Cyclical Behavior in Two-Speed Evolutionary Game Environments

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March 28, 2013

Abstract

We present a model of two-speed evolution in a generalized coordination game, in which preferences slowly adjust to changes in the aggregate behavior of the population. Under best-response and replicator dynamics the unique steady state of joint dynamics of strategies and preferences is repelling. The same result holds for logit dynamics with small noise levels, whereas for large noises the steady state becomes a sink.

JEL CODES: C72, C73.

1 Introduction

Economic models in evolutionary game theory study the dynamics of human behavior. A model consists of a game that agents in a certain population are matched to play, a payoff function that describes agents' preferences, and a revision protocol - a rule according to which players receive and act upon opportunities to revise their strategies in the game. Analysis of such models allows one to describe the evolution of the aggregate behavior of the population and to make predictions about the long-run behavior for a given initial population state.

Evolutionary models relax some of the assumptions that are common to the majority of standard models in game theory. Instead of having perfect foresight, the agents are only concerned with their short-term goals, and despite the fact that the agents are matched to play the game repeatedly, the play is anonymous and players cannot acquire a reputation. Under such circumstances the notion of time earns great significance: We are concerned not only with the outcome of the interaction within the population but also with the speed of evolution of agents' behavior. If the speed of convergence towards an equilibrium is relatively slow, it might be reasonable to assume that instead of being constant, preferences of agents are also subject to change but at even slower pace.

The model presented in this paper assumes a continuum of agents with homogenous preferences. These preferences adjust to changes in the aggregate behavior of the population. The agents are matched to play a 2x2 coordination game and are only allowed to choose pure strategies. Opportunities to revise strategies arrive randomly according to a Poisson process. Preferences evolve in an equilibrating manner as to offset the natural tendency of the population to coordinate on one of the two pure strategy equilibria of the underlying game. While there are still benefits to coordination, the more popular strategy loses its appeal over time, forcing the agents to switch away from it. Therefore neither pure strategy can be a steady state of the joint dynamics of strategies and preferences, and the only steady states are the ones for which the change in payoffs is the same for all strategies. Given that the payoff change is linear in aggregate behavior, for any revision protocol there exists a unique steady state. For best response, replicator, and logit dynamics with small noise levels the steady state is repelling. At large noise levels it becomes a sink. The difference in results in the last case can be explained by the fact that the higher the noise level the slower the rate of change of the aggregate behavior, thus at high noise levels equilibrating effect will completely dominate coordination benefits.

Previous models of two speed evolution, developed in Sandholm (2001) and Possajennikov (2005), suggest an alternative way of looking at the evolution of preferences. Instead of considering a homogenous population, both authors assume that each agent in the population has an idiosyncratic bias toward one of the two strategies so that preferences may differ from actual payoffs. Sandholm assumes that the distribution of biases is continuous. As the play evolves, preferences determine individual behavior, whereas aggregate behavior shapes the preference distribution: The proportion of players with biases toward less profitable strategy decreases. Sandholm shows that the solution trajectory of the two speed dynamic exists and is unique and that aggregate behavior changes continuously in equilibration games and exhibits jumps in coordination games. Possajennikov analyzes stability of preferences in a model with discrete set of biases under two informational assumptions. In complete information case the agents know the preferences of the opponent in a match; under incomplete information they only know the distribution of preferences. Under best response dynamics in the former case only fitness-efficient strategy profiles can be stable, whereas in the latter case only Nash equilibria of the fitness game are stable.

Two key aspects distinguish our model from those of Sandholm and Possajennikov. Firstly, we require the agents to have homogenous rather than heterogenous preferences and allow the payoff function to change over time. Secondly, while in Sandholm the population maintains equilibrium play at every moment in time because behavior adjusts infinitely faster than preferences, we assume a finite relative speed of adjustment, and this leads to instability of rest points of the joint dynamics of strategies and preferences.

The result of our paper can also be viewed in the light of fashion cycles literature, such as Pesendorfer (1995) and Matsuyama (1993). In the former paper, consumers with hidden characteristics get involved in a matching game, and fashion cycles emerge when fashion is used as a signaling device that helps distinguish between types. In the latter paper, agents in a population consisting of conformists and nonconformists are matched to play a symmetric strategy game in which conformists benefit from coordination, whereas nonconformists prefer noncoordination. Cyclical behavior is observed under certain conditions on proportion of conformists and intergroup matches. In our model agents act as conformists or nonconformists depending on the situation in which they receive a revision opportunity.

We proceed by introducing the game the agents are matched to play and describing joint behavior of strategies and preferences under best response, replicator, and and logit dynamic. The paper concludes with a discussion of a generalized model.

2 The Model

We begin by describing how players interact and what preferences they have. Then we introduce the rule according to which the preferences evolve and end up with a system of differential equations that describe the dynamics of the process. After laying out the model we investigate the existence and stability of steady states under different revision protocols.

2.1 Players and Preferences

There is a continuous population of agents with homogenous preferences. The agents are randomly matched to play a symmetric two strategy game. Time is also continuous, and the interaction starts at $t = 0$, at which point the game is described by a bimatrix below with $a > c$ and $d > b$. Agents are only allowed to play pure strategies and randomly receive opportunities to revise them. The revision rules vary for different protocols and will be discussed in corresponding sections of the paper.

		2	
		L	R
1	L	a, a	b, c
	R	c, b	d, d

Denote the strategies L and R for 'Left' and 'Right', correspondingly. If x is the proportion of agents that choose to play strategy L, then the payoff to each strategy can be expressed as

$$f_L(x) = ax + b(1-x)$$
$$f_R(x) = cx + d(1-x)$$

and the mixed equilibrium y of the game solves $f_L(y) - f_R(y) = 0$, yielding $y = \frac{d-b}{a-b-c+d}$. Despite the fact that the agents cannot change their strategies instantly, the payoffs to strategies only reflect their expected payoff at the present time. For a fixed payoff matrix payoffs only vary with the aggregate behavior x, but our next step is to allow the payoff matrix to change in time, too. Initially $y \in (0, 1)$ and the interaction is a coordination game, but as we allow payoff changes the type of the game might change as well, and the mixed equilibrium might leave the unit interval if one of the strategies becomes payoff dominant.

2.2 Evolution of Preferences

Instead of allowing multiple types of preferences and investigating which of them survive natural selection in a game with fixed payoffs, we would like to see what happens to a single type of preferences if payoffs in a game are dependent on aggregate behavior. In other words, while it is commonplace that individual decisions are determined by the environment, it is not always taken into account that the environment can be shaped by individual decisions. For instance, increased use of air conditioning as a reaction to global warming indirectly reinforces it. Reinforcement effects are, however, of little interest, since they only affect the speed but not the direction of change of aggregate behavior. We will restrict attention to situations in which the decisions will have an equilibrating impact on the environment. An example that can motivate such a setup is as follows: There are two substitute resources, and each player has to decide which resource to utilize. There are advantages to coordination on the same resource; however its over-utilization adversely affects benefits of its use. The former effect impacts agents' utility through strategy adjustments, whereas the latter changes payoffs in the underlying game and might work in the direction opposite to that of the former. Thus over time the resource that is more utilized becomes less attractive.

We begin the analysis by letting the payoff change be a linear function of the population state. Let A denote the initial payoff matrix and $\dot{A}$ be the matrix describing the change in A. Assume that the growth rate to payoffs within a strategy is the same:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \dot{A} = r \begin{pmatrix} \hat{x} - x & \hat{x} - x \\ \hat{x} - k(1-x) & \hat{x} - k(1-x) \end{pmatrix}$$

Parameter $r \geq 0$ relates the speed of payoff change to the speed of revision of strategies. Case $r = 0$ corresponds to standard evolutionary models when payoffs do not change at all. Parameter $k > 0$ defines relative 'wearing' of a resource. If $k > 1$, then payoffs to strategy *Right* change faster. Constant term $\bar{x}$ such that $0 < \bar{x} < \min\{1, k\}$ guarantees that the payoff to a less popular strategy increases or, in other words, that the less utilized resource has the ability to recover. If one plugs in $x = 1$ and $x = 0$ one can see that once the aggregate behavior is close to coordination on strategy $Left$, strategy *Right* becomes relatively more attractive and vice versa.

$$\dot{A}(1) = p \begin{pmatrix} \hat{x} - 1 & \hat{x} - 1 \\ \hat{x} & \hat{x} \end{pmatrix} \quad \dot{A}(0) = p \begin{pmatrix} \hat{x} & \hat{x} \\ \hat{x} - k & \hat{x} - k \end{pmatrix}$$

Just like the equilibration games' case in Sandholm (2001), the result will be determined by the interaction of coordination and equilibration effects. The magnitude of the coordination effect will depend on the properties of the revision protocol, whereas the equilibration effect will be the stronger the closer the population state is to 0 or 1. Equality of the two effects will define a rest point of the dynamic of aggregate behavior.

2.3 Joint Dynamics of Strategies and Preferences

No matter what revision protocol is used, we would like the agents in the model to switch away from suboptimal strategies. Therefore we need to track down the changes in the mixed equilibrium of the game since it corresponds to the state in which payoffs to both strategies are the same. Hence our next step is to derive its law of motion:

$$\dot{y} = \frac{d}{dt}\left[\frac{d-b}{a-b-c+d}\right] = \frac{\dot{d} - \dot{b}}{a-b-c+d} - \frac{(d-b)(\dot{a} - \dot{b} - \dot{c} + \dot{d})}{(a-b-c+d)^2}$$

We introduce a new variable $s = a - b - c + d = (a - c) + (d - b)$ that will measure the strength of the incentives to coordinate. Indeed, $(a - c)$ is the gain to coordination if agent's opponent plays $Left$, and $(d - b)$ is the gain if the opponent plays *Right*. Variable s allows one to connect the change in payoffs with the change in mixed equilibrium of the game. We are now able to describe the interactions in the model using a system of differential equations: First, changes in the mean dynamic of the aggregate behavior will depend on the mixed equilibrium; then changes in the mixed equilibrium will depend on the strength of the incentives to coordinate, which in turn will depend on the aggregate behavior:

$$\dot{x} = V(x, y, s)$$
$$\dot{y} = M(x, y, s)$$
$$\dot{s} = P(x, y, s)$$

We can simplify the general system by employing the assumptions from the previous section. By requiring same growth rates for payoffs within a strategy, we can guarantee that the incentives to coordinate remain unchanged ($\dot{s} = 0$ since $\dot{a} - \dot{b} = \dot{c} - \dot{d} = 0$). And by assuming linear dependence between aggregate behavior and payoff change, we can rewrite the gains to coordination on strategy *Right* as:

$$\dot{d} - \dot{b} = r[\hat{x} - k(1-x) - \hat{x} + x] = r[(1+k)x - k]$$

These relationships allow us to rewrite the law of motion for the mixed equilibrium as:

$$\dot{y} = \frac{\dot{d} - \dot{b}}{a - b - c + d} - \frac{(d-b)(\dot{a} - \dot{b} - \dot{c} + \dot{d})}{(a-b-c+d)^2} = \frac{\dot{d} - \dot{b}}{a - b - c + d} = \frac{r}{s}[(1+k)x - k]$$

And therefore the simplified system of equations will be as follows:

$$\dot{x} = V(x, y, s) \tag{1}$$
$$\dot{y} = \frac{r}{s}[(1+k)x - k] \tag{2}$$
$$\dot{s} = 0 \tag{3}$$

with some initial conditions x_0, y_0, and s_0. In the next three sections we will derive the explicit formulae for the mean dynamic of the aggregate behavior ($V(x, y, s)$) for best response, logit, and replicator dynamics, and investigate the stability of the resulting steady states.

3 Best-Response Dynamic

The best response dynamic, introduced by Gilboa and Matsui (1991), is a deterministic dynamic in which the only type of revision that occurs is the one in which players that are not playing the current best response switch to it. This dynamic requires the population state to be common knowledge, otherwise the agent who receives a revision opportunity will not be able to compare the available strategies. The obvious drawback of such a revision rule is that when the population state is the mixed equilibrium, there are multiple best responses, hence there can be multiple solution trajectories.

If preferences do not evolve ($r = 0$) this revision rule enables three types of behavior. If $x_0 > y_0$ there will be exponential decay toward the state $x = 1$. In other words, if initial bias is toward strategy *Left*, one should expect the population to coordinate on that strategy as $t \to \infty$. Same reasoning applies to the case in which strategy *Right* is initially more appealing: If $x_0 < y_0$, the population will move toward the state $x = 0$. Initial condition $x_0 = y_0$ gives rise to multiple solution trajectories since there are multiple best responses at that state. The system might spend arbitrary time at the mixed equilibrium before leaving it.

Since the agents that are not playing the best response switch to it with certainty, at each state different from mixed equilibrium the mean dynamics will be equal to the proportion of players not playing the the best response. In case $x = y$, any strategy is a best response, hence the dynamic can be any point from the interval $[-x, 1-x]$.

$$\dot{x} = \begin{cases} 1-x & \text{if} \quad x > y \\ [-x, 1-x] & \text{if} \quad x = y \\ -x & \text{if} \quad x < y \end{cases}$$

For any initial condition $x_0 \neq y_0$ one obtains the closed form solution for the dynamics on some interval as long as trajectories of the aggregate behavior and the mixed equilibrium do not intersect.

$$x = \begin{cases} 1-(x_0-1)e^{-t} & \text{if} \quad x_0 > y_0 \\ x_0 e^{-t} & \text{if} \quad x_0 < y_0 \end{cases} \tag{4}$$

If we integrate x over time we can also recover y on the same time interval:

$$y = \begin{cases} \frac{r}{s}t + \frac{r}{s}(1+k)(1-x_0)(e^{-t}-1) + y_0 & \text{if} \quad x_0 > y_0 \\ -\frac{r}{s}kt + \frac{r}{s}(1+k)x_0(1-e^{-t}) + y_0 & \text{if} \quad x_0 < y_0 \end{cases} \tag{5}$$

Given the equations (4) and (5) we can describe the behavior of the dynamic as long as the population state and the mixed equilibrium do not coincide. If $x_0 > y_0$, then strategy $Left$ is the only best response, and we should expect the population proportion to increase, as agents will be switching away from *Right*. But since strategy $Left$ becomes more utilized, at some point benefits to its use start to decrease, and the mixed equilibrium will start moving towards $Left$ as well. So, for $x_0 > y_0$ the population proportion is moving towards the state $x = 1$ and the mixed equilibrium is chasing it. If $x_0 < y_0$, both states move towards 0.

Now we need to investigate under which conditions the two trajectories intersect and what happens if they do so. It turns out that Achilles (y) always overtakes the tortoise (x).

Claim 1 *For $r > 0$ and any initial condition (x_0, y_0) there is time t: $0 < t < \infty$ for which $x = y$.*

Proof. First consider $x_0 > y_0$. Setting (4) equal to (5) in that case yields:

$$\begin{aligned} 1-(x_0-1)e^{-t} &= \frac{r}{s}t + \frac{r}{s}(1+k)(1-x_0)(e^{-t}-1) + y_0 \\ 1 - y_0 + \frac{r}{s}(1+k)(1-x_0) - \frac{r}{s}t &= [1 + \frac{r}{s}(1+k)](1-x_0)e^{-t} \\ \frac{1-y_0+\frac{r}{s}(1+k)(1-x_0)}{1-x_0+\frac{r}{s}(1+k)(1-x_0)} - \frac{\frac{r}{s}}{1-y_0+\frac{r}{s}(1+k)(1-x_0)}t &= e^{-t} \end{aligned}$$

The left-hand side is linear with the y-intercept greater than 1 and negative slope, the right-hand side is an exponent with negative power, so for $t > 0$ there will always exist a unique solution.
For $x_0 < y_0$ the resulting equation is similar:

$$\frac{y_0 + \frac{r}{s}(1+k)x_0}{x_0 + \frac{r}{s}(1+k)x_0} - \frac{\frac{r}{s}}{y_0 + \frac{r}{s}(1+k)x_0} t = e^{-t}$$

and the same reasoning applies. ■

The intuition for the proof is as follows: The population proportion x does not attain any pure equilibrium state in finite time. The speed of motion toward the desired state decays as the proportion gets closer to it as there are less agents who choose suboptimal strategy, whereas the mixed equilibrium adjustment gets faster the more players coordinate on the same strategy.

Case $x = y$ requires separate consideration. If $r = 0$, the population will spend arbitrary time at that state before leaving it. Agents are indifferent between the two strategies and randomly choose one to play whenever they get a revision opportunity. This process continues until a significant number of agents switch to the same strategy as a result of series of random events, and only one strategy retains as the best response. The population state escapes mixed equilibrium due to random fluctuations in x. If $r > 0$, then as long as $\dot{y} \neq 0$ the population state coincides with the mixed equilibrium for only one moment in time, and then payoffs adjustment forces the mixed equilibrium to change. Case $r > 0$ and $\dot{y} = 0$ is resolved in the same way as the one in which $r = 0$: If there are no payoff adjustments, the only way to escape $x = y$ is by waiting long enough for x to change.

Having these intuition in mind we can summarize the joint behavior of population proportion x and mixed equilibrium y under the best-response dynamic. From equation (2) we find out that the only rest point of the mixed equilibrium dynamic is the one at which $x = \frac{k}{k+1}$. If $x > \frac{k}{k+1}$, then $\dot{y} > 0$ and the mixed equilibrium moves towards strategy $Left$; if $x < \frac{k}{k+1}$, it goes in the opposite direction. Suppose initially $x_0 > y_0 > \frac{k}{k+1}$, so that $Left$ is the best response, but payoffs to it will decrease over time. Then the population proportion will be moving towards 1 until overtaken by the mixed equilibrium (see first parts of the equations (4) and (5)). At the moment the two trajectories intersect we will observe a switch: For a short while the agents will be indifferent between the two strategies, whereas the mixed equilibrium will keep growing, so at the next moment the situation will be described by $x_t < y_t$, and the population proportion starts decreasing since $Right$ is the new best response. The mixed equilibrium will continue to grow until x hits the level $\frac{k}{k+1}$, at which point y starts decreasing and will overtake the population proportion again at some new state below $\frac{k}{k+1}$. Thus whenever trajectories intersect, x will change the direction of motion until the next time y overtakes it. After the switch the trajectories will behave according to the second parts of equations (4) and (5). The same reasoning applies to other possible cases except for $x_0 = y_0 = \frac{k}{k+1}$ and can be summarized in a claim:

Claim 2 *If $x_0 \neq \frac{k}{k+1}$ or $y_0 \neq \frac{k}{k+1}$, there is an infinite sequence $\{t_n\}$ of times at which trajectories of x and y intersect and if $x(t_n) > \frac{k}{k+1}$, then $x(t_{n+1}) < \frac{k}{k+1}$.*

Case $x_0 = y_0 = \frac{k}{k+1}$ is the only one in which we face the same difficulty and in a model without preference evolution: The system could spend arbitrary time in that state until x changes and we observe the behavior described above.

The next point of interest is whether $x_n = x(t_n)$ converges to $\frac{k}{k+1}$. We know that the mixed equilibrium is at steady state whenever $x_t = \frac{k}{k+1}$, but there are no feasible steady states for the population dynamic x. Simulations suggest that the trajectories form an orbit around the state $(\frac{k}{k+1}, \frac{k}{k+1})$.

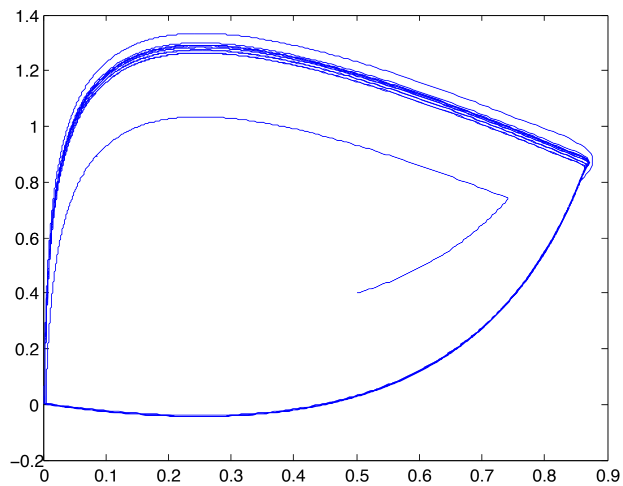

Figure 1: $k = \frac{1}{3}$, $r = 1$, $\frac{k}{k+1} = 0.25, x_0 = 0.5, y_0 = 0.4$

In the simulation illustrated by Figure 1 the initial conditions are $x_0 = 0.5$, $y_0 = 0.4$, $r = 1$, and $k = \frac{1}{3}$ so that $\frac{k}{k+1} = 0.25$. The inflection points of the trajectory must belong to the line $x = y$, and this is when there is a switch in the direction of change of x. Variable y changes direction whenever $x = \frac{k}{k+1} = 0.25$. A typical loop of the trajectory starts at $x = y$ very close to 0, then x grows up to 0.25 while y falls to some small negative number, at which point y changes direction, and both values increase until they equalize around 0.85. Then x starts decreasing, while y keeps growing until x reaches 0.25 again, and in the end both values fall to the starting point of the loop. We can now formulate a conjecture:

Conjecture 3 *For all values of parameters r and k the trajectory converges to an orbit around the point $(x^*, y^*) = (\frac{k}{k+1}, \frac{k}{k+1})$ from all initial conditions.*

We state this result as a conjecture because the best response dynamic is not smooth. We devote our next section to logit dynamic which can be treated as a smooth approximation to the best response dynamic, and we will see that the state $(x^*, y^*) = (\frac{k}{k+1}, \frac{k}{k+1})$ is the limiting state of the logit dynamic as the noise level goes to 0.

4 Logit Dynamic

Logit dynamic, introduced in Blume (1993), is an example of a perturbed best response dynamic. An agent who receives a revision opportunity switches to the best response strategy unless he chooses some other strategy by mistake. The chance of such mistake is a function of payoffs: The likelihood of a mistake should be lower the lower the payoff to the suboptimal strategy. In the logit case switch rates are exponents of payoff differences, and mistakes are introduced through the noise level $\eta > 0$. In a two strategy game the probability of switching to strategy $Left$ at the population state x would equal:

$$\text{P(choose Left)} = \frac{\exp(\eta^{-1} f_L(x))}{\exp(\eta^{-1} f_L(x)) + \exp(\eta^{-1} f_R(x))} = \frac{\exp(\eta^{-1}(f_L(x) - f_R(x)))}{\exp(\eta^{-1}(f_L(x) - f_R(x))) + 1}$$

If $Left$ is the only best response, then the probability it will be chosen tends to 1 as η approaches 0. If both strategies are best responses, then the likelihood of choosing either of them is $\frac{1}{2}$. Given the switch rates, we can obtain the mean dynamic by calculating the increment in the number of agents choosing to play $Left$:

$$\begin{aligned} \dot{x} &= \text{P(choose Left} \mid \text{currently Right)} - \text{P(choose Right} \mid \text{currently Left)} = \\ &= (1-x)\frac{\exp(\eta^{-1}(f_L(x) - f_R(x)))}{\exp(\eta^{-1}(f_L(x) - f_R(x))) + 1} - x\left(1 - \frac{\exp(\eta^{-1}(f_L(x) - f_R(x)))}{\exp(\eta^{-1}(f_L(x) - f_R(x))) + 1}\right) = \\ &= \frac{\exp(\eta^{-1}(f_L(x) - f_R(x)))}{\exp(\eta^{-1}(f_L(x) - f_R(x))) + 1} - x \end{aligned}$$

Although an individual's choice is stochastic, the average behavior can be well approximated by its mean dynamic, since the idiosyncratic noise is averaged away when the population size is large. To obtain the equations for the joint dynamic of the population state and the mixed equilibrium, we first need to express the payoff difference in terms of x, y, and s:

$$f_L(x) - f_R(x) = ax + b(1-x) - cx - d(1-x) = sx - (d-b) = s(x-y)$$

We are now able to rewrite the equation (1) from the system of equations (1)-(3). Since by assumption $\dot{s} = 0$ for all values of parameters, we will abandon equation (3) until section 6, in which we consider the general case. Thus the system reduces to:

$$\dot{x} = \frac{\exp(\eta^{-1}s(x-y))}{\exp(\eta^{-1}s(x-y)) + 1} - x \tag{6}$$

$$\dot{y} = \frac{r}{s}[(1+k)x - k] \tag{7}$$

and we can state our result for the logit dynamic:

Theorem 4 *For any $k > 0$, $s > 0$, $r > 0$*

1. *state $(x^*, y^*) = (\frac{k}{k+1}, \frac{k}{k+1} - \frac{\eta}{s}\log k)$ is the only steady state*
2. *there exists $\eta^* > 0$ such that for all $\eta \in (0, \eta^*)$ state (x^*, y^*) is repelling and for $\eta > \eta^*$ state (x^*, y^*) is a sink.*

Proof. 1. The steady states are the rest points of the dynamic. Setting equations (6) and (7) equal to 0 yields:

$$\frac{\exp(\eta^{-1}s(x-y))}{\exp(\eta^{-1}s(x-y))+1} - x = 0$$
$$\frac{r}{s}[(1+k)x - k] = 0$$

The second equation implies that y is only at rest when $x = \frac{k}{k+1}$. Then the first equation can be rewritten as:

$$\frac{\exp(\eta^{-1}s(x-y))}{\exp(\eta^{-1}s(x-y))+1} = \frac{k}{k+1}$$

so that $\exp(\eta^{-1}s(x-y)) = k$ and therefore $y = x - \frac{\eta}{s}\log k = \frac{k}{k+1} - \frac{\eta}{s}\log k$. This proves that the only steady state is:

$$(x^*, y^*) = (\frac{k}{k+1}, \frac{k}{k+1} - \frac{\eta}{s}\log k)$$

2. To investigate stability of the trajectory around the steady state, we calculate the Jacobian of $F(x,y) = (V(x,y), M(x,y))$ letting $R = \exp(\eta^{-1}s(x-y))$ to simplify notation:

$$DF = \begin{pmatrix} \frac{sR}{\eta(R+1)^2} - 1 & -\frac{sR}{\eta(R+1)^2} \\ \frac{r}{s}(1+k) & 0 \end{pmatrix}$$

and linearize the system around the steady state:

$$DF(x^*, y^*) = \begin{pmatrix} \frac{sk}{\eta(k+1)^2} - 1 & -\frac{sk}{\eta(k+1)^2} \\ \frac{r}{s}(1+k) & 0 \end{pmatrix}$$

The characteristic polynomial is:

$$\lambda^2 + (1 - \frac{sk}{\eta(k+1)^2})\lambda + \frac{rk}{\eta(k+1)} = 0$$

The Hurwitz stability criterion requires that the principle minors of the Hurwitz matrix are positive

$$H = \begin{pmatrix} 1 - \frac{sk}{\eta(k+1)^2} & 0 \\ 1 & \frac{rk}{\eta(k+1)} \end{pmatrix}$$

so we need $1 - \frac{sk}{\eta(k+1)^2} > 0$ and $(1 - \frac{sk}{\eta(k+1)^2})\frac{rk}{\eta(k+1)} > 0$. The second condition is satisfied whenever the first one is, so as long as $1 - \frac{sk}{\eta(k+1)^2} > 0$, the state (x^*, y^*) is a sink. If the sign is reversed, the steady state is repelling. We obtain $\eta^* = \frac{sk}{(k+1)^2}$. ∎

As η tends to 0, the steady state $(x^*, y^*) = (\frac{k}{k+1}, \frac{k}{k+1} - \frac{\eta}{s}\log k) \to (\frac{k}{k+1}, \frac{k}{k+1})$ and thus we should expect the actual best response dynamic to possess a unique repelling steady state. Figure 2 illustrates the trajectories of the dynamic with two different noise levels ($\frac{1}{9}$ on the left, $\frac{4}{9}$ on the right) for the same initial conditions $k = 2$, $r = 1$, $s = 1$, $x_0 = 0.5, y_0 = 0.6$. In this case the borderline noise level is $\eta^* = \frac{2}{9}$, and we can convince ourselves that at smaller noise levels the steady state is repelling while at larger ones it is a sink.

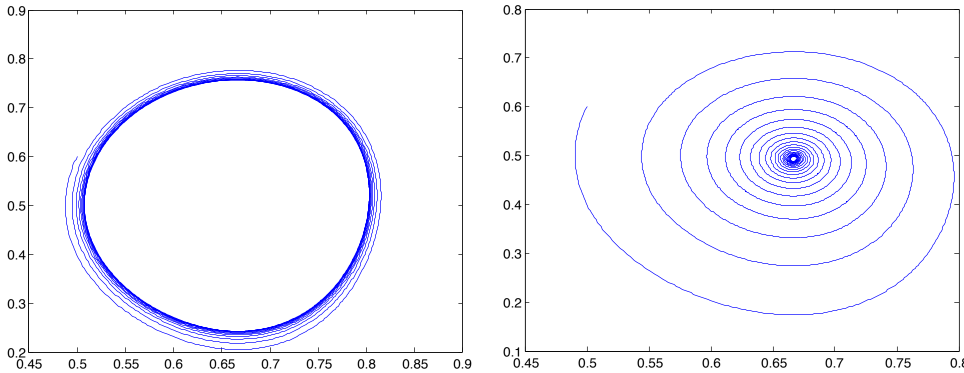

Figure 2: $k = 2$, $r = 1$, $\frac{k}{k+1} = \frac{2}{3}, x_0 = 0.5, y_0 = 0.6$, $s = 1$, $\eta_L = \frac{1}{9}$, $\eta_R = \frac{4}{9}$

5 Replicator Dynamic

In this section we employ a dynamic that requires less knowledge from the agents than the two examples we considered in previous sections. While best response and logit dynamics assume that the aggregate behavior is common knowledge, replicator dynamic (Taylor and Jonker (1978)) relaxes this assumption and instead suggests that strategy revision is based on peer comparison. Whenever an agent receives a revision opportunity, he learns about some other agent's strategy and switches in case that strategy brings higher payoff than his own. In a two strategy case an agent would only switch if he is not playing the best response. Therefore the direction of motion is the same for the best-response dynamic and the replicator dynamic, but the latter is slower since an agent switches only if he learns about someone already playing the optimal strategy.

Assuming that the switch rate is proportional to the difference between the strategy payoff and the average payoff, we can rewrite the payoff difference in terms of the population state and the mixed equilibrium:

$$f_L(x) - \bar{f}(x) = ax + b(1-x) - x(ax + b(1-x)) - (1-x)(cx + d(1-x)) = \\ = x(1-x)(a-b-c+d) - (d-b)(1-x) = s(1-x)(x-y)$$

and then plug the result into the equation describing the law of motion for x to obtain the following system:

$$\dot{x} = x[f_L(x) - \bar{f}(x)] = sx(1-x)(x-y)$$
$$\dot{y} = \frac{r}{s}[(1+k)x - k]$$

which only rest point is $(x^*, y^*) = (\frac{k}{1+k}, \frac{k}{1+k})$. Having obtained the system of equations, we can formulate the result for the replicator dynamic similar to that of Theorem 4.

Theorem 5 *For any $k > 0$, $s > 0$, $r > 0$ the only steady state $(x^*, y^*) = (\frac{k}{k+1}, \frac{k}{k+1})$ is repelling.*

Proof. We have already shown the existence and uniqueness of the rest point of the dynamic, so now we calculate the Jacobian of the function $F(x, y)$ to investigate stability:

$$DF = \begin{pmatrix} s(2x - 3x^2 - y + 2xy) & s(x^2 - x) \\ \frac{r}{s}(1+k) & 0 \end{pmatrix}$$

The value of the Jacobian at the steady state is:

$$DF(x^*, y^*) = \begin{pmatrix} \frac{sk}{(k+1)^2} & -\frac{sk}{(k+1)^2} \\ \frac{r}{s}(1+k) & 0 \end{pmatrix}$$

The characteristic polynomial is:

$$\lambda^2 - \frac{sk}{(k+1)^2}\lambda + \frac{rk}{s(k+1)} = 0$$

The Hurwitz stability criterion requires that the principle minors of the Hurwitz matrix are positive

$$H = \begin{pmatrix} -\frac{sk}{(k+1)^2} & 0 \\ 1 & \frac{rk}{s(k+1)} \end{pmatrix}$$

so we need $\frac{sk}{(k+1)^2} < 0$ and $\frac{rk^2}{(k+1)^3} < 0$ - the rest point is always repelling. ■

A sample trajectory of the replicator dynamic can be seen in Figure 3. For the same set of parameters as in the logit dynamic example and initial conditions $(x_0, y_0) = (0.5, 0.2)$, we observe that the dynamic spirals out, rather than in. We conclude the analysis with a

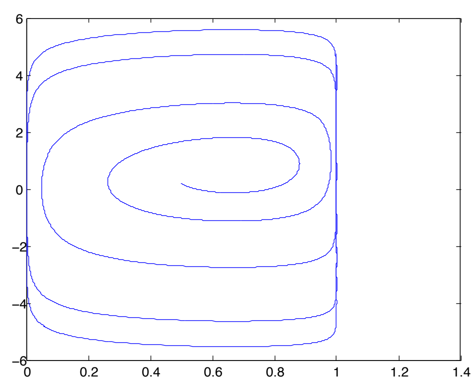

Figure 3: $k = 2$, $s = 1$, $r = 1$, $\frac{k}{k+1} = \frac{2}{3}, x_0 = 0.5, y_0 = 0.2$

discussion of existence and stability of rest points in a more general case.

6 Accounting for Incentives to Coordinate

In the previous sections we assumed that the incentives to coordinate remain the same ($\dot{s} = 0$) even as payoffs to strategies change. In general this assumption might not hold and the goal of this section is to investigate how much it shapes our previous results. To do so we introduce two new functions, $\alpha = \dot{a} - \dot{c}$ and $\beta = \dot{d} - \dot{b}$, which would represent the change in the gains to coordination on one of the two strategies (recall the definition of gains to coordination in subsection 2.3). Given the new notation, we can rewrite the equation describing the trajectory of the mixed equilibrium as

$$\dot{y} = \frac{d}{dt}[\frac{d-b}{a-b-c+d}] = \frac{\dot{d}-\dot{b}}{a-b-c+d} - \frac{(d-b)(\dot{a}-\dot{b}-\dot{c}+\dot{d})}{(a-b-c+d)^2} = \frac{\beta}{s} - \frac{y(\alpha+\beta)}{s}$$

and therefore the system $(1)-(3)$ can be transformed into

$$\dot{x} = V(x,y,s) \tag{8}$$

$$\dot{y} = \frac{\beta}{s} - \frac{y(\alpha+\beta)}{s} \tag{9}$$

$$\dot{s} = \alpha + \beta \tag{10}$$

For a rest point of the new system $(8)-(10)$ it must be that $\alpha+\beta=0$ from equation (10), hence $\beta = 0$ from equation (9), and so the rest point must be a solution to system

$$V(x,y,s) = 0 \tag{11}$$

$$\alpha(x,y,s) = 0 \tag{12}$$

$$\beta(x,y,s) = 0 \tag{13}$$

Therefore relaxing the assumption that the incentives to coordinate are unchanged does not increase the number of rest points.

7 Conclusion

In this paper we introduced a new model of two-speed evolution. We showed that under best response, replicator, and logit dynamic with small noise levels the steady state of the resulting system is unique and repelling and a sink in case of logit dynamic with large noise levels. The reason we obtain cyclical behavior is mostly due to the fact that the coordination effect is weak whenever the equilibration effect is strong and vice versa. All the results hold under the assumption that gains to coordination remain unchanged. If this assumption is relaxed, the number of steady states might reduce to zero. A possible question for future research is whether relaxing the gains to coordination assumption might change the properties of the steady states (for instance, turn a repelling state into a sink) and whether different types of behavior arise under some other relationships between aggregate behavior and mixed equilibrium.

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