Hybrid Procedures.*

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Abstract

We consider hybrid procedures: a first step of reducing the game by iterated elimination of weakly dominated strategies ($IEWDS$) and then applying an equilibrium refinement. We show that the set of perfect/proper outcomes of a reduced normal-form game might be larger than the set of the perfect/proper outcomes of the whole game by applying $IEWDS$. In dominance solvable games in which all the orders of $IEWDS$ select a unique singleton in the game, the surviving outcome need not be a proper equilibrium of the whole game. However, in generic dominance solvable games that satisfy the transference of decision maker indifference condition, the surviving outcome coincides with the unique stable one and hence is proper. We finally apply hybrid procedures in voting games and use them to evaluate coordination failures of strategic voters under plurality voting.

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1 Introduction

Whereas the iterated removal of strictly dominated strategies seems to be commonly accepted as an appealing procedure to simplify a game[1], the procedure of iterated removal of weakly dominated strategies ($IEWDS$) seems to be more controversial[2]. Indeed, $IEWDS$ is an order-dependent procedure that removes at each step some set of weakly dominated strategies; this order-dependency is among its least attractive features. In our paper, we ask a simple question: what can be inferred about the set of perfect/proper equilibria of the whole normal-form game from just focusing on the same set of equilibria of the fully reduced game(s) obtained through this procedure? In other words, does applying $IEWDS$ and then use a perfect/proper equilibrium in a normal-form game refine the set of perfect/proper equilibria? We answer this question in a negative way, unless one focuses on dominance solvable games.

Up to now, we have not addressed the question of why would one want to infer some information about the set of equilibria of the whole game by just focusing on the set of equilibria of the reduced game. From the point of view of computational complexity[3], one interesting venue of research could be to understand the properties of first applying $IEWDS$ and then solving the game. To the best of our knowledge, such an idea is seldom present in the literature with the notable exceptions of Kohlberg and Mertens [8] and Samuelson [16]. First, Kohlberg and Mertens [8] consider such a procedure[4] and then prove that such a method does not uniquely reach stability in a game in which a dominated strategy of a player is replaced with a constant-sum game that has a value equal to the initial payoff matrix and at the same time no dominated strategies. In a sense, they prove that such a method is too weak. Samuelson [16] also considers such a procedure[5] even though the focus of such a paper is the interaction between the common knowledge of admissibility and iterated dominance. Our results imply that applying

[1]The previous observation holds in finite games. When agents can choose among an infinite number of strategies, this need not be even the case (see Duwfenberg and Stegeman [3]).

[2]Such a result is present, for instance, in the strategic voting literature (see De Sinopoli [1] among others).

[3]See the recent advances in computation of equilibria in finite games (for instance von Stengel et al. [19].

[4]Kohlberg and Mertens [8] (p.1015) argue "that one might therefore conclude that strategic stability could be obtained by first reducing the normal form to some submatrix by iterative eliminations of dominated strategies, and then applying the relevant backwards induction solution (i.e. proper equilibrium)".

[5]Indeed, Samuelson [16] (p.287) states that "concepts such as properness perform well in all respects except admissibility calculations. In particular, the set of proper equilibria can be affected by the deletion of a dominated strategy from a game. One possible response is to construct a two-stage procedure. In the first step, the common knowledge of admissibility is applied to possibly eliminate some strategies. The second step then consists of the application of a solution concept such as properness to the resulting strategy sets".

(any order of) $IEWDS$ and then applying properness might simply lead to different results than properness (both in the strategy profiles and in the payoffs) so that the "hybrid" procedure does not ensure neither perfectness nor properness.

To the best of our knowledge, two results, well-known in the literature, can be considered as a benchmark to our work. First, the set of Nash equilibria of a game G contains the set of NE of any game G' obtained from G by deletion of a (weakly) dominated strategy. So is the case with Mertens' stable sets (connected components of perfect equilibria). We call this property *inclusion*. The surviving outcome in a dominance solvable game is hence a Nash equilibrium and is part of the unique stable set of the game. Therefore, it is perfect as any point in a stable set is a perfect equilibrium. The results get more icy when one scrutinizes the relation between perfect, proper equilibrium and $IEWDS$.

The problem for ensuring perfect and proper inclusion seems to be related to the existence of connected components of equilibria with a continuum of outcomes. Examples of such components can be found in Govindan and McLennan [5] and Kukushkin, Litan and Marhuenda [9][6] We slightly modify the previously mentioned examples, in order to prove that, removing weakly dominated strategies might enlarge the set of perfect and proper outcomes.

Nonetheless, we provide a positive result concerning dominance solvable-games, in which at least one order of $IEWDS$ selects a unique singleton from the game. Our question can be rephrased in dominance solvable games in the following terms: does the surviving outcome coincide with the outcome of a proper equilibrium? Indeed, as argued by Marx and Swinkels [10], "at an intuitive level, there seems to be an intimate relationship between backward induction and weak dominance". They prove that, in perfect information games, all orders of $IEWDS$ leave only strategy profiles that give rise to the unique BI payoff vector[7]. This result holds provided that when some player is indifferent between two strategy profiles that differ only in that player's choice of strategy, all other players are indifferent as well: this condition is denoted transfer of decision-maker indifference (TDI). Of course, as we deal with normal-form games, the precise definition of backward induction is elusive in contrast with perfect information games. The concept of proper equilibrium which is often associated to backward induction since van Damme [17] and Kohlberg and Mertens [8] established that a proper equilibrium of a normal form game induces a quasi-perfect/sequential equilibrium in every extensive form game with that normal form.

We first provide an example of a dominance solvable game in which all the orders of deletion lead to the same strategy profile; this profile does not lead to the same payoff that none of the proper equilibria of the whole game as it violates TDI. We then prove that in generic dominance solvable games satisfying TDI, that

[6]See also [15] that proves that such components do not exist in three-outcome bimatrix games.

[7]A related work (Hummel [6]) explores the relation of $IEWDS$ and backward induction in binary voting sequential games.

is games satisfying TDI^{*8}, the surviving outcome coincides with the unique stable one and hence is proper. More precisely, let Γ be a normal form game with associated strategy space S. Iteratively applying $IEWDS$ transforms S into a sequence of restrictions W. Note that the game is solvable then there is a unique stable set in the game. We prove that, if the solvable game satisfies TDI^*, this stable set is included within a connected component with a unique associated payoff. Hence, the singleton that survives $IEWDS$ leads to the stable outcome and hence its outcome is proper. Our contribution is related to Glazer and Rubistein [4], which underlines an interesting relationship between $IEWDS$ and backward induction. For dominance solvable games, it is proved that the elimination procedure is equivalent to backward induction in some appropriately chosen extensive game[9]. Their result holds provided that the agents are indifferent among the different outcomes which is stronger than assuming TDI^*.

The work is structured as follows. Section 2 introduces the canonical framework in which we work. Section 3 presents the results dealing with perfection and Section **??** is focused on the relation between properness and $IEWDS$.

2 The setting

Γ is an n-person game in normal form if $\Gamma = (S_1,\ldots,S_n;U_1,\ldots,U_n)$, where each S_i is a non-empty finite set, and each U_i is a real-valued function defined on the domain $S = S_1\times S_2\times\ldots\times S_n$ and w.l.o.g $S_i\cap S_j = \emptyset$ for any i and j. We let $U = \prod_{i=1}^{n} U_i$. The set of players in the game is $\{1,2,\ldots,n\}$. For each player i, S_i is the set of pure strategies which are available to player i. Each U_i is the utility function for player i, so that $U_i(s_1,\ldots,s_n)$ is the payoff to player i if $(s_1,\ldots,s_n)$ represents the combination of strategies chosen by the players.

For any finite set M, let $\Delta(M)$ the set of all probability distributions over M. Thus, $\Delta(S_i)$ is the set of mixed strategies for player i in Γ with $\Delta(S_i) = \{\sigma_i \in R^{S_i} | \sum_{s_i\in S_i}\sigma_i(s_i) = 1, \sigma_i(s_i) \geq 0\ \forall s_i \in S_i\}$. Similarly, $\Delta^0(S)$ and $\Delta^0(S_i)$ stand for the set of *completely* mixed strategies in S and for player i. Furthermore, for any mixed strategy σ, its support $\mathrm{Supp}(\sigma)$ is denoted by $\mathrm{Supp}(\sigma) = \{s \in S \mid \sigma(s) > 0\}$.

The utility functions are extended to mixed strategies in the usual way, i.e.

$$U_j(\sigma_1,\ldots,\sigma_n) = \sum_{(s_1,\ldots,s_n)\in S_1\times\ldots\times S_n}\left(\prod_{i=1}^{n}\sigma_i(s_i)\right)U_j(s_1,\ldots,s_n).$$

In other words, $U_j(\sigma_1,\ldots,\sigma_n)$ stands for the expected utility player j would get if each player i plays according to strategy σ_i.

[8]A game satisfies TDI^* if for all restrictions W and for all strategies s , if s is very weakly dominated on W, then it is either weakly dominated on W or redundant on W. It is generically equivalent to TDI as proved by Marx and Swinkels [10].)

[9]See Perea [14] for a summary of this literature.

Slightly abusing notation, let $U_j(s_j^*, \sigma_{-j})$ denote the expected utility for j if he plays the pure strategy s_j^* and all the others play according the mixed strategy combination $\sigma_{-j} = (\sigma_1, , \ldots, \sigma_{j-1}, \sigma_{j+1}, \ldots, \sigma_n)$.

The pure strategy s_j^* is a best response to σ_{-j} for player j iff

$$U_j(s_j^*, \sigma_{-j}) = \max_{s_j' \in S_j} U_j(s_j', \sigma_{-j}).$$

An ε-perfect equilibrium of a normal form game is a completely mixed strategy combination, such that whenever some pure strategy s_i is a worse reply than some other pure strategy t_i, the weight on s_i is smaller than ε. A perfect equilibrium of a normal form game is a limit of ε-perfect equilibria when $\varepsilon \to 0$.

An ε-proper equilibrium of a normal form game is a completely mixed strategy combination, such that whenever some pure strategy s_i is a worse reply than some other pure strategy t_i, the weight on s_i is smaller than ε times the weight on t_i. A proper equilibrium of a normal form game is a limit of ε-proper equilibria when $\varepsilon \to 0$.

The sets of perfect and proper equilibria of a game Γ are denoted by $Pre(\Gamma)$ and $Pro(\Gamma)$.

Iterated Dominance.

For $W \subseteq S = \prod_{i=1}^n S_i$, let the strategies in W that belong to i be denoted $W_i = W \cap S_i$. Say that $W \subseteq S$ is a restriction of S if $\forall i$, $W_i \neq \emptyset$. Note that any restriction W of S generates a unique game game given by strategy spaces W_i and the restriction of U_i to $\prod_{i=1}^n W_i$. We denote the restricted game by (W, U).

Γ^k denotes the reduced game after k rounds of successive restrictions of Γ and $S_i^k \subseteq S_i^{k-1}$, $S^k \subseteq S^{k-1}$ the corresponding strategy spaces.

Let $S^0 = S$ and $\lim_{k\to\infty} S^k = \cap_{k=0}^{\infty} S^k = S^\infty$. Γ^∞ denotes the reduced game with strategy space S^∞ and the restriction of U_i to S^∞.

Definition 1 (Weak Domination)**.** *For all $i \in N$, let V_i be a nonempty finite subset of $\Delta(S_i) \cup S_i$, and let $V = \cup_{i \in N} V_i$. Let $\sigma_i, \tau_i \in \Delta(S_i) \cup S_i$. Then,*
(i) σ_i very weakly dominates τ_i on V, denoted $\sigma_i \succeq_V \tau_i$, if $U_i(\sigma_i, \gamma_{-i}) \geq U_i(\tau_i, \gamma_{-i}) \forall \gamma_{-i} \in V_{-i} = \prod_{j \neq i} V_j$, and
(ii) σ_i weakly dominates τ_i on V, denoted $\sigma_i \succ_V \tau_i$, if $\sigma_i \succeq_V \tau_i$, and, in addition, $U_i(\sigma_i, \gamma'_{-i}) > U_i(\tau_i, \gamma'_{-i})$ for some $\gamma'_{-i} \in V_{-i}$.

Definition 2. *Let W be a restriction of S, and let r_i, $s_i \in S_i$. Then r_i is redundant to s_i on W, if $U_j(r_i, x_{-i}) = U_j(s_i, x_{-i})$ for any $j \in N$ and any $x_{-i} \in W_{-i}$. A strategy s_i is redundant on W if there is $r_i \in W_i \setminus s_i$ with r_i redundant to s_i.*

Following Marx and Swinkels [10], we define the TDI^* condition.

Definition 3. *Game Γ satisfies TDI^* if for all restrictions W, $\forall i \in N$, and $\forall s_i \in S_i$, if s_i is very weakly dominated on W by $\sigma_i \in \Delta(S_i \setminus s_i)$, then $\exists \sigma_i' \in \Delta(S_i \setminus s_i)$ such that either s_i is weakly dominated on W by σ_i' or s_i is redundant on W to σ_i'.*

If a game satisfies TDI^*, then whenever player i is indifferent between strategies s_i and σ_i, fixing the profile of opponents strategies s_{-i}, either all players are indifferent between profiles (s_i, s_{-i}) and (σ_i, s_{-i}) or there is some strategy σ_i' such that i strictly prefers σ_i' over s_i and σ_i given s_{-i}.

Marx and Swinkels [10] show that if a game satisfies the following condition on pure strategies, then it generically satisfies TDI^*: $\forall i \in N, \forall s_i, r_i \in S_i, U_i(s_i, s_{-i}) = U_i(r_i, s_{-i}) \Longrightarrow U_j(s_i, s_{-i}) = U_j(r_i, s_{-i})$.

3 Perfect equilibria

For any game $\Gamma = (S, U)$, let $Pe(\Gamma)$ denote its set of perfect equilibria and $Pro(\Gamma)$ denote its set of proper equilibria. The sets of (Nash) equilibria and undominated equilibria of Γ are respectively denoted $Ne(\Gamma)$ and $UNe(\Gamma)$. The set of weakly dominated strategies of any game Γ is denoted $Dom(S)$. Similarly, $Dom(S_i)$ stands for the set of pure and mixed strategies of player i that are weakly dominated by some (mixed) strategy.

By iterated weak dominance, there exists a finite number of orders (as there is a finite number of strategies and assuming that at least one strategy is deleted at each stage until the game is fully reduced). We denote such orders by $o, p, q, \ldots$, each order belongs to Θ. Hence the successive reductions of a game Γ due to order o are as follows

$$\Gamma_o^0 = \Gamma = (S, U), \Gamma_o^1 = (S_o^1, U), \Gamma_o^2 = (S_o^2, U), \ldots, \Gamma_o^\infty = (S_o^\infty, U),$$

with $S_o^i \supseteq S_o^{i+1}$.

Γ_o^∞ stands for the fully reduced game obtained by the order of reduction o by weak dominance by mixed strategies.

It is simple to understand that the set of perfect equilibria of a reduced game is not nested in the whole set of perfect equilibria. The next well-known example proves that removing either M, C or both M and C leads to different sets of perfect equilibria on the reduced games whereas the unique perfect equilibrium of the whole game is (T, L).

	L	C
T	2,1	1,1
M	2,1	0,0

However, despite this path-dependent procedure, we can state the following result.

Proposition 1. *For any order of deletion $o \in \Theta$, $Pe(\Gamma_o^1) \cap Pe(\Gamma) \neq \emptyset$.*

Proof. We omit the definition of Mertens' stable sets and refer to Mertens (1989) [11] for a complete definition. We simply use three of its properties. First, the existence property according to which stable sets always exist. Second, stable sets are connected sets of normal-form perfect equilibria (connectedness). Third, stable sets of a game contain stable sets of any game obtained by deleting a pure strategy which is at its minimum probability in any normal form ε-perfect equilibrium in the neighborhood of the stable set (iterated dominance and forward induction). Hence, the last property applies in particular to any weakly dominated strategy. So that the stable sets of Γ_o^k are included in the stable sets Γ_o^{k-1}. As any point in a stable set is a normal form perfect equilibrium, we can directly conclude. $\square$

As stable sets satisfy inclusion, there is always a common perfect equilibrium in both the fully reduced game and the whole game. We can therefore state the next corollary of Proposition 1,

Corollary 1. *For any order of deletion $o \in \Theta$, $Pe(\Gamma_o^\infty) \cap Pe(\Gamma) \neq \emptyset$.*

3.1 Bimatrix games

Within the set Θ, m stands for the maximal simultaneous reduction by weak dominance in which all mixed and pure strategies that are weakly dominated by some (mixed) strategy are removed at each step.

Proposition 2. *Let Γ be a bimatrix game. By maximal simultaneous deletion, $Pe(\Gamma_m^1) \subseteq Pe(\Gamma)$. Moreover, $Pe(\Gamma_m^\infty)) \subseteq Pe(\Gamma)$.*

The converse of Proposition 3 does not hold. To see this, let us consider the famous Myerson (1978) [12]'s example. Two players 1,2 with three strategies each. Two perfect equilibria (T,L) and (M,C); however the only equilibrium that survives maximal simultaneous deletion is (T,L).

	L	C	R
T	1,1	0,0	-9,-9
M	0,0	0,0	-7,-7
B	-9,-9	-7,-7	-7,-7

To see why, it suffices to understand that $M \succ_S B$ and that $C \succ_S R$ in Γ. Furthermore, in the game Γ^1 in which both B and R have been deleted, both $T \succ_{S^1} M$ and that $L \succ_{S^1} C$, hence only (T,L) is perfect in the fully reduced game and it is the unique proper equilibrium of the game.

We now state the proof of Proposition 3.

Proof. Let σ be a perfect equilibrium in the game Γ_m^1. In bimatrix games, an equilibrium is perfect if and only it is undominated. An equilibrium σ is undominated if each of its components σ_i of σ is undominated. Suppose that σ is not a perfect equilibrium in Γ.

Either σ is not an equilibrium in Γ or σ is an equilibrium in such a game but some of the strategies in σ are dominated in Γ. In the former case, this is a contradiction with the definition of iterated dominance as an equilibrium σ of a reduced game is an equilibrium of the whole game. In the latter case, some of the strategies in σ are dominated in Γ so that by maximal simultaneous deletion, the strategy σ is not present in Γ_m^1, a contradiction. □

To see why Proposition 3 does not hold with more than two players, let us consider the next example (p.29 Van Damme (1996) [18]).

	L	C
T	1,1,1	1,0,1
M	1,1,1	0,0,1

A

	L	C
T	1,1,0	0,0,0
M	0,1,0	1,0,0

B

In such a game, both $L \geq_S C$ and $A \geq_S B$. There is just one perfect equilibrium in Γ: (T, L, A). Nevertheless, applying maximal simultaneous deletion removes C and B from S, so that (T, L, A) and (M, L, A) are both perfect equilibria in the fully reduced game. So that reducing weakly dominated strategies may enlarge the set of perfect equilibria.

Proposition 3. *Let Γ be a bimatrix game satisfying TDI^*. For any order of deletion, the set of perfect outcomes of any fully reduced game is a subset of the set of perfect outcomes of Γ.*

Proof. By Proposition 3, the set of perfect equilibria of the fully reduced game Γ_m^∞ is a subset of the set of perfect equilibria of Γ. As stated by Marx and Swinkels [10], in any game satisfying TDI^*, any two full reductions by weak dominance, are the same up to the addition or removal of redundant strategies. Moreover, the set of perfect equilibria is invariant to the addition of redundant strategies (see for instance Kohlberg and Mertens [8]). It hence follows that the set of outcomes of any fully reduced game is a subset of the set of outcomes of the whole game. □

3.2 Finite Games

This example is a modified version of the one present in Govindan and McLennan [5] with the addition of a weakly dominated strategy X for player 3 (as long as the payoff for player 3 in each of the outcomes is strictly positive). This is an outcome game that satisfies TDI and TDI^*.

	L	R
T	a	a
M	b	b
B	a	b
D	e	f

U

	L	R
T	c	c
M	d	d
B	c	d
D	e	f

D

	L	R
T	c	c
M	d	d
B	0,0,0	0,0,0
D	e	f

X

There is a connected component of equilibria with a continuum of outcomes with support $\{T,M,B,D\}\times\{L,R\}\times\{U,D\}$. Hence, in the fully reduced game without X, this game has a continuum of perfect equilibria.

However, in the whole game, in any sequence of ε-perfect equilibria, $U_1(T,\sigma_{-1}^{\varepsilon}) > U_1(B,\sigma_{-1}^{\varepsilon})$ so that there is not a perfect equilibrium with both T and M in the support. There is not a continuum of outcomes anymore in the set of perfect equilibria. Hence, the perfect outcomes of the reduced game are a superset of the set of perfect outcomes of the whole game. Therefore, it is not even the case that $IEWDS$ restricts the set of perfect outcomes.

4 Proper Equilibria

4.1 A non-solvable game

This section presents an example that proves that the proper outcomes of the whole game and of the reduced game differ. This example is a modification of the nice one provided by Kukushkin, Litan and Marhuenda [9]: more precisely, two strictly dominated strategies (X and Y) have been added. Moreover, the game satisfies TDI and TDI^*. There are four outcomes a, b, c and d. We let s_i stand for the payoff for player i associated to outcome s.

	L	C	R	S
T	c	a	b	b
M	d	a	a	b
B	c	d	b	c
X	0,0	1,1	1,1	0,0
Y	1,1	0,0	0,0	1,1

Note that X and Y are strictly dominated by T, B and M as long as $a_1, b_1, c_1, d_1 > 1$ (a). We assume that this inequality holds. If we remove this pair of strategies, the reduced game $\Gamma^\infty = \Gamma \setminus \{X,Y\}$ has no dominated strategies. Moreover, there is a connected component $\mathcal{C}$ with a continuum of outcomes as proved by Kukushkin, Litan and Marhuenda [9] provided that

$$d_1, b_1 < a_1, c_1 \text{ and } d_2 < b_2 < a_2, c_2, \ (b)$$

and that

$$b_2(d_1 - c_1) + b_1(c_2 - d_2) + c_1 d_2 - c_2 d_1 \neq 0 \ (c).$$

This component is defined by the following strategies

$$\sigma_1(u_2) = \frac{1}{a_2 - b_2 + c_2 - d_2}(b_2 - d_2, c_2 - b_2, a_2 - d_2),$$

and

$$\begin{aligned}\sigma_2(u_1;t) = &(\frac{a_1 - b_1}{a_1 - b_1 + c_1 - d_1} - \frac{(a_1 - b_1)t}{a_1 - d_1}, \frac{(c_1 - b_1)t}{a_1 - d_1} \\ &\frac{c_1 - d_1}{a_1 - b_1 + c_1 - d_1} - \frac{(c_1 - d_1)t}{a_1 - d_1}, t).\end{aligned}$$

We assume that (a), (b) and (c) hold so that it is easy to check that the pair (σ_1, σ_2) defines a completely mixed strategy equilibrium in Γ^∞ provided t is positive and small enough.

We now prove that every equilibrium in $\mathcal{C}$ is not a proper equilibrium in Γ, proving that the set of proper equilibria of both games differ. Note that every equilibrium in $\mathcal{C}$ is an equilibrium in Γ and is also perfect as every undominated equilibrium is perfect in bimatrix games.

We consider the sequences $\sigma^\varepsilon = (\sigma_1^\varepsilon, \sigma_2^\varepsilon)$ of ε-proper equilibria converging towards the strategy combinations in $\mathcal{C}$.

By the definition of properness, $U_2(L, \sigma_1^\varepsilon) = U_2(S, \sigma_1^\varepsilon)$ as both are in the support of player 2's strategy. As the utility payoffs of L and S only differ when player 1 plays strategies T and M, it follows that in any ε-proper equilibrium, $\sigma_1^\varepsilon(M) = \frac{c_2 - b_2}{b_2 - d_2}\sigma_1^\varepsilon(T)$. Moreover, we must have that $U_2(C, \sigma_1^\varepsilon) = U_2(R, \sigma_1^\varepsilon)$ so that $\sigma_1^\varepsilon(B) = \frac{a_2 - b_2}{b_2 - d_2}\sigma_1^\varepsilon(T)$.

Hence, it follows that $\sigma_1^\varepsilon(B) = \frac{a_2 - b_2}{c_2 - b_2}\sigma_1^\varepsilon(M)$ (*).

Finally, in any equilibrium with full support for player 2, it must be the case that $U_2(R, \sigma_1^\varepsilon) = U_2(S, \sigma_1^\varepsilon)$. This implies that:

$$a_2\sigma_1^\varepsilon(M) + b_2\sigma_1^\varepsilon(B) + \sigma_1^\varepsilon(X) = b_2\sigma_1^\varepsilon(M) + c_2\sigma_1^\varepsilon(B) + \sigma_1^\varepsilon(Y).$$

Due to (*), one can check that the previous equality implies that $\sigma_1^\varepsilon(X) = \sigma_1^\varepsilon(Y)$. Hence, $U_1(X, \sigma_2^\varepsilon) = U_1(Y, \sigma_2^\varepsilon)$ as otherwise there is a contradiction with the definition of ε-properness. However, this implies that

$$\sigma_2^\varepsilon(C) + \sigma_2^\varepsilon(R) = \sigma_2^\varepsilon(L) + \sigma_2^\varepsilon(S).$$

It is clear that not every equilibrium in $\mathcal{C}$ satisfies this constraint, proving the claim.

4.2 Dominance Solvable Games

Dominance Solvability need not imply Properness

In this example, the unique strategy profile that survives all orders of deletion of $IEWDS$ need not be proper. Note that the game does not satisfy TDI. Furthermore, the outcomes by dominance solvability and properness need not coincide. We focus in a bimatrix game, each player has three strategies. Let us remark that L strictly dominates C.

	L	C	R
T	2,3	1,0	0,4
M	2,2	0,0	1,-1
B	2,3	1/2,-1	1/2,4

The set of Nash equilibria equals player 1 randomizing between his three strategies with the probability of M being higher or equal than 1/4 and player 2 playing L. Within this set, the unique pure strategy equilibrium is (M,L). Such an equilibrium is not proper since whenever the probability of player 1 playing M becomes sufficiently close to 1, player 2 strictly prefers to play C than R. Therefore, due to the definition of ε-properness, player 1 strictly prefers to play T than to play M for any $\varepsilon > 0$.

Furthermore, any order of deletion of $IEWDS$ singles out the singleton (M,L). To see this, it suffices to understand that it will first remove C then T and B (simultaneously or sequentially) and finally strategy R.

Hence, the strategy profile (M,L) satisfies three interesting features: (i) it is the unique strategy profile that survives all orders of deletion of $IEWDS$, (ii), it is not a proper equilibrium of the whole game and (iii) it does not lead to the same payoff outcome than any proper equilibrium of the whole game.

A Positive Result

Before stating our main positive result, we list four properties of stable sets (see Mertens [11] for a complete definition.).

1. Stable sets always exist (*Existence*).
2. Stable sets are connected sets of normal-form perfect equilibria (*Connectedness*).
3. Stable sets of a game contain stable sets of any game obtained by deleting a pure strategy which is at its minimum probability in any normal form ε-perfect equilibrium in the neighborhood of the stable set (*Iterated dominance and Forward Induction*).
4. Every stable set contains a proper (hence sequential) equilibrium (*Backwards induction.*).

Let us recall that the set of Nash equilibria consists of finitely many connected components (Kohlberg and Mertens [8]).

Theorem 1. *Let Γ be a normal-form game that satisfies TDI^*. If the game is dominance solvable, then there is a unique stable payoff. Moreover, the surviving outcome is proper.*

Proof. Let Γ be a normal-form game that satisfies TDI^*. As the game is dominance-solvable, we denote by o the order of deletion according to which the game is dominance solvable. We hence let $\Gamma = \Gamma_o^1$ denote the whole game and $\Gamma_o^\infty = \Gamma_o^k$ with $S_o^k = \{s_1, \ldots, s_n\}$ be the singleton that it is selected by $IEWDS$ for some finite k.

We now prove that all the equilibria in the component in which $\{s\}$ is included lead to the same outcome than $\{s\}$.

Let us consider a mixed strategy equilibria in Γ in which s is present.

As, by definition, there is some $s_i' \in \Gamma$ which is weakly dominated, such a strategy does not belong to the support of the mixed strategy equilibrium in which the rest of the players are using a completely mixed strategy. This remains true provided that given the other players' mixed strategies, the strategy s_i' is weakly dominated in the restriction of the game defined by the support of the players' mixed strategies.

Suppose that the players different from i play a mixed strategy with a support such that s_i' is not weakly dominated in the restriction defined by the support of the players different from i. Hence, the strategy s_i' becomes very weakly dominated. As s_i' is very weakly dominated, it is redundant to some (mixed) strategy in the game as Γ satisfies TDI^*.

Hence, s_i' is either not in the support of a mixed strategy equilibrium or in the support of an equilibrium while being redundant to some other strategy in the support.

In other words, s_i' is irrelevant to determine the outcome of the equilibria in the connected component of equilibria.

Note that we can iteratively apply the same argument at each reduced game Γ_o^i as we only need the existence of a weakly dominated strategy in each of these games which is ensured as the game is dominance solvable. Hence, it follows that the unique outcome in the connected component in which $\{s\}$ is included coincides with the one of $\{s\}$.

Finally, let us recall that the stable sets of a game contain stable sets of any game obtained by deleting a pure strategy which is at its minimum probability in any normal form ε-perfect equilibrium in the neighborhood of the stable set (and hence a weakly dominated strategy). Moreover, stable sets are connected components of perfect equilibria. Hence, as a stable set is a subset of a component of equilibria and as the game is dominance solvable, we can conclude that the unique stable outcome is the one corresponding to $\{s\}$.

As each stable set contains a proper equilibrium this ensures that the outcome of $\{s\}$ coincides with the one of a proper equilibrium. □

5 An Application to Voting Games

While we have proved that the hybrid procedure might be troublesome in generic normal-form games (as it might lead to different outcomes than applying a usual refinement in the whole game), there exist interesting classes of games in which the procedure seems to be more relevant. Indeed, the problems highlighted in this paper are a consequence of the continuum of equilibria outcomes, emphasized by Govindan and McLennan [5]. De Sinopoli [2] proves that generic plurality voting games are very particular as each connected component of Nash equilibria leads to a unique outcome so that these games have a finite number of Nash outcomes. More specifically, the claim holds for voting games à la Myerson and Weber [13] in which the voters' preferences are common knowledge. These games are represented by the following elements:

- the set of voters N, an element of which is denoted by i,
- the set of candidates K with $K = \{a, \dots, k\}$,
- the set of voters' pure strategies B which coincides with K jointly with the abstention ($B = K \cup \{0\}$) (as we deal with plurality voting),
- and the voters'preferences over the candidates represented by a vector $u^i = (u^i_1, \dots, u^i_k)$.

Each u^i_j represents the payoff that voter i gets if candidate j is the sole winner. It should be noted that each ballot can be written as a vector in $\mathbb{N}^k$; the total scores of the candidates simply coincide with the sum of the different strategy vectors of the voters.

If there is a unique candidate with the most votes, he is the unique winner of the election. In the event of a tie between the candidates with the most votes, one of the tied candidates is randomly selected among those that are tied. A voting game can hence be represented by normal-form game.

De Sinopoli [2] states for any game with

$$\forall i \in N, u^i_c \neq u^i_d, \forall c, d \in K, \ (IC)$$

the set of Nash outcomes is finite.

Hence, a proof analogous to the one of Theorem 1, it can be proved that if a plurality voting game is dominance-solvable and satisfies IC, then the surviving outcome is the unique stable outcome. We hence obtain a different proof for De Sinopoli [1]'s result (Proposition 10) concerning plurality voting dominance solvable games.

Moreover, according to our results, if one can ensure that the set of Nash outcomes of a game is finite, then applying $IEWDS$ and then perfectness/properness must lead to a set of outcomes, at least one of which must be stable. So this is the case for any generic PV game.

Nonetheless, it seems interesting to understand whether the current technique can be used in non-generic games. We prove that it is indeed the case. To show the interest of the procedure, we prove that the conjecture of Myerson and Weber [13] dealing with the set of PV equilibria outcomes in the divided majority situation does not hold. Interestingly, applying a first step of removal of weakly dominated strategies allows us to check for stability by "merely" computing the conditions for the existence of a completely mixed strategy equilibrium.

The Conjecture on the Divided Majority

Consider a PV game in which there are three candidates ($K = \{a, b, c\}$) and n voters. Furthermore, the voters have three different cardinal utilities vectors to be represented by a type in their type set ($T = \{\alpha, \beta, \gamma\}$). Each of the different utilities' vectors is as follows:

$$u^{\alpha} = (10, 9, 0),\ u^{\beta} = (9, 10, 0) \text{ and } u^{\gamma} = (0, 0, 10).$$

The number of voters of each type is denoted by n_{α}, n_{β} and n_{γ} with $n_{\alpha}+n_{\beta}+n_{\gamma} = n \geq 4$ and $n_{\alpha} = n_{\beta} = m$. We impose the restriction that $(n_{\alpha} + n_{\beta}) - n_{\gamma} = \phi$ for some $0 \leq \phi < m$ with $0 < n_{\alpha}, n_{\beta}, n_{\gamma}$. Hence, any divided majority game $D(m, \phi)$ depends on two parameters m and ϕ. These assumptions imply that only through cooperation between the voters in both of the majority types (α and β), one of majority preferred alternatives (a or b) might be implemented with positive probability.

Moreover, in social choice theory terms, candidate c is the Condorcet loser whereas a and b are both weak Condorcet winners. It seems reasonable to require for a voting rule to be desirable not to select Condorcet losers in equilibrium.

Myerson and Weber [13] analyze this game in a model of large elections and find that there are three equilibria:

- an equilibrium in which $\alpha \to \{a\}$, $\beta \to \{a\}$ and $\gamma \to \{c\}$ so that a is the unique winner,
- an equilibrium in which $\alpha \to \{b\}$, $\beta \to \{b\}$ and $\gamma \to \{c\}$ so that b is the unique winner,
- and finally an equilibrium in which $\alpha \to \{a\}$, $\beta \to \{b\}$ and $\gamma \to \{c\}$ so that c is the unique winner.

This third equilibrium proves the existence of deep coordination failures of strategic voters under PV. Moreover, it proves that Duverger's law need not be satisfied and conclude that "to eliminate equilibria of the type just illustrated" would "seem to require some additional assumption of dynamic stability (see e.g. Kalai and Samet [7])".

We prove that there is a Mertens-stable component in which there is a lack of cooperation between the majoritarian voters (the ones with type α and β). This coordination failure leads to an equilibrium in which c wins with positive probability. To do so, we apply first maximal deletion of weakly dominated strategies and then prove that there exists a completely mixed-strategy equilibrium σ which is an isolated quasi-equilibrium. Since we prove that the set of *undominated* Nash payoffs is finite, all the equilibria in the component lead to the same outcome. Hence, candidate c wining the election is a stable outcome proving that the conjecture is not valid. Hence, the coordination problems under PV seem to be more problematic as they are independent of the Myerson-Weber framework.

Note that De Sinopoli [1] proves that there might exist dominance-solvable plurality voting games in which the Duverger's law is violated. Our claim just concerns the coordination failure under PV in the divided majority situation.

The next proposition proves the finiteness of undominated Nash outcomes, its proof is included in the appendix.

Proposition 4. *For any pair (m,ϕ), the set of undominated Nash outcomes of the game $D(m,\phi)$ is finite.*

Each voter has a pure strategy vector $V = K \cup \{0\}$. Each vector is a vector of three components which are all zeros but for one in position c which denotes the vote for such candidate. Given a pure strategy vector $v \in V^n$, we let $s(v) = \sum_{i=1}^n v^i$ denote the total score vector. The probability that candidate c is elected if v is played by $p(c\,|\,v)$ equals 0 if $s_c(v)$ is lower than $s_d(v)$ for some d, otherwise

$$p(c\,|\,v) = 1/q \text{ if } s_c(v) \geq s_d(v) \text{ and } \#\{e \in K\,|\,s_c(v) = s_e(v)\} = q.$$

The vector $p(\cdot\,|\,v)$ extends to a mixed strategies vector σ by writing $p(c\,|\,\sigma) = \sum p(c|v)\sigma(v)$, in which $\sigma(v)$ denotes the probability of strategy combination v given σ. Denoting $p(\cdot\,|\,\sigma) = (p(a\,|\,\sigma), p(b\,|\,\sigma), p(c\,|\,\sigma))$, the vector $p(\sigma)$ is in $\Delta(K)$.

Let D^m denote the reduced game obtained by maximal simultaneous deletion of weakly dominated strategies.

The proof is done in two separate sections, when $\phi = 0$ and when $\phi > 0$

5.1 The Divided Majority with $\phi = 0$.

The game $D(3,0)$ Before stating the proof, let us take a simple example. Assume that $n_\alpha = n_\beta = 3$ and $n_\gamma = 6$ so that $\phi = 0$. W.l.o.g voters 1,2,3 are of type α, voters 4,5,6 are of type α and $7,\ldots,12$ are of type α. In this extremely simple case, the α and β voters might cooperate in order to ensure the existence of a tie between candidates a and c or b and c. These equilibria exist and are even strict hence stable.

Nonetheless, we prove that the coordination failure under PV is a stable outcome in which all voters randomize between their undominated strategies. This randomization leads to an equilibrium in which the probability of candidate c being the winner of the election is maximized. We denote by ρ the strategy according to which the voter i chooses the strategy a with probability $(1-p_i)$ and he votes b with probability p_i. As in the previous case, the voter i' s best response only differs when $s(\rho_{-i}) \in \{(5,0,6),(0,5,6)\}$.

Hence, for ρ to be an equilibrium, all voters must be indifferent between their two strategies a and b so that

$$U_i(a) = U_i(b) \iff 10p((5,0,6) \mid \rho_{-i}) = 9p((0,5,6) \mid \rho_{-i}) \text{ for all } i \in \{1,2,3\}. \quad (1)$$

and

$$U_i(a) = U_i(b) \iff 9p((5,0,6) \mid \rho_{-i}) = 10p((0,5,6) \mid \rho_{-i}) \text{ for all } i \in \{4,5,6\}. \quad (2)$$

with $p((5,0,6) \mid \rho_{-i}) = \prod_{j=1\&j\neq i}^{6}(1-p_j)$ and $p((0,5,6) \mid \rho_{-i}) = \prod_{j=1\&j\neq i}^{6} p_j$.

Note that there are 6 variables and 6 independent equations so that there must exist a unique solution. Setting $p_i = p$ for $i = 1,2,3$ and $p_i = q$ for $i = 4,5,6$, equations (1) and (2) can be respectively rewritten as

$$10(1-p)^2(1-q)^3 = 9p^2q^3, \quad (3)$$

and

$$9(1-p)^3(1-q)^2 = 10p^3q^2. \quad (4)$$

so that $p = 9/19$ and $q = 10/19$.

Hence, there exists a completely mixed strategy equilibrium in which c wins with positive probability. More specifically, there is a tie between a and c with probability $(9/19)^3(10/19)^3$ and a tie between b and c with the same probability. Hence, c wins alone with probability $1-2(9/19)^3(10/19)^3 \approx 0.985$.

The game $D(m,0)$.

Proposition 5. *For any $m \geq 3$, there is a stable outcome of the game $D(m,0)$ in which c wins with probability strictly higher than 1/2.*

Proof. The proof proceeds in two steps. The first step of the proof consists in proving that there exists a completely mixed strategy equilibrium $D^{max}(m,0)$ in which c wins the election with positive probability. The second step shows that this equilibrium is stable in $D^{max}(m,0)$ and hence is a stable set of $D(m,0)$.

Step 1: In the game $D^{max}(m,0)$, we denote by σ the strategy according to which the voter i chooses the strategy a with probability $(1-p_i)$ and he votes b with probability p_i. The voter i' s best response only differs when $s(\sigma_{-i}) \in \{(2m-1,0,2m),(0,2m-$

$1, 2m)\}$. If σ is an equilibrium, each voter must be indifferent between his two strategies. Setting $p_i = p$ for $i = 1, \ldots, m$ and $p_i = q$ for $i = m+1, \ldots, 2m$, the indifference conditions for the α-voters are as follows

$$10(1-p)^{m-1}(1-q)^m = 9p^{m-1}q^m,$$

whereas the ones for the β-voters can be written as

$$9(1-p)^m(1-q)^{m-1} = 10p^m q^{m-1}.$$

As the strategic incentives for these voters are complementary, it follows that $q = 1-p$. Hence, the unique solution for the previous equalities is $p = 9/19$ and $q = 10/19$. Therefore, there exists a completely mixed strategy equilibrium in which c wins with probability $1 - 2(9/19)^m(10/19)^m$. Hence, as $m \geq 2$, the probability of c winning is strictly higher than $1/2$.
Step 2: In order to prove that σ is a stable equilibrium of $D^{max}(m, 0)$ we prove that σ is an isolated quasi-equilibrium. It is trivial that σ is a quasi-equilibrium (each voter uses his strict best replies) since each voter plays each pure action with positive probability. The fact that σ is isolated follows from the uniqueness of the solution of the indifference conditions. In other words, assume that some voter i deviates from σ. Hence, there must exist a voter j that also deviates from σ as $p((0, 2m-1, 2m) \mid \sigma_{-i}) \neq p((2m-1, 0, 2m) \mid \sigma_{-i})$. Therefore, σ is isolated. Since σ is an isolated quasi-strict equilibrium we can conclude that it is a strongly stable equilibrium (van Damme [18] Th. 3.4.4). Hence $\{\sigma\}$ is a stable set of $D^{max}(m, 0)$. Since stable sets of a reduced game are included in the stable sets of the whole, we can conclude that $\{\sigma\}$ is a stable set of $D(m, 0$, since by Proposition 4, there is one outcome per connected component of equilibria.

□

5.2 The Divided Majority with $\phi > 0$.

The game $D(3,1)$**.** Assume that $m = n_\alpha = n_\beta = 3$ and $\phi = 1$ so that $n_\gamma = 5$. W.l.o.g voters 1,2,3 are of type α, voters 4,5,6 are of type α and $7, \ldots, 12$ are of type α. We want to determine whether there exists a perfect equilibrium in which c wins. As with $\phi = 0$, we work directly in the fully reduced game $D^m ax$.

We consider the completely mixed strategy equilibrium in which each player plays each of his two strategies with positive probability.

We denote by ρ the strategy according to which the voter i chooses the strategy a with probability $(1 - p_i)$ and he votes b with probability p_i. As in the previous case, the voter i' s best response only differs when

$$s(\rho_{-i}) \in \{(4, 1, 5), (5, 0, 5), (1, 4, 5), (0, 5, 5)\}.$$

Hence, for ρ to be an equilibrium, all voters must be indifferent between their two strategies a and b so that

$$10\,[p((5,0,5)\,|\,\rho_{-i})+p((4,1,5)\,|\,\rho_{-i})]=9\,[p((1,4,5)\,|\,\rho_{-i})+p((0,5,5)\,|\,\rho_{-i})]. \quad (5)$$

for all $i \in \{1,2,3\}$ and

$$9\,[p((5,0,5)\,|\,\rho_{-i})+p((4,1,5)\,|\,\rho_{-i})]=10\,[p((1,4,5)\,|\,\rho_{-i})+p((0,5,5)\,|\,\rho_{-i})], \quad (6)$$

for all $i \in \{4,5,6\}$.

Note that there are 6 variables and 6 independent equations so that there must exist a unique solution (if any). Setting $p_i = p$ for $i = 1,2,3$ and $p_i = q$ for $i = 4,5,6$.

Voters $i = 1,2,3$.

$p((5,0,5)\,|\,\rho_{-i}) = (1-p)^2(1-q)^3$.
$p((4,1,5)\,|\,\rho_{-i}) = 2p(1-p)(1-q)^3 + 3q(1-p)^2(1-q)^2$.
$p((0,5,5)\,|\,\rho_{-i}) = p^2q^3$.
$p((1,4,5)\,|\,\rho_{-i}) = 2p(1-p)q^3 + 3(1-q)p^2q^2$.

Voters $i = 4,5,6$.

$p((5,0,5)\,|\,\rho_{-i}) = (1-p)^3(1-q)^2$.
$p((4,1,5)\,|\,\rho_{-i}) = 2q(1-q)(1-p)^3 + 3p(1-p)^2(1-q)^2$..
$p((0,5,5)\,|\,\rho_{-i}) = p^3q^2$
$p((1,4,5)\,|\,\rho_{-i}) = 2q(1-q)p^3 + 3(1-p)p^2q^2$.

Therefore, the indifference conditions depicted by (5) and (6) lead to $p = 9/19 - \xi$ and $q = 10/19 + \xi$ with $\xi \approx 0.013$.

Hence, there exists a completely mixed strategy equilibrium in which c wins with positive probability. In this equilibrium, a is the unique winner with probability $(1-p)^3(1-q)^3$ whereas b wins alone with probability p^3q^3. The tie between a and c occurs with probability $3p(1-p)^2(1-q)^3 + 3q(1-q)^2(1-p)^3$ and the one between b and c with probability $3(1-p)p^2q^3 + 3(1-q)q^2p^3$. Hence, the probability of c winning alone is roughly equal to $1 - 0.217008 = 0.782992$.

The game $D(3,2)$. Assume that $m = n_\alpha = n_\beta = 3$ and $\phi = 2$ so that $n_\gamma = 4$. W.l.o.g voters 1,2,3 are of type α, voters 4,5,6 are of type α and $7,\ldots,12$ are of type α. We want to determine whether there exists a perfect equilibrium in which c wins. As with $\phi = 1$, we work directly in the fully reduced game D^max.

We consider the completely mixed strategy equilibrium in which each player plays each of his two strategies with positive probability.

We denote by ρ the strategy according to which the voter i chooses the strategy a with probability $(1-p_i)$ and he votes b with probability p_i. As in the previous case, the voter i' s best response only differs when

$$s(\rho_{-i}) \in \{(4,1,4),(3,2,4),(2,3,4),(1,4,4)\}.$$

Hence, for ρ to be an equilibrium, all voters must be indifferent between their two strategies a and b so that

$$10\,[p((4,1,4)\,|\,\rho_{-i}) + p((3,2,4)\,|\,\rho_{-i})] = 9\,[p((1,4,4)\,|\,\rho_{-i}) + p((2,3,4)\,|\,\rho_{-i})]. \qquad (7)$$

for all $i \in \{1,2,3\}$ and

$$9\,[p((4,1,4)\,|\,\rho_{-i}) + p((3,2,4)\,|\,\rho_{-i})] = 10\,[p((1,4,4)\,|\,\rho_{-i}) + p((2,3,4)\,|\,\rho_{-i})], \qquad (8)$$

for all $i \in \{4,5,6\}$.

We set $p_i = p$ for $i = 1,2,3$ and $p_i = q$ for any $i = 4,5,6$.

Voters $i = 1,2,3$**.**

$p((4,1,4)\,|\,\rho_{-i}) = 2p(1-p)(1-q)^3 + 3q(1-p)^2(1-q)^2$.
$p((3,2,4)\,|\,\rho_{-i}) = p^2(1-q)^3 + 6pq(1-p)(1-q)^2 + 3q^2(1-p)^2(1-q)$.
$p((1,4,4)\,|\,\rho_{-i}) = 2p(1-p)q^3 + 3(1-q)p^2q^2$.
$p((2,3,4)\,|\,\rho_{-i}) = (1-p)^2q^3 + 6(1-p)(1-q)pq^2 + 3(1-q)^2p^2q$.

Voters $i = 4,5,6$**.**

$p((4,1,4)\,|\,\rho_{-i}) = 2q(1-q)(1-p)^3 + 3p(1-p)^2(1-q)^2$.
$p((3,2,4)\,|\,\rho_{-i}) = q^2(1-p)^3 + 6pq(1-p)^2(1-q) + 3p^2(1-p)(1-q)^2$.
$p((1,4,4)\,|\,\rho_{-i}) = 2q(1-q)p^3 + 3(1-p)p^2q^2$.
$p((2,3,4)\,|\,\rho_{-i}) = (1-q)^2p^3 + 6(1-p)(1-q)p^2q + 3(1-p)^2pq^2$.

Therefore, the indifference conditions depicted by (7) and (8) lead to $p = 9/19 - \xi$ and $q = 10/19 + \xi$ with $\xi \approx 0.052$ implying that there exists a completely mixed strategy equilibrium in which c wins with positive probability.

The game $D(m,\phi)$

Theorem 2. *For any $m \geq 3$, there is a stable outcome of the game $D(m,\phi)$ in which c wins with positive probability.*

Proof. The proof proceeds in two steps. The first step of the proof consists in proving that there exists a completely mixed strategy equilibrium $D^{max}(m,0)$ in which c wins the election with positive probability. The second step shows that this equilibrium is stable in $D^{max}(m,0)$ and hence is a stable set of $D(m,\phi)$.

Step 1: In the game $D^{max}(m,\phi)$, we denote by σ the strategy according to which the voter i chooses the strategy a with probability $(1-p_i)$ and he votes b with probability p_i. The voter i' s best response only differs when

$$s(\sigma_{-i}) \in \{(2m-\phi,\phi-1,2m-\phi),(2m-\phi-1,\phi,2m-\phi),\\ (\phi-1,2m-\phi,2m-\phi),(\phi,2m-\phi-1,2m-\phi)\}.$$

We set $p_i = p$ for $i = 1,\ldots,m$ and $p_i = q$ for any $i = m+1,\ldots,2m$.

The probability of the event $(2m-\phi,\phi-1,2m-\phi)$ given σ_{-i} for any voter i equals

$$\sum_{j=0}^{\phi-1} C_{m-1}^{j} C_{m}^{\phi-j-1} p^j q^{\phi-j-1}(1-p)^{m-j-1}(1-q)^{m-\phi+j+1},$$

whereas the one corresponding to the pivotal outcome $(2m-\phi-1,\phi,2m-\phi)$ equals:

$$\sum_{j=0}^{\phi} C_{m-1}^{j} C_{m}^{\phi-j} p^j q^{\phi-j}(1-p)^{m-j-1}(1-q)^{m-\phi+j}.$$

By symmetry, the probabilities perceived by this type of player of the outcomes $(\phi-1,2m-\phi,2m-\phi)$ and $(\phi,2m-\phi-1,2m-\phi)$ are respectively equal to

$$\sum_{j=0}^{\phi-1} C_{m-1}^{j} C_{m}^{\phi-j-1}(1-p)^j(1-q)^{\phi-j-1} p^{m-j-1} q^{m-\phi+j+1},$$

and

$$\sum_{j=0}^{\phi} C_{m-1}^{j} C_{m}^{\phi-j}(1-p)^j(1-q)^{\phi-j} p^{m-j-1} q^{m-\phi+j}.$$

As previously argued, the strategic incentives of both α and β voters are complementary so that $q = 1-p$.

Applying this equality, the indifference condition of any voter $i = 1,\ldots,2m$ equals

$$10\left(\sum_{j=0}^{\phi-1} C_{m-1}^{j} C_{m}^{\phi-j-1} p^{m-\phi+2j+1}(1-p)^{m+\phi+-2j-2} + \sum_{j=0}^{\phi} C_{m-1}^{j} C_{m}^{\phi-j} p^{m-\phi+2j}(1-p)^{m+\phi-2j-1}\right) =$$
$$9\left(\sum_{j=0}^{\phi-1} C_{m-1}^{j} C_{m}^{\phi-j-1} p^{m+\phi-2j-2}(1-p)^{m-\phi+2j+1} + \sum_{j=0}^{\phi} C_{m-1}^{j} C_{m}^{\phi-j} p^{m+\phi-2j-1}(1-p)^{m-\phi+2j}\right).$$

In order to ensure that there exists some $p \in (0,1)$ for which the indifference condition holds, we let, given m and ϕ, $U_{m,\phi} : [0,1] \to \mathbb{R}$ and $L_{m,\phi} : [0,1] \to \mathbb{R}$ in which $U_{m,\phi}(p)$ and $L_{m,\phi}(p)$ respectively stand for the left and the right side of the previous equality. Both $U_{m,\phi}$ and $L_{m,\phi}$ are continuous functions with respect to p. Hence, if we can prove that there exists two points x and y in $(0,1)$ in which $U_{m,\phi}(x) < L_{m,\phi}(x)$ and $U_{m,\phi}(y) > L_{m,\phi}(y)$, the Intermediate Value Theorem proves our claim. The existence of both x and y are ensured by Propositions 6 and 7 included in the appendix.

Step 2: This step of the proof consists in proving that σ is an isolated quasi-strict equilibrium and hence is a stable singleton. The logic is identical to the Step 2 in the proof of Proposition 5 and hence is omitted.

Therefore, we can conclude that σ is a stable outcome of the game $D(m,\phi)$. $\square$

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A Proof of Proposition 4

Following De Sinopoli [2], a strategy combination is non-degenerate if at least two candidates have a strictly positive probability of winning. Conversely, in a degenerate strategy combination, only one candidate wins with positive probability.

Note that as voters have only two weakly undominated strategies: voting for their first or for their second preferred candidate. More formally, in any undominated Nash equilibrium, the support C_l of any voter with type l is as follows:

$$C_\alpha \subseteq \{a, b\},\ C_\beta \subseteq \{a, b\},\ \text{and } C_\gamma = \{c\}.$$

Hence, as in the game D, we have $n_\alpha + n_\beta \geq n_\gamma$ and $n_\alpha, n_\beta < n_\gamma$, it is simple to see that the set of undominated outcomes equals

$$\{V_a, V_b, V_c, V_{a,b}, V_{b,c}\},$$

in which V_J denotes for any $J \subset K$, the victory of all candidates in J.

We focus on the non-degenerate strategy combinations. The claim is trivially true if there are only equilibria associated to degenerate strategy combinations in which only one candidate wins.

Given the set of undominated outcomes, there are three cases for the vector $p(\cdot \mid \sigma)$ to be non-degenerate. In the first case, a and c have strictly positive probability of wining (case A) whereas in the second case, b and c win with positive probability (case B). Finally, in the third case, three candidates have a possibility of winning the election.

Case A:

In this case, voters' best responses are uniquely determined. Indeed, no voter is indifferent between candidates a and c. Hence, if these candidates are the unique ones with positive probability of victory, it follows that $\alpha \to \{a\}$, $\beta \to \{a\}$ and $\gamma \to \{c\}$.

Hence if $\phi = 0$, then $n_\alpha + n_\beta = n_\gamma$ and therefore there is a strict equilibrium in which a and c are tied.

On the contrary, if $\phi > 0$, then $n_\alpha + n_\beta > n_\gamma$ and therefore $s_a(v) > s_c(v)$ so that a wins with probability one. Hence, if $\phi > 0$, there is no equilibrium in which both a and c win with positive probability.

Case B:

In this case too, voters' best responses are uniquely determined. Indeed, no voter is indifferent between candidates b and c. Hence, if these candidates are the unique ones with positive probability of victory, it follows that $\alpha \to \{b\}$, $\beta \to \{b\}$ and $\gamma \to \{c\}$.

Hence if $\phi = 0$, then $n_\alpha + n_\beta = n_\gamma$ and therefore there is a strict equilibrium (hence isolated) in which b and c are tied.

On the contrary, if $\phi > 0$, then $n_\alpha + n_\beta > n_\gamma$ and therefore $s_b(v) > s_c(v)$ so that b wins with probability one. Hence, if $\phi > 0$, there is no equilibrium in which both b and c win with positive probability.

Case C:

In this case, the three candidates can win the election. This case must involve the use of mixed strategies as in any pure undominated Nash equilibrium, either c wins or there is a tie between a and c or b and c. However, as we focus on undominated equilibria, the support C_l of any voter with type l includes at most two pure strategies, as previously discussed. **Hence, there cannot exist a continuum of outcomes in such a connected component. to be justified**.

Conclusion:

We have proved that whenever the strategy combination is non-degenerate there are two cases. If $\phi = 0$, there are two strict equilibria. As they are strict, their associated component of equilibria is a singleton and then leads to a unique outcome. Moreover, if $\phi > 0$, there are no equilibria in which two candidates win with positive probability. Therefore, every connected component of equilibria is associated to one outcome: the set of undominated Nash outcomes in D is finite.

B Proofs of Propositions 6 and 7

Take $x = \frac{1}{t+1}$ for some integer $t \in \mathbb{N}^+$ so that $1 - x = \frac{t}{t+1}$. It follows that

$$U_{m,\phi}(x) = \frac{10}{(t+1)^{2m-1}} \left(\sum_{j=0}^{\phi-1} C_{m-1}^{j} C_{m}^{\phi-j-1} t^{m+\phi-2j-2} + \sum_{j=0}^{\phi} C_{m-1}^{j} C_{m}^{\phi-j} t^{m+\phi-2j-1} \right),$$

and

$$L_{m,\phi}(x) = \frac{9}{(t+1)^{2m-1}} \left(\sum_{j=0}^{\phi-1} C_{m-1}^{j} C_{m}^{\phi-j-1} t^{m-\phi+2j+1} + \sum_{j=0}^{\phi} C_{m-1}^{j} C_{m}^{\phi-j} t^{m-\phi+2j} \right).$$

Proposition 6. *Let $x = \frac{1}{t+1}$. There exists a t large enough for which $U_{m,\phi}(x) < L_{m,\phi}(x)$.*

Proof. Note first that rearranging the terms in the expression

$$\sum_{j=0}^{\phi-1} C_{m-1}^{j} C_{m}^{\phi-j-1} t^{m+\phi-2j-2} + \sum_{j=0}^{\phi} C_{m-1}^{j} C_{m}^{\phi-j} t^{m+\phi-2j-1} \tag{9}$$

leads to

$$\sum_{h=1}^{2\phi+1} t^{m+\phi-h} \delta_h, \text{ with } \delta_h = \begin{cases} C_{m-1}^{l-1} C_{m}^{\phi-l} & \text{if } h = 2l, \\ C_{m-1}^{l} C_{m}^{\phi-l} & \text{if } h = 2l+1. \end{cases}$$

Similarly, it can be proven that

$$\sum_{j=0}^{\phi-1} C_{m-1}^{j} C_{m}^{\phi-j-1} t^{m-\phi+2j+1} + \sum_{j=0}^{\phi} C_{m-1}^{j} C_{m}^{\phi-j} t^{m-\phi+2j}, \tag{10}$$

leads to

$$\sum_{h=0}^{2\phi} t^{m+\phi-h} \psi_h, \text{ with } \psi_h = \begin{cases} C_{m-1}^{\phi-l} C_{m}^{l} & \text{if } h = 2l, \\ C_{m-1}^{\phi-l-1} C_{m}^{l} & \text{if } h = 2l+1. \end{cases}$$

Before continuing the proof of Proposition 6, we introduce some lemmata dealing with the properties of ψ_i and δ_i.

Note first that $\delta_1 = \psi_1 + \psi_0$ and $\delta_{2\phi} + \delta_{2\phi+1} = \psi_{2\phi}$.

Lemma 1. *For any $l \in \{1, \ldots, 2\phi - 1\}$, $\delta_{2l} + \delta_{2l+1} = \psi_{2l} + \psi_{2l+1}$.*

Proof. Take any $l \in \{1, \ldots, 2\phi - 1\}$. By definition, $\delta_{2l} + \delta_{2l+1} = C_{m-1}^{l-1} C_m^{\phi-l} + C_{m-1}^{l} C_m^{\phi-l}$ so that $\delta_{2l} + \delta_{2l+1} = C_m^{\phi-l}(C_{m-1}^{l-1} + C_{m-1}^{l}) = C_m^{\phi-l} C_m^{l}$. Similarly, $\psi_{2l} + \psi_{2l+1} = C_{m-1}^{\phi-l} C_m^{l} + C_{m-1}^{\phi-l-1} C_m^{l}$ so that $\psi_{2l} + \psi_{2l+1} = C_m^{\phi-l} C_m^{l}$ proving the claim. □

Lemma 2. *For any $l \in \{1, \ldots, 2\phi - 1\}$, $\psi_{2l} > \delta_{2l}$ and hence $\psi_{2l+1} < \delta_{2l+1}$.*

Proof. Take any $l \in \{1, \ldots, 2\phi - 1\}$. Due to the definition of ψ_{2l} and δ_{2l}, it follows that

$$\begin{aligned}
\psi_{2l} > \delta_{2l} &\Longleftrightarrow C_{m-1}^{\phi-l} C_m^{l} > C_{m-1}^{l-1} C_m^{\phi-l} \\
&\Longleftrightarrow \frac{(m-1)!}{(m-\phi+l-1)!(\phi-l)!} \frac{m!}{l!(m-l)!} > \frac{(m-1)!}{(l-1)!(m-l)!} \frac{m!}{(\phi-l)!(m-\phi+l)!} \\
&\Longleftrightarrow (l-1)!(m-\phi+l)! > l!(m-\phi+l-1)! \\
&\Longleftrightarrow m > \phi,
\end{aligned}$$

which holds by assumption. Hence $\psi_{2l} > \delta_{2l}$. Moreover, due to Lemma 1, for any $l \in \{1, \ldots, 2\phi - 1\}$, $\delta_{2l} + \delta_{2l+1} = \psi_{2l} + \psi_{2l+1}$. Therefore, $\psi_{2l+1} < \delta_{2l+1}$ must to ensure that the previous equality holds. □

Once equipped with Lemmata 1 and 2, we proceed with the proof of Proposition 6.

Given that $\delta_1 = \psi_1 + \psi_0$ and that $t > 0$, it follows that $t^{m+\phi}\psi_0 + t^{m+\phi-1}\psi_1 > t^{m+\phi-1}\delta_1 >$. Similarly, as $\delta_{2\phi} + \delta_{2\phi+1} = \psi_{2\phi}$, we can write that $t^{m-\phi}\psi_{2\phi} > t^{m+\phi-2}\delta_{2\phi} + t^{m-\phi-1}\delta_{2\phi+1}$. Moreover, due to Lemma 1 and 2, $\delta_{2l} + \delta_{2l+1} = \psi_{2l} + \psi_{2l+1}$ with $\psi_{2l} > \delta_{2l}$. Hence, we can write that

$$t^{m+\phi-2l}\psi_{2l} + t^{m+\phi-2l-1}\psi_{2l+1} > t^{m+\phi-2l}\delta_{2l} + t^{m+\phi-2l-1}\delta_{2l+1}.$$

Therefore, it follows that (9) < (10), the different between both expressions being strictly increasing on t.

Hence, as $U_{m,\phi}(x) < L_{m,\phi}(x)$ is equivalent to

$$9\left(\sum_{h=0}^{2\phi} t^{m+\phi-h}\psi_h\right) > 10\left(\sum_{h=1}^{2\phi+1} t^{m+\phi-h}\delta_h\right),$$

we can deduce that there must exist a t high enough for which the inequality holds, concluding the proof. □

Proposition 7. *Let $y = \frac{t}{t+1}$. There exists a t large enough for which $U_{m,\phi}(y) > L_{m,\phi}(y)$.*

Proof. The proof is analogous to the one of Proposition 6 and hence is omitted. □