

Equilibria in Finite Games with Imperfect Recall

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Abstract

We generalize traditional equilibrium concepts for finite games in extensive form with behavioral strategies so that they apply to all games, including games of imperfect recall. Adapting and augmenting previous definitions (in particular, by Piccione and Rubinstein, and by Battigalli), we define four notions: *Distributed Agent Equilibrium* (DAE), *Distributed Nash Equilibrium* (DNE), *Distributed Sequential Equilibrium* (DSE), and *Distributed Perfect Equilibrium* (DPE). We show that, in a precise sense, these extend the classical equilibrium notions: (a) they form a strict inclusion hierarchy (e.g., every DNE is a DAE but not necessarily vice versa, and so on up the hierarchy), (b) every game has a DPE (and therefore also a DSE, DNE and DAE), and (c) in the subclass of games of perfect recall, DAE, DNE, DSE and DPE collapse, respectively, to agent equilibrium, Nash equilibrium, sequential equilibrium, and perfect equilibrium. In service of these results we introduce several novel notions – including *partial symmetry*, the *distributed agent form* and *phantom strategies* – which may be interesting in their own right.¹

1 Introduction

Games of imperfect recall are those in which players may not remember their entire experience during the game (specifically, when in some information set, a player may not know the information sets she visited previously, nor the actions she took). Although historically under-researched in game theory, imperfect recall is a key notion if one hopes to apply game theoretic models in practice. In the real world, humans and computer agents alike do not have unlimited memory, and requiring players to remember the entire history of play is often unrealistic. Even when realistic, it can be counterproductive. A good example of this can be seen in computer poker. The state of the art in computer poker for the past few years has been for players to abstract the full extensive form game of poker (by losing the distinction among certain cards) and then play an approximate Bayes-Nash equilibrium of this abstract game. Recently, it has

¹While this paper covers equilibria in imperfect recall, we have a companion paper that tackles complexity in imperfect recall.

been shown that allowing imperfect recall strategies improves the quality of the resulting strategy [7]. There is every reason to think that this lesson applies broadly in many application domains.

Games of imperfect recall present several novel fundamental challenges. In this paper we address the modeling challenges, specifically solution concepts that are appropriate for all extensive-form games, including those of imperfect recall. In a companion paper we address computational challenges.

In order to understand imperfect recall, it is a good idea to understand the state of the art of perfect recall. A good starting point for this discussion is provided by games in extensive forms with perfect information. For this class of games the two most common solution concepts are the *Bayes Nash equilibrium* (BNE)² [12], and its refinement *subgame-perfect equilibrium* (SPE) [10]. When one moves to games of imperfect information, the notion of SPE must be replaced by the more involved notions of sequential equilibrium (SE) [9] and perfect equilibrium (PE) [14]. But so long as one restricts the discussion to games with *mixed strategies* rather than the more natural *behavioral strategies*, the intuitions carry over from the perfect-information case, as do the formal properties: The solution concepts form a strict hierarchy (every SE is a NE but not vice versa; and every PE is a SE but not vice versa), and every game has a PE (and thus also a SE and NE).

All this is still true with behavioral strategies in the subclass of games of perfect recall, that is, games in which agents in fact do remember their history in the game. It is well known that in such games behavioral strategies can be emulated by mixed strategies, and vice versa [10], and thus in general these games do not present new issues. But all this changes in games of imperfect recall. In this broader class neither the standard intuitions nor the standard technical results hold, and the elegant map of solution concepts breaks down. The goal of this paper is to extend the elegance to games with imperfect recall.

To gain appreciation for the challenges associated with imperfect recall, we first note that the basic properties of the standard solution concepts cease to hold in this class. For one thing, games with imperfect recall may not have a NE, as seen in the game depicted in Figure 1. It is not hard to see that this game has no NE. This has been known for a while; it was shown formally by Wichardt [16], but observed informally earlier (cf. [2], example 2.4).

Other standard properties also break down in games of imperfect recall; for example, it is shown by Kline that there exist SE that are not NE and PE that are not SE [8]. However, it is not only the formal properties that start to break down, but also the intuition about the very concepts involved. This was most strikingly demonstrated in Piccione and Rubinstein's 1997 paper on imperfect recall in extensive decision problems (that is, extensive-form games with a single player) [13]. That paper provides a paradoxical example demonstrating that naive reasoning about beliefs and behavioral strategies can lead to inconsistent conclusions.

The literature following the Piccione and Rubinstein paper (e.g., [1, 3, 5, 4])

²We will simply refer to this as the Nash equilibrium (NE) for the remainder of the article.

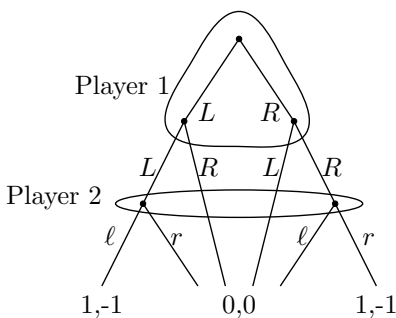


Figure 1: An extensive form game with no Nash equilibrium in behavioral strategies.

offered a variety of formal constructs that shed light on the situation, but did not culminate in a map of solution concepts that naturally extends the elegant map from the perfect-recall case. We will provide exactly this extension, which is summarized in Figure 2.

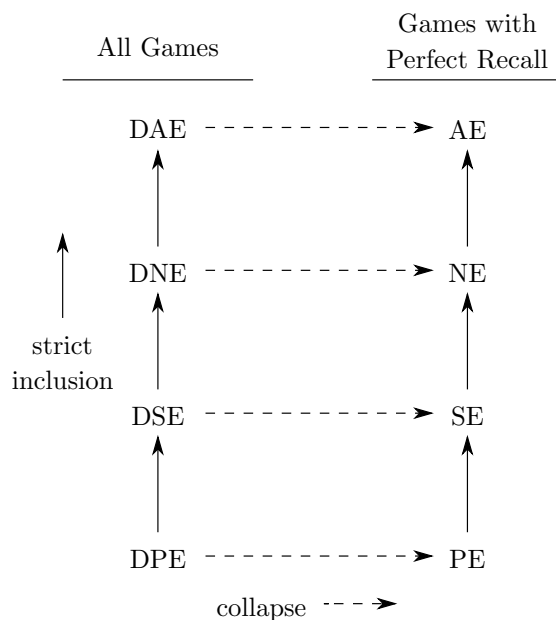


Figure 2: The map of solution concepts for extensive form games.

In this symmetric picture, we add to the perfect-recall class the solution concept of *agent equilibrium* (AE). Introduced by Kuhn [10], in games of perfect information AE considers each player as being composed of multiple agents (or “selves”), all sharing the same payoff function, and each controlling a single information set. The four solution concepts form a strict inclusion hierarchy

(that is, every PE is a SE but not vice versa, and so on up the hierarchy (Figure 6 shows examples that demonstrate the strictness of the inclusion)). All four notions are generalized to the class of games with imperfect recall, in the form of Distributed Agent Equilibrium (DAE), Distributed Nash Equilibrium (DNE), Distributed Sequential Equilibrium (DSE), and Distributed Perfect Equilibrium (DPE). The generalized versions form their own hierarchy (that is, every DAE is a DNE, etc.), and collapse to the standard notions in games with perfect recall. Note that from (a) the inclusion among the generalized notions, (b) the collapse to classical notions in case of perfect recall, and (c) the strictness of inclusion in the case of perfect recall, follows the strictness of inclusion of these generalized solution concepts as well. Most importantly, we will prove the existence of this hierarchy; we will show that every game has a DPE (and therefore also a DSE, DNE and DAE).

Two remarks are in order. First, although as we said previously, research has not resulted in a clean picture such as the one we present here, it definitely produced many of the ingredients needed to construct this picture. We will discuss that literature in more detail in Section 6, but let us highlight two papers in particular. Piccione and Rubinstein themselves discuss approaches to resolving the apparent paradox, one of which they call *modified multiselves consistent strategy* (or MMCS), so called since it is a modification of Strotz's original multiselves approach [15]. Our notion of DAE is essentially the multi-player generalization of MMCS. Battigalli, among other things, defines the Modified Multiselves Sequential Equilibrium (MMSE). Our notion of DSE is essentially MMSE, again generalized to the multi-player case. So the solution concepts we lay out have their origins in these two earlier papers (as well as some others we will discuss in Section 6), although some nuanced changes and additional constructs are needed in order to have the clean inclusion relations among them, and the collapse to the traditional notions in the special case of perfect recall. The hardest technical contribution in this paper consists of existence proofs, specifically existence of DPE. This part has no precedent in the papers mentioned; they are concerned almost exclusively with the single agent case, in which existence is trivial (any optimal strategy is outcome equivalent to a DPE). Our proof appeals to a novel game form we call the *distributed agent form*, which may have interest in its own right. The distributed agent form can be viewed as further breaking up the strategic entities; if the agent form assigns a strategic decision maker to each information set, the distributed agent form assigns a strategic decision maker to certain subsets of nodes within each information set.

The second comment is that Figure 2 tells only part of the story. Additional solution concepts can be defined, and relationships among them proved. Section 6 presents some of this expanded map. However, in our view, while the expanded map is useful to explain precisely the connections to previous work, Figure 2 captures the most salient concepts and relationships.

The rest of the paper is organized as follows. Section 2 will go over background definitions and notation, and then define the notions of DAE, DNE, DSE, and DPE. Section 3 will establish the inclusion hierarchy among these four equilibrium notions. Section 4 will prove the existence of a DPE (and thus

also DAE, DNE, and DSE). Section 5 will prove the collapse to standard equilibrium notions in the case of perfect recall. Section 6 will expand the map of Figure 2, and use the expanded map to position relevant prior work. Finally, Section 7 will discuss the implications of our results and some open questions.

2 Background and Definitions

2.1 Standard Definitions and Notation

In this section we cover the necessary material to understand AE, NE, SE, and PE and define these four equilibrium concepts.

Definition 1 (Extensive Form). *An extensive form game is a six-tuple $\Gamma = \langle N, H, P, \rho, u, I \rangle$, where*

- *N is a finite set of players.*
- *H is a finite set of sequences that represent the possible histories of actions. H must contain the empty sequence and if $(a_1, \dots, a_K) \in H$ and $K \neq 0$ then $(a_1, \dots, a_{K-1}) \in H$. A history (a_1, \dots, a_K) is terminal if there is no action a such that $(a_1, \dots, a_K, a) \in H$. The set of terminal histories of H is denoted Z . The set of actions available at a particular history h is defined as $A(h) = \{a : (h, a) \in H\}$.*
- *$P : H/Z \rightarrow N \cup \{c\}$ (where H/Z is the set of all non-terminal histories) is the player function that assigns a player or Nature to each non-terminal history. We will denote C to be the set of non-terminal histories assigned to Nature and D to be the set of non-terminal histories assigned to players.*
- *ρ is a prior on Nature's actions (i.e. for all $h \in C$, $\rho(h) \in \Delta(A(h))$).*
- *$u_i : Z \rightarrow \mathbb{R}$ is the utility function for player i .*
- *I is the set of information sets. I is a partition of D such that if $X \in I$ then for all $h, h' \in X$, $A(h) = A(h')$ and $P(h) = P(h')$. Because of this requirement we will overload the action and player functions; for an information set X we define $A(X) = A(h)$, $P(X) = P(h)$ where $h \in X$. For a player i , $I_i = \{X : P(X) = i\}$ denotes the set of information sets assigned to player i .*

The *experience* of player i at history h , denoted $\text{exp}_i(h)$, is the sequence of information sets and actions of player i along the history h . An extensive form game has *perfect recall* if for every information set X and for every $h, h' \in X$ it is the case that $\text{exp}_i(h) = \text{exp}_i(h')$, where i is the player assigned to X . A game of *imperfect recall* is one without perfect recall. Given a history $h = (a_1, \dots, a_K)$ and $L \leq K$ the history $h' = (a_1, \dots, a_L)$ is said to *precede* h , which is denoted $h' \leq h$. Furthermore, a history h' *strictly precedes* history h , written $h' < h$, if $h' \leq h$ and $h' \neq h$. A game exhibits *absentmindedness* if there is an information

set X and $h, h' \in X$ such that $h' < h$; thus, absentmindedness is a special case of imperfect recall.

A *behavioral strategy*³ for player i , b_i , is a distribution for each information set assigned to i over the actions available at that information set (i.e. $b_i \in \prod_{X \in I_i} \Delta(A(X))$). The set of strategies for player i will be denoted Σ_i . A *strategy profile* $b = \prod_{i \in N} b_i$ is a set of strategies, one for each player. Likewise the set of strategy profiles for a game will be denoted Σ . Given a strategy profile b and an information set X we let $b(X)$ denote the distribution over $A(X)$ defined by b (that is, $b(X) = b_i(X)$ where $b_i \in B$ and i is the player assigned to X). Thus given a particular action $a \in A(X)$, $b(X)(a)$ gives the probability of playing a upon reaching information set X . We will often break up a strategy profile into two parts; $b = (b_i, b_{-i})$ where b_i is strategy for player i and b_{-i} is the set of strategies for all other players.

A strategy profile b induces a probability distribution over terminal histories, $p_b \in \Delta(Z)$. p_b can be extended to all (not necessarily terminal) histories $h \in H$ by

$$p_b(h) = \sum_{z \in Z: h \leq z} p_b(z)$$

Note that the resulting function is not a distribution over H (i.e. $p_b \notin \Delta(H)$).

We use this distribution to extend the utility function for player i to a strategy profile b linearly as follows

$$u_i(b) = \sum_{z \in Z} p_b(z) u_i(z)$$

Definition 2 (Nash Equilibrium [12]). *A strategy profile b is a Nash equilibrium (NE) if for every player i and every strategy b'_i .*

$$u_i(b'_i, b_{-i}) \leq u_i(b)$$

The following definition has its roots in much prior work, in particular [10].

Definition 3 (Agent Form). *The agent form of a game $\Gamma = \langle N, H, P, \rho, u, I \rangle$ is a derivative game Γ' where:*

- $N' = I$.
- $H' = H$.
- $P'(h) = \begin{cases} c & : \text{If } h \in C \\ X & : \text{If } h \in X \text{ for } X \in I \end{cases}$
- $\rho' = \rho$.
- For $X \in I$, $u'_X = u_{P(X)}$.
- $I' = I$.

³We will refer to behavioral strategies simply as strategies for the remainder of the paper.

In other words, the agent form gives control of every information set to an independent agent who receives the same payoff as the player that originally controlled that information set. Notice that because the set of histories and the set of information sets remains unchanged there is a clear bijective correspondence between strategy profiles of the agent game and strategy profiles of the original game. This fact allows us to define the agent equilibrium (AE) of a game.

Definition 4 (Agent Equilibrium). *An agent equilibrium of a game Γ is a strategy profile that corresponds to a Nash equilibrium of the agent form of Γ .*

Before we define sequential equilibrium, we need to define the notion of beliefs [13]. For each information set X , a belief $\mu(X)$ is an distribution over the histories in X . A belief μ is said to be consistent with a strategy profile b if for every information set X with positive probability of being reached and every $h \in X$ we have

$$\mu(X)(h) = \frac{p_b(h)}{\sum_{h' \in X} p_b(h')}$$

We can also consider the distribution over terminal histories induced by strategy profile b when we assume play starts at a particular history h : $p_b(\cdot|h) \in \Delta(Z)$. We can similarly define utility for player i conditioned on history h as

$$u_i(b|h) = \sum_{z \in Z} p_b(z|h) u_i(z)$$

Combining beliefs and conditional utility, we can develop a subjective version of utility. That is to say for a strategy profile b , a player i , and an information set X the *subjective utility* is given by

$$SU_i(b; X, \mu) = \sum_{h \in X} \mu(X)(h) u_i(b|h)$$

Armed with this machinery we can define sequential equilibrium (SE).

Definition 5 (Sequential Equilibrium [9]⁴). *A strategy profile b is a sequential equilibrium if there exists a sequence of completely mixed strategy profiles b^1, b^2, \dots that converge to b and a sequence of positive real numbers $\epsilon_1, \epsilon_2, \dots$ that converge to 0 such that for every k , for every belief μ consistent with b^k , for every information set X (assigned to player i), and for every strategy b'_i*

$$SU_i(b'_i, b^k_{-i}; X, \mu) \leq SU_i(b^k; X, \mu) + \epsilon_k$$

The last standard definition we will present is perfect equilibrium. In order to define this concept we must give a definition of a perturbed game [14]. Given a game Γ , a *perturbation* η of Γ is a function that associates with every information

⁴The reader may be more familiar with another common characterization of SE involving strongly consistent beliefs also given by Kreps and Wilson [9].

set X and action $a \in A(X)$ a positive probability $\eta(X)(a) > 0$ such that for every information set X , $\sum_{a \in A(X)} \eta(X)(a) < 1$. A *perturbed game* is then a pair (Γ, η) consisting of a game Γ a perturbation η . In a perturbed game players are not allowed play strategies that assign less than η probability to each of their actions. That is to say for every player i strategy b_i of the perturbed game (Γ, η) , it must be that for each information set X assigned to player i and for each action $a \in A(X)$, we have $b_i(X)(a) \geq \eta(X)(a)$. In order to distinguish these more restricted strategies from strategies of Γ , we call the strategies allowed in (Γ, η) *perturbed strategies*. We can analogously define *perturbed strategy profiles* in the obvious way.

Definition 6 (Perfect Equilibrium [14]). *Given a game Γ , a strategy profile b is a perfect equilibrium if there exists a sequence of perturbed games $(\Gamma, \eta_1), (\Gamma, \eta_2), \dots$ where $\eta_k \rightarrow 0$ and a sequence of correspondingly perturbed strategy profiles b^1, b^2, \dots that converge to b such that b^k is a Nash equilibrium of (Γ, η_k) for all k .⁵*

Finally, having presented the definitions of these four concepts, we note that it is also well known that they form an inclusion hierarchy [10, 9]. Furthermore, as demonstrated in Figure 6, this hierarchy is strict.

2.2 Main Definitions

We now move on from these well known definitions to our new definitions that are the focus of this paper: DAE, DNE, DSE, and DPE.

All of these notions rely on the hypothetical construct we will call *phantom strategies*. Whereas a standard strategy specifies a distribution for every information set (i.e., a strategy b_i is in $\prod_{X \in I_i} \Delta(A(X))$), a *phantom strategy* β_i specifies for each node h assigned to player i a distribution over the actions available at h . A *phantom strategy profile* $\beta = \prod_{i \in N} \beta_i$ is a set of phantom strategies, one for each player. Like a normal strategy profile, a phantom strategy profile induces a distribution of terminal nodes (both ex-ante and from a particular history h). Thus we can further extend the notions of utility to apply to phantom strategy profiles in the obvious way.

Phantom strategies allow us represent the ex-interim reasoning of the agent. Of particular interest to us will be a subclass of phantom strategy profiles called *single-deviation phantom strategy*. Specifically, given a player i , a (standard) strategy b_i , a history h assigned to player i , and a deviation distribution $s \in \Delta(A(h))$, we denoted the single-deviation phantom strategy profile $b_i[h/s]$ and formally define it given a player i history h' (in information set X) by

$$b_i[h/s](h') = \begin{cases} s & : \text{If } h' = h \\ b_i(X) & : \text{otherwise} \end{cases}$$

Intuitively, in a single-deviation phantom strategy the agent is saying “just this once I will deviate from the strategy prescribed for this information set.”

⁵A Nash equilibrium of a perturbed game is a perturbed strategy profile of such that no player can profit by deviating to some other perturbed strategy.

Of course the agent does not know at which node in the information set she is, hence the name “phantom”, but she may have a belief regarding the nodes, in which case she can reason about the effect of this “just this one time” deviation on an expected basis.

The role of phantom strategies becomes apparent when we define a new form of subjective utility, called *distributed subjective utility*. Given a strategy profile b , a player i , an information set X , a distribution $s \in A(X)$, and belief μ , we define distributed subjective utility to be

$$DSU_i(b; X, s, \mu) = \sum_{h \in X} \mu(X)(h) u_i((b_i[h/s], b_{-i})|h)$$

In words, distributed subjected utility first calculates for each node in information set X the expected payoff given strategy profile b and assuming that the deviation occurs only at that node, and then takes the expectation over the nodes given the belief μ . Note that $SU_i(b; X, \mu) = DSU_i(b; X, b(X), \mu)$.

We can now move on the definitions of our equilibrium concepts.

Definition 7 (Distributed Agent Equilibrium (DAE)). *A strategy profile b is a Distributed Agent Equilibrium if there exists a belief μ consistent with b such that for every information set X assigned to player i and reached with positive probability, and for every distribution over actions in X , $s \in \Delta(A(X))$, it is the case that*

$$DSU_i(b; X, s, \mu) \leq SU_i(b; X, \mu)$$

In other words, if a player reaches a history but only knows what information set she is in, it is not beneficial to change the action of a DAE at that history subject to her uncertainty about where she is.

A DSE is defined similarly to a DAE only we no longer ignore the information sets that are not visited.

Definition 8 (Distributed Sequential Equilibrium (DSE)). *A strategy profile b is a Distributed Sequential Equilibrium if there exists a sequence of completely mixed strategy profiles b^1, b^2, \dots that converge to b and a sequence of positive real numbers $\epsilon_1, \epsilon_2, \dots$ that converge to 0 such that for every k , for every belief μ consistent with b^k , for every information set X (assigned to player i), and for every $s \in \Delta(A(X))$, it is the case that*

$$DSU_i(b^k; X, s, \mu) \leq SU_i(b^k; X, \mu) + \epsilon_k$$

In order to define DPE we must extend DAE to perturbed games. A DAE of a perturbed game (Γ, η) is a perturbed strategy profile that satisfies the definition of a DAE with the addition provision that the deviations considered must also be perturbed. By this we mean that for an information set X , deviations $s \in \Delta(A(X))$ must satisfy for all $a \in A(X)$, $s(a) \geq \eta(X)(a)$.

Definition 9 (Distributed Perfect Equilibrium (DPE)). *Given a game Γ , a strategy profile b is a Distributed Perfect Equilibrium if there exists a sequence*

of perturbed games $(\Gamma, \nu_1), (\Gamma, \eta_2), \dots$ where $\eta_k \rightarrow 0$ and a sequence of correspondingly perturbed strategy profiles b^1, b^2, \dots that converge to b such that b^k is a DAE of (Γ, η_k) for all k .

DNE is a bit of a contrived notion, and for this reason we present it last. To define DNE we first have to present the notion of equivalent strategy profiles, as defined by Battigalli [3]: Two strategy profiles b and b' are *equivalent*, written $b \cong b'$, if they induce the same probability for every history. We extend this definition slightly to apply to strategies, not only strategy profiles. Given a strategy profile b , two strategies b'_i and b''_i of player i are equivalent with respect to b , written $b'_i \cong_b b''_i$, if $(b'_i, b_{-i}) \cong (b''_i, b_{-i})$.

Definition 10 (Distributed Nash Equilibrium (DNE)). *A strategy profile b is an Distributed Nash Equilibrium if it is a DAE and for every player i there exists a strategy $b'_i \cong_b b_i$, a sequence of completely mixed player i strategies b_i^1, b_i^2, \dots that converge to b'_i , and a sequence of positive real numbers $\epsilon_1, \epsilon_2, \dots$ that converge to 0 such that for all k , for every belief μ consistent with $b^k = (b_i^k, b_{-i})$, for every information set X (assigned to player i) with positive probability in b^k , and for every $s \in \Delta(A(X))$, it is the case that*

$$DSU_i(b^k; X, s, \mu) \leq SU_i(b^k; X, \mu) + \epsilon_k$$

3 Establishing the Hierarchy of Distributed Equilibrium Concepts

We begin by proving the inclusion relationships that are analogous to the inclusion relationships among AE, NE, SE, and PE.

Proposition 1. *Every DPE is a DSE.*

Proof. See Appendix. □

Proposition 2. *Every DSE is a DNE.*

Proof. See Appendix. □

Proposition 3. *Every DNE is a DAE.*

Proof. This is true by definition. □

The strictness of these inclusion relations is implied by the collapse of the distributed equilibrium concepts to their standard counterparts (as will be shown in Section 5) and the strictness of the inclusion relations for these standard concepts (as shown, for example, in Figure 6).

4 Proof of Existence

This section will be concerned with proving the following theorem:

Theorem 1. *All games of imperfect recall have a DPE.*

From this theorem, we will immediately get the corollary:

Corollary 2. *All games of imperfect recall have a DSE, a DNE, and a DAE.*

Before we tackle the existence of DPE directly we must present two new concepts. First, we define the notion of *partial symmetry* and show the existence of partially symmetric perfect equilibria in partially symmetric games. We then go on to define a generalization of the agent form of a game which we dub the *distributed agent form*. Finally, we bring these two concepts together to and show that a partially symmetric perfect equilibrium of this distributed agent form corresponds to a DPE of the original game.

4.1 Partial Symmetry

A *symmetry* of a game Γ is a permutation χ over the players of Γ such that the strategy spaces of permuted players remains the same and the utility to each player of permuting all players' roles by χ remains the same. In other words χ is a symmetry of Γ if for every player i and every strategy profile b we have

1. $\Sigma_i = \Sigma_{\chi(i)}$
2. $u_i(b) = u_{\chi(i)}(\chi(b))$

Note that the second line overloads the permutation χ to permute the strategies of a strategy profile (which produces a well defined strategy profile so long so long as the first line holds).⁶

Let G be a subgroup of S_n (the symmetric group on n symbols). A game Γ with n players is G -symmetric if for every $\chi \in G$, χ is a symmetry of the game Γ . Furthermore, a G -symmetric strategy profile of a G -symmetric game Γ is a strategy profile $b \in \Sigma$ such that for all $\chi \in G$, $\chi(b) = b$ (we will denote the set of G -symmetric strategy profiles of Γ as Σ_G). Lastly, a G -symmetric Nash equilibrium is a G -symmetric strategy profile that is also a Nash equilibrium of Γ .

Thus an n -player symmetric game (in the well known sense) is a S_n -symmetric game and a symmetric equilibrium of this game is a S_n -symmetric Nash equilibrium. Also note that all games are still symmetric with respect to the trivial subgroup. Partial symmetry, therefore, bridges the gap between fully symmetric games and games with no explicit symmetry.

A natural question to ask is whether a partially symmetric Nash equilibrium is guaranteed to exist in a partially symmetric game. Thankfully, just as with

⁶This notion of symmetry is inspired by Nash's seminal work showing the existence of symmetric equilibria [12]. Note that while Nash originally defines a symmetry to be over all pure actions he mainly concerns himself with the permutation that this induces over players.

standard Nash and fully symmetric Nash, a partially symmetric Nash equilibrium exist in every game. Formally, this is shown in the following proposition.

Proposition 4. *Every G -symmetric game has a G -symmetric Nash equilibrium.*

Proof. See Appendix. □

We now turn our attention to partially symmetric perfect equilibria. For our purposes it won't be sufficient to simply show the existence of a perfect equilibrium that is also partially symmetric. Instead we will opt for a notion of perfectness with partial symmetry built in.

To this end we need to define the notion of a partially symmetric perturbation. A perturbation η is considered a G -symmetric perturbation if for all $\chi \in G$ $\chi(\eta) = \eta$. In this definition we again overload χ to permuted perturbations as though they were strategy profiles (that happens to not sum to 1 for every information set). Similarly, a G -symmetric perturbed game is a perturbed game (Γ, η) where η is a G -symmetric perturbation.

Before proving the existence of partially symmetric perfect equilibria we must first show the existence of partially symmetric Nash equilibria in partially symmetric perturbed games.

Proposition 5. *Let Γ be a G -symmetric game and η a G -symmetric perturbation of Γ . There exists a G -symmetric Nash equilibrium of (Γ, η) .*

Proof. See Appendix. □

We now have sufficient vocabulary to define a partially symmetric perfect equilibrium.

Definition 11. *Let Γ be a G -symmetric game. b is a G -symmetric perfect equilibrium if there exists a sequence of G -symmetric perturbed games $(\Gamma, \eta_1), (\Gamma, \eta_2), \dots$ and a sequence of G -symmetric Nash equilibria b^1, b^2, \dots of these perturbed games such that $b^k \rightarrow b$ and $\eta_k \rightarrow 0$.*

Our final proposition of the section follows easily from this definition and our other previous result.

Proposition 6. *Let Γ be a G -symmetric game. There exists a G -symmetric perfect equilibrium of Γ .*

Proof. See Appendix. □

4.2 The Distributed Agent Form

The standard agent form of a game gives control of every information set to an independent agent. The distributed agent form further breaks down control of information sets. Before we talk about the distributed agent form, however, we must first take an aside and define a few crucial constructions. First, we

must stratify an information set into partitions that have no absentmindedness. To do this we define a more general version of the Grove and Halpern upper frontier of an information set [5].

Let $d_X(h)$ denote the depth of h within information set X . Formally, let us recursively define depth within an information set as follows:

Definition 12. For a history h , Let $H_{<h} = \{h' : h' < h\}$. The depth of history h within information set X is given by

$$d_X(h) = \begin{cases} 0 & : \text{If } h \notin X \\ 1 & : \text{If } h \in X \text{ and } \neg \exists h' : h' < h \\ \max_{h' \in H_{<h}} (d_X(h')) + 1 & : \text{If } h \in X \end{cases}$$

Definition 13. Let the n th frontier of an information set X , denoted \hat{X}^n , be defined as follows:

$$\hat{X}^n = \{h : d_X(h) = n\}$$

Finally let us define the depth of an information set $D(X)$.

Definition 14. The depth of an information set X is defined as the highest depth value of any history for that information set. In other words,

$$D(X) = \max_{h \in X} (d_X(h))$$

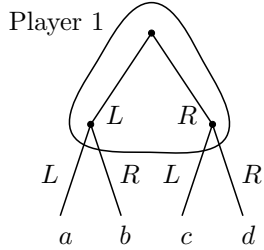
Now that we have the definition of a frontier, we can give an informal description of the distributed agent form. Just as in the agent form each information set is controlled by an independent agent, in the distributed agent form each frontier is controlled by an independent agent. However, this game allows for strategy profiles not allowed in the original game. To mitigate this we will have Nature randomly choose which frontier within an information set an agent controls without any agent knowing what that choice is. Applying a specific form of partial symmetry to this game then eliminates the excess strategy profiles. This is made significantly clearer in Figure 3.

Formally, the distributed agent form is defined as follows.

Definition 15 (distributed agent form). The agent form of a game $\Gamma = \langle N, H, P, \rho, u, I \rangle$ is a derivative game Γ'' where:

- $\tilde{N} = \bigcup_{X \in I} \{X\} \times \{1, \dots, D(X)\}$. There is an agent for every frontier and if $(X, j) \in \tilde{N}$ then the agent represented by (X, j) is assigned to frontier \hat{X}^j . Additionally, let $\tilde{n} = |\tilde{N}|$.
- Consider the following set of permutations over \tilde{N} which will be denoted Ψ . A permutation ψ is in Ψ if (and only if) for every agent $i = (X, j)$, if $\psi(i) = (Y, \ell)$ then it must be the case that $X = Y$. It is easy to see that this set of permutations in fact composes a subgroup of $S_{\tilde{n}}$ and that $\Psi \cong \prod_{X \in I} S_{D(X)}$.

Original Game



Distributed Agent Form

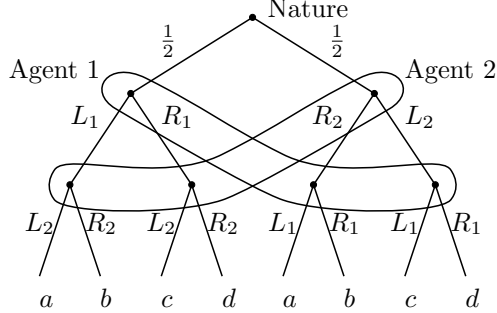


Figure 3: This figure demonstrates the relationship between the original game and the distributed agent form. Note there is only one set of payoffs in the distributed agent form since both agents receive the same payoff.

If $h \in H$ and $h = (a_1, \dots, a_k)$ then for all $\psi \in \Psi$, $\tilde{h} = (\psi, a_1, \dots, a_k) \in \tilde{H}$. Additionally we add the empty sequence, ϕ , to \tilde{H} , in order to make \tilde{H} a properly defined set of histories. In other words, at the beginning of play a permutation from Ψ is chosen, then play resumes as in H .

- Given a history $\tilde{h} \notin \tilde{Z}$

$$\tilde{P}(\tilde{h}) = \begin{cases} c & : \text{If } \tilde{h} = \phi \\ c & : \text{If } \tilde{h} = (\psi, h) \text{ and } P(h) = c \\ (X, \psi(j)) & : \text{If } \tilde{h} = (\psi, h) \text{ and } h \in \hat{X}^j \end{cases}$$

- Given a history $\tilde{h} \in \tilde{C}$ and an action $a \in A(\tilde{h})$

$$\tilde{\rho}(\tilde{h})(a) = \begin{cases} 1/|\Psi| & : \text{If } \tilde{h} = \phi \\ \rho(h)(a) & : \text{If } \tilde{h} = (\psi, h) \end{cases}$$

- For $i = (X, j) \in \tilde{N}$, and $\tilde{z} = (\psi, z) \in \tilde{Z}$, $u_i(\tilde{z}) = u_{P(X)}(z)$.
- \tilde{I} is simply the partition induced by the player function over \tilde{D} . That is to say for all $i \in \tilde{N}$, $\tilde{X}_i = \{\tilde{h} : \tilde{P}(\tilde{h}) = i\}$ and $\tilde{I} = \{\tilde{X}_i : i \in \tilde{N}\}$.

Thus each agent will assigned a frontier of a particular information set uniformly at random, but will never have knowledge of which frontier she has been assigned. Notice that the distributed agent form does have perfect recall as claimed since each agent only has one information set and it cannot be visited twice in the course of the game. Note further that when the original game does not have absentmindedness Ψ is the trivial subgroup and thus collapses to the standard agent form (with an additional single action move by Nature at the beginning of the game).⁷ On the other hand, if there is only one information

⁷We would like to point out that most of the desired properties of the distributed agent form are preserved when using any subpartition of information sets that does not exhibit

		Agent 2	
		L_2	R_2
Agent 1	L_1	a	$\frac{b+c}{2}$
	R_1	$\frac{b+c}{2}$	d

Figure 4: The canonical normal form game of the distributed agent form as seen in Figure 3. This form of the distributed agent form provides a good demonstration of the symmetry of the distributed agent form.

set (and therefore only one player) then $\Psi = S_n$ and as will be shown the distributed agent form is a fully symmetric game.

There are two crucial observations to make about the distributed agent form. First, notice that the distributed agent form is Ψ -symmetric. This can easily be seen since agents' roles are determined by the permutation Nature chooses uniformly at random from Ψ and agents receive the same payoff as any agent they could be mapped to by a permutation in Ψ .

Second, note that there is a bijection between Ψ -symmetric strategy profiles and strategy profiles of the original game. This follows immediately from the fact that any strategy profile of the distributed agent form respects Ψ if and only if agents assigned to the same information set are using the same strategy. Thus there is a well defined strategy for each information set which is identical to saying that it defines a strategy profile of the original game.

Remark: We will consistently be using the term “player” to refer to someone playing the original game and “agent” to refer to someone playing the distributed agent form (on behalf of some player). Also because of the natural bijection between strategy profiles of the original game and Ψ -symmetric strategy profiles of the distributed agent form, where appropriate due to context adding or removing a tilde will change the game that strategy refers to (i.e. if b is strategy profile for the original game \tilde{b} is the corresponding Ψ -symmetric strategy profile for the distributed agent form and visa versa).

4.3 Final Proof

The following lemma will help form a bridge between utility of the distributed agent form and the original game.

Lemma 1. *Let b be a Ψ -symmetric strategy profile for the original game, let μ be a belief consistent with b , let X be an information set with positive prob-*

absentmindedness. However, one advantage of using the subpartition based on frontiers is that the distributed agent form reduces to the agent form in games without absentmindedness as mentioned.

ability of being reached under b , and let $i = (X, j)$ be an agent assigned to X . Furthermore, let s and s' be distributions over $A(X)$ and let $\ell = P(X)$. Then the following two statements are equivalent:

$$\begin{aligned} \tilde{u}_i(s, \tilde{b}_{-i}) &\geq \tilde{u}_i(s', \tilde{b}_{-i}) & (1) \\ DSU_\ell(b; X, s, \mu) &\geq DSU_\ell(b; X, s', \mu) & (2) \end{aligned}$$

Proof. See Appendix. \square

This bridge lemma can be used to give a characterization of the DPE using the distributed agent form.

Proposition 7. *A strategy profile is a distributed perfect equilibrium if and only if it corresponds to a Ψ -symmetric perfect equilibrium of the distributed agent form.*

Proof. See Appendix. \square

We now have all the tools needed to prove Theorem 1.

of Theorem 1. By Proposition 6 we know that a Ψ -symmetric perfect equilibrium exists for the distributed agent form and by Proposition 7 this must be a DPE of the original game. Therefore every game has a distributed perfect equilibrium. \square

5 Collapse to Standard Notions

We now present our final set of results: the collapse of our four equilibrium notions, DAE, DNE, DSE, and DPE, to their standard versions in games of perfect recall.

Before tackling the collapse of DAE to AE we present a characterization in terms of the distributed agent form as we did for DPE.

Proposition 8. *A strategy profile is a DAE if and only if it corresponds to a Ψ -symmetric Nash equilibrium of the distributed agent form.*

Proof. See Appendix. \square

The collapse of DAE to AE follows quickly from this proposition.

Proposition 9. *Let Γ be a game with perfect recall. A strategy profile is a DAE if and only if it is a Nash equilibrium of the agent form of Γ .*

Proof. See Appendix. \square

Proposition 10. *Let Γ be a game with perfect recall. A strategy profile is a DPE if and only if it is a perfect equilibrium of Γ .*

Proof. See Appendix. \square

Proposition 11. *Let Γ be a game with perfect recall. A strategy profile is a DSE if and only if it is a sequential equilibrium of Γ .*

Proof. See Appendix. □

We now turn to the collapse of the last distributed equilibrium concept, DNE.

Proposition 12. *Let Γ be a game with perfect recall. A strategy profile is a DNE if and only if it is a Nash equilibrium of Γ .*

Proof. See Appendix. □

6 Connection to Previous Work

Now that we have presented our main results, we can explain their connection to previous research. As was said in the introduction, our work builds on this previous work very directly. To properly assign the credit, it is useful to expand the map of solution concepts from Figure 2 into the one in Figure 5.

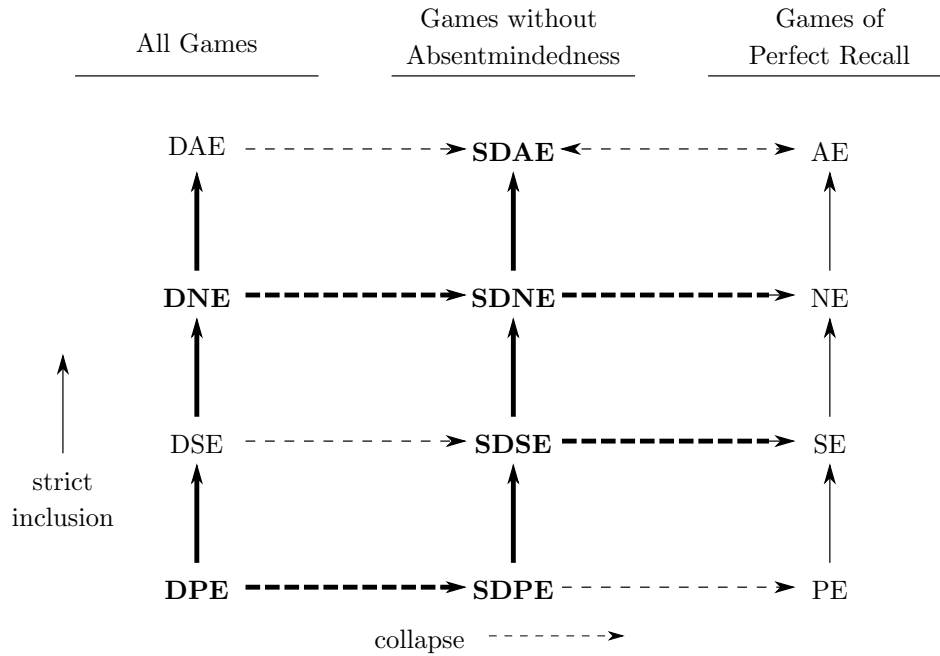


Figure 5: An expanded map of solution concepts for extensive form games. **Bold** solution concepts and arrows indicate our contributions.

In this expanded picture we add a third column, for the class of games that may exhibit imperfect recall but not absent-mindedness (that is, a player in an

information set may not recall her entire game experience, but will never visit that information set more than once).

We populate this column with a new set of novel equilibrium concepts. These concepts (which we denote with an “SD” moniker for “semi-distributed”) are very similar to their distributed (“D”) versions. Both reflect an ex-interim perspective; in both we consider each node in turn, perform a payoff calculation with respect to that node given a certain strategy profile, and take an expectation with respect to some belief function. The difference is in the calculation. In the D version, as we discussed, we assume a one-time deviation, assuming all other nodes (histories) obey the given strategy profile. In the SD version, we assume a deviation at that node as well as all nodes downstream from it in the information set.

More formally, given a player i , a strategy b_i , an information set X assigned to player i , a history $h \in X$, and a deviation $s \in \Delta(A(X))$, we can define a *persistent-deviation phantom strategy*, denoted $b_i[h//s]$, for a player i history h' (in information set X') by

$$b_i[h//s](h') = \begin{cases} s & : \text{If } h' \in X \text{ and } h \leq h' \\ b_i(X') & : \text{otherwise} \end{cases}$$

The definition of SD-solution concepts are identical to the D ones, except the notion of single deviation is replaced by that of persistent deviation. Specifically, where D concepts use distributed subjective utility, will use semi-distributed subjective utility which is defined for a player i , a strategy profile b , an information set X , a deviation $s \in \Delta(A(X))$, and a belief μ by

$$SDSU_i(b; X, s, \mu) = \sum_{h \in X} \mu(X)(h) u_i((b_i[h//s], b_{-i})|h)$$

As an explicit example of SD-based solution concepts, we define Semi-Distributed Agent Equilibrium.

Definition 16 (Semi-Distributed Agent Equilibrium). *A strategy b is a semi-distributed agent equilibrium if there exists a belief μ consistent with b such that for every information set X assigned to player i and reached with positive probability, and for every distribution over actions in X , $s \in \Delta(A(X))$*

$$SDSU_i(b; X, s, \mu) \leq SU_i(b; X, \mu)$$

Notice that the only difference between DAE and SDAE is that the $DSU_i(b; X, s, \mu)$ term has been replaced with $SDSU_i(b; X, s, \mu)$. For this reason, it is easy to see that in games without absentmindedness our distributed concepts collapse to their semi-distributed counterparts. On the other hand, the absentminded driver game described by Piccione and Rubinstein cleanly distinguishes the distributed concepts from the semi-distributed ones (i.e. all distributed concepts prescribe one strategy, while the semi-distributed concepts yield a different

one⁸).

In Figure 5, the **bold** solution concepts and (subset or collapse) relationships between them denote our novel contributions; the rest are either produced by previous research, or easily obtained from it. The following list briefly acknowledges the sources of these previous contributions; we discuss a few of them in more detail below.

- NE defined by Nash [12].
- AE defined by Kuhn [10].
- PE defined by Selten [14].
- SE defined by Kreps and Wilson [9].
- DAE is a straightforward multiagent generalization of Modified Multiselves Consistency as defined by Piccione and Rubinstein [13].
- DSE is a straightforward multiagent generalization of Modified Multiselves Sequential Equilibrium as defined by Battigalli [3].
- $NE \subseteq AE$ shown by Kuhn [10].
- $PE \subseteq NE$ shown by Selten [14].
- SDPE collapse to PE shown by Selten [14].
- $PE \subseteq SE$ shown by Kreps and Wilson [9].
- $SE \subseteq NE$ shown by Kreps and Wilson [9].
- $DSE \subseteq DAE$ shown by Battigalli [3].
- DAE collapse to SDAE shown by Battigalli [3].
- DSE collapse to SDSE shown by Battigalli [3].

Notice that in Figure 5 there are a few non-bold arrows that either originate or end in bold solution concepts. For example, we say that Battigalli showed the collapse from DAE to SDAE and from DSE to SDSE. In reality, Battigalli showed in games without absentmindedness that DAE collapses to AE and DSE collapses to SE of the agent form. However, it makes sense to attribute these results to others because it is trivial to show that SDAE is equivalent to AE in games without absentmindedness and SDSE is trivially equivalent to SE of the agent form for all games. Likewise, we say that Selten showed the collapse from SDPE to PE, when he actually showed the equivalence of PE and PE of the agent form in games with perfect recall. Again, our claim is justified by the

⁸Specifically, the distributed concepts give the same solution as Piccione and Rubinstein's Modified Multiselves Consistency (which is also the ex-ante optimum), while the semi-distributed concepts give the same solution as Piccione and Rubinstein's Time Consistency [13].

trivial observation that SDPE is equivalent to PE of the agent form for games without absentmindedness.

As mentioned in the introduction, this work is a natural outgrowth from a series of papers sparked by Piccione and Rubinstein. Piccione and Rubinstein initially defined two equilibrium notions for games⁹ of imperfect recall [13]. The first –*time consistency*– is similar to our notion of SDAE. It does not play a major role in our story, nor in the previous literature; Piccione and Rubinstein themselves, as well as the literature that followed them, found this notion problematic, for a number of reasons. Their second definition, however –*modified multiselves consistency* or MMCS – seemed less problematic. MMCS took hold in the ensuing literature; for example, Aumann, Hart, and Perry explicitly advocate the mathematically equivalent notion of *action optimality* [1]. MMCS is central to our paper, as DAE is its straightforward generalization to the multiagent case.

The previous literature establishes several important properties of MMCS. Piccione and Rubinstein themselves show that all optimal solutions to the extensive decision problem must be MMCS. Still focusing on the single-agent case, Battigalli [3] shows that, when restricted to games without absentmindedness, a MMCS is a Nash equilibrium of the agent form of the game. Among other things Battigalli defines *modified multiselves sequential equilibrium* (MMSE). Our definition of DSE is a straightforward generalization of MMSE to the multiagent case.

These two papers – by Piccione and Rubinstein, and by Battigalli – are the ones we drew on the most, in addition to the seminal papers by Nash, Kuhn, Selten, and Kreps and Wilson. In addition, we have benefited from the insights in the papers by Gilboa and by Halpern and Pass [4, 6]. Overall, though, in order to synthesize the pictures depicted in Figures 2 and 5 we have needed to not only modify and extend the definitions, but add more of our own (the central ones being DPE and DNE). Most of the specific (inclusion and collapse) relations are novel, and most fundamentally so is the existence result (of DPE, and therefore also of all the weaker notions). As mentioned in the introduction, the fact that existence was not addressed in the previous literature is not surprising, since that literature focused primarily on the single-agent case, in which existence is trivial.

7 Final Thoughts

We have extended the classical scheme of solution concepts for (finite) extensive-form games with perfect recall to the full class of games, including games with imperfect recall. Specifically, we defined the four notions of Distributed Agent Equilibrium (DAE), Distributed Nash Equilibrium (DNE), Distributed Sequential Equilibrium (DSE) and Distributed Perfect Equilibrium (DPE) as distributed versions of their classical counterparts. We showed that (like the

⁹Although strictly speaking they only addressed single-agent games, or decision problems.

classical notions) the distributed notions form a strict hierarchy, that they exist in every game, and that they collapse to their classical counterparts in the special class of games with imperfect recall.

In developing the scheme we defined novel notions such as the distributed agent form, phantom strategies, and partial symmetry, which may merit interest beyond the specific results they enable in this paper. In general, this work suggests a deeper structure of games which underlies the classical view of them, somewhat akin to the way in which quantum physics underlies classical mechanics.¹⁰ It suggests that an individual decision maker can be broken down into even more basic units than “agents” or “selves” corresponding to information sets; within a given state of knowledge (the information set) the decision maker is made up of multiple – phantom, counterfactual – decision makers, each operating in smaller contexts, as small as a single node.

Our results have made use of certain restricting assumptions. These include the probabilistic assumption of, given a strategy profile, setting the subjective probability (belief) of a node within an information set proportionally to its probability of being visited under the given strategy profile. This assumption, which we inherit from previous models (in particular, Piccione and Rubinstein’s), is substantive, and it is interesting to contemplate whether it can be relaxed while retaining some or all of our results. Similarly, we have made a specific use of phantom strategies, namely those corresponding to frontiers of an information set. Our essential results also hold in the context of other subpartitions of information sets, so long as those do not exhibit absentmindedness. It is interesting to see whether these alternate partitions can give rise to interesting theories.

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¹⁰To be clear, the analogy is merely suggestive; we don’t mean to imply any connection to quantum game theory or to physics in general.

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APPENDIX

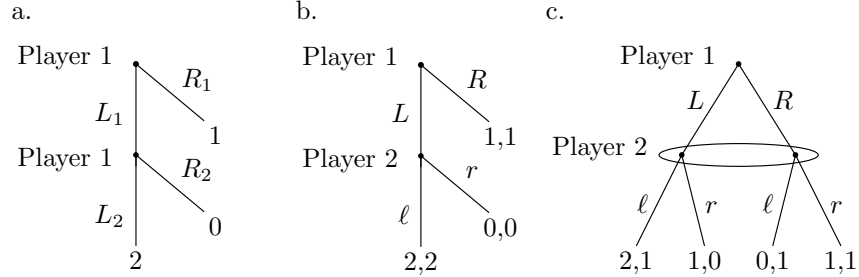


Figure 6: In game a. (R_1, R_2) is an AE but not an NE, in game b. (R, r) is an NE but not an SE, and in game c. (R, r) is an SE but not a PE. Thus all inclusion relations are strict.

of Proposition 1. Let b be a DPE with corresponding sequence of perturbations η_1, η_2, \dots and corresponding sequence of perturbed DAE b^1, b^2, \dots . Given a perturbation η , an information set X (assigned to player i), and a distribution $s \in \Delta(A(X))$, consider the following function

$$\pi_\eta(X, s) = \sum_{a \in A(X)} \eta(X)(a) \cdot a + \left(1 - \sum_{a \in A(X)} \eta(X)(a)\right) \cdot s$$

As can easily be seen, this function shifts the distribution s according to η . More importantly if b is a perturbed strategy profile for perturbation η then for all $h \in X$, $b[h/\pi_\eta(X, s)]$ is a valid perturbed phantom strategy profile. Moreover, $\pi_\eta(X, s)$ is clearly continuous with respect to η and if $\eta = 0$ it maps back to the initial distribution. Because of this and the fact that $\eta_k \rightarrow 0$, given $\epsilon > 0$ we can choose a K such that for all $k \geq K$ for all strategy profiles b , for all information sets X (assigned to player i), and for all $s \in \Delta(A(X))$.

$$|u_i(b[x/\pi_{\eta_k}(s, X)]|x) - u_i(b[x/s]|x)| < \epsilon$$

Now let $\epsilon_1, \epsilon_2, \dots$ be a sequence of positive real numbers that converge to 0. Let ℓ be an arbitrary positive integer. Using the above argument and the definition of distributed subjective utility, we can choose a K_ℓ such that for all $k \geq K_\ell$, for all beliefs μ consistent with b^k , for all information sets X (assigned to player i), and for all $s \in \Delta(A(X))$.

$$|DSU_i(b^k; X, \pi_{\eta_k}(X, s), \mu) - DSU_i(b^k; X, s, \mu)| < \epsilon$$

Furthermore, because b^k is a DAE of (Γ, η_k) and because $b^k[h/\pi_{\eta_k}(X, s)]$ is a valid perturbed phantom strategy profile for all $h \in X$, we have

$$DSU_i(b^k; X, \pi_{\eta_k}(X, s), \mu) \leq SU_i(b^k; X, \mu)$$

combining these two inequalities yields

$$DSU_i(b^k; X, s, \mu) \leq SU_i(b^k; X, \mu) + \epsilon$$

Without loss of generality, assume that for positive integers $\ell < m$ implies $K_\ell < K_m$. This provision ensures that the sequence of completely mixed strategies b_{K_1}, b_{K_2}, \dots converges to b . Thus this sequence, along with $\epsilon_1, \epsilon_2, \dots$, satisfies the necessary conditions to prove that b must be a DSE. \square

of Proposition 2. First, note that a DSE must be a DAE and thus the fact that a DNE is defined to be a DAE plays no part in the remaining proof. This fact follows easily from the definitions and will not be explicitly proven.

Now, Let b be a DSE with corresponding sequences b^1, b^2, \dots and $\epsilon_1, \epsilon_2, \dots$. Since strategy equivalence is, as its name suggests an equivalence relation, we have for all i $b_i \cong_b b_i$. Furthermore, for player i let us choose the sequences b_i^1, b_i^2, \dots (the sequence of player i strategies drawn from the original sequence of strategy profiles) and $\epsilon_1, \epsilon_2, \dots$. It is clear from the definition of DSE that for each player i , b_i and these two sequences satisfy the definition of DNE. \square

of Proposition 4. Let us consider the set-valued best response function $B : \Sigma \rightarrow 2^\Sigma$. It is well known that this function satisfies the requirements of Kakutani's fixed point theorem [11]. It is sufficient to show that for all $b \in \Sigma_G$, $B(b) \cap \Sigma_G$ is convex and non-empty. Nash already noted that convex combinations of symmetric strategies are also symmetric (all with respect to a particular symmetry χ) [12]. Thus the set of χ -symmetric strategies, Σ_χ is convex and because $\Sigma_G = \bigcap_{\chi \in G} \Sigma_\chi$, Σ_G is convex as well. Finally, because $B(b)$ is convex for all $b \in \Sigma$, $B(b) \cap \Sigma_G$ is convex.

Let b' be a best response to some G -symmetric strategy profile b (i.e. $b' \in B(b)$). Notice that for $\chi \in G$ that $\chi(b')$ is also a best response to b since b is G -symmetric. Now consider the following strategy

$$b'' = \sum_{\chi \in G} \frac{\chi(b')}{|G|}$$

Clearly b'' is a G -symmetric strategy profile and because b'' is the average of best responses it is a best response. Hence $B(b) \cap \Sigma_G$ is non-empty. \square

of Proposition 5. The argument used to prove proposition 4 is valid provided that for any perturbed strategy profile b and any symmetry $\chi \in G$, $\chi(b)$ must be a valid perturbed strategy profile. Fortunately, this follows immediately from the fact that η is a G -symmetric perturbation and thus

$$\begin{aligned} b &\geq \eta \\ \implies \chi(b) &\geq \chi(\eta) \\ \implies \chi(b) &\geq \eta \end{aligned}$$

\square

of Proposition 6. Choose an arbitrary infinitely long sequence of G -symmetric perturbations η_1, η_2, \dots that converge to 0. For each perturbation η_k choose a G -symmetric Nash equilibrium b^k (which exists by Proposition 5). By the compactness of Σ_G , there exist a subsequence of the sequence b^1, b^2, \dots that converges to a G -symmetric strategy profile. It is clear this strategy profile satisfies the criterion of a G -symmetric perfect equilibrium. \square

of Lemma 1.

$$\begin{aligned}
& \tilde{u}_i(s, \tilde{b}_{-i}) \geq \tilde{u}_i(s', \tilde{b}_{-i}) \\
\iff & \frac{1}{D(X)} \sum_{j=1}^{D(X)} u_\ell(b[\hat{X}^j/s]) - \frac{1}{D(X)} \sum_{j=1}^{D(X)} u_\ell(b[\hat{X}^j/s']) \geq 0 \\
\iff & \sum_{j=1}^{D(X)} \sum_{z \in Z} \left(p_{b[\hat{X}^j/s]}(z) u_\ell(z) - p_{b[\hat{X}^j/s']} (z) u_\ell(z) \right) > 0 \\
\iff & \sum_{j=1}^{D(X)} \sum_{x \in \hat{X}^j} p_b(x) \sum_{z \in Z} \left(p_{b[x/s]}(z|x) u_\ell(z) - p_{b[x/s']} (z|x) u_\ell(z) \right) \geq 0 \\
\iff & \frac{1}{\sum_{x \in X} p_b(x)} \sum_{x \in X} p_b(x) \sum_{z \in Z} \left(p_{b[x/s]}(z|x) u_\ell(z) - p_{b[x/s']} (z|x) u_\ell(z) \right) \geq 0 \\
\iff & \sum_{x \in X} \mu_X(x) \sum_{z \in Z} \left(p_{b[x/s]}(z|x) u_\ell(z) - p_{b[x/s']} (z|x) u_\ell(z) \right) \geq 0 \\
\iff & \sum_{x \in X} \mu_X(x) \left(u_\ell(b[x/s]|x) - u_\ell(b[x/s']|x) \right) \geq 0 \\
\iff & DSU_\ell(b; X, s, \mu) \geq DSU_\ell(b; X, s', \mu)
\end{aligned}$$

\square

of Proposition 7. Consider the fact that bijective relationship between Ψ -symmetric strategy profiles of the distributed agent form and strategy profiles of the original game also holds for perturbations. That is to say, there is a bijective mapping between Ψ -symmetric perturbations of the distributed agent form and perturbations of the original game. As with strategy profiles, for a perturbation of the original game η , we will denote $\tilde{\eta}$ to be the corresponding Ψ -symmetric perturbation of the distributed agent form.

Now let b be a perturbed strategy profile of the original game, let X be an information set, and let $s = b(X)$. Notice that if statement 1 of Lemma 1 is true for \tilde{b} , s and arbitrary $i = (X, j)$ and perturbed deviation s' , then \tilde{b} is a Nash equilibrium of the perturbed distributed agent form. Moreover, it is a Ψ -symmetric Nash equilibrium since \tilde{b} is a Ψ -symmetric strategy profile. Likewise, notice that if statement 2 of Lemma 1 is true for b , X , s , $\ell = P(X)$ as given and for arbitrary perturbed deviation s' , then b is a DAE of the original perturbed game. Hence Lemma 1 implies that b is a DAE of a perturbed game (Γ, η) if and only if \tilde{b} is a Nash equilibrium of $(\tilde{\Gamma}, \tilde{\eta})$. It follows immediately from this that a strategy profile b is a DPE of a game Γ if and only if it is a Ψ -symmetric perfect equilibrium of the distributed agent form of Γ . \square

of Proposition 8. Let b, X, i, ℓ, s , and s' be chosen as in the preamble of Lemma 1.

Note that if we consider information sets X (and an agent i assigned to X) with zero probability of being reached when using b all payoffs are the same. In

other words, for all pairs of player i strategies s, s' we have

$$\forall h \in X, p_b(h) = 0 \Rightarrow \tilde{u}_i(s, \tilde{b}_{-i}) = \tilde{u}_i(s', \tilde{b}_{-i})$$

Thus if statement 1 of Lemma 1 is true for all X (and any agent i assigned to X) with positive probability of being reached it is equivalent to stating that \tilde{b} is a Ψ -symmetric Nash equilibrium of the distributed agent form since \tilde{b} is a Ψ -symmetric strategy profile. Furthermore, if statement 2 is true for all X with positive probability of being reached it is equivalent to stating that b is a DAE. Therefore \tilde{b} is a Ψ -symmetric Nash equilibrium if and only if b is a DAE. \square

of Proposition 9. Proposition 8 tells us that a strategy profile is a DAE if and only if it is a Ψ -symmetric Nash of the distributed agent form. In games that don't have absentmindedness (which of course include games of perfect recall) Ψ is the trivial subgroup and the distributed agent form collapses to the standard agent form of the game. Thus in games without absentmindedness a strategy profile is a DAE if and only if it is a Nash equilibrium of the agent form and hence an agent equilibrium. \square

of Proposition 10. By a similar argument to the one used in Proposition 9, Proposition 7 implies that in games without absentmindedness a strategy profile is a DPE if and only if it is a perfect equilibrium of the agent form. By Theorem 4 of [14], for games of perfect recall a strategy profile is a perfect equilibrium of the agent form if and only if it is a PE.¹¹ \square

of Proposition 11. Before we tackle this proposition head on we must first establish a couple of lemmas. Our first lemma requires a couple new definitions. First, we will define a *uniform-deviation strategy* much like we defined single-deviation phantom strategies. Namely, given a player i , a strategy b_i , an information set X assigned to player i , and a deviation distribution $s \in \Delta(A(X))$, we denote the single-deviation strategy $b_i[X/s]$ and formally define it given an information set X' by

$$b_i[X/s](X') = \begin{cases} s & : \text{If } X' = X \\ b_i(X') & : \text{otherwise} \end{cases}$$

Second is a slight variant of subjective utility that explicitly includes this type of deviation. Namely, given a strategy profile b , a belief μ consistent with b , a player i , an information set X , and a deviation $s \in \Delta(A(X))$ the subjective utility of this deviation is given by

$$SU_i(b; X, s, \mu) = SU_i((b_i[X/s], b_{-i}); X, \mu)$$

Lemma 2. *Let b be a strategy profile of a game without absentmindedness and μ be a belief consistent with b . Furthermore, let i be a player, X be an information set, and let $s \in \Delta(A(X))$. Then the following statement is true:*

$$SU_i(b; X, s, \mu) = DSU_i(b; X, s, \mu)$$

¹¹This was actually shown for the agent *normal* form of a game, but because the Nash equilibria of the agent form and the agent normal form coincide, even for perturbed variants, this is equivalent to what Selten stated.

Proof. The proof of this lemma follows easily from the fact that for any $h \in X$ $u_i(b[X/s]|h) = u_i(b[h/s]|h)$ in games without absentmindedness. \square

Lemma 3. *Let b be a strategy profile of a game Γ with perfect recall and μ be a belief consistent with b . Let X be an information set (controlled by player i) and J be the set of information sets that come after X that are still controlled by player i (Note that this is well defined because Γ does not have absentmindedness). Given ϵ_Y such that for all $Y \in J$ and for all strategy profiles b'*

$$SU_i((b'_i, b_{-i}); Y, \mu) \leq SU_i(b; Y, \mu) + \epsilon_Y$$

Also given ϵ_X such that for all $a \in A(X)$

$$SU_i(b; X, a, \mu) \leq SU_i(b; X, \mu) + \epsilon_X$$

Then for all strategy profiles b'

$$SU_i((b'_i, b_{-i}); X, \mu) \leq SU_i(b; X, \mu) + \epsilon_X + \max_{Y \in J} \epsilon_Y$$

where max of the empty set evaluates to 0.

Proof. Assume that information set X has been reached and action $a \in A(X)$ taken. Now consider the set of information sets \hat{J} that are reached first after action a is taken (again this is unambiguous since Γ has perfect recall). $Y \in \hat{J}$ will have a probability of $p(Y|X, b)$ of being reached. Moreover because this is a game of perfect recall $\sum_{Y \in \hat{J}} p(Y|X, b) \leq 1$. Furthermore, because for all strategy profiles b'

$$SU_i((b'_i, b_{-i}^k); Y, \mu) \leq SU_i(b; Y, \mu) + \epsilon_Y$$

the most that changing strategy at information set Y can contribute to increasing the payoff is $p_b(Y|X)\epsilon_Y$. Therefore we have

$$\begin{aligned} SU_i((b'_i[X/a], b_{-i}); X, \mu) &\leq SU_i(b[X/a]; X, \mu) + \sum_{Y \in \hat{J}} p_b(Y|X)\epsilon_Y \\ \implies SU_i((b'_i[X/a], b_{-i}); X, \mu) &\leq SU_i(b[X/a]; X, \mu) + \max_{Y \in J} \epsilon_Y \\ \implies SU_i((b'_i[X/a], b_{-i}); X, \mu) &\leq SU_i(b; X, \mu) + \epsilon_X + \max_{Y \in J} \epsilon_Y \end{aligned}$$

For a game with perfect recall there is always a pure strategy best response at every information set. As a result, it is sufficient to consider deviations of the form $b'_i[X/a]$ instead of b'_i in the above inequality. Taking this into account leads to the statement

$$SU_i((b'_i, b_{-i}); X, \mu) \leq SU_i(b; X, \mu) + \epsilon_X + \max_{Y \in J} \epsilon_Y$$

as desired. \square

Lemma 4. *Let Γ be a game with perfect recall. b is a sequential equilibrium if and only if it is a sequential equilibrium of the agent form.*

Proof. \implies Clearly any sequential equilibrium is a sequential equilibrium of the agent game since it expands the set of strategies under consideration at every information set.

\Leftarrow Let $\epsilon > 0$ and let b be a sequential equilibrium of the agent form with associated sequences b^1, b^2, \dots and $\epsilon_1, \epsilon_2, \dots$. Furthermore, let μ be a belief consistent with b . Finally, let m be the most number of decisions made by any one player in the course of play (i.e. the number of information sets passed through). Because $\epsilon_k \rightarrow 0$ we can choose a k such that $\epsilon_k \leq \epsilon/m$.

For a particular information set X (assigned to player i), let d be the maximum number of moves left for player i including a decision at X . An inductive application of Lemma 3 yields that for all strategy profiles b'

$$SU_i((b'_i, b^k_{-i}); X, \mu) \leq SU_i(b^k; X, \mu) + d\epsilon/m$$

Because $d \leq m$ we have

$$SU_i((b'_i, b^k_{-i}); X, \mu) \leq SU_i(b^k; X, \mu) + \epsilon$$

Furthermore, if a sequence of such ϵ are chosen that converge to 0, the resulting sequence of completely mixed b_k will converge to b . Thus b is a sequential equilibrium. \square

We can now return to the proof of proposition 11.

First notice that if we replace the DSU term with the comparable SU term defined for Lemma 2 in the definition of DSE then the resulting equilibrium concepts is the same as a sequential equilibrium of the agent form. This observation combined with Lemma 2 implies that a DSE must collapse to a sequential equilibrium of the agent form in games without absentmindedness.¹² By Lemma 4 this implies that DSE collapses to SE in games of perfect recall. \square

of Proposition 12. First note that the fact that a DNE is stipulated to be a DAE will be irrelevant to this proof. This is because DAE collapse to AE in games of perfect recall, as shown in Proposition 9, and all NE are AE in games of perfect recall. Thus if we show that DNE collapse to NE without using the fact that all DNE are stipulated to be DAE (as will be shown), the fact that it is a DAE is redundant for games of perfect recall.

It follows easily from the definition of a DNE that a strategy profile b is a DNE if and only if for every player i there exists a DSE of player i 's induced decision problem that is equivalent to player i 's strategy in b (where player i 's induced decision problem is the game where all other players actions are regarded as fixed acts of Nature). By Proposition 11 this leads to the following characterization of a DNE. A strategy profile b is a DNE if for every player i there exists a SE of player i 's induced decision problem, b'_i , such that $b_i \cong_b b'_i$.

¹²This fact was first observed without proof by Battigalli for single agent games [3].

Furthermore, a strategy profile is a Nash equilibrium if and only if every player is playing an optimum in her respective decision problem. Thus to prove this proposition, it is sufficient to show that a strategy is an optimum of an extensive decision problem if and only if it is equivalent to a sequential equilibrium.

\Leftarrow An SE involves choosing optimal strategies at all of the initial information sets (that is only Nature can precede them) and thus for games of perfect recall it is clearly an optimal strategy ex-ante. Furthermore, because two equivalent strategies induce the same distribution over histories they must have the same payoff and if one is optimal, the other must be as well. Hence if a strategy is equivalent to a sequential equilibrium it is optimal.

\Rightarrow It follows from Battigalli's work (Propositions 3.5 and 4.3 [3]) that for every optimal strategy b to a decision problem, there exists a DSE b' such that $b \cong b'$.¹³ Because all DSE collapse to SE in games of perfect recall, it follows that for every optimal strategy there exists an equivalent SE. \square

¹³Battigalli actually showed this for what he calls a Modified Multiselves Sequential Equilibrium, which amounts to a single player version of a DSE.