The Marginality Approach for the Shapley Value in Games with Externalities

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Extended Abstract

One of the long-debated issues in coalitional game theory is how to extend the Shapley to games with externalities (partition-function games). When externalities are present, not only can a player's marginal contribution to a coalition—a central notion to the Shapley value—be defined in a variety of ways, but it is also not obvious which axiomatization should be used. Consequently, a number of authors extended the Shapley value using complex and often unintuitive axiomatizations.

Of Shapley's original four axioms, *Efficiency*, *Symmetry* and *Additivity* can straightforwardly be extended so as to apply to games with externalities.[1] This is, unfortunately, not also the case for the concept of a null-player in the *Null-Player Axiom*. One approach, leading to the *weakest* version of this axiom, *Weak Null-Player Axiom*, is to give a very *strict* condition on a null-player. In particular, in this approach, a null-player has no impact on the value of *any* coalition [Bolger (1989), Macho-Stadler et al. (2007), McQuillin (2009)].

If i is a strict null-player, then $\varphi_i(v) = 0$. (*Weak Null-Player Axiom)*

Let v denote a partition function mapping each coalition S embedded in partition P to a real value in $\mathbb{R}$. Formally, a player i is null-player in a strict sense if for all partitions P, all coalitions $S, T \in P \cup \{\emptyset\}$ with $i \in S$

$$v(S, P) - v(S \setminus \{i\}, \tau_i^T(P)) = 0,$$

where $\tau_i^T(P) \stackrel{\text{def}}{=} P \setminus \{S, T\} \cup \{S \setminus \{i\}, T \cup \{i\}\}$ denotes the partition resulting from the transfer of i from its original coalition S to T. We refer to the value $v(S, P) - v(S \setminus \{i\}, \tau_i^T(P))$ as to *elementary marginal contribution of a player i leaving S and joining T in P*. Thus, intuitively, the Weak Null-Player Axiom requires that a null-player leaving a coalition S does not change the coalition's value, no matter which outside coalition the null player joins.[2] Note that—because of externalities—player

[1] Additivity is sometimes strengthened to Linearity. Linearity is satisfied by all known extensions of the Shapley value to games with externalities [Bolger (1989), Macho-Stadler et al. (2007), McQuillin (2009), Skibski (2011)].

[2] Note that typically the intuition behind the Shapley value is presented as a process of entering coalitions. We find that the concept of players leaving coalitions more convenient to deal with games with externalities.

i leaving coalition (S, P) may cause different changes to the value of S depending which coalition in $P \setminus S$ player i joins.

Now, the **marginality approach**—adopted by Bolger (1989) PhamDo and Norde (2007), De Clippel and Serrano (2008), and Skibski (2011)—comes from the observation, that although particular transfers may change the value of the coalition, the aggregated *marginal contribution* may still be equal zero. Formally, we define, for each weight α:

$$[mc_i^\alpha(v)](S,P) = \sum_{T \in P \setminus \{S\} \cup \{\emptyset\}} \alpha_i(S_{-i}, \tau_i^T(P)) \cdot [v(S,P) - v(S \setminus \{i\}, \tau_i^T(P))].$$

Then, an α*-null-player* is defined as a player i with $[mc_i^\alpha(v)](S,P) = 0$ for all coalitions S in P and for each α we obtain a separate axiom.

If i is an α-null-player, then $\varphi_i(v) = 0$. (*Null-Player*$^\alpha$*Axiom*)

Five specific values of α (i.e., definitions of marginal contributions) have been proposed in the literature [Bolger (1989), PhamDo and Norde (2007), Macho-Stadler et al. (2007), Hu and Yang (2010), Skibski (2011)]. For two of these values it has been demonstrated that *Efficiency*, *Symmetry*, *Additivity* and *Null-Player*$^\alpha$ *Axiom* imply uniqueness of the value [PhamDo and Norde (2007), Hu and Yang (2010)]. However, a general result that holds for all weights α has remained elusive to date.

Against this background, the contributions of out paper can be summarised as follows:

- We prove that for every value of α, *Efficiency*, *Symmetry*, *Additivity* and *Null-player*$^\alpha$ *Axiom* yield a unique value, which we will refer to as the α-value (Theorem 1).

- A natural question arises: which values proposed in the literature can be defined as an α-value? We show that a value is an α-value if and only if it satisfies *Efficiency*, *Symmetry*, *Linearity* and *Weak Null-Player Axiom* (Theorem 2). Thus, every linear value that satisfies the weakest extension of the Shapley's axioms can be described using the marginality approach and obtained by strengthening the Null-Player Axiom. In particular, this includes Myerson's value [Myerson (1977)].

- Furthermore, *we analyze how properties of a value translate into properties of weights* α. Specifically, we focus on the following properties of values: Weak Monotonicity, Strong Symmetry, and the Strong Null-player Axiom. Specifically:
  - *Weak Monotonicity*: $v_1(\tilde{S}, \tilde{P}) > v_2(\tilde{S}, \tilde{P}) \wedge \forall_{(S,P) \neq (\tilde{S}, \tilde{P})} v_1(S,P) = v_2(S,P)$ $\Rightarrow \varphi_i(v_1) \geq \varphi_i(v_2)$ for every $(\tilde{S}, \tilde{P})$ such that $i \in \tilde{S}$. This axiom states that if we increase value of a coalition containing a player, the payoff of this player will not decrease. We prove that α-value satisfies Weak Monotonicity if and only if $\alpha_i(S,P) \geq 0$ for every (S,P) (Theorem 3). If we limit ourselves to non-negative weights, $\alpha_i(S,P)$ has a natural interpretation as the probability that i transfers from S_{+i} forming partition P.
  - *Strong Symmetry*: $(S \cap \{i,j\} \neq \emptyset \Rightarrow v(S,P) = 0) \Rightarrow \varphi_i(v) = \varphi_j(v)$. Intuitively, Strong Symmetry states that if two players do not appear in any coalition with a non-zero payoff then they both should get the same payoff. We prove that α-value satisfies Strong Symmetry if and only if weights α have the following condition: $\alpha_i(S_{+j}, \tau_j^S(P)) \cdot \alpha_j(S,P) = \alpha_j(S_{+i}, \tau_i^S(P)) \cdot \alpha_i(S,P)$ for every (S,P) such that $i,j \notin S$ (Theorem 4). We

say that weights α satisfying this property are **interlace resistant**. Interestingly, Theorem 4 combined with Theorem 2 imply that average approach proposed by Macho-Stadler *et al.* [Macho-Stadler et al. (2007)] is equivalent to the marginality approach used with interlace resistant weights.

- *Strong Null-Player Axiom*: *i is a strict null-player* $\Rightarrow \varphi_j(v) = \varphi_j(v_{-i})$. This strengthening of the Weak Null-player Axiom states that player which does not have an impact on coalition values in the game also does not have an impact on the payoff (thus, if we remove null-player the payoffs will stay the same). We prove that α-value satisfies Strong Null-Player Axiom if and only if weights α satisfy the following condition: $\alpha_i(S, P) = \alpha_i(S_{-j}, P)$ for every (S, P) such that $i \notin S$ and $j \in S$ (Theorem 5). We say that weights α satisfying this property are **expansion resistant**.

Although Shapley's axiomatization is a golden standard for games without externalities, a couple of alternative values have been proposed. Two most important are (i) Young's axiomatization based on the Marginality Axiom (combined with Efficiency and Symmetry) [Young (1985)] and (ii) Myerson's axiomatization based on the Balanced Contributions property (combined with Efficiency) [Myerson (1980)]. In an important paper, De Clippel and Serrano [de Clippel and Serrano (2008)] proved that all three axiomatizations are equivalent for the externality-free definition of marginal contribution. Given this:

- As a corollary of our Theorem 1 and the brilliant work of Fujinaka [Fujinaka (2004)] we have that Young's axiomatization parametrized with weights α (Efficiency, Symmetry and Marginality Axiom$^\alpha$) is equivalent to the marginality-based version of Shapley's axioms (Efficiency, Symmetry, Additivity and Null-Player Axiom$^\alpha$ (Corollary 1).

- Finally, we define a game without player i as the difference between game with player i and i's marginal contribution (i.e., $v^\alpha_{-i}(S_{-i}, P_{-i}) \stackrel{\text{def}}{=} v(S, P) - [mc^\alpha_i(v)](S, P)$). Then, we prove for all α, that the α-value satisfies Myerson's axioms (Efficiency, Balanced Contribution$^\alpha$) if and only if α is interlace and expansion resistant (Theorem 6).

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