there is an mixed strategy equilibrium where player 1 chooses the actions 0.3428, 0.5817 and 0.7335 with probabilities 0.3093, 0.3531 and 0.3375; while player 2

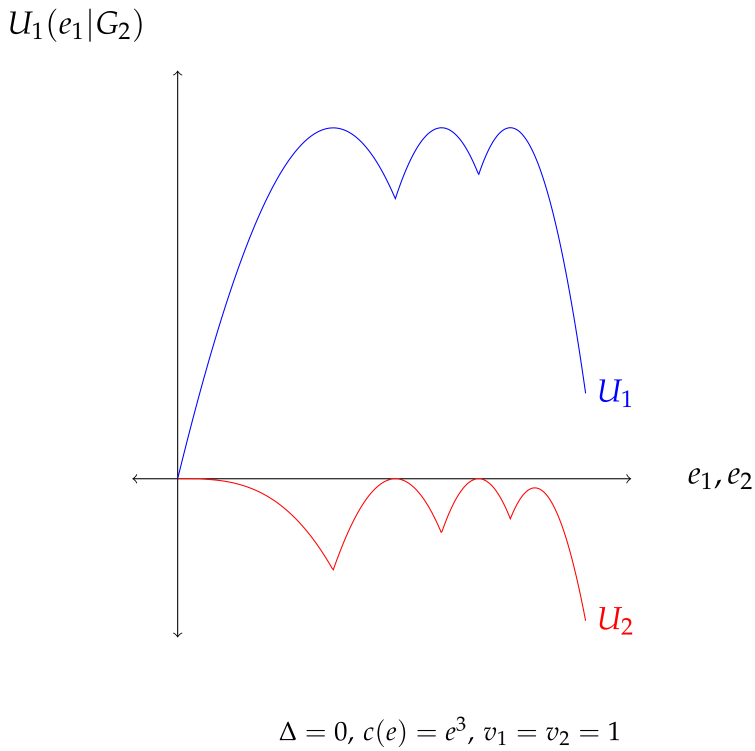

FIG. 6. The asymmetric equilibrium with cubic costs of Example 5.

mixes between 0, 0.4799 and 0.6636 with probabilities 0.2723, 0.3604 and 0.3673. Figure 6 shows the corresponding expected payoffs.

7.2. **Asymmetric Valuations.**

Proposition 5. *For the asymmetric case, $\Delta \neq 0$ or $v_1 > v_2$, and quadratic costs, there is no mixed strategy Nash equilibrium where players use a continuous distributions (with possibly an atom at* 0*).*

Proof. See the appendix C.1 ∎

We conjecture the proposition is true for any polynomial cost function. Its proof for the case of power cost functions with integer coefficients should be nearly identical to the quadratic case but notationally cumbersome.

Moreover, in the asymmetric case the relatively more competitive equilibria break down when asymmetries are sufficiently high:

Example 6. Assume quadratic costs, $v_1 = 1$ and $\Delta = 0$. For $v_2 < \frac{1}{2}$ there is no equilibrium σ where player 2 gets zero payoff, supp $(()\,\sigma_1) = 1$, and supp $(()\,\sigma_2) = 2$. In contrast, the collusive equilibria where player 2 gets zero payoff, always exists.

Proof. We use notation from Appendix B.2. If in equilibrium player 2 mixes between $y_1 = 0$ and y_2 and player 1 chooses x_1 with probability 1, by solving `sys(m,n)`, we obtain the probability of y_0 is $\frac{2v-1-\sqrt{v}+\sqrt{2}v}{1-v+2\sqrt{v}}$, which is negative for $v < \frac{1}{2}$. ■

8. Conclusions

We showed that for dynamic, all-pay auctions where the substitution and the dead-line effects are present, the clear cut prediction given by the unique (collusive-like) equilibrium of existing models goes away. One may have multiple equilibria exhibiting various degrees of rent dissipation. Also, we may have equilibria with the property that the underdog obtains positive payoffs and the favorite obtains 0 payoff. However, we showed that the indeterminacy is not absolute as the primitives (degree of valuations asymmetries and inter-temporal marginal rate of effort substitution) constraint the equilibria set: Relatively moderate valuation asymmetries are not compatible with relatively high levels of rent dissipation. Also, relatively low levels for the effort elasticity are not compatible with relatively low levels of rent dissipation.

Using Proposition 1 is trivial to construct examples of contests with three or more periods that exhibit a *continuum* of subgame perfect equilibria. Thus, in this paper, we did not attempt to investigate comparative statics regarding the horizon's length. In a companion work, however, we consider an infinite-horizon version of this model where we investigate comparative statics regarding the expected duration of the contests.

One obvious direction for future research is to test the empirical implications of our comparative statics results.

As the indeterminacy of equilibria raises the possibility of *sunspot* equilibria, another interesting direction is to characterize the set of correlated equilibria in contests and establish how it depends on the primitives: valuation asymmetries, effort elasticity, etc...

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Appendix A. Collusive Equilibria

A.1. **Proof of Proposition 1 .** First, we derive the players' best response functions. Please refer to Definition 1 for the definition of the dead, active and safe zones. For e_1 in the active zone, 1's payoff is $v_1 - c\left(c^{-1}(v_2) + e_2 - \Delta_{T-1} - e_1\right) - c(e_1)$; which is strictly concave in the interior of the active zone. The first-order conditions yield a unique global maximum in the active zone: $\beta_1(e_2) = \dfrac{c^{-1}(v_2) + e_2 - \Delta_{T-1}}{2}$. We refer to β_1 as 1's *pseudo-best response*. Also we define analogously, player 2's pseudo-best response. For the pseudo-best response to coincide with the best-response, it needs to satisfy two additional requirements:

$$2c^{-1}(v_1/2) - c^{-1}(v_2) - e_2 + \Delta_{T-1} \geq 0 \qquad (\mathtt{IR}_1(e_2))$$

$$2c^{-1}(v_1) - c^{-1}(v_2) - e_2 + \Delta_{T-1} \geq 0 \text{ and} \qquad (\mathtt{Range}_1(e_2))$$

$$c^{-1}(v_2) + e_2 - \Delta_{T-1} \geq 0$$

The condition $\mathtt{IR}_1(e_2)$ states $\beta_1(e_2)$ is a better response than 0; while $\mathtt{Range}_1(e_2)$ states $\beta_1(e_2)$ lies in the active-zone. Simplifying the above conditions, we write the best-response of player 1 as:

$$BR_1(e_2, \Delta_{T-1}) = \begin{cases} \dfrac{c^{-1}(v_2) + e_2 - \Delta_{T-1}}{2} & \text{if } 2c^{-1}(v_1/2) \geq \\ & \geq c^{-1}(v_2) + e_2 - \Delta_{T-1} \geq 0, \\ \text{and} & \\ 0 & \text{otherwise.} \end{cases}$$

A tortoise-hare equilibrium with player 1 active exists if and only if: $BR_2(BR_1(0)) = 0$ and $BR_1(0) > 0$. Using the expression for the best-response, we can reformulate these two conditions as requiring that

$$\Delta \geq \max\left(\underbrace{4c^{-1}(v_2/2) - 2c^{-1}(v_1) - c^{-1}(v_2)}_{IC_2}, \underbrace{c^{-1}(v_2) - 2c^{-1}(v_1)}_{IR_1}\right).$$

Using a mirror argument, we establish a tortoise-hare equilibrium with 2 active exists if and only if: $\Delta \leq \min(IC_1, IR_2)$.

Appendix B. Mixed Strategy

B.1. **Proof of proposition 2 .** Assume a power cost function, $c(e) = e^\theta$ with $\theta \in \mathbb{Z}_+$ and symmetric players, $v_1 = v_2$ and $\Delta = 0$; We want to characterize the symmetric, mixed strategy equilibrium in continuous strategies, G. Symmetry implies G is non-atomic which considerably simplifies expected payoffs:

$$U_i(e|G) = \int_0^e \left(v - \left(v^{\frac{1}{\theta}} + x - e\right)^\theta\right) g(x)\, dx - e^\theta$$

We further simplify payoffs by normalizing of the effort levels (change of variables): Players choose to bid a fraction of the maximum individually rational effort level, $b = e \,/\, v^{\frac{1}{\theta}} \in [0, 1]$. Thus if B is equilibrium CDF of b, the equilibrium CDF of effort levels is recovered by setting, $G(x) = B\left(\frac{x}{v^{\frac{1}{\theta}}}\right)$. Using this change of variables, we write:

$$\begin{aligned} U_i(b|B) &= \int_0^b \left(v - \left(v^{\frac{1}{\theta}} + v^{\frac{1}{\theta}} b_{-i} - v^{\frac{1}{\theta}} b\right)^\theta\right) dB(b_{-i}) - v\, b^\theta \\ &= v\left(\int_0^b \left(1 - (1 + b_{-i} - b)^\theta\right) dB(b_{-i}) - b^\theta\right) \end{aligned}$$

Without loss of generality, consider the normalized ($v = 1$) payoff, $U(x) = \int\limits_0^x \left(1 - (1 + b - x)^\theta\right) dB(b) - b^\theta$. As the support of the symmetric equilibrium has no gaps, $\operatorname{supp}(B) = [0, \overline{b}]$, and moreover since in equilibrium, players are indifferent between actions in the support, all j_{th}-derivatives of $U(x)$ must vanish. Considering the first $\theta - 1)$ derivatives, we obtain the $j = 1, \ldots, \theta - 1$ equilibrium restrictions:

$$D^j U(x) = -\frac{\theta!\, x^{\theta-j}}{(\theta-j)!} + \frac{(-1)^{j+1}\,\theta!}{(\theta-j)!} \int_0^x (1+b-x)^{\theta-j}\, dB(b) +$$

$$+ \sum_{k=1}^{j-1} \frac{(-1)^{\theta-1-k}\,\theta!}{(\theta-k)!} D^{j-k} B(x) = 0 \text{ for } j = 1, \ldots, \theta-1 \qquad (j\text{-indiff.})$$

We write the last restriction, $j = \theta - 1$, as a linear ODE with constant coefficients:

$$-\theta!\, x + (-1)^{\theta} \theta! \int_0^x (1+b-x)\, dB(b) + \sum_{k=1}^{\theta-1} (-1)^{\theta-1-k} \frac{\theta!}{k!} D^{k-1} B = 0$$

$$-\theta!\, x + (-1)^{\theta} \theta! \left[(1-x)B + \int_0^x b\, dB(b) \right] + \sum_{k=1}^{\theta-1} (-1)^{\theta-1-k} \frac{\theta!}{k!} D^{k-1} B = 0$$

$$\int_0^x b\, dB(b) = x \cdot B(x) - \int_0^x B(b) db \qquad \text{(Integration by parts)}$$

$$H(x) \stackrel{\text{def}}{=} \int_0^x B(b) db \qquad \text{(Change of variables)}$$

$$D^{\theta-1} U(x) = -\theta!\, x + \sum_{k=0}^{\theta-1} (-1)^{\theta-1-k} \frac{\theta!}{k!} D^k H(x) = 0 \qquad (\text{ODE}_H)$$

Notice that $(-1)^{\theta+1}(1+x)$ is a particular solution of ODE_H. Any solution of ODE_H can be expressed as a sum of this particular solution and linear combinations of solutions of the homogenous ODE,

$$\sum_{k=0}^{\theta-1} (-1)^{\theta-1-k} \frac{\theta!}{k!} D^k H(x) = 0, \qquad \text{(hom.)}$$

And in turn, any solution of hom. has the form $\sum_{k=1}^{\theta-1} C_k \cdot \exp(c_k \cdot x)$, where c_k is the k_{th} root of the characteristic polynomial, $p(Z) = \sum_{k=0}^{\theta-1} (-1)^{\theta-1-k} \frac{\theta!}{k!} Z^k$, and C_k are *arbitrary* constants. As a result, we finally obtain the general solution for ODE_H:

$$H(x) = (-1)^{\theta+1}(1+x) + \sum_{k=1}^{\theta-1} C_k \cdot \exp(c_k \cdot x) \text{ such that} \tag{sol.}$$

Since we have $\theta - 1$ arbitrary constants, the space of solutions has dimension $\theta - 1$. To pin-down the values of the constants, we use the additional $\theta - 2$ restrictions given by j-indiff.

The 1–indiff. constraint implies any its solution B of 1–indiff. is increasing, which is a necessary condition for dB to be an increasing PDF. To prove this fact, write 1–indiff. as:

$$\int_0^x (1+b-x)^{\theta-1} \, dB(b) = x^{\theta-1}.$$

As the right-hand side of this version of 1–indiff. is increasing, the left-hand size must be as well. Thus there is a $\hat{x}$ such that for for all $x < \hat{x} < 1$, we have $dB(x) > 0$. So, if there was the case that $dB(x) \leq 0$ for some range $x \in (\hat{x}, \tilde{x})$ with $\tilde{x} < 1$, we get the following contradiction: $0 < \tilde{x}^{\theta-1} - \hat{x}^{\theta-1} = \underbrace{\int_{\hat{x}}^{\tilde{x}} (1+b-\tilde{x})^{\theta-1} \, dB(b)}_{\leq 0} +$

$$+ \underbrace{\int_0^{\hat{x}} \left[(1+b-\tilde{x})^{\theta-1} - (1+b-\hat{x})^{\theta-1} \right] dB(b)}_{<0} < 0.$$

Now it follows that as $DH(x)$ is a non-atomic CDF function, we also have an additional restriction, $DH(0) = 0$. Finally, we compute the upper bound of the support solving $H(\overline{x}) = 1$ for $\overline{x}$.

B.2. **Proof of Proposition 3 .** As before we can normalize valuations in the symmetric case, $v_1 = v_2 = 1$,Let σ be an asymmetric, mixed strategy Nash equilibrium with finite support where player 2's payoff is 0. For $k = 1, \ldots, m$ denote by x_k be highest k_{th} effort level that is chosen with positive probability by player 1. Analogously define y_k for $k = 1, \dots, n$. And let p_k and q_k be the respective associated probabilities.

Standard arguments imply that ties can only happen with 0 probability. Moreover, since 1's payoff is strictly concave in $(y_k, y_k + 1)$, we must have that player 1 will pick exactly one effort level in the range (y_k, y_{k+1}). After applying the same argument to player 2, we obtain that $0 = y_1 < x_1 < y_2 < \dots < x_k < y_k < \dots < 1$. This already proves part (b) of Proposition 3 and establishes that either $n = m$ or $n = m + 1$; and allow us to write payoffs as: $U_1(x_k, \sigma_2) = \sum_{j=1}^{k} q_j \cdot \left(v_1 - \left(v_2^{\frac{1}{\theta}} + y_k - \Delta - x_k \right)^{\theta} \right) - (x_k)^{\theta}$ and $U_2(\sigma_1, y_k) = \sum_{j=1}^{k-1} q_j \cdot \left(v_2 - \left(v_1^{\frac{1}{\theta}} + x_k + \Delta - y_k \right)^{\theta} \right) - (y_k)^{\theta}$. In the next step, we consider the necessary equilibrium conditions for σ:

$$U_1(x_k, \sigma_2) - u_1 = 0, \quad U_2(\sigma_1, y_l) = 0, \quad \frac{\partial U_1}{\partial x_k}(x_k, \sigma_2) = 0, \quad \frac{\partial U_2}{\partial y_l}(\sigma_1, y_l) = 0$$
$$1 - \sum_{k=1}^{m} p_k = 0 \quad \text{if } n = m + 1 \quad \text{and} \quad 1 - \sum_{k=1}^{m} q_k = 0 \quad \text{otherwise.} \tag{sys(m,n)}$$

The system is square: it has $2n + 2m - 1$ unknowns and the same number of equations. Notice that the probability of the highest action, which is p_m when $m = n$ and q_n when $m = n + 1$, does not appear directly in the payoffs. Thus, for the case $m = n$, the vector of unknowns is $(x_1, \dots, x_m, p_1, \dots, p_{m-1}, u_1, y_2, \dots, y_m, q_1, \dots, q_m)$. As for the $n = m + 1$ case, we add p_m but not q_{m+1} to the vector of unknowns.

We are interested in solutions of `sys(m,n)` that also satisfy the additional constraints:

$$x_1 > 0, y_2 - x_1 > 0, \dots, x_m - y_m > 0, p_1 > 0, \dots, p_{m-1} > 0,$$
$$q_1 > 0, \dots, q_m > 0, \text{ and}$$
$$\begin{cases} 1 - \sum_{k=1}^{m-1} p_k > 0 & \text{if } n = m \\ 1 - \sum_{k=1}^{m} q_k > 0 \text{ and } p_m > 0 & \text{if } n = m+1. \end{cases} \tag{mon.}$$

Therefore, fixed points/topological methods are not useful to establish existence. First, our constraint set is an open set. Second, let $x_1 = BR_1(0)$ be the collusive strategy of player 1, and letting $x_k = x_1$, $y_k = 0$, $p_1 = q_1 = 1$ we can always solve `sys(m,n)` for any m and n but clearly mon. does not hold.

An algebraic geometry[4] approach is needed: we use Xia and Yang (2002)'s algorithm, which is implemented in *Maple*, to count the number of real solutions of `sys(m,n)` that satisfies mon..

The code and an output example is available on-line at `http://www.unc.edu/sergiop/countsolutions.mws`. It requires Maple version 16 or 17. In a Linux cluster using $4G$ of memory ($47G$ max. swap) and 4 processors, for the largest case ($m = n = 5$), it takes 5417.04 seconds of CPU time to successfully run the code. For larger cases ($m, n > 5$) memory requirements 'blowup'.

For $(m,n) = (1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,4), (4,5)$ and $(5,5)$ there is only one solution of `sys(m,n)` that satisfies mon..

Next, we use numerical methods to compute the values of the solution. The code and an output example for this part is available on-line at `http://www.unc.edu/sergiop/checkdeviation.mws`.

[4]See Cox et al. (2005, pp. 69-76) for methods of root isolation and Kubler and Schmedders (2010) for related examples of application in economics.

We use the competitive equilibrium as the 'initial condition' for the numeric algorithm; and with these values we show that: player 2 does not have an incentive to deviate by choosing an effort above x_m for the case $m = n$; and player 1 does not have an incentive to deviate by choosing an effort above y_{m+1} for the case $n = m + 1$.

APPENDIX C. COMPARATIVE STATICS

C.1. **Proof of Proposition 5 .** We consider the case $c^{-1}(v_2) - c^{-1}(v_1) - \Delta_{T-1} \geq 0$ and omit the proof for the complementary case as it is analogous. We first need the following lemma:

Lemma 2. *Assume costs are strictly convex and that in equilibrium, player i mixes continuously in the interval $[\underline{e_i}, \overline{e_i}]$ and $\Pr[e_i \in \{0\} \cup [\underline{e_i}, \overline{e_i}]] = 1$ for $i = 1, 2$. Then $\overline{e}_2 - \overline{e_1} = \underline{e}_2 - \underline{e_1} = c^{-1}(v_1) + \Delta_{T-1} - c^{-1}(v_2)$.*

Proof. First let's prove $\overline{e}_2 - \overline{e_1} = c^{-1}(v_1) + \Delta_{T-1} - c^{-1}(v_2)$. If this were not true, say if $\overline{e}_2 - \overline{e_1} > c^{-1}(v_1) + \Delta_{T-1} - c^{-1}(v_2)$ then it would imply the existence of a neighborhood of $\overline{e}_2$ where player 2 always faces herself outside her *dead-zone* (see definition 1). Since outside the dead-zone, the marginal payoff of player 2 is continuous; and since for each e_1, it is strictly concave, it follows that player 2's expected payoff is strictly concave as well. Which implies player 2 is not willing to mix. Also, an analogous argument shows that the case $\overline{e}_2 - \overline{e_1} < c^{-1}(v_1) + \Delta_{T-1} - c^{-1}(v_2)$ is ruled out.

Now assume $\underline{e}_2 - \underline{e_1} > c^{-1}(v_1) + \Delta_{T-1} - c^{-1}(v_2)$. Unless player 2 chooses 0, for efforts in a neighborhood of $\underline{e_1}$ player 1 will always face herself in the dead-zone and so, her payoff is:
$G_2(0) \cdot \left[v_1 - c\left(c^{-1}(v_2)\Delta_{T-1} - e_1\right)\right] - c(e_1)$, which is strictly concave.

A similar argument takes care of the case where $\underline{e}_2 - \underline{e}_1 < c^{-1}(v_1) + \Delta_{T-1} - c^{-1}(v_2)$. ■

Notice the lemma 2 does not depend upon the specification of the cost function; we conjecture Proposition 5 holds in general but its proof is considerably simplified if one is able to compute the candidate equilibrium explicitly. For this reason, we assume quadratic costs.

Player 2's is then:

$$(\sqrt{v_1} + \Delta_{T-1} - e_2)\, G_1(\sqrt{v_2} - \sqrt{v_1} - \Delta_{T-1} + e_2) + \int\limits_{\underline{e}_1}^{\sqrt{v_2}-\sqrt{v_1}-\Delta_{T-1}+e_2} e\, dG_1(e) - e_2 = 0, \tag{C.1}$$

let $x = \sqrt{v_2} - \sqrt{v_1} - \Delta_{T-1} + e_2$ then

$$(\sqrt{v_2} - x)\, G_1(x) + \int\limits_{\underline{e}_1}^{x} e\, dG_1(e) + \sqrt{v_2} - \sqrt{v_1} - \Delta_{T-1} - x = 0$$

Integrating by parts the integral term and simplifying the resulting expression:

$$\sqrt{v_2} G_1(x) - \underline{e}_1\, G_1(\underline{e}_1) - \int_{\underline{e}_1}^{x} G_1(e_1) de_1 + \sqrt{v_2} - \sqrt{v_1} - \Delta_{T-1} - x = 0$$

Let H_1 be G_1's primitive, that is $H_1(x) \stackrel{\text{def}}{=} \int_{\underline{e}_1}^{x} G_1(e)\, de$ then H_1 must solve the following differential equation:

$$\sqrt{v_2}\, H_1'(x) - \underline{e}_1\, H_1'(\underline{e}_1) - H_1(x) + H_1(\underline{e}_1) + \sqrt{v_2} - \sqrt{v_1} - \Delta_{T-1} - x = 0 \tag{ODE}$$

The general solution of ODE is:

$$H_1(x) = -\underline{e}_1\, H_1'(\underline{e}_1) - \sqrt{v_1} - \Delta_{T-1} - x + C \exp\left(\frac{x}{\sqrt{v_2}}\right),$$

we differentiated it to obtain the CDF of player 1's effort :

$$G_1(x) \equiv H_1'(x) = -1 + \frac{C}{\sqrt{v_2}} \exp\left(\frac{x}{\sqrt{v_2}}\right).$$

At first glance it seems we may have a continuum of solutions (index by C) but, as a matter of fact, given $\underline{e}_1$ we can pin-down the value of C by solving the system:

$$H_1(\underline{e}_1) = -\underline{e}_1\, H_1'(\underline{e}_1) - \sqrt{v_1} - \Delta_{T-1} - \underline{e}_1 + C\, \exp(\frac{\underline{e}_1}{\sqrt{v_2}}) = 0 \text{ and}$$

$$H_1'(\underline{e}_1) = -1 + \frac{C}{\sqrt{v_2}}\, \exp(\frac{\underline{e}_1}{\sqrt{v_2}}).$$

Since $C = \frac{\sqrt{v_2}\,(\sqrt{v_1} + \Delta_{T-1})}{\sqrt{v_2} - \underline{e}_1} \exp\left(-\frac{\underline{e}_1}{\sqrt{v_2}}\right)$,

$$G_1(x) = \frac{\sqrt{v_1} + \Delta_{T-1}}{\sqrt{v_2} - \underline{e}_1} \exp\left(\frac{x - \underline{e}_1}{\sqrt{v_2}}\right) - 1.$$

A mirror argument establishes that:

$$G_2(x) = \frac{\sqrt{v_2} - \Delta_{T-1}}{\sqrt{v_1} - \underline{e}_2} \exp\left(\frac{x - \underline{e}_2}{\sqrt{v_1}}\right) - 1.$$

The condition $G_i(\bar{e}_i) = 1$ implies $\bar{e}_i = \underline{e}_i + \sqrt{v_{-i}} \cdot \ln\left(2\frac{\sqrt{v_{-i}} - \underline{e}_i}{\sqrt{v_i} + (-1)^{i+1}\Delta_{T-1}}\right)$, using the other equations given by Lemma 2, we have:

$$\sqrt{v_2} \cdot \ln\left(2\frac{\sqrt{v_2} - \underline{e}_1}{\sqrt{v_1} + \Delta_{T-1}}\right) = \sqrt{v_1} \cdot \ln\left(2\frac{\sqrt{v_1} - \underline{e}_2}{\sqrt{v_2} - \Delta_{T-1}}\right)$$

$$\sqrt{v_2} \cdot \ln\left(2\frac{\sqrt{v_2} - \underline{e}_1}{\sqrt{v_1} + \Delta_{T-1}}\right) = \sqrt{v_1} \cdot \ln\left(2\frac{\sqrt{v_2} - \underline{e}_1 - \Delta_{T-1}}{\sqrt{v_2} - \Delta_{T-1}}\right)$$

$$\left(2\frac{\sqrt{v_2} - \underline{e}_1}{\sqrt{v_1} + \Delta_{T-1}}\right)^{\sqrt{v_2}} = \left(2\frac{\sqrt{v_2} - \underline{e}_1 - \Delta_{T-1}}{\sqrt{v_2} - \Delta_{T-1}}\right)^{\sqrt{v_1}} \quad \text{(a)}$$

Moreover, since $\bar{e}_i > \underline{e}_i$, we also must have:

$$2\frac{\sqrt{v_2} - \underline{e}_1}{\sqrt{v_1} + \Delta_{T-1}} > 1. \quad \text{(b)}$$

Clearly the symmetric case, $v_1 = v_2$ and $\Delta_{T-1} = 0$ solves (a) and (b). Also it is easy to verify that for $v_1 = v_2$ and $\Delta_{T-1} \neq 0$, equation (a) has a unique solution that violates (b). Thus, for even for arbitrarily small head-starts and symmetric valuations, there is no competitive equilibria. Also, for the additonal cases: 1) $\Delta_{T-1} = 0$ and $v_1 \neq v_2$

or 2) $\Delta_{T-1} = \sqrt{v_2} - \sqrt{v_1} \neq 0$, one can show that any $\underline{e}_1$ solution of (a) either violates (b) or falls outside the range $[0, \sqrt{v_2})$. Numerical analysis confirms that these results are robust; only for non-generic parameters values, a competitive equilibria exists. ■

E-mail address: sergiop@unc.edu