Common-Value All-Pay Auctions with Asymmetric Information

Ezra Einy, Ori Haimanko, Ram Orzach, Aner Sela *

November 23, 2012

Abstract

We study two-player common-value all-pay auctions with asymmetric information, assuming that one of the players has an information advantage over his opponent. We characterize the unique equilibrium of this contest, and examine the role of information on the players' expected efforts, probabilities of winning, and expected payoffs. In particular, we show that the players always have the same probability of winning the contest, their expected efforts are the same, but they do not have the same expected payoffs. We also show that budget constarints may have non-trivial effect on the players' expected payoffs.

Keywords*:* Common-value all-pay auctions, asymmetric (differential) information, information advantage, budget constraints.

JEL Classification: C72*,* D44, D82.

****Ezra Einy, Ori Haimanko, Aner Sela**: Department of Economics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel; their respective e-mail addresses are: einy@bgu.ac.il, ori@bgu.ac.il, anersela@bgu.ac.il. **Ram Orzach**: Department of Economics, Oakland University, Rochester, MI 48309, USA; orzach@oakland.edu

1 Introduction

All-pay auctions appear in various areas of economics e.g., lobbying in organizations, R&D races, political contests, promotions in labor markets, trade wars, and biological wars of attrition. In the all-pay auction each player submits a bid (effort) and the player who submits the highest bid wins the contest, but, independently of success, all players bear the cost of their bids. All-pay auctions have been studied both in the complete information framework, where each player's type (value of winning the contest or ability) is common knowledge (see, for example, Hillman and Samet (1987), Hillman and Riley (1989), Baye et al. (1993, 1996), Che and Gale (1998) and Siegel (2009)), and in the incomplete information framework, where each player's type is private information and only the distribution from which the players' types is drawn is common knowledge (see, for example, Amann and Leininger (1996), Moldovanu and Sela (2001, 2006) and Moldovanu et al. (2010)). In most of the literature on all-pay auctions with incomplete information it is assumed that the players' types are independent. However, in several competitive environments the players' types are not necessarily independent (see Milgrom and Weber 1982). Krishna and Morgan (1997) analyzed the equilibrium strategies of the all-pay auction with interdependent types in the Harsanyi-type formulation of Bayesian games. They assumed that the players' types are affiliated and symmetrically distributed. A generalization of their work to a model where players' types are asymmetrically distributed is usually not tractable. Thus, in order to study the value of information in a contest with ex-ante asymmetrically informed players, we consider a two-player common-value all-pay auction with asymmetric (differential) information where the value of winning is the same for all players in the same state of nature, but the information about which state of nature was realized can be different. In other words, the information of a player about the value of winning is described by a partition of the space of states of nature, which is assumed to be finite. Jackson (1993) and Vohra (1999) showed that this partition representation is equivalent to the more common Harsanyi-type formulation of Bayesian games. The framework of partitions is more suitable to for expressing the information advantage some players may have over others, which will figure prominently in our model as we explain below.

In our two-player model of differential information, we assume that information sets of each player are connected with respect to the value of winning the contest (see, Einy et al. 2001, 2002 and Forges and Orzach

2011). This means that if a player's information partition does not enable him to distinguish between two possible values of winning, then he also cannot distinguish between these values and all other possible values of winning between them. This assumption seems plausible in environments where the information of a player allows him to put upper and lower bounds on the actual value of winning, without being able to rule out any outcome within these bounds. We also assume that one player has information advantage over the other, which is reflected by his finer information partition. Then, without loss of generality, we can assume that the information partition of one player is the coarsest one possible, and he will be referred to as the uninformed player, and that the information partition of the other player is the finest one possible, and he will be referred as the informed player.

We characterize the unique equilibrium in mixed strategies, which turn out to be monotonic (i.e., more favorable signals do not lead to lower efforts). In equilibrium, the expected payoff of the uninformed player is zero, while the informed payer has a positive expected payoff. Our results show that, although the players have asymmetric strategies that yield different expected payoffs, the expected efforts of both players are the same. Moreover, the probabilities of each player to win the contest are equal in equilibrium. Hence, we find that asymmetry of information between the players does not result in different expected efforts or different chances to win the contest, but it does affect the allocation of payoffs between the players.

Next, we examine how the relation between players' information sets affect their expected total effort. We find that the expected total effort is maximized when the players have the same information sets (namely, the players are ex-ante symmetric). Furthermore, if there are three players (a,b and c) where a has an information advantage over b who has an information advantage over c, then the expected total effort in the contest between a and c is necessarily lower than in the contest between b an c. In other words, when the players' information are similar to each other, their total efforts are larger. Thus, we can conclude that a contest designer who wishes to maximize the expected total effort has an incentive to reveal information about the value of winning to the player with inferior information.

Finally we assume that players face budget constraints, which implies that there will be caps on the bids that the players are able to place. A budget constraint changes the players' equilibrium behavior compared to the same contests without budget constraints. This was shown, among others, by Che and Gale (1998) and Gavious, Moldovanu and Sela (2003) in the standard all-pay auction under complete and incomplete information[1]. We show that this observation is also valid in common-value all-pay auctions, and the budget constraint may drastically change the relation between the players' expected payoffs. In particular, we show that the budget constraint may imply that the player with information advantage will have a lower expected payoff than his opponent. In other words, information advantage may turn into payoff disadvantage.

Several researchers used the same framework as ours to analyze common-value second-price auctions and common-value first price auctions (see Einy et al. 2001, 2002, Forges and Orzach 2011, Malueg and Orzach 2011 and Abraham et al. 2012). To the best of our knowledge we are the first to take advantage of this framework in order to study the role of information in asymmetric all-pay auctions. Without budget constraints, in the all-pay auctions, as well as in the first-price and second-price auctions, the player with information advantage has a higher expected payoff than his opponent. However, in contrast to first-price and second-price auctions, in all-pay auctions the players' bids (effort) as well as their chances of winning are the same despite their asymmetric information.

Although we analyze two-player common-value all-pay auctions, our results can be generalized to any number of players as long as the players' information sets can be ranked, namely, for each pair of players we can say who is the player with the information advantage. In such a case, as well as in the complete information all-pay auction (see Baye et al. 1996), given any set of players, there will be an equilibrium in which only the two more informed players will participate, and the other players will stay out of the contest (or alternatively will place a bid of zero).

The rest of the paper is organized as follows. In Section 2 we present the model, followed by an example in Section 3. In Section 4.1 we characterize the equilibrium and prove its uniqueness. In Section 4.2 we analyze the players' expected effort, their probabilities of winning and their expected payoffs. In Section 4.3 we examine the effect of the information on the players' total effort. In section 5 we study the model with budget constrained players. Section 6 concludes. The proof of Proposition 1 is in the appendix.

[1] Che and Gale (1998) and Gavious, Moldovanu and Sela (2003) deal with all-pay auctions with bid caps. The bid cap is a budget constraint that the contest designer imposes on the contestants.

2 The model

Consider the set $\mathcal{N} = \{1, 2, ..., N\}$ of $N \geq 2$ players, who compete in an all-pay auction where the player with the highest effort (output) wins the contest, but all the players bear the cost of their effort. The uncertainty is described by a finite set Ω of states of nature, and a probability distribution p over Ω – the common prior belief about the realized state of nature (w.l.o.g. $p(\omega) > 0$ for every $\omega \in \Omega$). A function $v : \Omega \to \mathbb{R}_+$ represents the common value of winning the contest, i.e., if $\omega \in \Omega$ is realized then the value of winning is $v(\omega)$ for every player.

The private information each player $n \in \mathcal{N}$ is described by a partition Π_n of Ω. We assume that each Π_n is *connected* w.r..t. the common value function v, i.e., for every element $\pi_n \in \Pi_n$, if $\omega_1, \omega_2 \in \pi_n$ and $\omega \in \Omega$ are such that $v(\omega_1) \leq v(\omega) \leq v(\omega_2)$, then $\omega \in \pi_n$.

A common-value all-pay auction starts with the move of nature, which selects a state ω form Ω according to the distribution p. The players do not observe the state ω, but each $n \in \mathcal{N}$ is informed of the element $\pi_n(\omega)$ of Π_n which contains ω – thus, $\pi_n(\omega)$ constitutes the information set of player n at ω – and then he chooses an effort $x_n \in \mathbb{R}_+$. The players will typically have different information partitions, and thus be ex-ante asymmetric.

The utility (payoff) of player $n \in \mathcal{N}$ is given by the function $u_n : \Omega \times \mathbb{R}_+^N \to \mathbb{R}$:

$$u_n(\omega, x) = \begin{cases} \frac{1}{m(x)} v(\omega) - x_n, & \text{if} \quad x_n = \max\{x_k\}_{k \in \mathcal{N}}, \\ -x_n, & \text{if} \quad x_n < \max\{x_k\}_{k \in \mathcal{N}}, \end{cases}$$

where $m(x)$ denotes the number of players who exert the highest effort, namely, $m(x) = |n \in N : x_n = \max\{x_k\}_{k \in \mathcal{N}}|$. A common-value all-pay auction with differential information is fully described by, and identified with, the collection $G = (N, (\Omega, p), \{u_n\}_{n \in \mathcal{N}}, \{\Pi_n\}_{n \in \mathcal{N}})$.

In all-pay auctions, there is usually no equilibrium in pure strategies, and our attention will thus be given to mixed strategy equilibria. A mixed strategy of player $n \in \mathcal{N}$ is a function $F_n : \Omega \times \mathbb{R}_+ \to [0, 1]$, such that, for every $\omega \in \Omega$, $F_n(, \cdot)$ is a cumulative distribution function (c.d.f.) on $\mathbb{R}_+$, and for all $x \in \mathbb{R}_+$, $F_n(\cdot, x)$ is a Π_n-measurable function (that is, $F_n(\cdot, x)$ is constant on every element of Π_n). If player n plays pure strategy given π_n, i.e., if the distribution represented by $F_n(\pi_n, \cdot)$ is supported on some $y \in \mathbb{R}_+$, we will identify between $F_n(\pi_n, \cdot)$ and y wherever appropriate.

Given a mixed strategy profile $F = (F_1, ..., F_N)$, denote by $E_n(F)$ the expected payoff of player n when players use that strategy profile, i.e.,

$$E_n(F) \equiv E(\int_0^\infty ... \int_0^\infty u_n(\cdot, (x_1, ..., x_N))dF_1(\cdot, x_1), ..., dF_N(\cdot, x_N)).$$

If $\pi_n \in \Pi_n$, $E_n(F \mid \pi_n)$ will denote the conditional expected payoff of player n given his information set π_n, i.e.,

$$E_n(F \mid \pi_n) \equiv E(\left[\int_0^\infty ... \int_0^\infty u_n(\cdot, (x_1, ..., x_n))dF_1(\cdot, x_1), ..., dF_i(\cdot, x_i), ..., dF_N(\cdot, x_N)\right] \mid \pi_n).$$

An N-tuple of mixed strategies $F^* = (F_1^*, ..., F_N^*)$ constitutes a Bayesian equilibrium in the common-value all-pay auction G if for every player n, and every mixed strategy F_n of that player, the following inequality holds:

$$E_n(F^*) \geq E_n(F_1^*, ..., F_n, ..., F_N^*).$$

3 An Example

We begin with a simple example to illustrate the players' behavior in our model. Consider a common-value all-pay auction with two players. Assume that there are three states of nature such that in state ω_i the value of winning is $v(\omega_i) = i$ with probability of $p_i = \frac{1}{3}, i = 1, 2, 3$. Player 1 knows *only* the prior distribution p, and hence he has the trivial information partition, $\Pi_1 = \{\{\omega_1, \omega_2, \omega_3\}\}$, while player 2 is completely informed of the value of winning, hence $\Pi_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$ partitions Ω into singleton states.

It can be easily verified that the corresponding common-value all-pay auction does not have an equilibrium in pure strategies. However, there exists a mixed strategy equilibrium. In this equilibrium, player 1's mixed strategy F_1^* is a state-independent c.d.f. given by

$$F_1^*(x) = \begin{cases} 0 & \text{if } x < 0, \\ x, & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{x}{2} + \frac{1}{6}, & \text{if } \frac{1}{3} < x \leq 1, \\ \frac{x}{3} + \frac{1}{3} & \text{if } 1 < x \leq 2, \\ 1, & \text{if } 2 < x. \end{cases}$$

Player 2's mixed strategy F_2^* does depend on the state of nature (of which he is informed):

$$F_2^*(\omega_1, x) = \begin{cases} 0 & \text{if } x < 0, \\ 3x, & \text{if } 0 \le x \le \frac{1}{3}, \\ 1 & \text{if } x > \frac{1}{3}, \end{cases}$$

$$F_2^*(\omega_2, x) = \begin{cases} 0, & \text{if } x < \frac{1}{3}, \\ \frac{3}{2}x - \frac{1}{2}, & \text{if } \frac{1}{3} < x \le 1, \\ 1, & \text{if } x > 1, \end{cases}$$

$$F_2^*(\omega_3, x) = \begin{cases} 0, & \text{if } x < 1, \\ x - 1, & \text{if } 1 < x \le 2, \\ 1, & \text{if } x > 2. \end{cases}$$

In order to see that the strategies above are in equilibrium, note that, given player 2's mixed strategy F_2^*, player 1's expected payoff if he exerts effort $x \in [1, 2]$ is

$$E_1(x, F_2^*) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 3 \cdot (x - 1) - x = 0$$

When $x \in [\frac{1}{3}, 1]$,

$$E_1(x, F_2^*) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 \cdot (\frac{3}{2}x - \frac{1}{2}) - x = 0,$$

and when $x \in [0, \frac{1}{3}]$,

$$E_1(x, F_2^*) = \frac{1}{3} \cdot 1 \cdot (3x) - x = 0.$$

As any effort above 2 would result in a negative expected payoff, $[1, 2]$ is the set of player 1's pure strategy best responses to to F_2^*, and in particular his mixed strategy F_1^* is a best response to F_2^* (resulting in the expected payoff of zero).

Now, fix payer 1's mixed strategy F_1^*, and assume that ω_3 is the realized state of nature. If player 2 exerts effort $x \in [1, 2]$, then his conditional expected payoff is

$$E_2(\{\omega_3\}, F_1^*, x) = 3 \cdot (\frac{x}{3} + \frac{1}{3}) - x = 1.$$

If he exerts $x \in [\frac{1}{3}, 1)$ or $x \in [0, \frac{1}{3}]$, his expected payoff is, correspondingly,

$$E_2(\{\omega_3\}, F_1^*, x) = 3 \cdot (\frac{x}{2} + \frac{1}{6}) - x = \frac{x}{2} + \frac{1}{2} < 1,$$

or

$$E_2(\{\omega_3\}, F_1^*, x) = 3 \cdot x - x = 2x < 1,$$

and thus, conditional on the realization of ω_3, $[1, 2]$ is the set of player 2's pure strategy best responses to F_1^*. In particular, conditional on ω_3, $F_2^*(\omega_3, \cdot)$ is a mixed strategy best response to F_1^*.

If ω_2 is the realized state, by exerting $x \in [\frac{1}{3}, 1]$ player 2 obtains the expected payoff

$$E_2(\{\omega_2\}, F_1^*, x) = 2 \cdot (\frac{x}{2} + \frac{1}{6}) - x = \frac{1}{3}.$$

As before, it can be seen that all effort levels outside $[\frac{1}{3}, 1]$ lead to a lower expected payoff, and thus conditional on ω_2, $F_2^*(\omega_2, \cdot)$ is a mixed strategy best response to F_1^*.

If ω_1 is the realized state, by exerting $x \in [0, \frac{1}{3}]$ player 2 in expectation obtains

$$E_2(\{\omega_1\}, x) = 1 \cdot x - x = 0,$$

and effort levels outside $[0, \frac{1}{3}]$ lead to negative expected payoffs. Thus, also conditional on ω_1, $F_2^*(\omega_1, \cdot)$ is a mixed strategy best response to F_1^*. We conclude that F_2^* is a best response of player 2 also w.r.t. the unconditional expected payoff, and hence the pair (F_1^*, F_2^*) is a mixed strategy equilibrium. The expected payoff of player 2 is

$$E_2(F_1^*, F_2^*) = \frac{1}{3}(E_2(\{\omega_1\}, F_1^*, F_2^*) + E_2(\{\omega_2\}, F_1^*, F_2^*) + E_2(\{\omega_3\}, F_1^*, F_2^*)) = \frac{4}{9}.$$

In the next section, we will characterize the players' mixed-strategy equilibrium in general two-player common-value all-pay auctions, and prove its uniqueness.

4 Results

4.1 Equilibrium analysis

We will consider the case of two players, where player 2 has information advantage over player 1 (i.e., information partition Π_2 of player 1 is finer than Π_1). Without loss of generality, we assume that $\Pi_1 = \{\Omega\}$

and $\Pi_2 = \{\{\omega_1\}, \{\omega_1\}, ..., \{\omega_n\}\}$.[2] That is, player 1 has no information on the realized state of nature (other than the common prior distribution p) and thus has the trivial information partition, while player 2 knows the realized state precisely, and thus his information partition is the finest one possible.

For each state of nature $\omega_i \in \Omega$, denote $v_i = v(\omega_i)$ and $p_i = p(\omega_i) > 0$. Assume that the possible values are strictly ranked: $0 < v_1 < v_2 < ... < v_n$. In what follows we will describe a mixed strategy equilibrium (F_1^*, F_2^*) of the all-pay auction.

Let $x_0 \equiv 0$, and

$$x_i \equiv \sum_{j=1}^{i} p_j v_j \tag{1}$$

for each $i = 1, ..., n$. Thus $x_0 < x_1 < ... < x_n$. Consider a function F_1^* on $\mathbb{R}_+$ given by

$$F_1^*(x) = \frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right] \tag{2}$$

whenever $x \in [x_{i-1}, x_i]$, $i = 1, ..., n$, with $F_1^*(x) \equiv 0$ for $x < 0$ and $F_1^*(x) \equiv 1$ for $x > x_n$. It is easy to see that $F_1^*(x)$ is well defined, strictly increasing and continuous. Moreover, $F_1^*(x_0) = 0$ and $F_1^*(x_n) = 1$. Thus, $F_1^*(x)$ is a c.d.f. of a continuous probability distribution supported on the interval $[x_0, x_n]$. (Such a distribution is obtained by assigning probability p_i to each interval $[x_{i-1}, x_i]$, randomly choosing an interval, and then selecting a point w.r.t. the uniform distribution on the chosen interval). The function F_1^*, being state-independent, can be viewed as a mixed strategy of the uninformed player 1.

Note next that

$$\begin{aligned} E_2(\{\omega_i\}, F_1^*, x) &= v_i F_1^*(x) - x &(3) \\ &= v_i F_1^*(x_{i-1}) - x_{i-1} = E_2(\{\omega_i\}, F_1^*, x_{i-1}) &(4) \end{aligned}$$

[2] In the general case of Π_2 being finer than Π_1, note the following. Given $\pi_1 \in \Pi_1$, the event π_1 is common knowledge at any $\omega \in \pi_1$. Thus the equilibrium analysis can be carried out separately for each $\pi_1 \in \Pi_1$, as the auction G conditional on the occurence of π_1 can be viewed as a distinct common-value all-pay auction G', where the set of states of nature is $\Omega' = \pi_1$ and the conditional distribution $p(\cdot \mid \pi_1)$ is serves the common prior distribution p'. In G', player 1 has the trivial information partition, $\Pi_1' = \{\pi_1\}$. Furthermore, since the mixed strategies of both players are constant on every $\pi_2 \subset \pi_1$, $\pi_2 \in \Pi_2$, there will be no payoff distinction between G' and its variant G'', where the set of states of nature $\Omega'' = \Pi_2 |_{\pi_1}$ consists of those elements of Π_2 that are subsets of π_1 (i.e., all states of nature in Ω' that are contained in the same element of π_1 are lumped into one state). By definition, the information partition of player 2 in G'' will be the finest possible, consisting of singletons in $\Pi_2 |_{\pi_1}$.

for every $x \in [x_{i-1}, x_i]$, and $i = 1, ..., n$. Thus, given that ω_i was realized, the informed player 2 is indifferent between all efforts in the interval $[x_{i-1}, x_i]$, provided his rival acts according to F_1^*. Since the slopes of the function $v_i F_1^*(x) - x$ are positive when $x < x_{i-1}$ and negative when $x > x_{i-1}$, the set of player 2's pure strategy best responses is the interval $[x_{i-1}, x_i]$.

Now, for each $i = 1, ..., n$, consider a function $F_2^*(\omega_i, x)$ on $\mathbb{R}_+$ given by

$$F_2^*(\omega_i, x) = \frac{x - \sum_{j=1}^{i-1} p_j v_j}{p_i v_i} \tag{5}$$

for $x \in [x_{i-1}, x_i]$, with $F_2^*(\omega_i, x) \equiv 0$ for $x_{i-1} < 0$ and $F_2^*(\omega_i, x) \equiv 1$ for $x > x_i$. Note that $F_2^*(\omega_i, x)$ is well defined, strictly increasing, continuous, $F_2^*(\omega_i, x_{i-1}) = 0$ and $F_2^*(\omega_i, x_i) = 1$. Thus $F_2^*(i, x)$ is a c.d.f. of a probability distribution supported on $[x_{i-1}, x_i]$, and in particular F_2^* constitutes a mixed strategy of player 2. Moreover,

$$E_1(x, F_2^*) = \sum_{j=1}^{i-1} p_j v_j + p_i v_i F_2^*(\omega_i, x) - x = 0 \tag{6}$$

for every $x \in [x_{i-1}, x_i]$. Thus, player 1 is (in expectation) indifferent between all efforts in $[x_0, x_n]$ (and is obviously worse off with efforts outside $[x_0, x_n]$) provided his rival 2 acts according to F_2^*.

We conclude that (F_1^*, F_2^*) is a mixed strategy equilibrium. It turns out that it is the only one:

Proposition 1 *The mixed strategy equilibrium (F_1^*, F_2^*) described above is the unique equilibrium in G.*

Proof. See Appendix. ∎

We have assumed thus far that there are only two players. This entails no loss of generality, in the following sense. Suppose that there are $N > 2$ players, such that players' information endowments are ranked: player 2 has information advantage over player 1, and player 1 has information advantage over (or the same information endowment as) players $3, ..., N$. Let (F_1^*, F_2^*) be the unique equilibrium in the contest between 1 and 2 (that exists by Proposition 1 and Footnote 1). We claim that in the contest between $1, 2, ..., N$, strategy profile $(F_1^*, F_2^*, 0, 0, ...0)$ constitutes a Bayesian equilibrium. That is, all but the two players with the best information submit bids of zero – effectively staying out of contest – while players 1 and 2 behave as if they were engaged in a two-player contest. This will ensure that any N-player contest in which information endowments are ranked, possesses a reduction to the two-player case.

In order to see that $(F_1^*, F_2^*, 0, 0, ...0)$ is a Bayesian equilibrium, note first that players 1 and 2 have no incentive to unilaterally deviate from their strategies in $(F_1^*, F_2^*, 0, 0, ...0)$, since their payoffs are identical to those in a two-player contest, where such deviations are not profitable in expectation. Note next that, if any of the remaining players (say, player 3) had a profitable deviation F_3 from bid 0, we would have had

$$E_3(F_1^*, F_2^*, F_3, 0, ..., 0) > E_3(F_1^*, F_2^*, 0, 0, ..., 0) = 0,$$

and hence

$$E_3(F_1^*, F_2^*, F_3, 0, ..., 0) > 0.$$

Since player 1 has information advantage over (or the same information as) player 3, F_3 is also a Bayesian strategy of player 1. As F_1^* is 1's best response to $(F_1^*, F_2^*, 0, 0, ...0)$, it follows that

$$\begin{aligned} E_1(F_1^*, F_2^*, 0, 0, ...0) &\geq E_1(F_3, F_2^*, 0, 0, ..., 0) \\ &= E_3(0, F_2^*, F_3, 0, ..., 0) \\ &\geq E_3(F_1^*, F_2^*, F_3, 0, ..., 0) > 0. \end{aligned}$$

Thus

$$E_1(F_1^*, F_2^*, 0, 0, ...0) > 0,$$

and in particular $E_1(F_1^*, F_2^*) > 0$ in the two-player contest between 1 and 2. However, it follows from (6), Proposition 1, and Footnote 1 that the expected payoff to player 1 in the unique equilibrium is zero, a contradiction. We conclude that players $3, .., N$ cannot unilaterally deviate from bid 0 and make profit, and hence that $(F_1^*, F_2^*, 0, 0, ...0)$ is a Bayesian equilibrium of the N-player contest, as claimed.

In the next section, using the characterization of equilibrium in Proposition 1, we study the effect of information on players' payoffs, efforts and probabilities of winning.

4.2 Expected payoffs and efforts

We have seen that equilibrium strategies in a two-payer common-value all-pay auction are determined uniquely, given our assumptions. The expected equilibrium payoff of player 1 is zero: it follows from (6) that

$$E_1(F_1^*, F_2^*) = 0. \tag{7}$$

And it follows from (3)-(4) that player 2's expected payoff is

$$\begin{aligned} E_2(F_1^*, F_2^*) &= \sum_{i=1}^{n} p_i(v_i F_1(x_{i-1}) - x_{i-1}) \\ &= \sum_{i=1}^{n} p_i(\sum_{j=1}^{i-1} p_j(v_i - v_j)). \end{aligned} \tag{8}$$

The equilibrium strategies F_1^*, F_2^* of the two players are quite different. Among other distinctions, F_2^* is state-dependent, while F_1^* is not. However, the following result shows that although the asymmetry of information leads to marked differences in equilibrium strategies, the *expected* equilibrium efforts of both players are identical.

Proposition 2 *In every two-player common-value all-pay auction, both players exert the same expected effort in equilibrium* (F_1^*, F_2^*).

Proof. By (2) and (5) the players' expected efforts are

$$TE_1 = \int_{x_0}^{x_n} x dF_1^*(x) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \frac{x}{v_i} dx = \sum_{i=1}^{n} \frac{x_i^2 - x_{i-1}^2}{2v_i} = \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{i-1} p_j v_j + \frac{1}{2} p_i v_i \right),$$

for player 1, and

$$TE_2 = \sum_{i=1}^{n} p_i \int_{x_{i-1}}^{x_i} x dF_2^*(\omega_i, x) = \sum_{i=1}^{n} p_i \int_{x_{i-1}}^{x_i} \frac{x}{p_i v_i} dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \frac{x}{v_i} dx = \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{i-1} p_j v_j + \frac{1}{2} p_i v_i \right),$$

for player 2. Thus, the expected effort is the same for both players. *Q.E.D.* ■

The next result shows that, although the asymmetry of the players' information affects their equilibrium strategies, their probability to win the contest are equal.

Proposition 3 *In equilibrium* (F_1^*, F_2^*) *of every two-player common-value all-pay auction, each player has probability* $\frac{1}{2}$ *to win the contest.*

Proof. The probability of player 2 to win is

$$P_2 = \sum_{i=1}^{n} p_i \int_{x_{i-1}}^{x_i} F_1(x) dF_2(\omega_i, x).$$

Since $dF_2(\omega_i, x) = \frac{1}{p_i v_i} dx$, we have

$$P_2 = \sum_{i=1}^{n} \frac{1}{v_i} \int_{x_{i-1}}^{x_i} F_1(x) dx$$

As $x_i \equiv \sum_{j=1}^{i} p_j v_j$ for $i = 1, ..., n$,

$$
\begin{aligned}
P_2 &= \sum_{i=1}^{n} \frac{1}{v_i} \int_{x_{i-1}}^{x_i} F_1(x)dx \\
&= \sum_{i=1}^{n} \frac{1}{v_i} \int_{\sum_{j=1}^{i-1} p_j v_j}^{\sum_{j=1}^{i} p_j v_j} \left(\frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i} \right] \right) dx \\
&= \sum_{i=1}^{n} \frac{1}{v_i} (\sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i} \right] (p_i v_i) + \frac{(p_i v_i)^2 + 2 p_i v_i \sum_{j=1}^{i-1} p_j v_j}{2 v_i}) \\
&= \sum_{i=1}^{n} (\sum_{j=1}^{i-1} p_j p_i + \frac{(p_i)^2}{2}) = \sum_{i=1}^{n} (\frac{(p_i)^2}{2} + p_i \sum_{j=1}^{i-1} p_j \\
&= \frac{(\sum_{i=1}^{n} p_i)^2}{2} = \frac{1}{2}
\end{aligned}
$$

Q.E.D. ■

According to Proposition 2, the asymmetry in information does not affect the ratio of the two players' expected efforts, as the expected efforts are equal. However, the asymmetric information does affect the players' expected *total* effort. In the next section we will examine the effect of the asymmetry in information on the expected total effort and the expected total payoff.

4.3 Comparative results

It has been shown that the expected payoff of player 1, over whom player 2 has information advantage, is zero in equilibrium. We will now examine how the *extent* of information advantage affects the expected payoff of player 2. Assume, as before[3], that $\Pi_1 = \{\Omega\}$ and $\Pi_2 = \{\{\omega_1\}, \{\omega_1\}, ..., \{\omega_n\}\}$. Also consider an additional player 2' with an "intermediate" connected information partition Π_2', which is a strict coarsening of Π_2 and a strict refinement of Π_1. Then we have the following comparative result.

Proposition 4 *In a two-player common-value all-pay auction, the expected payoff of player 2 (when he competes against player 1) is higher than the expected payoff of player 2' (when he competes against player 1).*

[3] See Footnote 1.

Proof. By (8) the expected payoff of player 2, when he competes against player 1, is given in equilibrium by

$$E_2 = \sum_{i=1}^{n} p_i (\sum_{k=1}^{i-1} p_k (v_i - v_k)).$$

Regarding player 2', assume first that Π'_2 is different from Π_2 only in that player 2' cannot distinguish between the states ω_j and ω_{j+1}, for some $1 \leq j < n$, and thus $\Pi'_2 = \{\{\omega_1\}, \{\omega_2\}, ...\{\omega_{j-1}\}, \{\omega_j, \omega_{j+1}\}, \{\omega_{j+2}\}, ...\{\omega_n\}\}$. The auction in which player 2' competes against player 1 is amenable to our previous analysis, with a minor modification: the set of states of nature must be redefined as $\Omega' = (\Omega \backslash \{\omega_j, \omega_{j+1}\}) \cup \{\omega_{j,j+1}\}$, where the new state $\omega_{j,j+1}$ is the amalgamation of ω_j and ω_{j+1}, occurring with probability $p_{j,j+1} = p_j + p_{j+1}$ and having the common value of $v_{j,j+1} = \frac{p_j}{p_j + p_{j+1}} v_j + \frac{p_{j+1}}{p_j + p_{j+1}} v_{j+1}$. In this modified contest (payoff-equivalent to the original), player 1 has the trivial information, and player 2' has the finest possible information partition. Applying (8) to this contest, the expected payoff of player 2 is equilibrium is given by

$$\begin{aligned} E'_2 &= \sum_{i=1}^{j-1} p_i (\sum_{k=1}^{i-1} p_j (v_i - v_j)) \\ &\quad + p_{j,j+1} \sum_{k=1}^{j-1} p_k (v_{j,j+1} - v_k) \\ &\quad + \sum_{i=j+2}^{n} p_i (\sum_{k \leq i-1, k \neq j, k \neq j+1} p_k (v_i - v_k) + p_{j,j+1} (v_i - v_{j,j+1})). \end{aligned}$$

Then we have

$$E_2 - E'_2 = p_j p_{j+1} (v_{j+1} - v_j) + \sum_{i=j+2}^{n} p_i (\sum_{k=j}^{j+1} p_k (v_i - v_k) - p_{j,j+1} (v_i - v_{j,j+1})).$$

Since $p_j p_{j+1} (v_{j+1} - v_j) > 0$ and $\sum_{i=j+2}^{n} p_i (\sum_{k=j}^{j+1} p_k (v_i - v_k) - p_{j,j+1} (v_i - v_{j,j+1})) = 0$ we obtain that $E_2 - E'_2 > 0$.

We have thus shown that 2' obtains in expectation less than 2 (when competing against 1 in a two-player auction) if Π'_2 is a connected partition, which is a strict coarsening of Π_2 with $|\Pi'_2| = |\Pi_2| - 1$ $(= n - 1)$. Inductively, the claim can be extended to *any* connected partition Π'_2 with $|\Pi'_2| < n$. Q.E.D. ■

The next result shows that there is an opposite relation between the players' total expected payoff and their total expected effort (bid).

Proposition 5 *In a two-player common-value all-pay auction, the expected total effort when player 2 competes against player 1 is lower than the expected total effort when player 2' competes against player 1.*

Proof. In every common-value all-pay auction, the relation between the players' expected total effort and their expected total payoff is

$$\text{Expected total effort} \ = \ \text{Expected reward} \ - \ \text{Expected total payoff}$$

Since the expected payoff of player 1 when he competes against player 2 or against 2' is zero (see (7)), in both auctions

$$\text{Expected total effort} = \text{Expected reward} - \text{Expected payoff of player 2 (or, 2')}$$

By Proposition 4 we know that the expected payoff of player 2 is higher than that of player 2' (when competing against player 1). On the other hand, both contests clearly have the same expected reward, $E(v)$. Thus, the expected total effort when player 2 competes against player 1 is lower than when player 2' competes against 1. *Q.E.D.* ■

The above propositions demonstrate that increasing asymmetry between players in a two-player common-value all-pay auction has a positive effect on the expected payoff of the player with information advantage, and a negative effect on the expected total effort.

5 Budget Constraints

We have thus far assumed that players submit any bids they wish, without constraint. In this section we will note that having (even identical) budget constraints can change our results in a significant way. We shall assume, as in Section 4, that there are two players, and that $\Pi_1 = \{\Omega\}$ and $\Pi_2 = \{\{\omega_1\}, \{\omega_1\}, ..., \{\omega_n\}\}$, i.e., player 1 has no information on the realized states of nature (other than the common prior distribution p), while player 2 knows the realized state precisely. As before, for each state of nature $\omega_i \in \Omega$, denote $v_i = v(\omega_i)$ and $p_i = p(\omega_i)$. Each player can submit any bid which is lower than or equal to a given budget constraint $d > 0$. The following example demonstrates the effect of the budget constraint on players' equilibrium strategies, and in particular shows that having budget constraint may imply higher expected payoff to the uninformed player 1 compared to the informed player 2.

Example 1 *Assume that $n = 3$, and that in state ω_i the value of winning is $v(\omega_i) = i$ with probability of $p_i = \frac{1}{3}, i = 1, 2, 3$. Assume also that the players have the same budget constraint, $d = \frac{5}{6}$. Then the content* *possesses the following pure strategy Bayesian equilibrium: player 1's bid is independent of the state of nature,* $x_1^* \equiv \frac{5}{6}$*, and player 2's state-dependent bid is given by*

$$x_2^*(\omega) = \begin{cases} 0, & \text{if } \omega = \omega_1 \\ \frac{5}{6}, & \text{if } \omega \neq \omega_1 \end{cases}.$$

The expected payoff of player 1 is then

$$E_1 = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot \frac{1}{2} \cdot 3 - \frac{5}{6} = \frac{1}{3},$$

and the expected payoff of play player 2 is

$$E_2 = \frac{1}{3} \cdot (\frac{1}{2} \cdot 2 - \frac{5}{6}) + \frac{1}{3} \cdot (\frac{1}{2} \cdot 3 - \frac{5}{6}) = \frac{5}{18}.$$

Thus, the expected payoff of the uninformed player 1 is higher than that of the informed player 2.

In the following we generalize the above example and present a pure strategy equilibrium when players have identical budget constraints.

Proposition 6 *Consider a two-player common-value all-pay auction with a budget constraint d for both players. Suppose that there exists $1 \leq j \leq n-1$ such that*

$$\frac{1}{2} \sum_{m=j+1}^{n} p_j v_j \geq d, \tag{9}$$

and

$$\frac{v_j}{2} \leq d < \frac{v_{j+1}}{2}. \tag{10}$$

Then the contest possesses a pure-strategy Bayesian equilibrium (x_1^*, x_2^*), in which

$$x_1^* \equiv d, \tag{11}$$

and

$$x_2^*(\omega_k) = \begin{cases} 0, & \text{for } \ k = 1, ..., j, \\ d, & \text{for } k = j+1, ..., n \end{cases}. \tag{12}$$

The expected total effort in this equilibrium is

$$TE = d(1 + \sum_{m=j+1}^{n} p_m).$$

Proof. If the players use the strategies given by (11) and (12), the expected payoff of player 2 is

$$E_2\left(x_1^*, x_2^*\right) = \sum_{m=j+1}^{n} p_m\left(\frac{1}{2}v_m - d\right), \tag{13}$$

and the expected payoff of player 1 is then

$$E_1\left(x_1^*, x_2^*\right) = \sum_{m=1}^{j} p_m(v_m - d) + \sum_{m=j+1}^{n} p_m\left(\frac{1}{2}v_m - d\right). \tag{14}$$

If player 1 submits a bid of $x_1 = \varepsilon < d$, then

$$E_1\left(x_1, x_2^*\right) = \sum_{m=1}^{j} p_m v_m - \varepsilon < E_1\left(x_1^*, x_2^*\right)$$

by (9). Thus x_1^* is player 1's (unique) best response to x_2^*.

If player 2 unilaterally deviates from x_2^* to a strategy x_2 with $0 < x_2(\omega_k) = \varepsilon \leq d$ for some $1 \leq k \leq j$, then

$$\begin{aligned} E_2\left(\{\omega_k\}, x_1^*, x_2\right) &\leq \max\left(-\varepsilon, \frac{v_k}{2} - d\right) \\ &\leq 0 = E_2\left(\{\omega_k\}, x_1^*, x_2^*\right), \end{aligned}$$

where the second inequality is implied by (10). And, if 2 unilaterally deviates from x_2^* to a strategy x_2 with $0 \leq x_2(\omega_k) = \varepsilon < d$ for some $j+1 \leq k \leq n$, then

$$\begin{aligned} E_2\left(\{\omega_k\}, x_1^*, x_2\right) &= -\varepsilon \\ &\leq 0 \leq \frac{v_k}{2} - d = E_2\left(\{\omega_k\}, x_1^*, x_2^*\right), \end{aligned}$$

also by (10). By taking expectation over ω_k, it follows that

$$E_2\left(x_1^*, x_2\right) \leq E_2\left(x_1^*, x_2^*\right)$$

for *any* pure strategy x_2 of player 2 that obeys his budget constraint, and thus x_2^* is player 2's best response to x_1^*. Q.E.D. ■

In Example 1, we have shown that the uninformed player 1's expected payoff could be higher than that of the informed player 2. By comparing the players' expected payoffs given by (13) and (14), we obtain that the expected payoff of player 1 is higher than that of player 2 if and only if

$$\sum_{m=1}^{j} p_m(v_m - d) \geq 0.$$

The existence of pure strategy equilibrium, established in Proposition 6, is based on assumptions (9) and (10). Furthermore, if the budget constraint d is below $\frac{v_1}{2}$, both players would make a bid equal to the bid cap in all the states of nature in a pure strategy equilibrium. But our model with identical budget constraints may also have a mixed strategy equilibrium, and for sufficiently large cap d the equilibrium would in fact be unique, identical to the one in Proposition 1.

6 Concluding remarks

In models with asymmetric information, differences in players' information usually result in different equilibrium strategies, probabilities of winning, and expected payoffs. In this model we show that even when the players' information can be ranked, with one player having information advantage over his opponent, the players' expected efforts, as well as their probabilities of winning the contest, are the same. The difference in information only manifests itself in the different expected payoffs. We also show that the highest expected total effort is obtained when the difference of the players' information is as small as possible. Thus, a contest designer who wishes to maximizes the players' expected total effort, has an incentive to provide information to an inferiorly informed player, or to restrict the use of information by a superiorly informed player, so that the players' information endowments become nearly comparable. We also showed that if players face budget constraints the information advantage might become disadvantage such that the player with the information advantage may have a lower expected payoff than that of his opponent. This result implies that a contest designer by imposing bid caps can control the relations of the players' expected payoffs.

We establish our results under the assumptions that information sets of each player are connected with respect to the value of winning the contest, and that the different information endowments can be ranked. These assumptions are found to be sufficient for the existence of equilibrium with monotonically increasing strategies. When information cannot be ranked, however, equilibrium with monotonically increasing strategies may not exist.

It would be interesting to examine whether the results in this paper can, at least partially, carry over to other contest forms with common value and asymmetric information. In particular, the question of how the asymmetric information is reflected in the relation between the players' expected efforts, and in their probabilities of winning, seems worthy of attention.

6.1 Proof of Proposition 1

Fix an equilibrium (F_1, F_2) in the auction G. We will prove that $(F_1, F_2) = (F_1^*, F_2^*)$.

In what follows, for $k = 1, 2$ and $\omega \in \Omega$, $F_k(\omega, \cdot)$ will be treated either as a *probability distribution* on $\mathbb{R}_+$, or as the corresponding *c.d.f.*, depending on the context. Also, as F_1 is state-independent, $F_1(\omega, \cdot)$ will be shortened to $F_1(\cdot)$, whenever convenient.

Notice that $F_k(\cdot, \{c\}) \equiv 0$ for any effort $c > 0$ and $k = 1, 2$. Indeed, if $F_k(\omega, \{c\}) > 0$ for some k and ω, then $F_m(\omega', (c - \varepsilon, c]) = 0$ for the other player m and every $\omega' \in \Omega$, and some sufficiently small $\varepsilon > 0$. But then k would be strictly better off by shifting the probability from c to $c - \frac{\varepsilon}{2}$, a contradiction to F_k being an equilibrium strategy. Thus, $F_1(\cdot), F_2(\omega, \cdot)$ are non-atomic on $(0, \infty)$ for every $\omega \in \Omega$. Notice also that there is no interval $(a, b) \subset (0, \infty)$ on which, in some state of nature, *only one* player places positive probability according to his equilibrium strategy. Indeed, otherwise there would exist $a' > a$ such that only one player places positive probability on (a', b), and it would then be profitable for that player to deviate (in at least one state of nature, if this is the informed player 2) by shifting positive probability from (a', b) to a'.

Suppose now that there is a bounded interval $(a, b) \subset (0, \infty)$ such that $F_1((a, b)) = 0$ (and thus $F_2(\omega, (a, b)) = 0$ for every $\omega \in \Omega$, by the previous paragraph), but $F_1([0, a]) > 0$ and $F_1([b, \infty)) > 0$. By extending this interval if necessary, it can also be assumed that (a, b) is maximal with respect to this property, i.e., that $F_1([\max(a - \varepsilon, 0), a]) > 0$ and $F_1([b, b + \varepsilon]) > 0$ for every small enough $\varepsilon > 0$. However, using the fact that $F_2(\omega, \cdot)$ is non-atomic on $(0, \infty)$ for every $\omega \in \Omega$, the expected payoff of player 1 at $\frac{a+b}{2}$ is strictly bigger than his payoff for any effort in $[b, b + \varepsilon]$, if $\varepsilon > 0$ is small enough. This contradicts the assumption that $F_1([b, b + \varepsilon]) > 0$. This contradiction shows that there exists no interval (a, b) as above, meaning that $F_1(\cdot)$ must have *full support* on some closed interval. Denote this interval[4] by $[c, d]$. Notice also that, for every $\omega \in \Omega$, $F_2(\omega, \cdot)$ must be supported on the interval $[c, d]$ (though there need not be full

[4] The interval must be bounded as no efforts above v_n will be made in equilibrium, due to the associated negative payoff.

support), since otherwise there would be an interval where only player 2 places positive probability, and this was ruled out.

Note next that $c = 0$. Indeed, if $c > 0$ then $F_2(\cdot, \{c\}) \equiv 0$, and thus player 1 has negative expected payoff for efforts in $[c, c + \varepsilon]$ for all small enough $\varepsilon > 0$ (because with efforts in $[c, c + \varepsilon]$ he loses the contest almost for sure, while expending positive effort of at least c). He would then profitably deviate from F_1 by shifting the probability from $[c, c + \varepsilon]$ to effort 0. Thus, indeed, $c = 0$. Note also that the interval $[0, d]$ is non-degenerate, i.e., $0 < d$, since otherwise the equilibrium strategies would prescribe the constant effort 0, and it is clear that each player would have a profitable unilateral deviation to some $\varepsilon > 0$.

Given i, $i = 1, ..., n$, we will now show that $F_2(\omega_i, \cdot)$ has full support on a (possibly degenerate) *subinterval* of $[0, d]$. Indeed, if not, there would exist an open subinterval $(a, b) \subset [0, d]$ such that $F_2(\omega_i, (a, b)) = 0$, but $F_2(\omega_i, [0, a]) > 0$ and $F_2(\omega_i, [b, d]) > 0$. Since $F_1((a, b)) > 0$, there must be $j \neq i$ such that $F_2(\omega_j, (a, b)) = 0 > 0$. Assume that $i < j$ (the opposite case is treated similarly). Then there are $x \in [b, d]$ and $y \in (a, b)$ such that

$$\begin{aligned} v_i F^1(x) - x &= E_2\left(\{\omega_i\}, F^1, x\right) && (15) \\ &\geq E_2\left(\{\omega_i\}, F^1, y\right) = v_i F^1(y) - y && (16) \end{aligned}$$

and

$$\begin{aligned} v_j F^1(x) - x &= E_2\left(\{\omega_j\}, F^1, x\right) && (17) \\ &\leq E_2\left(\{\omega_j\}, F^1, y\right) = v_j F^1(y) - y. && (18) \end{aligned}$$

But $x > y$, and therefore

$$(v_j - v_i)F^1(x) > (v_j - v_i)F^1(y) \tag{19}$$

since $v_i < v_j$ and the c.d.f. F^1 is strictly increasing on $[0, d]$. Adding (19) to the inequality in (15)-(16) contradicts the inequality obtained in (17)-(18), however, and therefore no such (a, b) exists. Consequently, each $F_2(\omega_i, \cdot)$ has full support on some subinterval[5] $[a_i, b_i]$ of $[0, d]$. Moreover, if $i < j$ then $[a_i, b_i]$ lies below $[a_j, b_j]$ (barring boundary points), since otherwise it would have been possible to find $x > y$, where

[5] All these subintervals are either non-degenerate (of positive length), or $\{0\}$, as only the latter can be an atom of $F_2(\omega_i, \cdot)$.

$x \in [a_i, b_i]$ and $y \in [a_j, b_j]$, such that inequalities (15)-(16) and (17)-(18) hold. As above, this would lead to contradiction via (19).

Thus, the intervals $\{[a_i, b_i]\}_{i=1}^n$ are disjoint (barring boundary points), and "ordered" according to the index i on the interval $[0, d]$. Moreover, $\cup_{i=1}^n [a_i, b_i] = [0, d]$, since otherwise there would be a "gap" (a, b) on which only player 1 places positive probability, which is impossible as we have seen earlier. It follows that there are points $0 = x_0 \leq x_1 \leq ... < x_n \equiv d$ such that $[a_i, b_i] = [x_{i-1}, x_i]$ for every $i = 1, 2, ..., n$, i.e., $F^1(\cdot)$ has full support on $[0, x_n]$, and, for $i = 1, ..., n$, $F_2(\omega_i, \cdot)$ has full support on $[x_{i-1}, x_i]$. Denote by i_0 the smallest integer with $x_{i_0} > 0$.[6]

Since $F^1(\cdot)$ has full support on $[0, x_n]$ and $F_2(\omega, \cdot)$ has no atoms (except, possibly, at 0), player 1 is indifferent between any two efforts in $(0, x_n]$. Thus, the following equality must hold for every $i = i_0, ..., n$ and every positive $x \in [x_{i-1}, x_i]$:

$$\sum_{j=1}^{i-1} p_j v_j + p_i v_i F_2(\omega_i, x) - x = E_1(x, F_2) = \lim_{y \searrow 0} E_1(y, F_2) \equiv e_1 \geq 0.$$

In particular,

$$F_2(\omega_i, x) = \frac{x - \sum_{j=1}^{i-1} p_j v_j + e_1}{p_i v_i} \tag{20}$$

for every $i = i_0, ..., n$ and every positive $x \in [x_{i-1}, x_i]$. Since $F_2(\omega_i, \cdot)$ is supported on $[x_{i-1}, x_i]$, we have $F_2(\omega_i, x_i) = 1$, and thus

$$x_i = \sum_{j=1}^{i} p_j v_j - e_1 \tag{21}$$

for every $i = i_0, ..., n$.

Since, for $i = i_0, ..., n$, $F_2(\omega_i, \cdot)$ has full support on $[x_{i-1}, x_i]$ and $F_1(\cdot)$ has no atoms (except, possibly, at 0), player 2 is indifferent between all positive efforts in $[x_{i-1}, x_i]$. Thus, the following equality must hold for every positive $x \in [x_{i-1}, x_i]$:

$$\begin{aligned} v_i F_1(x) - x &= E_2(\{\omega_i\}, F_1, x) \\ &= E_2(\{\omega_i\}, F_1, x_i) = v_i F_1(x_i) - x_i. \end{aligned}$$

In particular,

$$F_1(x) = \frac{x}{v_i} + F_1(x_i) - \frac{x_i}{v_i},$$

[6] Since each interval $[x_{i-1}, x_i]$ is either non-degenerate or $\{0\}$, $0 = x_0 = ... = x_{i_0 - 1} < x_{i_0} < ... < x_n$.

and using the fact that $F_1(x_n) = 1$ and (21), we obtain

$$F_1(x) = \frac{x + e_1}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right] \tag{22}$$

for every $i = i_0, ..., n$, and every positive $x \in [x_{i-1}, x_i]$.

If $e_1 > 0$, it follows from (22) that $F_1(\cdot)$ has an atom at effort 0. Then, obviously $F_2(\omega_i, \cdot)$ cannot have an atom at 0, for any i, since otherwise each player would have a profitable unilateral deviation that shifts the probability from zero to an effort slightly above zero. In particular, all intervals $\{[x_{i-1}, x_i]\}_{i=1}^n$ are non-degenerate, i.e., $i_0 = 1$. But then, by (20), $F_2(\omega_1, \cdot)$ has an atom at 0, a contradiction. We conclude that $e_1 = 0$.

If $i_0 > 1$, $x_{i_0-1} = 0$, and thus (20) should hold for $i = i_0$ and any sufficiently small x. But then, if $x < p_1 v_1$

$$F_2(\omega_{i_0}, x) = \frac{x - \sum_{j=1}^{i_0-1} p_j v_j}{p_{i_0} v_{i_0}} \leq \frac{x - p_1 v_1}{p_{i_0} v_{i_0}} < 0,$$

and thus $F_2(\omega_{i_0}, x)$ is not a c.d.f., a contradiction. Consequently, $i_0 = 1$.

It now follows from (21), (20), and (22) that

$$x_i = \sum_{j=1}^{i} p_j v_j$$

for every $i = 1, ..., n$, that

$$F_1(x) = \frac{x}{v_i} + \sum_{j=1}^{i-1} p_j \left[1 - \frac{v_j}{v_i}\right]$$

for every $i = 1, ..., n$ and every $x \in [x_{i-1}, x_i]$, and that

$$F_2(\omega_i, x) = \frac{x - \sum_{j=1}^{i-1} p_j v_j}{p_i v_i}$$

for every $i = 1, ..., n$ and positive $x \in [x_{i-1}, x_i]$. Thus, (F_1, F_2) coincides with (F_1^*, F_2^*) described in (2) and (5).

Q.E.D.

References

[1] Abraham, I., Athey, S., Babaiof, M., Grubb, M.: Peaches, Lemons, and Cookies: Designing auction markets with dispersed information. Mimeo

[2] Amman, E., Leininger,W.: Asymmetric all-pay auctions with incomplete information: the two-player case. *Games and Economic Behavior* 14, 1-18 (1996)

[3] Baye, M. R., Kovenock, D., de Vries, C.: Rigging the lobbying process: an application of the all-pay auction. *American Economic Review* 83, 289-294 (1993)

[4] Baye, M., Kovenock, D., de Vries, C.: The all-pay auction with complete information. *Economic Theory* 8, 291-305 (1996)

[5] Che, Y-K., Gale, I.: Caps on political lobbying. *American Economic Review* 88(3), 643-651 (1998)

[6] Einy, E., Haimanko, O., Orzach, R., Sela, A.: Dominant strategies, superior information,, and winner's curse in second-price auctions. *International Journal of Game Theory* 30, 405-419 (2001)

[7] Einy, E., Haimanko, O., Orzach, R., Sela, A.: Dominance solvability of second-price auctions with differential information. *Journal of Mathematical Economics* 37, 247-258 (2002)

[8] Forges, F., Orzach, R.: Core-stable rings in second price auctions with common values. *Journal of Mathematical Economics* 47, 760-767 (2011)

[9] Gavious, A., Moldovanu, B., Sela, A.: Bid costs and endogenous bid caps. *Rand Journal of Economics* 33(4), 709-722 (2003)

[10] Hillman, A., Riley, J.: Politically contestable rents and transfers. *Economics and Politics* 1, 17-40 (1989)

[11] Hillman, A., Samet, D.: Dissipation of contestable rents by small numbers of contenders. *Public Choice* 54(1), 63-82 (1987)

[12] Jackson, M.: Bayesian implementation. *Econometrica* 59, 461-477 (1993)

[13] Krishna, V., Morgan, J.: An analysis of the war of attrition and the all-pay auction. *Journal of Economic Theory* 72(2), 343-362 (1997)

[14] Malueg, D., Orzach, R.: Equilibrium and revenue in a family of common-value first-price auctions with differential information. *International Journal of Game Theory*, forthcoming (2011)

[15] Milgrom, P., Weber, R.: A theory of auctions and competitive bidding. *Econometrica* 50(5), 1089-1122 (1982)

[16] Moldovanu, B., Sela, A.: The optimal allocation of prizes in contests. *American Economic Review* 91(3), 542-558 (2001)

[17] Moldovanu, B., Sela, A.: Contest architecture. *Journal of Economic Theory* 126(1), 70-97 (2006)

[18] Moldovanu, B., Sela, A., Shi, X.: Carrots and sticks: prizes and punishments in contests. *Economic Inquiry* 50(2), 453-462 (2012)

[19] Siegel, R.: All-pay contests. *Econometrica* 77 (1), 71-92 (2009)

[20] Vohra, R.: Incomplete information, incentive compatibility and the core. *Journal of Economic Theory* 54, 429-447 (1999)