Bargaining and Buyout*

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Abstract

Extending the random-proposer model (Baron and Ferejohn, 1989; Okada, 2011) and the random-bilateral-meeting model (Gul, 1989), we introduce a non-cooperative coalitional bargaining model in which agents can strategically buy out other agents. With buyout options, due to agents' strategic alliance behaviors, an efficient coalition may not be formed immediately in stationary subgame perfect equilibria and the equilibrium payoff vector is not necessarily in the core. We characterize conditions for a grand-coalition equilibrium and an efficient stationary equilibrium. Two applications are studied. For simple games with veto players, non-winning intermediate coalitions may be formed and the equilibrium winning coalition is not always minimal. For employer-employee games, workers form a union and the equilibrium wage is higher than the marginal product of each worker.

1 Introduction

When three or more agents bargain over their joint surplus, forming a transitional coalition is common phenomena though such a coalition is inefficient. Rather than immediately forming an efficient coalition,[1] agents can increase their bargaining power by forming a transitional inefficient coalition. In wage bargaining, for instance, workers form a labor union even though the union itself produces nothing; similarly, in legislative bargaining, minor parties form a coalition though the coalition is still minor.

In the theoretical coalitional bargaining models, however, agents immediately form an efficient coalition, especially when the gain from cooperation is substantial and commonly

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[1]In characteristic function form games, an efficient coalition maximizes the characteristic function.

known.[2] To investigate agents' strategic alliance behaviors and gradual agreement phenomena, we propose a noncooperative bargaining model, combining the random-proposer model (Baron and Ferejohn, 1989; Okada, 2011) and the random-bilateral-meeting model (Gul, 1989). More precisely, we allow the agents to strategically buy out other agents. If an agent makes an offer to a subcoalition and all the members in the subcoalition accept the offer, then the proposer represents the subcoalition and participates the subsequent bargaining game for the remaining surplus.[3] [4]

With buyout options, we characterize a condition for a grand-coalition equilibrium, in which all the agents always form the grand-coalition. When the agents are sufficiently patient, a grand-coalition equilibrium exists if and only if the underlying characteristic function form game is *unanimous*.[5] Interestingly, even if the gain from forming the grand-coalition is substantial like convex games, the grand-coalition will not be immediately formed with positive probability and hence delay may be occurred in an equilibrium.

In addition to the impossibility result on grand-coalition equilibria, we provide a more strong impossibility result on efficient stationary equilibria. If there is an *essential player*,[6] then not only the grand-coalition but also other efficient coalitions will not be immediately formed with positive probability unless the underlying characteristic function form game is

[2] With complete information, delayed equilibria in coalitional bargaining have been studied, but those equilibria rely on some restrictive bargaining protocols and some specific structures of coalitional worths (Chatterjee et al., 1993; Cai, 2000).

[3] This is in contrast to Baron and Ferejohn (1989) , Chatterjee et al. (1993), and Okada (1996, 2011), in which even a proposer must leave the game, and Selten (1988) and Compte and Jehiel (2010), in which the game is terminated with only one coalition. Such assumptions in the literature simplify the analysis so that forming inferior coalitions is precluded. As a result, for simple games only minimal winning coalitions formed and for convex games, the grand-coalition is always formed immediately. Our approach is similar to Gul (1989), Serrano (1993); Krishna and Serrano (1995); Serrano (1995); Krishna and Serrano (1996); Dagan et al. (1997); Serrano and Vohra (1997). However, in Gul (1989), coalition formation is based on random pairwise meetings and agents do not choose their partner strategically. Serrano and his coworkers considered some restricted class of games such as three-person games, a pie-splitting game, bankruptcy problems, but not general characteristic function form games.

[4] Our model is close to Seidmann and Winter (1998) and Okada (2000), which allow *renegotiation*s in noncooperative coalitional bargaining games. Seidmann and Winter (1998) considered the rejector-proposer model with semi-strict superadditive games; while our model is based on a random-proposer model and does not need superadditivity. Okada (2000) considers a random-proposer model, but it is based on a very strong assumption; once a subcoalition formed, any other disjoint subcoalition cannot be formed.

[5] A characteristic function form game (N, v) is *unanimous* if $v(S) = 0$ for all $S \subsetneq N$.

[6] In a characteristic function form game (N, v), a player $i \in N$ is *essential* if $v(N) > v(N \setminus \{i\})$.

unanimous.

This inefficiency result is applied to simple games. For simple games with veto players, transitional non-winning coalitions may be formed and the equilibrium winning coalition is not always minimal unless all the players are veto.[7] That is, non-veto players form a union each other in an equilibrium, even though that is not a winning coalition. Furthermore, the equilibrium expected payoff vector is not necessarily in the core of the underlying characteristic function form game. Conversely, either if there is no veto player or if all the players are veto, then minimal winning coalitions form immediately. As specific examples, three-party weighted majority games are studied.

We also investigate workers' strategic solidarity behaviors with employer-employee games and the effect of the discount factor and the recognition probability. When the common discount factor is higher than a certain level, workers endogenously form a union; and as the discount factor increases, the equilibrium wage also increases. The workers' recognition probability also plays an important role. The more likely the workers make a proposal, the less likely they form a union. These results highlight the role of buyout options, compared to Okada (2011)'s result, in which the equilibrium wage converges to the marginal product of each worker as the discount factor increases no matter what the recognition probability is.

The paper is organized as follows. Section 2 describes a non-cooperative coalitional bargaining model with buyout options. In Section 3, we define a stationary subgame perfect equilibrium, in short SSPE, and a cutoff strategy equilibrium as a special form of SSPE. Then we show that for any arbitrary SSPE, there exists a corresponding cutoff strategy equilibrium which yields the same expected payoff vector. Section 4 and Section 5 characterize conditions for a grand-coalition equilibrium and an efficient stationary equilibrium. In Section 6, as an application to simple games, we fully describe SSPE in three-parties weighted majority

[7] Aumann and Myerson (2003) informally argued that forming a non-minimal winning coalition could be a stable equilibrium. In the existing noncooperative model including Montero (2002), Morelli and Montero (2003), Montero (2006) and Montero and Vidal-Puga (2011), however, only a minimal winning coalition occurs in stationary subgame perfect equilibria.

games. In Section 7, the effect of agents' buyout options and the workers' solidarity behaviors are discussed in a wage bargaining model. Appendix presents an outline to show the existence of SSPE in this model.

2 A Model

Let N be a set of agents and n be the cardinality of N. For any $S \subseteq N$ and $i \in S$, define $\mathcal{P}(S) = \{S' \subseteq S \mid S' \neq \emptyset\}$ and $\mathcal{P}_i(S) = \{S' \subseteq S \mid i \in S'\}$.[8] Let $v : \mathcal{P}(N) \to \mathbb{R}^n_+$ be a *characteristic function* on N. We assume that v is zero-normalized, monotone, and essential, that is, $v(\{i\}) = 0$ for all $i \in N$; $v(S) \leq v(S')$ for all $S \subseteq S' \subseteq N$; and $v(N) > 0$. A tuple (N, v) is an *underlying characteristic function form game*, or shortly an *underlying game*. Given an underlying game (N, v), the super-additive cover of v is the characteristic function $\hat{v}$ defined by: for all $S \subseteq N$,

$$\hat{v}(S) = \max \left\{ \sum_{S' \in \varphi} v(S') \;\middle|\; \varphi \text{ is a partition of } S \right\}.$$

Let $X \equiv \{x \in \mathbb{R}^n_+ \mid \sum_{i \in N} x_i \leq v(N)\}$ be a set of feasible and individually rational allocations.

A *coalitional state* π consists of a set of active players $N(\pi) \in \mathcal{P}(N)$ and a partition $\{\pi_i\}_{i \in N(\pi)}$ of N such that $i \in \pi_i$.[9] Let $n(\pi)$ be the number of active players in π and let $\pi_S = \cup_{k \in S} \pi_k$. The initial coalitional state π° consists of $N(\pi^\circ) = N$ and $\pi^\circ_i = \{i\}$ for all $i \in N$. Given coalitional state π, let $X(\pi) \equiv \{x \in X \mid (\forall i \in N(\pi))\ x_i \geq v(\pi_i) \text{ and } (\forall i \notin N(\pi))\ x_i = 0\}$ be a set of feasible allocations restricted on $N(\pi)$. A characteristic function $v^\pi : \mathcal{P}(N(\pi)) \to X(\pi)$, restricted on $N(\pi)$, is defined by, for all $S \in \mathcal{P}(N(\pi))$,

$$v^\pi(S) = \hat{v}(\pi_S) - \sum_{k \in S} v(\pi_k).$$

[8] $\mathcal{P}_i(S)$ can be interpreted as the set of subcoalitions which can be proposed by i. When there is a communication restriction, we extend this definition. A communication restriction among agents is represented by a connected directed graph E over N. Then we define $\mathcal{P}_i(S) = \{S' \subseteq S \mid (\forall j \in S')\ j = i \text{ or } ij \in E\}$.

[9] A coalition structure, proposed in Aumann and Dreze (1974), is a partition of the set of agents N. On the other hand, a coalitional state, defined in this paper, consists of a partition of N indexed by active players. Thus, a coalitional state specifies the owner or the representative agent for each coalition in the partition.

A coalitional state π is *efficient* if there is no unrealized surplus, that is, $v^\pi(N(\pi)) = 0$. A coaltional state π is *inefficient* if π is not efficient, that is, $v^\pi(N(\pi)) > 0$. Let $\bar{\Pi}$ be the set of all efficient coalitional states and Π be the set of all inefficient coalitional states. Note that if a game (N, v) is strictly grandcoalition superadditive[10], then any efficient coalitional state $\pi \in \bar{\Pi}$ involves the grandcoalition, that is, $N(\pi) = \{i\}$ and $\pi_i = N$.

For $\pi \in \Pi$, if $i \in N(\pi)$ forms a coalition $S \in \mathcal{P}_i(N(\pi))$, then the subsequent coalitional state, denoted by $\pi(i, S)$, consists of $N(\pi(i, S)) = N(\pi) \setminus S \cup \{i\}$ and $\pi_i(i, S) = \pi_S$ and $\pi_j(i, S) = \pi_j$ for all $j \in N(\pi) \setminus S$. For a pair of coalitional states π and π', π *precedes* π', denoted by $\pi \prec \pi'$, or π' *succeeds* π, denoted by $\pi' \succ \pi$, if for all $j \in N(\pi')$ there exists $S \in \mathcal{P}(N(\pi))$ such that $j \in S$ and $\pi'_j = \pi_S$. Given $\pi \in \Pi$, define $\Pi_{|\pi} = \{\pi' \in \Pi \mid \pi' \succ \pi\}$, which is the set of *succeeding states* of π.

A *non-cooperative coalitional bargaining game*, or shortly, a *bargaining game* is a tuple $\Gamma = (N, v, p, \delta)$, where $p \in \triangle(N)$ is the initial recognition probability, and δ is the common discount factor. A bargaining game proceeds as follows. In the initial state π°, an agent $i \in N$ is selected as a proposer with probability p_i. The proposer i makes a proposal, a pair of $S \in \mathcal{P}_i(N)$ and $y \in X$. By an exogenously given order, each respondent $j \in S \setminus \{i\}$ sequentially either accepts the proposal or rejects it. If any $j \in S \setminus \{i\}$ rejects then move to the next period and a new proposer will be selected. If all $j \in S \setminus \{i\}$ accept the proposal, then $S \setminus \{i\}$ leaves the game with receiving $\{y_j\}_{j \in S \setminus \{i\}}$ from i and move to the subsequent coalitional state $\pi^\circ(i, S)$.

In any subgame with $\pi \in \Pi$, a new proposer $i \in N(\pi)$ is selected with probability $p_i^\pi = \sum_{k \in \pi_i} p_k$ and the proposer makes an proposal (S', y') such that $S' \in \mathcal{P}_i(N(\pi))$ and $y' \in X(\pi)$. By the same manner, the active players in $N(\pi)$ continue to bargain and move to a subsequent coalitional state until the coalitional state is efficient. If the coalitional state π is efficient, then the game ends and each active agent $i \in N(\pi)$ gets the final payoff $v(\pi_i)$. Once the game ends, then the last coalitional state continues forever.

[10] A characteristic function v is *strictly grandcoalition superadditive* if v is superadditive and $[S \subsetneq N \implies v(S) < v(N)]$

For a characteristic function form game (N, v), an infinite sequence of coalitional states $\tilde{\pi} = \{\pi^t\}_{t=0}^{\infty}$, and a sequence of tranfers among players $\tilde{\tau} = \{\tau^t\}_{t=0}^{\infty}$, an agent $i \in N$'s discounted sum of payoffs is

$$U_i(N, v, \tilde{\pi}, \tilde{\tau}) = \sum_{t=0}^{\infty} \delta^t \left[(1-\delta)v(\pi_i^t) + \tau_i^t\right].$$

3 Stationary Subgame Perfect Equilibria

A stationary strategy depends only on the current coalitional state and *within-period* histories, but not the histories of past periods. We define a simple stationary strategy, namely *cutoff strategy*, and we show that for any SSPE there exists a corresponding cutoff strategy equilibrium which produces the same expected payoff vector. This section generalizes Yan (2003) and Eraslan and McLennan (2011).

3.1 Stationary strategies

Fix an inefficient coalitional state $\pi \in \Pi$. At the beginning of each period with π, a proposer is selected randomly with the recognition probability p^π, that is, each within-period history specifies a current proposer. After a proposer is selected, the proposer makes a proposal, a pair of a subset of active players and a feasible allocation. Then the nominated players, the members in the proposed subset of active players, are supposed to response sequentially.

Let $H^0(\pi) = N(\pi) \times \mathcal{P}(N(\pi)) \times X(\pi)$ be a set of histories right after the proposer makes an offer; and for all $i \in N$, $H^i(\pi) = H^{i-1}(\pi) \times \{0, 1\}$ be a set of histories right after an agent i responses. The set of all possible *within-period histories* is

$$H(\pi) = N(\pi) \cup H^0(\pi) \cup H^1(\pi) \cup \cdots \cup H^n(\pi).$$

For all $h \in H(\pi)$, typically denoted by $(\phi, S, y, r_1, r_2, \cdots, r_i)$, the first element of the history specifies the current proposer, denoted by $\phi(h) = \phi \in N(\pi)$. For all $h \in \cup_{\ell=0}^{n} H^\ell(\pi)$, the second element and the third element specify the proposed coalition and the proposed allocation, denoted by $S(h) = S \in \mathcal{P}(N(\pi))$ and $y(h) = y \in X(\pi)$. For all $i \in N$ and all

$h \in \cup_{\ell=i}^{n} H^{\ell}(\pi)$, the $(i+3)$th element specifies the agent i's response, denoted by $r_i(h) = r_i \in \{0, 1\}$, where 0 represents i's rejection and 1 represents i's acceptance.

For a measurable space $(\Omega, \mathcal{A})$, let $\Delta(\Omega)$ be the set of probability measures on Ω. For a pair of measurable spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$, let $\Delta(\Omega_1, \Omega_2)$ be the set of transition probabilities from Ω_1 to Ω_2. An agent i's *(stationary) proposal strategy* in π is $\alpha_i^{\pi} \in \Delta(\mathcal{P}_i(N(\pi)) \times X(\pi))$. Define a *proposal transition probability* $\alpha^{\pi} \in \Delta(N(\pi), \mathcal{P}(N(\pi)) \times X(\pi))$ so that $\alpha^{\pi}(i)(S, y) = \alpha_i^{\pi}(S, y)$. An agent i's *(stationary) response strategy* in π is $\beta_i^{\pi} \in \Delta(H^{i-1}(\pi), \{0, 1\})$ such that $\beta_i^{\pi}(h)(1) = 1$ if $\phi(h) = i$ or $i \notin S(h)$.

For a fixed π, the three stochastic components, the recognition probability p^{π}, the proposal transition probability α^{π}, and the response strategy profile $\beta^{\pi} \equiv \{\beta_i^{\pi}\}_{i \in N}$, induce a unique probability measure on H^n.[11] The induced probability measure is denoted by $p^{\pi} \otimes \alpha^{\pi} \otimes \beta_1^{\pi} \otimes \cdots \otimes \beta_n^{\pi}$.

Let $\mathcal{O}(\pi) = [\mathcal{P}(N(\pi)) \times X(\pi)] \cup \{\pi\}$ be the outcome space in π. Define an outcome function $o : H^n(\pi) \to \mathcal{O}(\pi)$, such that for all $h \in H^n(\pi)$,

$$o(h) = \begin{cases} (S(h), y(h)) & \text{if } \times_{j \in N} r_j(h) = 1 \\ \pi & \text{otherwise.} \end{cases}$$

Let $(p^{\pi} \otimes \alpha^{\pi} \otimes \beta_1^{\pi} \otimes \cdots \otimes \beta_n^{\pi}) \circ o^{-1}$ be the induced measure on $\mathcal{O}(\pi)$ by $(p^{\pi}, \alpha^{\pi}, \beta^{\pi})$. We also define the induced measures on $\mathcal{O}(\pi)$ by any history $h \in H(\pi)$ and the agents' stationary strategies (α, β):

$$\kappa(h, \alpha^{\pi}, \beta^{\pi}) = \begin{cases} (\delta_h \otimes \alpha^{\pi} \otimes \beta_1^{\pi} \otimes \cdots \otimes \beta_n^{\pi}) \circ o^{-1} & \text{if } h \in N(\pi) \\ \left(\delta_h \otimes \beta_{\ell+1}^{\pi} \otimes \cdots \otimes \beta_n^{\pi}\right) \circ o^{-1} & \text{if } h \in H^{\ell}(\pi) \quad \ell = 0, 1, \cdots, n, \end{cases}$$

where δ_h is the Dirac probability measure on $H(\pi)$.

Given π, let $\mathbf{x}_{|\pi} \equiv \{x^{\pi'}\}_{\pi' \in \Pi_{|\pi}}$ be the collection of values of succeeding states, where $x^{\pi'} \in X(\pi')$ for all $\pi' \in \Pi_{|\pi}$. Denote $\mathbf{x}^{\pi} \equiv \mathbf{x}_{|\pi} \cup \{x^{\pi}\}$. We define an active agent i's expected

[11] This is known as a generalized Fubini Theorem. See Eraslan and McLennan (2011).

payoffs from the stationary strategy (α^π, β^π) at $h \in H(\pi)$ with respect to $\mathbf{x}^\pi$:

$$\begin{aligned} w_i(h, \alpha^\pi, \beta^\pi, \mathbf{x}^\pi) &= \kappa(h, \alpha^\pi, \beta^\pi)(\pi) x_i^\pi \\ &+ \sum_{S \in \mathcal{P}_i(N(\pi))} \int_{y \in X(\pi)} \left(y_i + \left(x_i^{\pi(i,S)} - \sum_{j \in S} y_j \right) \mathbb{1}(i = \phi(h)) \right) \kappa(h, \alpha^\pi, \beta^\pi)(S, dy) \\ &+ \sum_{S \in \mathcal{P}(N(\pi)) \setminus \mathcal{P}_i(N(\pi))} x_i^{\pi(\phi(h),S)} \int_{y \in X(\pi)} \kappa(h, \alpha^\pi, \beta^\pi)(S, dy). \end{aligned}$$

A within-period stationary strategy profile (α^π, β^π) is a *within-period stationary subgame perfect* $\mathbf{x}^\pi$*-equilibrium* if, for all $h \in H(\pi)$, all $i \in N(\pi)$, and i's all possible within-period stationary strategies $\hat{\alpha}_i^\pi$ and $\hat{\beta}_i^\pi$,

$$w_i(h, \alpha^\pi, \beta^\pi, \mathbf{x}^\pi) \geq w_i(h, (\hat{\alpha}_i^\pi, \alpha_{-i}^\pi), (\hat{\beta}_i^\pi, \beta_{-i}^\pi), \mathbf{x}^\pi). \tag{1}$$

Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be a stationary strategy profile, where $\boldsymbol{\alpha} \equiv \{\alpha^\pi\}_{\pi \in \Pi}$ and $\boldsymbol{\beta} \equiv \{\beta^\pi\}_{\pi \in \Pi}$; and $\mathbf{x} \equiv \{x^\pi\}_{\pi \in \Pi}$ be a stationary value profile.

Lemma 1. *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an arbitrary stationary strategy profile. There exists a value profile $\mathbf{x}$ induced by $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.*

Proof. Fix $i \in N$ and $(\boldsymbol{\alpha}_{-i}, \boldsymbol{\beta}_{-i})$, that is all the other players except for i play the given stationary strategy. Then the player i's problem is to find i's optimal strategy for a stationary discounted dynamic programming. By the fundamental theorem of stochastic dynamic programming, for every state $\pi \in \Pi$, i has a optimal strategy and it induces a value for i. Furthermore, the optimal strategy of i maximizes the expectation of the sum of the current payoff and the discounted value of next period's state, that is the optimal strategy of i is also stationary. □

Note that any SSPE is represented by $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Now we show that any stationary strategy profile $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is an SSPE if and only if it induces a value profile $\mathbf{x}$ and for each state it constructs a within-period stationary subgame perfect equilibrium with respect to the value profile.

Proposition 1. *If $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is an SSPE, then there exists $\mathbf{x}$ such that, for all $\pi \in \Pi$, the partial strategy profile (α^π, β^π) is a within-period stationary subgame perfect $\mathbf{x}^\pi$-equilibrium.*

Proof. (Sketch) Denote $\boldsymbol{\alpha}_{|\pi} \equiv \{\alpha^{\pi'}\}_{\pi' \in \Pi_{|\pi}}$, $\boldsymbol{\beta}_{|\pi} \equiv \{\beta^{\pi'}\}_{\pi' \in \Pi_{|\pi}}$, $\boldsymbol{\alpha}^{\pi} \equiv \boldsymbol{\alpha}_{|\pi} \cup \{\alpha^{\pi}\}$, and $\boldsymbol{\beta}^{\pi} \equiv \boldsymbol{\beta}_{|\pi} \cup \{\beta^{\pi}\}$. The fixed strategy profile $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ induces a value profile $\mathbf{x}$. For each state π, x^{π} depend only on $\mathbf{x}_{|\pi}$, $\boldsymbol{\alpha}^{\pi}$, and $\boldsymbol{\beta}^{\pi}$. Since $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is an SSPE, the corresponding partial strategy profile $(\alpha^{\pi}, \beta^{\pi})$ satisfies the period optimality condition (1) with respect to $\mathbf{x}^{\pi}$. Thus, $(\alpha^{\pi}, \beta^{\pi})$ is a within-period stationary subgame perfect $\mathbf{x}^{\pi}$-equilibrium. □

Proposition 2. *If there exist $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathbf{x}$ such that for all $\pi \in \Pi$, $(\alpha^{\pi}, \beta^{\pi})$ is a within-period stationary subgame perfect $\mathbf{x}^{\pi}$-equilibrium, then $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is an SSPE.*

Proof. (Sketch) If a coalitional state π is efficient, then the game ends and each active player $i \in N(\pi)$ gets $v(\pi_i)$ from that period on. If a coalitional state π is inefficient, then it must be $n(\pi) \geq 2$. For $\pi \in \Pi$ with $n(\pi) = 2$, there exists a unique subgame perfect equilibrium of Γ^{π}, for all $i \in N(\pi)$,

$$\begin{aligned} \alpha_i^{\pi}(h) &= (N(\pi), x^{\pi}) \qquad \text{for all } h \in H(\pi) \text{; and} \\ \beta_i^{\pi}(h) &= \begin{cases} 1 & \text{if } y_i(h) \geq x_i^{\pi} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $x_i^{\pi} = (1-\delta)v(\pi_i) + \delta p_i^{\pi} v(N)$ for each $i \in N$.

Now we consider an arbitrary inefficient state $\pi \in \Pi$ such that $n(\pi) \geq 3$. Suppose, for all the succeeding states $\pi' \in \Pi_{|\pi}$, $(\boldsymbol{\alpha}^{\pi'}, \boldsymbol{\beta}^{\pi'})$ is an SSPE of the subgame with π' and it induces the value of state $x^{\pi'}$. To show that $(\boldsymbol{\alpha}^{\pi}, \boldsymbol{\beta}^{\pi})$ is an SSPE of the subgame with π, suppose all the active agents except for an arbitrary i follow the stationary strategy profile $(\boldsymbol{\alpha}_{-i}^{\pi}, \boldsymbol{\beta}_{-i}^{\pi})$. The agent i faces a stochastic dynamic programming and hence i has an optimal strategy which maximizes the current return plus the sum of discounted future values, or equivalently, solves the condition (1). Therefore, if $(\alpha^{\pi}, \beta^{\pi})$ is a within-period stationary subgame perfect $\mathbf{x}^{\pi}$-equilibrium, then $(\boldsymbol{\alpha}^{\pi}, \boldsymbol{\beta}^{\pi})$ is an SSPE of Γ^{π}.

Induction argument completes the proof. □

Lemma 2. *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an SSPE and $\mathbf{x}$ be the induced value profile. For all $\pi \in \Pi$, $\sum_{j \in N(\pi)} x_j^{\pi} < v(N)$.*

Proof. Fix $\pi \in \Pi$. (α^π, β^π) is a within-period stationary subgame perfect $\mathbf{x}^\pi$-equilibrium. Thus, for all $j \in N(\pi)$, it must be $w_j(h, \alpha^\pi, \beta^\pi, \mathbf{x}^\pi) \geq x_j^\pi$ and hence we have

$$\sum_{j \in N(\pi)} w_j(h, \alpha^\pi, \beta^\pi, \mathbf{x}^\pi) \geq \sum_{j \in N(\pi)} x_j^\pi.$$

The left-hand side, the sum of expected payoffs of all the active players must be less than or equal to $v(N)$. Suppose, for contradiction, $\sum_{j \in N(\pi)} x_j^\pi \geq v(N)$. This yields that $\sum_{j \in N(\pi)} w_j(h, \alpha^\pi, \beta^\pi, \mathbf{x}^\pi) = 1$, and hence we have $w_j(h, \alpha^\pi, \beta^\pi, \mathbf{x}^\pi) = x_j^\pi$ for all $j \in N(\pi)$, since $w_j(h, \alpha^\pi, \beta^\pi, \mathbf{x}^\pi) \geq x_j^\pi$ for all $j \in N(\pi)$. However, this contradicts that π is inefficient. $\square$

3.2 Cutoff strategy SSPE

Even in the class of stationary subgame perfect equilibria, agents' strategies may depend on within-period histories, which involve the identity of the proposer, the proposed coalition and the proposed payoffs, and preceding respondents' reactions. We define a *cutoff strategy* as a special form of stationary strategies, and we show the equivalence between cutoff strategy SSPE and general SSPE in terms of equilibrium payoffs.

A cutoff strategy profile $(\mathbf{x}, \mathbf{q})$ consists of a cutoff value profile $\mathbf{x} = \{\{x_i^\pi\}_{i \in N(\pi)}\}_{\pi \in \Pi}$ and a coalition formation strategy profile $\mathbf{q} = \{\{q_i^\pi\}_{i \in N(\pi)}\}_{\pi \in \Pi}$, where $x_i^\pi \in \mathbb{R}$ and $q_i^\pi \in \Delta(\mathcal{P}_i(N(\pi)))$ for each $\pi \in \Pi$ and it specifies the behaviors of an active agent $i \in N(\pi)$ in any coalitional state π in the following way:

- An agent i proposes (S, y) with probability $q_i^\pi(S)$ such that

$$y_k = \begin{cases} x_k^\pi & \text{if } k \in S \\ 0 & \text{otherwise;} \end{cases}$$

- An agent i accepts any proposal (S, y) if and only if $y_i \geq x_i^\pi$.

Given $\mathbf{x}$, define an active agent i's demand set in π:

$$D_i^\pi(\mathbf{x}) = \operatorname*{argmax}_{S \in \mathcal{P}_i(N(\pi))} \left[x_i^{\pi(i,S)} - \sum_{j \in S} x_j^\pi \right]$$

and its maximized value $m_i^\pi(\mathbf{x})$. Given a cutoff strategy profile $(\mathbf{x}, \mathbf{q})$, define an active agent i's continuation payoff in π:

$$u_i^\pi(\mathbf{x}, \mathbf{q}) = p_i^\pi m_i^\pi(\mathbf{x}) + \sum_{j \in N(\pi)} p_j^\pi \sum_{S \in \mathcal{P}_j(N(\pi))} q_j^\pi(S) \left[\mathbb{1}(i \in S) x_i^\pi + \mathbb{1}(i \notin S) x_i^{\pi(j,S)} \right]. \tag{2}$$

Now we show that, for any SSPE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, there exists a cutoff strategy SSPE $(\mathbf{x}, \mathbf{q})$ such that the SSPE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ induces the value profile $\mathbf{x}$. Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an SSPE. Due to Proposition 1, there exists a value profile $\mathbf{x}$ such that, for all $\pi \in \Pi$, (α^π, β^π) is a within-period stationary subgame perfect $\mathbf{x}^\pi$-equilibrium.

Lemma 3. *For all $\pi \in \Pi$, $i \in N$, and $h \in H^i(\pi)$ such that*

i) $\bigtimes_{1 \le \ell \le i-1} r_\ell(h) = 1$*; and*

ii) $(\forall \ell \in S(h) \setminus \{\phi(h)\})\ \ell \geq i \implies y_\ell(h) > x_\ell^\pi,$

the current proposal $(S(h), y(h))$ will be implemented for sure in any SSPE.

Proof. Fix $\pi \in \Pi$ and we divide the proof into two cases.

Case 1: $i = n$.

subcase 1-1: Suppose that $n \notin S(h) \setminus \{\phi(h)\}$. It must be $\beta_n^\pi(h) = 1$ no matter what $y_n(h)$. Thus, the outcome at the history of $h' = (h, r_n)$ must be $o(h') = (S(h), y(h))$, that is, $(S(h), y(h))$ will be implemented for sure.

subcase 1-2: Suppose that $n \in S(h) \setminus \{\phi(h)\}$. If $\beta_n^\pi(h) = 1$, then the outcome at the history of $h' = (h, r_n)$ is $o(h') = (S(h), y(h))$, and hence the player n's payoff at the state is $y_n(h)$. If $\beta_n^\pi(h) < 1$, then the outcome at the history of $h' = (h, r_n)$ will be $o(h') = \pi$ with positive probability, and hence they face the same state π in which the player n's value is x_n^π. Thus, for the player n, $\beta_n^\pi(h) = 1$ is optimal at the history h and hence $(S(h), y(h))$ is implemented.

Case 2: $i \leq n - 1$.

As induction hypothesis, suppose that, for all $j > i$ and all $h \in H^j(\pi)$ if $\left[\bigtimes_{1 \le \ell \le j-1} r_\ell(h) = 1\right]$ and $\left[(\forall \ell \in S(h) \setminus \{\phi(h)\})\ \ell \geq j \implies y_\ell(h) > x_\ell^\pi\right]$, then $(S(h), y(h))$ is implemented for sure.

subcase 2-1: Suppose that $i \notin S(h) \setminus \{\phi(h)\}$. It must be $\beta_i^\pi(h) = 1$ no matter what $y_i(h)$. Thus, the outcome at the history of $h' = (h, r_i, \cdots, r_n)$ must be $o(h') = (S(h), y(h))$, that is, $(S(h), y(h))$ will be implemented for sure.

subcase 2-2: Suppose that $i \in S(h) \setminus \{\phi(h)\}$. If $\beta_i^\pi(h) = 1$, then the outcome at the history of $h' = (h, r_i, \cdots, r_n)$ is $o(h') = (S(h), y(h))$, and hence the player i's payoff at the state is $y_n(h)$. If $\beta_i^\pi(h) < 1$, then the outcome at the history of $h' = (h, r_i, \cdots, r_n)$ will be $o(h') = \pi$ with positive probability, and hence they face the same state π in which the player i's value is x_i^π. Thus, for the player i, $\beta_i^\pi(h) = 1$ is optimal for the player i at the history h and hence $(S(h), y(h))$ is implemented. □

Lemma 4. *For all $\pi \in \Pi$, $i \in N$, and $h \in H^i(\pi)$ such that*

i) $\bigtimes_{1 \leq \ell \leq i-1} r_\ell(h) = 1$*; and*

ii) $(\exists \ell \in S(h) \setminus \{\phi(h)\})\ \ell \geq i$ *and* $y_\ell(h) < x_\ell^\pi$,

the current proposal $(S(h), y(h))$ will never be implemented in any SSPE.

Proof. Fix $\pi \in \Pi$ and we divide the proof into two cases.

Case 1: $i = n \in S(h) \setminus \{\phi(h)\}$ and $y_n(h) < x_n^\pi$.

If $\beta_n^\pi(h) < 1$, then the outcome at the history of $h' = (h, r_n)$ will be $o(h') = (S(h), y(h))$ with positive probability, and hence the player n's payoff at the state is $y_n(h)$, which less than the stationary value x_n^π. Thus, $\beta_n^\pi(h) = 0$ is optimal for the player n and the current proposal $(S(h), y(h))$ is not implemented.

Case 2: $i \leq n-1$. As induction hypothesis, suppose that, for any $h \in H^{i+1}(\pi)$, if $\bigtimes_{1 \leq \ell \leq i} r_\ell(h) = 1$ and there exists $j \in S(h) \setminus \{\phi(h)\} \cap \{j \geq i+1\}$ such that $y_j(h) < x_j^\pi$, then the proposal $(S(h), y(h))$ will not be implemented.

subcase 2-1: If there exists $j \in S(h) \setminus \{\phi(h)\} \cap \{j \geq i+1\}$ such that $y_j(h) < x_j^\pi$, then by the induction hypothesis, the proposal $(S(h), y(h))$ will not be implemented no matter what $\beta_i^\pi(h)$ is.

subcase 2-2: Suppose that $y_j(h) \geq x^\pi_j$ for all $j \in S(h) \setminus \{\phi(h)\} \cap \{j \geq i+1\}$. It must be $i \in S(h) \setminus \{\phi(h)\}$ and $y_i(h) < x^\pi_i$. For all continuation histories of $(h, r_i = 1)$, $(h, r_i = 1, r_{i+1} = 1)$, $\cdots$, $(h, r_i = 1, \cdots, r_{n-1} = 1)$, if $\beta^\pi_\ell(h, r_i = 1, \cdots, r_{\ell-1} = 1) > 0$ for all $\ell = i+1, i+2, \cdots, n$, then $\beta^\pi_i(h) = 0$ is optimal for the player i and the proposal $(S(h), y(h))$ will not be implemented. If there exists $\ell = i+1, i+2, \cdots, n$, such that $\beta^\pi_\ell(h, r_i = 1, \cdots, r_{\ell-1} = 1) = 0$, again the proposal $(S(h), y(h))$ will not be implemented no matter what $\beta^\pi_i(h)$ is. □

For any $S \in \mathcal{P}(N)$, define an allocation $\bar{y}^S \in X$ as $\bar{y}^S_j = x^\pi_j$ for all $j \in S$ and $\bar{y}^S_j = 0$ otherwise.

Lemma 5. *Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an SSPE and $\mathbf{x}$ be the induced value profile. For all $\pi \in \Pi$ and $h \in H(\pi)$, if $\alpha^\pi_{\phi(h)}(S, y) > 0$ then $S \in D^\pi_{\phi(h)}(\mathbf{x})$ and $y = \bar{y}^S$. Furthermore, every proposal is implemented for sure and the proposal gain of $\phi(h)$ is $m^\pi_{\phi(h)}(\mathbf{x})$.*

Proof. Fix $\pi \in \Pi$ and $h \in H(\pi)$. First, by Lemma 4, the proposal gain of $\phi(h)$ in an SSPE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is less than or equals to $m^\pi_{\phi(h)}(\mathbf{x})$. Suppose, for contraction, that the proposal gain of $\phi(h)$ in an SSPE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is strictly less than $m^\pi_{\phi(h)}(\mathbf{x})$ and let $\alpha^\pi_{\phi(h)}$ be the proposal strategy. There must exist (S, y) such that $\alpha^\pi_{\phi(h)}(S, y) > 0$ and $y_j \geq x^\pi_j$ for all $j \in S$ and $y_{j'} > x^\pi_{j'}$ for some $j' \in S$. By Lemma 3, $\phi(h)$ can be strictly better off by slightly decreasing j' share in the proposal, which is a contradiction. Thus, for all player $i \in N(\pi)$, the proposal gain of the player i in an SSPE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ equals to $m^\pi_i(\mathbf{x})$. For the proposal gain $m^\pi_i(\mathbf{x})$ in order to be obtained, the player i must make a proposal $(S, \bar{y}^S)$ for any $S \in \mathcal{P}_i(N(\pi))$, that is, the player i chooses S in $D^\pi_i(\mathbf{x})$. □

Theorem 1. *For an arbitrary SSPE, there exists a cutoff strategy SSPE which yields the same value for each agent.*

Proof. Let $\mathbf{x}$ be the value profile induced by an arbitrary SSPE $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Case 1: For $\pi \in \Pi$ with $n(\pi) = 2$, $\Gamma^\pi = (N(\pi), v, p^\pi, \delta)$ has a unique subgame perfect Nash equilibrium, in which the active agents play cutoff strategies.

Case 2: Consider $\pi \in \Pi$ with $n(\pi) \geq 3$. As induction hypothesis, suppose for all the succeeding states $\pi' \in \Pi_{|\pi}$, there exists a cutoff strategy SSPE $(\mathbf{x}^{\pi'}, \mathbf{q}^{\pi'})$ of $\Gamma^{\pi'} = (N(\pi'), v, p^{\pi'}, \delta)$. Now we show that $(\mathbf{x}^{\pi}, \mathbf{q}^{\pi})$ is a cutoff strategy SSPE of Γ^{π}. By Lemma 5, since the proposed allocation is determined by $\mathbf{x}$, a player i's proposal strategy in π can be represented by $q_i^{\pi} \in \Delta(D_i^{\pi}(\mathbf{x}))$, which is a proposal cutoff strategy. Since this proposal must be implemented for sure by Lemma 5, an active player i's expected payoff when i is not selected as a proposer must equal to the value of current state, that is, $x_i^{\pi} = (1-\delta)v(\pi_i) + \delta u_i^{\pi}(\mathbf{x}^{\pi}, \mathbf{q}^{\pi})$. Since all the active proposer plays a cutoff proposal strategy, for all $i \in N(\pi)$ and all $h \in H^{i-1}(\pi)$ such that $i \in S(h) \setminus \{\phi(h)\}$, an agent i's optimal response strategy is $\beta_i^{\pi}(h) = 1$ if $y_i(h) \geq x_i^{\pi}$ and otherwise $\beta_i^{\pi}(h) = 0$. Thus all the respondents follows the cutoff strategies. □

By Theorem 1, when we are interested in the agents' equilibrium payoffs, without loss of generality, we can focus on a cutoff strategy SSPE. The next proposition characterizes a cutoff strategy SSPE in terms of the value profile and coalition forming strategy profile.

Proposition 3. *Let $0 < \delta < 1$. A cutoff strategy profile $(\mathbf{x}, \mathbf{q})$ is an SSPE if and only if for all $\pi \in \Pi$ and $i \in N(\pi)$,*

i) $x_i^{\pi} = (1-\delta)v(\pi_i) + \delta u_i^{\pi}(\mathbf{x}, \mathbf{q})$*; and*

ii) $q_i^{\pi} \in \Delta(D_i^{\pi}(\mathbf{x}))$.

Proof. **(only if part)** Suppose $(\mathbf{x}, \mathbf{q})$ is an SSPE. Consider $\pi \in \Pi$ and $i \in N(\pi)$. By Lemma 5, the agent i's equilibrium proposal strategy must maximizes the proposal gain $x_i^{\pi(i,S)} - \sum_{j \in S} x_j^{\pi}$, and hence it must be $q_i^{\pi} \in \Delta(D_i^{\pi}(\mathbf{x}))$. When i is supposed to response, i can get at most $(1-\delta)v(\pi_i) + \delta u_i^{\pi}(\mathbf{x}, \mathbf{q})$ by rejecting any proposal. Thus in equilibrium, each respondent must indifferent between accepting and rejecting, which requires that $x_i^{\pi} = (1-\delta)v(\pi_i) + \delta u_i^{\pi}(\mathbf{x}, \mathbf{q})$.

(if part) Suppose all the agents except i follow the given cutoff strategies $(\mathbf{x}_{-i}, \mathbf{q}_{-i})$. For any π such that $i \in N(\pi)$, if $x_i^{\pi} = (1-\delta)v(\pi_i) + \delta u_i^{\pi}(\mathbf{x}, \mathbf{q})$, then it is impossible for i to deviate profitably from the given response strategy. When i is supposed to propose, forming a subcoalition which is not in $D_i^\pi(\mathbf{x})$ is not optimal for i. On the other hand, a proposer i can propose a grand coalition, in which i's proposal gain is

$$x_i^{\pi(i,N(\pi))} - \sum_{j\in N(\pi)} x_j^\pi = v(N) - \sum_{j\in N(\pi)} x_j^\pi > v(N) - v(N) = 0,$$

where the first equality is from the fact $\pi(i, N(\pi))$ is efficient; and the inequality is from Lemma 2. Thus, given $\mathbf{x}$, we have $m_i^\pi(\mathbf{x}) > 0$ and the proposer i always has a strictly positive proposal gain as long as the current state is inefficient. That is, making an acceptable proposal is strictly better than a proposal which will be rejected. Therefore, in an SSPE, a proposal i makes a proposal $(S, \bar{y}^S)$ with $S \in \Delta(D_i^\pi(\mathbf{x}))$, which is the proposal cutoff strategy. □

4 Grand-coalition Equilibria

A grand-coalition equilibrium is a special SSPE in which the grand-coalition is always formed immediately. In the literature, a grand-coalition equilibrium exists when the grand-coalition produces a relatively large surplus. Okada (2011) characterized the condition for a grand-coalition equilibrium to exist in the random-proposer model: when the agents are sufficiently patient, they get the payoffs proportional to their recognition probabilities via a grand-coalition equilibrium, if the payoff vector belongs to the core. Chatterjee et al. (1993) show that a grand-coalition will be formed immediately under a certain condition, *Condition M*, which is implied by convexity.

In this section, we show that a grand-coalition equilibrium does not exist in general if the agents have buyout options. That is, even though the underlying characteristic function is convex so they can produce much higher output, however, some agents might form an intermediate subcoalition in equilibrium. The only case in which a grand-coalition equilibrium exists is a unanimity game, that is, all the subcoalitions except for the grand coalition produce nothing.

Definition 1 (Grand-coalition equilibrium)**.** A strategy profile is a *grand-coalition equilibrium* if it is a subgame perfect Nash equilibrium, in which for any history a proposer proposes to all the active player and the respondents accept the offer.

Definition 2 (Unanimity game)**.** A characteristic function form game (N, v) is unanimous if $v(N) > 0$ and $v(S) = 0$ for all $S \subsetneq N$.

Clearly, a grand-coalition equilibrium is a cutoff strategy SSPE, and hence a grand-coalition equilibrium can be represented by a cutoff strategy profile $(\mathbf{x}, \mathbf{q})$.

Lemma 6. *Suppose* $(\mathbf{x}, \mathbf{q})$ *is a grand-coalition equilibrium. For any coalitional state* $\pi \in \Pi$*,*

A. $\displaystyle\sum_{j \in N(\pi)} u_j^\pi(\mathbf{x}, \mathbf{q}) = v(N)$.

B. $\displaystyle\sum_{j \in N(\pi)} x_j^\pi = (1 - \delta) \sum_{j \in N(\pi)} v(\pi_i) + \delta v(N)$.

Proof. Consider an arbitrary coalitional state $\pi \in \Pi$ in a grand-coalition equilibrium. In grand-coalition equilibrium, an active agent $i \in N(\pi)$ makes an offer to a grand-coalition if i is supposed to propose; and all the other active agent $j \in N(\pi) \setminus \{i\}$ offers i to x_i^π. Thus, for all $i \in N(\pi)$, we have

$$m_i^\pi = v(N) - \sum_{j \in N(\pi)} x_j^\pi \tag{3}$$

and

$$u_i^\pi(\mathbf{x}, \mathbf{q}) = p_i^\pi m_i^\pi + x_i^\pi. \tag{4}$$

Plugging (3) into (4) and summing (4) over $N(\pi)$, we have

$$\sum_{j \in N(\pi)} u_j^\pi(\mathbf{x}, \mathbf{q}) = \sum_{j \in N(\pi)} p_j^\pi \left[v(N) - \sum_{j \in N(\pi)} x_j^\pi \right] + \sum_{j \in N(\pi)} x_j^\pi. \tag{5}$$

Since $\sum_{j \in N(\pi)} p_j^\pi = 1$, (5) completes the proof of the part A.

By Proposition 3, summing x_j^π over $N(\pi)$, we have

$$\sum_{j \in N(\pi)} x_j^\pi = (1 - \delta) \sum_{j \in N(\pi)} v(\pi_i) + \delta \sum_{j \in N(\pi)} u_j^\pi(\mathbf{x}, \mathbf{q}). \tag{6}$$

Thus, the part A of this lemma implies the part B. □

Lemma 7. *Suppose* $(\mathbf{x}, \mathbf{q})$ *is a grand-coalition equilibrium. For any coalitional state* $\pi \in \Pi$ *and all active player* $i \in N(\pi)$,

A. $m_i^\pi(\mathbf{x}) = (1-\delta)v^\pi(N(\pi))$.

B. $u_i^\pi(\mathbf{x}, \mathbf{q}) = v(\pi_i) + p_i^\pi v^\pi(N(\pi))$.

C. $x_i^\pi = v(\pi_i) + \delta p_i^\pi v^\pi(N(\pi))$

D. $q_i^\pi(N(\pi)) = 1$.

Proof. Plugging Lemma 6.B into (3), we have

$$m_i^\pi = v(N) - \left[(1-\delta) \sum_{j \in N(\pi)} v(\pi_i) + \delta v(N)\right] = (1-\delta)\left[v(N) - \sum_{j \in N(\pi)} v(\pi_i)\right],$$

implies the part A.

Plugging the part A into (4), we have $u_i^\pi(\mathbf{x}, \mathbf{q}) = (1-\delta)p_i^\pi v^\pi(N(\pi)) + x_i^\pi$, or equivalently, due to Proposition 3,

$$u_i^\pi(\mathbf{x}, \mathbf{q}) = (1-\delta)p_i^\pi v^\pi(N(\pi)) + (1-\delta)v(\pi_i) + \delta u_i^\pi(\mathbf{x}, \mathbf{q}). \tag{7}$$

Rearranging (7) completes the part B.

By Proposition 3 and the part B, we have

$$\begin{aligned} x_i^\pi &= (1-\delta)v(\pi_i) + \delta\left[v(\pi_i) + p_i^\pi v^\pi(N(\pi))\right] \\ &= v(\pi_i) + \delta p_i^\pi v^\pi(N(\pi)), \end{aligned}$$

which implies the part C.

The part D is directly from the definition of a grand-coalition equilibrium. □

Proposition 4. *For any* $p \in \Delta^\circ(N)$, *there exists* $\bar{\delta} < 1$ *such that, for all* $\delta > \bar{\delta}$, *a bargaining game* (N, v, p, δ) *has no grand-coalition equilibrium unless* (N, v) *is a unanimity game.*

Proof. Suppose that a bargaining game (N, v, p, δ) has a grand-coalition equilibrium $(\mathbf{x}, \mathbf{q})$. For any $i \in N$, since $m_i^{\pi^\circ} = x_i^{\pi^\circ(i,N)} - \sum_{j \in N(\pi)} x_j^{\pi^\circ}$, for all $k \in N \setminus \{i\}$,

$$x_i^{\pi^\circ(i,N)} - \sum_{j \in N} x_j^{\pi^\circ} \geq x_i^{\pi^\circ(i,N\setminus\{k\})} - \sum_{j \in N\setminus\{k\}} x_j^{\pi^\circ}. \tag{8}$$

Recall that $x_i^{\pi^\circ(i,N)} = v(N)$ and by Lemma 7

$$\begin{aligned} x_i^{\pi^\circ(i,N\setminus\{k\})} &= v(\pi^\circ(i, N \setminus \{k\})) + \delta p_i^{\pi^\circ(i,N\setminus\{k\})} v^{\pi^\circ(i,N\setminus\{k\})}(N(\pi^\circ(i, N \setminus \{k\}))) \\ &= v(N \setminus \{k\}) + \delta(1 - p_k)\left[v(N) - v(N \setminus \{k\})\right]. \end{aligned} \tag{9}$$

Thus, rearranging (8) yields

$$v(N) \geq v(N \setminus \{k\}) + \delta(1 - p_k)\left(v(N) - v(N \setminus \{k\})\right) + x_k^{\pi^\circ}, \tag{10}$$

Since v is assumed to be zero-normalized, plugging $x_k^{\pi^\circ} = \delta p_k v(N)$, (10) implies

$$v(N) \geq (1 - \delta(1 - p_k))v(N \setminus \{k\}) + \delta v(N).$$

Thus, for all $k \in N$,

$$v(N) \geq \left(\frac{1 - \delta(1 - p_k)}{1 - \delta}\right) v(N \setminus \{k\}). \tag{11}$$

Since $p_k > 0$ for all $k \in N$ and $v(N) \geq v(N\setminus\{k\})$, define $\bar{\delta} = \max_{k \in N} \left(\frac{v(N) - v(N \setminus \{k\})}{v(N) - (1 - p_k)v(N \setminus \{k\})}\right)$. If there exists $k \in N$ such that $v(N \setminus \{k\}) > 0$, then $\bar{\delta} < 1$. Thus, for all $\delta > \bar{\delta}$,

$$\left(\frac{v(N) - v(N \setminus \{k\})}{v(N) - (1 - p_k)v(N \setminus \{k\})}\right) < \delta,$$

which violates (11). Therefore, in order for (N, v, p, δ) to have a grand-coalition equilibrium, it must be $v(N \setminus \{k\}) > 0$ for all $k \in N$. Since v is monotone, it must be $v(S) > 0$ for all $S \subsetneq N$. □

Proposition 5. *If (N, v) is a unanimity game, then a bargaining game (N, v, p, δ) has a grand-coalition equilibrium for all $p \in \Delta(N)$ and all $\delta \in [0, 1]$.*

Proof. If (N, v) is a two-person unanimity game, then the grand-coalition equilibrium is the unique subgame perfect equilibrium for all p and δ. Suppose that, for any less-than-n-person unanimity game, the corresponding bargaining game has a grand-coalition equilibrium for all p and δ. Now we prove that, for a n-person unanimity game, the corresponding bargaining game has a grand-coalition equilibrium for all p and δ. Assume that all the agents except for $i \in N$ follow the grand-coalition equilibrium strategies $(\mathbf{x}, \mathbf{q})$. By Lemma 7, for all $j \in N$, $x_j^{\pi^\circ} = \delta p_j v(N)$ and $q_j^{\pi^\circ}(N) = 1$.

Cases 1: proposal strategy.

If i follows the grand-coalition equilibrium strategy, then i's proposal gain is

$$v(N) - \sum_{j \in N} x_j^{\pi^\circ} = v(N) - \sum_{j \in N} \delta p_j v(N) = (1-\delta)v(N).$$

Suppose i forms a subcoalition $S \subsetneq N$. Since $(N(\pi(i,S)), v^{\pi(i,S)})$ is a less-than-n-person unanimity game, the subgame has a grand-coalition equilibrium and by Proposition 7 we have

$$x_i^{\pi(i,S)} = \delta p_i^{\pi(i,S)} v^{\pi(i,S)}(N(\pi(i,S))) = \delta \sum_{j \in S} p_j v(N).$$

Thus, i's proposal gain from the deviation is

$$x_i^{\pi(i,S)} - \sum_{j \in S} x_j^{\pi^\circ} = \delta \sum_{j \in S} p_j v(N) - \sum_{j \in S} \delta p_j v(N) = 0$$

Thus, the proposer i has no incentive to form any subcoalition.

Cases 2: response strategy.

For any $\pi \in \Pi$, a respondent i's expected payoff is x_i^π by rejecting any proposal since all other players are supposed to play the grand-coalition equilibrium. Thus it is optimal for i to accept any offer (S, y) if and only $y_i \geq x_i^\pi$. □

The following theorem is a direct consequence of Proposition 4 and Proposition 5

Theorem 2. *Suppose $p \in \Delta^\circ(N)$. A bargaining game (N, v, p, δ) has a grand-coalition equilibrium for all $\delta \in (0,1)$ if and only if (N, v) is a unanimity game.*

5 Efficient Equilibria

When the underlying characteristic function form game is not superadditive, forming a grand-coalition is not necessarily efficient and an efficient coalition structure cannot be formed immediately. In this section, we present a negative result on efficiency: even though the underlying characteristic function form game is superadditive, if there exists an *essential player*, then an efficient equilibrium does not exist unless it is a unanimity game.

In this section, we assume that v is zero-normalized, essential, and superadditive. A coalition $S \subseteq N$ is *efficient* if $v(S) \geq v(S')$ for all $S' \subseteq N$. Let $\mathbf{E}$ be a set of *efficient coalitions*. Denote $\bar{v} = v(S)$ for all $S \in \mathbf{E}$. Given the three assumptions on v, $\mathbf{E}$ has following properties:

(E1) $\{i\} \notin \mathbf{E}$ for all $i \in N$;

(E2) $S \in \mathbf{E}$ and $S \subset S'$ imply $S' \in \mathbf{E}$; and

(E3) $N \in \mathbf{E}$.

Let $\mathbf{E}^m = \{S \in \mathbf{E} \mid (\forall i \in S)\ S \setminus \{i\} \notin \mathbf{E}\}$ be a set of *minimal efficient coalitions*, $K = \cap \mathbf{E}$ a set of *essential players*, and $D = N \setminus (\cup \mathbf{E}^m)$ a set of *dummy players*. We also define a set of *auxiliary coalitions* $\mathbf{A} = \{A \subseteq N \setminus K \mid A \cup K \in \mathbf{E}^m\}$. For notational simplicity, for any $z \in \mathbb{R}^n$ and $S \in \mathcal{P}(N)$, denote $z_S = \sum_{j \in S} z_j$.

Definition 3 (Efficient equilibrium)**.** A strategy profile is an *efficient equilibrium* if it is an SSPE, in which for any history a proposer makes an offer to form an efficient coalition and respondents accept the offer.

Lemma 8. *Let* $(\mathbf{x}, \mathbf{q})$ *is an efficient equilibrium. For all* $\pi \in \Pi$*,*

$$\sum_{i \in N(\pi)} u_i(\mathbf{x}, \mathbf{q}) = v(N).$$

Proof. It is clear from the definition of an efficient equilibrium. □

Lemma 9. *Following properties for auxiliary coalitions hold.*

A. If $\emptyset \in \mathbf{A}$*, then* $\mathbf{A} = \{\emptyset\}$*.*

B. If $\emptyset \notin \mathbf{A}$*, then there exist* $A, A' \in \mathbf{A}$ *such that* $A' \neq A$*.*

Proof. If $\emptyset \in \mathbf{A}$, then $K \in \mathbf{E}$. Furthermore, eliminating any essential player $k \in K$, $K \setminus \{k\}$ is not a winning coalition any more, and hence $K \in \mathbf{E}^m$. Suppose there exists $A \in \mathbf{A}$ such that $A \neq \emptyset$. Then it must be $K \cup A \in \mathbf{E}^m$, which contradicts to $K \in \mathbf{E}^m$ and this completes the proof of the first part.

Suppose that $\emptyset \notin \mathbf{A}$ to prove the second part. By the definition of an auxiliary coalition and the existence of a winning coalition, there must be $A \in \mathbf{A}$. If A is the unique auxiliary coalition, then $\mathbf{E}^m = \{K \cup A\}$. Eliminating any player $i \in A$, $K \cup A \setminus \{i\}$ is not a winning coalition as more. Thus i must be an essential player and $A \subseteq K$, which contradicts to the condition $A \subseteq N \setminus K$ and this completes the proof of the second part. $\square$

Lemma 10. *Suppose that there exists $k \in K$ such that $p_k > 0$. If an efficient equilibrium exists, then there exists $\bar{\delta} < 1$ such that, for all $\delta > \bar{\delta}$, $x_K^{\pi^\circ} = \delta$ and $x_{N\setminus K}^{\pi^\circ} = 0$.*[12]

Proof. If an efficient equilibrium exists, then, due to Theorem 1, there exists a cutoff strategy SSPE $(\mathbf{x}, \mathbf{q})$ in which all the players form an efficient coalition immediately. Let $S^* \in \underset{S \in \mathcal{P}(N)}{\operatorname{argmax}} \left(v(S) - x_S^{\pi^\circ}\right)$. Since all the players form an efficient coalition, we have $v(S^*) = \bar{v}$. Take an essential player $k \in K$. Since k will be nominated by all the players in the equilibrium, k's expected payoff is:

$$\begin{aligned} u_k^{\pi^\circ}(\mathbf{x}, \mathbf{q}) &= p_k \max_{S \in \mathcal{P}(N)} \left(v(S) - x_S^{\pi^\circ}\right) + x_k^{\pi^\circ} \\ &= p_k \left(\bar{v} - x_{S^*}^{\pi^\circ}\right) + x_k^{\pi^\circ} \\ &= p_k \left(\bar{v} - \delta u_{S^*}^{\pi^\circ}(\mathbf{x}, \mathbf{q})\right) + \delta u_k^{\pi^\circ}(\mathbf{x}, \mathbf{q}), \end{aligned}$$

where the last equality follows from Proposition 3 and the fact $v(\pi_i^\circ) = 0$ for all $i \in N$. Rearranging the terms and summing k over K, we have

$$u_K^{\pi^\circ}(\mathbf{x}, \mathbf{q}) - p_K \bar{v} = \delta \left(u_K^{\pi^\circ}(\mathbf{x}, \mathbf{q}) - p_K u_{S^*}^{\pi^\circ}(\mathbf{x}, \mathbf{q})\right), \tag{12}$$

or equivalently,

$$(1 - \delta) u_K^{\pi^\circ}(\mathbf{x}, \mathbf{q}) = p_K \left(\bar{v} - \delta u_{S^*}^{\pi^\circ}(\mathbf{x}, \mathbf{q})\right). \tag{13}$$

[12] This lemma generalizes Winter (1996), which is restricted on simple games with a uniform recognition probability and no dummy player.

Since $u^{\pi^\circ}_{S^*}(\mathbf{x},\mathbf{q}) \le \bar{v}$, the right-hand side of (13) is greater than or equals to $p_K\bar{v}(1-\delta)$ and this yields

$$u^{\pi^\circ}_K(\mathbf{x},\mathbf{q}) \ge p_K\bar{v}. \tag{14}$$

If $p_K = 1$, then $u^{\pi^\circ}_K(\mathbf{x},\mathbf{q}) = 1$ and due to Proposition 3, we have $x^{\pi^\circ}_K = \delta$, as desired. Now we assume that $p_K < 1$ and let

$$\bar{\delta} \equiv \frac{u^{\pi^\circ}_K(\mathbf{x},\mathbf{q}) - p_K\bar{v}}{u^{\pi^\circ}_K(\mathbf{x},\mathbf{q}) - p_K u^{\pi^\circ}_{S^*}(\mathbf{x},\mathbf{q})}. \tag{15}$$

If $\bar{\delta} < 1$, then there exists $\delta > \bar{\delta}$ which violates condition (12), and hence an efficient equilibrium does not exist for all $\delta > \bar{\delta}$. We divide into two cases to show that $u^{\pi^\circ}_K(\mathbf{x},\mathbf{q}) < \bar{v}$ implies $\bar{\delta} < 1$.

Case 1: $\emptyset \in \mathbf{A}$. Due to Lemma 9, for all $A \in \mathbf{A}$, it must be $x_A = u^{\pi^\circ}_A(\mathbf{x},\mathbf{q}) = 0$ and $u^{\pi^\circ}_K(\mathbf{x},\mathbf{q}) = u^{\pi^\circ}_{S^*}(\mathbf{x},\mathbf{q})$. Therefore, $u^{\pi^\circ}_K(\mathbf{x},\mathbf{q}) < \bar{v}$ implies $\bar{\delta} < 1$.

Case 2: $\emptyset \notin \mathbf{A}$. By Lemma 9, there exist $A^*, A' \in \mathbf{A}$ such that $S^* = K \cup A^*$ and $A^* \cap A' = \emptyset$. Since $S^* \in \underset{S \in \mathcal{P}(N)}{\operatorname{argmax}} \left(v(S) - x^{\pi^\circ}_S\right)$ and $v(S^*) = v(K \cup A')$, it must be $x^{\pi^\circ}_{A^*} \le x^{\pi^\circ}_{A'}$, or $u^{\pi^\circ}_{A^*}(\mathbf{x},\mathbf{q}) \le u^{\pi^\circ}_{A'}(\mathbf{x},\mathbf{q})$. If $u^{\pi^\circ}_{A^*}(\mathbf{x},\mathbf{q}) = 0$, then $u^{\pi^\circ}_K(\mathbf{x},\mathbf{q}) = u^{\pi^\circ}_{S^*}(\mathbf{x},\mathbf{q})$ and hence $u^{\pi^\circ}_K(\mathbf{x},\mathbf{q}) < \bar{v}$ implies $\bar{\delta} < 1$. Suppose that $u^{\pi^\circ}_{A^*}(\mathbf{x},\mathbf{q}) > 0$. Since $u^{\pi^\circ}_K(\mathbf{x},\mathbf{q}) + u^{\pi^\circ}_{A^*}(\mathbf{x},\mathbf{q}) + u^{\pi^\circ}_{A'}(\mathbf{x},\mathbf{q}) = u^{\pi^\circ}_{S^*}(\mathbf{x},\mathbf{q}) + u^{\pi^\circ}_{A'}(\mathbf{x},\mathbf{q}) \le \bar{v}$ and $0 < u^{\pi^\circ}_{A^*}(\mathbf{x},\mathbf{q}) \le u^{\pi^\circ}_{A'}(\mathbf{x},\mathbf{q})$, it must be $u^{\pi^\circ}_K(\mathbf{x},\mathbf{q}) < u^{\pi^\circ}_{S^*}(\mathbf{x},\mathbf{q}) < \bar{v}$ and $\bar{\delta} < 1$. □

For any $\pi \in \Pi$, let $K(\pi)$ the set of essential players in π, that is, $K(\pi) = \cap \mathbf{E}(\pi)$, where $\mathbf{E}(\pi) = \{S \subseteq N(\pi) \mid (\forall S' \subseteq N(\pi)) \ v(S) \ge v(S')\}$.

Lemma 11. *Let* $(\mathbf{x},\mathbf{q})$ *be a cutoff strategy equilibrium. For all* $\pi \in \Pi$ *such that* $N(\pi) = K(\pi)$ *and all* $i \in N(\pi)$,

$$u^{\pi}_i(\mathbf{x},\mathbf{q}) \ge v(\pi_i) + \delta^{n(\pi)-2} p^{\pi}_i \left(\bar{v} - v(N \setminus \pi_i)\right).$$

Proof. When $n(\pi) = 2$, it is clear that $u^{\pi}_i(\mathbf{x},\mathbf{q}) \ge v(\pi_i) + p^{\pi}_i\left[\bar{v} - v(N \setminus \pi_i)\right]$. Consider $\pi \in \Pi$ such that $n(\pi) \ge 3$ and $N(\pi) = K(\pi)$. As induction hypothesis, for all $\pi' \in \Pi$ such that $n(\pi') < n(\pi)$ and $N(\pi') = K(\pi')$ and for all $i \in N(\pi')$, suppose that $u^{\pi'}_i(\mathbf{x},\mathbf{q}) \ge v(\pi'_i) + \delta^{n(\pi')-2} p^{\pi'}_i\left[\bar{v} - v(N \setminus \pi'_i)\right]$. For all $j \in N(\pi)$ and $S \in \mathcal{P}_j(N(\pi))$ such that $i \notin S$, this assumption implies that $u_i^{\pi(j,S)}(\mathbf{x},\mathbf{q}) \geq v(\pi_i) + \delta^{n(\pi)-3} p_i^\pi \left[\bar{v} - v(N \setminus \pi_i)\right]$. By the definition of $u_i^\pi(\mathbf{x},\mathbf{q})$ or (2) and Proposition 3, we have

$$\begin{aligned}
u_i^\pi(\mathbf{x},\mathbf{q}) &\geq \sum_{j\in N(\pi)} p_j^\pi \sum_{S\in\mathcal{P}_j(N(\pi))} q_j^\pi(S)\left[\mathbb{1}(i\in S)x_i^\pi + \mathbb{1}(i\notin S)x_i^{\pi(j,S)}\right] \\
&\geq Q_i^\pi\left[(1-\delta)v(\pi_i) + \delta u_i^\pi(\mathbf{x},\mathbf{q})\right] \\
&\quad +(1-Q_i^\pi)\left[(1-\delta)v(\pi_i) + \delta\left[v(\pi_i) + \delta^{n(\pi)-3}p_i^\pi\left(\bar{v} - v(N\setminus\pi_i)\right)\right]\right] \\
&= \delta Q_i^\pi u_i^\pi(\mathbf{x},\mathbf{q}) + (1-\delta Q_i^\pi)v(\pi_i) + (1-Q_i^\pi)\delta^{n(\pi)-2}(\bar{v} - v(N\setminus\pi_i)),
\end{aligned}$$

where $Q_i^\pi = \sum_{j\in N(\pi)} p_j^\pi \sum_{S\in\mathcal{P}_j(N(\pi))} q_j^\pi(S)\mathbb{1}(i\in S)$. Rearranging the terms, the inequality yields

$$\begin{aligned}
u_i^\pi(\mathbf{x},\mathbf{q}) &\geq v(\pi_i) + \frac{1-Q_i^\pi}{1-\delta Q_i^\pi}\delta^{n(\pi)-2}(\bar{v} - v(N\setminus\pi_i)) \\
&\geq v(\pi_i) + \delta^{n(\pi)-2}(\bar{v} - v(N\setminus\pi_i)),
\end{aligned}$$

as desired. □

Lemma 12. *Consider a bargaining game (N, v, p, δ) with $p \in \Delta^\circ(N)$. In a cutoff strategy equilibrium $(\mathbf{x},\mathbf{q})$, there exists $\delta' < 1$ such that for all $\delta > \delta'$ and all $\pi \in \Pi$,*

$$[N(\pi) = K(\pi)] \implies (\exists i \in N(\pi))\;\; u_i^\pi(\mathbf{x},\mathbf{q}) \geq p_i^\pi \bar{v}.$$

Proof. This is from the proof of Proposition 4. □

Theorem 3. *Let (N, v) be a superadditive game with an essential player. A bargaining game (N, v, p, δ) with $p \in \Delta^\circ(N)$ has an efficient equilibrium for all $\delta \in (0, 1)$ if and only if (N, v) is a unanimity game.*

Proof. First, it is clear that if (N, v) is a unanimity game, then there exists an efficient equilibrium. To prove "only if" part, suppose there exists an efficient equilibrium. By Theorem 1, there exists a cutoff strategy SSPE $(\mathbf{x},\mathbf{q})$ in which all the agents form an efficient coalition immediately. Let $\delta > \bar{\delta}$, and hence, by Lemma 10, $x_j^{\pi^\circ} = 0$ for all $j \in N \setminus K$. Define $\tilde{v} = \max_{S\in\mathcal{P}(N)\setminus\mathbf{E}} v(S)$ so that $\tilde{v} < \bar{v}$. The proof is divided into three cases.

Case 1: $\emptyset \notin \mathbf{A}$.

Note that $v(K) < \bar{v}$ if $\emptyset \notin \mathbf{A}$. Let $i \in A \in \mathbf{A}$. If i forms an efficient coalition, then i's proposal surplus is:

$$\bar{v} - x_K^{\pi^\circ} - x_A^{\pi^\circ} = \bar{v} - x_K^{\pi^\circ} = \bar{v}(1-\delta). \tag{16}$$

Now suppose that i forms $N \setminus K$. If i forms $N \setminus K$, then $N(\pi^\circ(i, N \setminus K)) = K(\pi^\circ(i, N \setminus K))$. By Proposition 3 and Lemma 11, we have

$$\begin{aligned} x_i^{\pi^\circ(i,N\setminus K)} &= (1-\delta)v(\pi_i^\circ(i, N \setminus K)) + \delta u_i^{\pi^\circ(i,N\setminus K)}(\mathbf{x}, \mathbf{q}) \\ &= v(\pi_i^\circ(i, N \setminus K)) + \delta^{\#K} p_i^{\pi^\circ(i,N\setminus K)} (\bar{v} - v(K)) \\ &\geq \delta^{\#K}(1-p_K)(\bar{v} - v(K)), \end{aligned}$$

where $\#K$ is the cardinality of K. Thus, i's expected proposal surplus by forming $N \setminus K$ is:

$$\begin{aligned} x_i^{\pi^\circ(i,N\setminus K)} - x_{N\setminus K}^{\pi^\circ} &\geq \delta^{\#K}(1-p_K)(\bar{v} - v(K)) - x_{N\setminus K}^{\pi^\circ} \\ &= \delta^{\#K}(1-p_K)(\bar{v} - v(K)). \end{aligned} \tag{17}$$

Let

$$\delta^* = \max\left\{\bar{\delta}, \left(\frac{\bar{v}}{\bar{v} + (1-p_K)(\bar{v} - v(K))}\right)^{\frac{1}{\#K}}\right\}.$$

Since $p_K < 1$, $\bar{v} > v(K)$, and $\bar{\delta} < 1$, it must be $\delta^* < 1$. If $\delta > \delta^*$, then (17) is strictly greater than (16). Thus for all $\delta > \delta^*$, an efficient equilibrium is impossible.

Case 2: $\emptyset \in \mathbf{A}$ and $D \neq \emptyset$.

Let $k \in K$. If k forms a efficient coalition, then k's proposal surplus is:

$$\bar{v} - x_K^{\pi^\circ} = \bar{v}(1-\delta). \tag{18}$$

Furthermore, in the equilibrium $(\mathbf{x}, \mathbf{q})$, since all the essential players $k \in K$ are always nominated, $u_k^{\pi^\circ}(\mathbf{x}, \mathbf{q}) = p_k(\bar{v} - x_K^{\pi^\circ}) + x_i^{\pi^\circ}$. By Proposition 3, we have

$$(1-\delta)u_k^{\pi^\circ}(\mathbf{x}, \mathbf{q}) = p_k(\bar{v} - x_K^{\pi^\circ}), \tag{19}$$

and $(1-\delta)u_K^{\pi^\circ}(\mathbf{x}, \mathbf{q}) = p_K(\bar{v} - x_K^{\pi^\circ})$ by summing k over K. By Lemma 10, we have $u_k^{\pi^\circ}(\mathbf{x}, \mathbf{q}) = \frac{p_k}{p_K} u_V^{\pi^\circ}(\mathbf{x}, \mathbf{q}) = \frac{p_k}{p_K}\bar{v}$, and hence $x_k^{\pi^\circ} = \delta \frac{p_k}{p_V}\bar{v}$. Now suppose that k forms $D \cup \{k\}$. If k forms

$D \cup \{k\}$, then $N(\pi^\circ(k, D \cup \{k\})) = K(\pi^\circ(k, D \cup \{k\}))$. By Proposition 3 and Lemma 12, there exists δ' and $k \in K$ such that when $\delta > \delta'$

$$\begin{aligned} x_k^{\pi^\circ(k,D\cup\{k\})} &= (1-\delta)v(\pi_k^\circ(k, D \cup \{k\})) + \delta u_k^{\pi^\circ(k,D\cup\{k\})}(\mathbf{x}, \mathbf{q}) \\ &\geq v(\pi_k^\circ(k, D \cup \{k\})) + \delta p_k^{\pi^\circ(k,D\cup\{k\})}\bar{v} \\ &\geq \delta(p_D + p_k)\bar{v}. \end{aligned}$$

Thus, k's expected proposal surplus by forming $D \cup \{k\}$ is:

$$\begin{aligned} x_k^{\pi^\circ(k,D\cup\{k\})} - x_{D\cup\{k\}}^{\pi^\circ} &\geq \delta(p_D + p_k)\bar{v} - x_{D\cup\{k\}}^{\pi^\circ} \\ &= \delta(p_D + p_k)\bar{v} - x_k^{\pi^\circ} \qquad (20) \\ &= \delta\bar{v}\left(p_D + p_k - \frac{p_k}{p_K}\right). \qquad (21) \end{aligned}$$

Let

$$\delta^* = \max\left\{\bar{\delta},\ \delta',\ \frac{1}{1 + p_D + p_k - \frac{p_k}{p_K}}\right\}.$$

To show $\delta^* < 1$, suppose for contraction that $1 + p_D + p_k - \frac{p_k}{p_K} \leq 1$, or equivalently,

$$p_D + p_k \leq \frac{p_k}{p_K}. \qquad (22)$$

Since $p_D + p_K = 1$, (22) can be rewritten as $(1 - p_K)(p_K - p_k) \leq 0$. Since $p_K < 1$, in order that (22) holds, it must be $p_K = p_k$. However, if $p_k = p_K$, then $\{k\} = K \in \mathbf{E}$, which contradicts to ($\mathbf{E}$1). We have shown that $\delta^* < 1$. If $\delta > \delta^*$, then (20) is strictly greater than (18). Thus for all $\delta > \delta^*$, an efficient equilibrium is impossible.

Case 3: $\emptyset \in \mathbf{A}$ and $D = \emptyset$.

In this case, it must be $\mathbf{E} = \{N\}$, that is, the grand-coalition is the unique efficient coalition. By Theorem 2, an efficient equilibrium is impossible unless (N, v) is a unanimity game.

By all the cases, the assumption of an efficient equilibrium yields contractions for sufficiently high δ unless (N, v) is a unanimity game. □

The following corollary is a direct consequence of Case 1 and Case 2 in the proof of Theorem 3.

Corollary 1. *Let (N, v) be a super-additive game with an essential player and suppose that it has multiple efficient coalitions. There exists $\delta^* < 1$ such that, for all $\delta > \delta^*$, a bargaining game (N, v, p, δ) with $p \in \Delta^\circ(N)$ ends up with non-minimal efficient coalition with positive probability.*

5.1 Simple Games

Given a set of agents N, a class of subsets $\mathbf{W} \subset \mathcal{P}(N)$ is a set of *winning coalitions* if

i) $\{i\} \notin \mathbf{W}$ for all $i \in N$; and

ii) $S \in \mathbf{W}$ and $S \subset S'$ imply $S' \in \mathbf{W}$.

A characteristic function form game (N, v) is *simple* if $v(S) = 1$ for all $S \in \mathbf{W}$ and $v(S) = 0$ otherwise.[13] Let $\mathbf{W}^m = \{S \in \mathbf{W} \mid (\forall i \in S)\ S \setminus \{i\} \notin \mathbf{W}\}$ be a set of *minimal winning coalitions*, $V = \cap \mathbf{W}$ a set of *veto players*, and $D = N \setminus (\cup \mathbf{W}^m)$ a set of *dummy players*. We also define a set of *auxiliary coalitions* $\mathbf{A} = \{A \subseteq N \setminus V \mid A \cup V \in \mathbf{W}^m\}$. For notational simplicity, for any $z \in \mathbb{R}^n$ and $S \in \mathcal{P}(N)$, denote $z_S = \sum_{j \in S} z_j$. For simplicity, assume that $p \in \Delta^\circ(N)$.

Definition 4 (Winning-coalition equilibrium)**.** A strategy profile is a *winning-coalition equilibrium* if it is a subgame perfect Nash equilibrium, in which for any history a proposer makes an offer to form a winning coalition and respondents accept the offer.

Corollary 2. *Let (N, v) be a simple game with a veto player. A bargaining game (N, v, p, δ) has a winning-coalition equilibrium for all $\delta \in (0, 1)$ if and only if it is unanimous.*

Corollary 3. *Let (N, v) be a simple game with a veto player and multiple winning coalitions. There exist $\delta^* < 1$ such that, for all $\delta > \delta^*$, a non-minimal winning coalition forms with positive probability in an SSPE of the bargaining game (N, v, p, δ). Furthermore, the equilibrium expected payoff vector is not in the core.*

[13] Simple games are introduced by von Neumann and Morgenstern (1944). See Shapley (1962) for mathematical properties of simple games.

6 Three-party weighted majority games

Let $N = \{1, 2, 3\}$ be a set of parties. A weight vector $p \equiv \{p_i\}_{i \in N}$ with $p_i > 0$ and $p_N = 1$ represents each party's weight. Without loss of generality, assume that $p_1 \geq p_2 \geq p_3$. Let $w^* \in (1/2, 1]$ be a winning quota. A characteristic function form game (N, v) is a *weighted majority game* with (p, w^*) if $v(S) = 1$ for all $S \subseteq N$ such that $p_S \geq w^*$ and $v(S) = 0$ otherwise. We assume that each party's recognition probability equals to its weight. We divide cases depend on the number of veto parties. If all the parties are veto party, then the game is unanimous.

For any $\pi \in \Pi$ such that $n(\pi) = 2$, by a standard two-party random proposer model, there exists a unique subgame perfect equilibrium, which is a cutoff strategy equilibrium with $x_i^\pi = \delta p_i^\pi$ and $q^\pi(N(\pi))_i = 1$ for all $i \in N(\pi)$. Thus specifying strategies for the initial coalitional state π° is enough for stationary subgame perfect equilibria. In this section, for notational simplicity, we omit the superscript π° for the initial coalitional state.

6.1 No veto party

No veto party implies that any two-party coalition is a winning coalition, that is, $\mathbf{W}^m = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Furthermore, in an SSPE (x, q), no matter who proposes first, the winning coalition will be formed immediately, and hence $u_N(x, q) = 1$, or equivalently, $x_N = \delta$.

Lemma 13. *Let (x, q) is an SSPE. For all $\delta \in (0, 1]$ and all $i \in N$, $x_i > 0$.*

Proof. First, suppose for contradiction $x_1 > 0$ and $x_2 = x_3 = 0$. It must be $D_2(x) = \{\{2, 3\}\}$ and party 2 can form a winning coalition with party 3 without any cost. Hence $u_2(x, q) \geq p_2 > 0$ is strictly positive and hence the expected payoff $x_2 = \delta u_2(x, q) > 0$ is also strictly positive, which yields a contradiction. Now suppose that $x_1 > 0$, $x_2 > 0$, and $x_3 = 0$. Then it must be $u_3(x, q) \geq p_3(1 - x_2)$. Since $x_2 < \delta$, we have $u_3(x, q) > 0$, which contradicts that $x_3 = 0$. □

Proposition 6. *Let* (N, v) *be a three-party weighter majority game with* (w, w^*)*. If there is no veto party, then all the parties have the same expected payoff in any SSPE. That is,* $x_1 = x_2 = x_3 = \frac{\delta}{3}$*.*

Proof. Suppose for contraction that $x_1 > x_2$ and $x_1 > x_3$. It must be that $D_2(x) = D_3(x) = \{\{2, 3\}\}$ and $N \notin D_1(x)$. Denote that $q_{12} = q_1(\{1, 2\})$ and $q_{13} = q_1(\{1, 3\})$. Then we have

$$\begin{aligned} u_1(x, q) &= p_1(1 - q_{12}x_2 - q_{13}x_3), \\ u_2(x, q) &= p_2(1 - x_3) + (p_3 + q_{12}p_1)x_2, \\ u_3(x, q) &= p_3(1 - x_2) + (p_2 + q_{13}p_1)x_3. \end{aligned}$$

Summing up, since $q_{12} + q_{13} = 1$, we have $u_N(x, q) = 1 - p_3x_2 - p_2x_3 < 1$, which contradicts to Lemma 8. □

It is worth to remark that for any weight vector w and any discount factor δ, all the parties expect the same payoff in an SSPE.

Example 1. Consider the following three-party voting problem. Each party has 4,3, and 2 votes and 5 votes are required to win, that is, $p = \left(\frac{4}{9}, \frac{3}{9}, \frac{2}{9}\right)$ and $w^* = \frac{5}{9}$. Since all the parties are veto, the equilibrium expected payoffs are $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Furthermore, this does not depend on δ.

6.2 Single veto party

By Theorem 3, if there exists a veto player, then an intermediate non-winning coalition can be formed. Thus, in an SSPE (x, q), it must be $u_N(x, q) < 1$ if $\delta < 1$ and the game is not unanimous. For ease of exposition, we concentrate on the cases $\delta = 1$. Since there always exists a strictly positive proposal gain even at $\delta = 1$, the equilibrium payoff vector is unique in the class of SSPE.

Lemma 14. *Suppose* $\delta = 1$*. For an SSPE* (x, q)*,* $x_N = u_N(x, q) = 1$*.*

Proof. It follows directly from Proposition 3. □

For any $S \subseteq N$ and $x \in X$, let $e(S,x) \equiv x_j^{\pi^\circ(j,S)} - x_S$ be an excess surplus for $j \in S$ to form S.[14]

If there exists a single veto party, then it must be $\mathbf{W}^m = \{\{1,2\},\{1,3\}\}$. That is party 1 is the veto party and party 2 and party 3 are auxiliary.

Lemma 15. *Let (N,v) be a three-party weighter majority game with (p,w^*) and there is a single veto party. Let (x,q) be an cutoff SSPE. If $\delta = 1$, then $x_2 = x_3$.*

Proof. Suppose $x_2 > x_3$. Only three cases are possible.

Case 1: $e(\{23\},x) > e(\{13\},x) > e(\{12\},x)$.

It must be $q_{13} = q_{23} = q_{32} = 1$. Thus, the players expected payoffs are:

$$\begin{aligned} x_1 &= p_1(1-x_3); \\ x_2 &= p_2(p_2+p_3-x_3)+p_3x_2; \\ x_3 &= p_3(p_2+p_3-x_2)+(p_1+p_2)x_3. \end{aligned}$$

The second equation yields $(p_1+p_2)x_2 = p_2(p_2+p_3-x_3)$ and the third equation yields $x_3 = p_2+p_3-x_2$. Combining two conditions, we have $p_1 = 0$, which is a contradiction.

Case 2: $e(\{13\},x) > e(\{23\},x) > e(\{12\},x)$.

It must be $q_{13} = q_{23} = q_{32} = 1$. Thus, player 3's expected payoff is

$$x_3 = p_3(1-x_1)+(p_1+p_2)x_3.$$

Rearranging the terms, we have $(1-p_1-p_2)x_3 = (1-x_1)p_3$, which implies $x_1+x_3 = 1$, or $x_2 = 0$. However, this contradicts to Lemma 13.

Case 3: $e(\{13\},x) > e(\{12\},x) > e(\{23\},x)$.

It must be $q_{13} = q_{21} = q_{31} = 1$ and this implies that a winning coalition must be formed immediately. By Theorem 3, the underlying game must be unanimous, which is a contraction.

□

Proposition 7. *Let (N,v) be a three-party weighter majority game with (p,w^*) and there is a single veto party. Let (x,q) be an cutoff SSPE.*

[14] Note that $x_j^{\pi^\circ(j,S)} - x_S$ does not depend on $j \in S$, and hence $e(S,x)$ is uniquely defined.

i. (Strong Solidarity.) If the single veto party's weight is greater than or equals to $\frac{1}{2}$, then two other auxiliary party form an intermediate coalitional each other with probability 1 and their expected payoffs are:

$$x_1 = \frac{p_1(3-2p_1)}{2-p_1}, \quad \text{and} \quad x_2 = x_3 = \frac{(1-p_1)^2}{2-p_1}. \tag{23}$$

ii. (Weak Solidarity.) If the single veto party's weight is less than $\frac{1}{2}$, then two other auxiliary party form an intermediate coalitional each other with strictly positive probability but less than 1 and their expected payoffs are:

$$x_1 = \frac{1+2p_1}{3}, \quad \text{and} \quad x_2 = x_3 = \frac{1-p_1}{3}. \tag{24}$$

Proof. i. (Strong Solidarity.) Suppose $q_{23} = q_{32} = 1$. It must be $e(\{2,3\}, x) \geq e(\{1,2\}, x)$, or $x_1 - x_2 \geq p_1$. Since $x_2 = x_3$ by Lemma 15, the veto party's expected payoff is:

$$\begin{aligned} x_1 &= p_1(1-x_2) + p_2 x_1^{\pi^\circ(2,\{2,3\})} + p_3 x_1^{\pi^\circ(3,\{2,3\})} \\ &= p_1\left(1 - \left(\frac{1}{2} - \frac{x_1}{2}\right)\right) + (1-p_1)p_1, \end{aligned}$$

which yields (23). The condition $x_1 - x_2 \geq p_1$ requires that $\frac{p_1(3-2p_1)}{2-p_1} - \frac{1-p_1}{3} \geq p_1$. Solving this inequality, p_1 must satisfy $-2p_1^2 + 3p_1 - 1 \geq 0$, or $\frac{1}{2} \leq p_1 \leq 1$. This completes the proof of the first part.

ii. (Weak Solidarity.) Suppose $0 < q_{23} < 1$ and $0 < q_{32} < 1$. It must be $e(\{2,3\}, x) = e(\{1,2\}, x) = e(\{1,3\}, x)$, or $x_1 - x_2 = p_1 = x_1 - x_3$. Solving these equations with $x_N = 1$, we have (24). In this case, the veto party's expected payoff is:

$$\begin{aligned} x_1 &= p_1(1-x_2) + p_2\left(q_{21}x_1 + q_{23}x_1^{\pi^\circ(2,\{2,3\})}\right) + p_3\left(q_{31}x_1 + q_{32}x_1^{\pi^\circ(3,\{2,3\})}\right) \\ &= p_1(1-x_2) + rx_1 + (1-r)p_1 - p_1^2, \end{aligned} \tag{25}$$

where $r = p_2q_{21} + p_3q_{31} > 0$ is the probability that the veto party is nominated by other parties. Plugging (24) into (25), it follows that

$$\frac{1+2p_1}{3} = p_1\left(1 - \frac{1-p_1}{3}\right) + r\frac{1+2p_1}{3} + (1-r)p_1 - p_1^2,$$

which yields $r = 1 - 2p_1$. Since $r > 0$, it must be $r = 1 - 2p_1 > 0$, or $p_1 < \frac{1}{2}$. This completes the proof of the second part. □

Example 2. Consider following three-party voting problems.

i. (Weak Solidarity.) Each party has 4,3, and 2 votes and 6 votes are required to win, that is, $p = \left(\frac{4}{9}, \frac{3}{9}, \frac{2}{9}\right)$ and $w^* = \frac{2}{3}$. Since the veto party's weight is less than $\frac{1}{2}$, the smaller two parties form a union each other with positive probability whenever they are supposed to propose. The equilibrium expected payoffs are $\left(\frac{17}{27}, \frac{5}{27}, \frac{5}{27}\right)$.

ii. (Strong Solidarity.) Each party has 3,2, and 1 votes and 4 votes are required to win, that is, $p = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$ and $w^* = \frac{2}{3}$. Since the veto party's weight is $\frac{1}{2}$, the smaller two parties always form a union each other whenever they are supposed to propose. The equilibrium expected payoffs are $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$.

6.3 Two veto parties

If there are exactly two veto parties, then it must be $\mathbf{W}^m = \{\{1, 2\}\}$. That is party 1 and party 3 are the veto parties and party 3 is dummy.

Lemma 16. *Let (N, v) be a three-party weighter majority game with (p, w^*) and there are two veto parties. Let (x, q) be an cutoff SSPE. If $\delta = 1$, then*

$$e(\{1, 2\}, x) = e(\{1, 3\}, x) = e(\{2, 3\}, x).$$

Proof. **Step 1:** Suppose that $e(\{1, 2\}, x) > e(\{1, 3\}, x)$. It must be $e(\{2, 3\}, x) \geq e(\{1, 2\}, x)$, otherwise Theorem 3 is violated. Thus, party 2 is always nominated by other parties, and hence we have $x_2 \geq p_2(1 - x_1) + (p_2 + p_3)x_2$, or equivalently, $x_1 + x_2 \geq 1$ and $x_3 \leq 0$, which is a contraction.

Step 2: Suppose that $e(\{1, 3\}, x) > e(\{1, 2\}, x)$.

- If $e(\{2, 3\}, x) > e(\{1, 2\}, x)$, then party 3 is always nominated by other parties, and hence $x_3 = p_3(1 - x_1) + (1 - p_3)x_3$, or $x_1 + x_3 = 1$, which is a contradiction.

- If $e(\{1,2\},x) > e(\{2,3\},x)$, then party 1 is always nominated by other parties, and hence $x_1 = p_1(1-x_3)+(1-p_1)x_1$, or $x_1+x_3 = 1$, which is a contradiction.

- If $e(\{1,2\},x) = e(\{2,3\},x)$, then it must be $1-x_1 = p_2+p_3-x_3$, or $p_1+x_2 = 1$. Since party 1 and party 3 do not nominate party 2, $x_2 = p_2(1-x_1)+(1-p_2)p_2 = p_2(2-x_1-p_2)$. Plugging $p_1+x_2 = 1$, it follows $x_2 = p_2$. Thus, we have $e(\{1,3\},x) = p_1+p_3-x_1-x_3 = (1-p_2)-(1-x_2) = 0$. However, the assumption $e(\{1,3\},x) > e(\{1,2\},x)$ implies $1-x_1-x_2 < 0$, which is a contradiction.

Step 3: Suppose that $e(\{1,2\},x) = e(\{1,3\},x)$. This condition implies that

$$1-x_2 = p_1+p_3-x_3 = (1-p_2)-(1-x_1-x_2) = x_1+x_2-p_2. \tag{26}$$

If $e(\{1,2\},x) = e(\{1,3\},x) > e(\{2,3\},x)$ then party 1 is always nominated by other parties, which is a contradiction again. If $e(\{1,2\},x) = e(\{1,3\},x) < e(\{2,3\},x)$, party 1 is not dominated by other parties, and hence $x_1 = p_1(1-x_2)+(1-p_1)p_1$. Thus, with (26) and $x_N = 1$, we have $x_1 = \frac{3(1-p_1)p_1}{2-p_1}$, $x_2 = \frac{1-p_1+p_1^2}{2-p_1}$, and $x_3 = \frac{1-3p+2p^2}{2-p}$. Plugging them into the condition $e(\{1,2\},x) < e(\{2,3\},x)$, that is, $1-x_1 < p_2+p_3-x_3$, it must be

$$1-\frac{3(1-p_1)p_1}{2-p_1} < 1-p_1-\frac{1-3p+2p^2}{2-p},$$

or equivalently, $(2p-1)^2 < 0$, which is a contradiction. By Theorem 4, there exists a cutoff strategy SSPE, and hence, it must be $e(\{1,2\},x) = e(\{1,3\},x) = e(\{2,3\},x)$. □

Proposition 8. *Let (N,v) be a three-party weighter majority game with (p,w^*) and there are two veto parties. Let (x,q) be an cutoff SSPE. If $\delta = 1$, then the veto players form an intermediate coalition with strictly positive probability and their expected payoffs are:*

$$x_1 = p_1+\frac{p_3}{3}, \qquad x_2 = p_2+\frac{p_3}{3}, \quad \textit{and} \qquad x_3 = \frac{p_3}{3}. \tag{27}$$

Proof. By Lemma 16, we have $e(\{1,2\},x) = e(\{1,3\},x) = e(\{2,3\},x)$. The first equation implies that $1-x_1-x_2 = p_1+p_3-x_1-x_3$, or $x_2-x_3 = p_2$. The second equation implies that $p_1+p_3-x_1-x_3 = p_2+p_3-x_2-x_3$, or $2x_2+x_3 = p_2+p_3$. Solving two conditions $x_2-x_3 = p_2$ and $2x_2+x_3 = p_2+p_3$ with $x_N = p_N = 1$ yields (27). □

Example 3. Consider the following three-party voting problem. Each party has 3,2, and 1 votes and 5 votes are required to win, that is, $p = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$ and $w^* = \frac{5}{6}$, that is, the two large parties are veto and the smallest party is dummy. The equilibrium expected payoffs are $\left(\frac{10}{18}, \frac{7}{18}, \frac{1}{18}\right)$.

7 Wage Bargaining and Labor Union

An employer-employee game is a three-person game with a set of players is $N = \{1, 2, 3\}$ and a characteristic function $v : \mathcal{P}(N) \to \mathbb{R}^3$ is:

$$v(S) = \begin{cases} 1 & \text{if } S = N \\ a & \text{if } S = \{1,2\}, \{1,3\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \leq a \leq 1$. Player 1 represents an employer or a firm; and player 2 and player 3 represent employees or workers. a refers the first worker's product and $1 - a$ reflects the second worker's marginal product. For extreme cases, if $a = 0$, then the game is unanimous; and if $a = 1$, then the game is a simple game with a single veto player.

In the first subsection, we study stationary subgame perfect equilibria, varying the common discount factor $0 < \delta \leq 1$, fixing $a = 1$ and the uniform recognition probability $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. When $\delta > \frac{6}{7}$, the workers form a union with positive probability. In the second subsection, we investigate the effect of the recognition probability, for fixed $a = 1$ and $\delta = 1$. The workers' recognition probability is lower than a certain level, then they form a union with probability 1, whenever they are supposed to propose. We compare this result with Okada (2011) to highlight the effect of buyout options.

7.1 The effect of the common discount factor δ

If $a = 1$, then there exists a unique core allocation, in which the market clearing wage is zero and the firm takes all the surplus. We assume a uniform recognition probability, that is, each player can be selected as a proposer with probabilities $\frac{1}{3}$. Again we focus on specifying strategies for the initial coalitional state π°. Suppose two workers are identical, we assume symmetric strategies for the workers. A cutoff strategy SSPE (x, q) is symmetric if $x_2 = x_3$, $q_2 = q_3$ and $q_{12} = q_{13}$. In a symmetric equilibrium (x, q), the excess surplus for each coalition is:

- $e(\{1,2\}, x) = e(\{1,3\}, x) = 1 - (x_1 + x_2)$;
- $e(\{2,3\}, x) = x_2^{\pi^\circ(2,\{2,3\})} - 2x_2 = \frac{2}{3}\delta - 2x_2$; and
- $e(N, x) = 1 - (x_1 - 2x_2)$.

Since $x_2 > 0$ in an equilibrium, forming N is strictly dominated by forming either $\{1,2\}$ or $\{1,3\}$ and hence $q_{12} = q_{13} = \frac{1}{2}$. Note that q_{23} be the probability that a worker makes a proposal to the other worker. If the proposal between workers is accepted, then a union is formed. Note that the union itself produces nothing, but it could increase workers' bargaining power by unifying their negotiation channel to the firm.

Lemma 17. *(Possibility of solidarity.) If $\delta > \frac{6}{7}$, each worker forms a union with a strictly positive probability.*

Proof. Suppose, for contradiction, $q_{23} = 0$. It must be $e(12, x) \geq e(23, x)$, that is,

$$1 - \frac{2}{3}\delta \geq x_1 - x_2 = \delta(u_1 - u_2). \tag{28}$$

Since $q_{21} = q_{31} = 1$ and $q_{12} = q_{13} = \frac{1}{2}$, their expected payoffs are:

$$u_1 = \frac{1}{3}(1 - \delta u_2) + \frac{2}{3}\delta u_1; \text{ and} \tag{29}$$

$$u_2 = \frac{1}{3}(1 - \delta u_1) + \frac{1}{3}\frac{1}{2}\delta u_2. \tag{30}$$

Solving (29) and (30), we have $u_1 = \frac{2-\delta}{6-5\delta}$ and $u_2 = \frac{2-2\delta}{6-5\delta}$. Plugging u_1 and u_2 into (28), it follows $7\delta^2 - 27\delta + 18 \geq 0$, which is $\delta \leq \frac{6}{7}$. $\square$

Lemma 18. *(Impossibility of strong solidarity.) For all $0 \leq \delta \leq 1$, each worker forms a union with probability less than 1.*

Proof. Suppose, for contradiction, $q_{23} = 1$, then it must be $e(12, x) \leq e(23, x)$. Since $q_{23} = q_{32} = 1$ and $q_{12} = q_{13} = \frac{1}{2}$, their expected payoffs are:

$$u_1 = \frac{1}{3}(1 - \delta u_2) + \frac{2}{3}\delta\frac{1}{3}; \text{ and} \tag{31}$$

$$u_2 = \frac{1}{3}(\delta\frac{2}{3} - \delta u_2) + \frac{1}{3}\delta u_2 + \frac{1}{3}\frac{1}{2}\delta u_2. \tag{32}$$

which yields $u_1 = \frac{6+3\delta-2\delta^2}{18-3\delta}$ and $u_2 = \frac{4\delta}{18-3\delta}$. With u_1 and u_2, the condition $e(12, x) \leq e(23, x)$, that is, $1 - \frac{2}{3}\delta \leq \delta(u_1 - u_2)$, implies that $\delta^2 + 3\delta - 6 \geq 0$. However, this contradicts to $0 \leq \delta \leq 1$. □

Proposition 9. *There are two types of cutoff strategy symmetric SSPE depend on δ.*

i. (No Solidarity.) If $\delta \leq \frac{6}{7}$, then each worker always makes an offer only to the firm and the equilibrium expected payoff is $u_1(\delta) = \frac{2-\delta}{6-5\delta}$ for the firm and $u_2(\delta) = \frac{2-2\delta}{6-5\delta}$ for each worker.

ii. (Weak Solidarity.) If $\delta > \frac{6}{7}$, then each worker makes an offer to each other with probability $q_{23}(\delta)$ and the equilibrium expected payoff is $u_1(\delta)$ and $u_2(\delta)$, where

$$q_{23}(\delta) = \frac{3 - \delta^2 - \sqrt{\delta^4 - 28\delta^3 + 130\delta^2 - 180\delta + 81}}{4\delta(1-\delta)}$$

$$u_1(\delta) = \frac{-9 + 26\delta - 19\delta^2 + 4\delta^3 - \sqrt{\delta^4 - 28\delta^3 + 130\delta^2 - 180\delta + 81}}{3\delta\left(3 - 2\delta + \delta^2 - \sqrt{\delta^4 - 28\delta^3 + 130\delta^2 - 180\delta + 81}\right)}$$

$$u_2(\delta) = \frac{8\delta(1-\delta)}{9 - 10\delta - \delta^2 + 3\left(\sqrt{\delta^4 - 28\delta^3 + 130\delta^2 - 180\delta + 81}\right)}.$$

Proof. The first part is directly from Lemma 17. By Lemma 17 and Lemma 18, if $\delta > \frac{6}{7}$, then it must be $0 < q_{23} = q_{32} < 1$ and hence $e(12, x) = e(23, x)$, or equivalently

$$1 - \frac{2}{3}\delta = x_1 - x_2 = \delta(u_1 - u_2). \tag{33}$$

With $q_{23} = q_{32}$ and $q_{12} = q_{13} = \frac{1}{2}$, their expected payoffs are:

$$u_1 = \frac{1}{3}(1 - \delta u_2) + \frac{2}{3}\left((1 - q_{23})\delta\frac{1}{3} + q_{23}\delta u_1\right); \text{ and} \tag{34}$$

$$u_2 = \frac{1}{3}(\delta\frac{2}{3} - \delta u_2) + \frac{1}{3}(q_{23}\delta u_2) + \frac{1}{3}\frac{1}{2}\delta u_2. \tag{35}$$

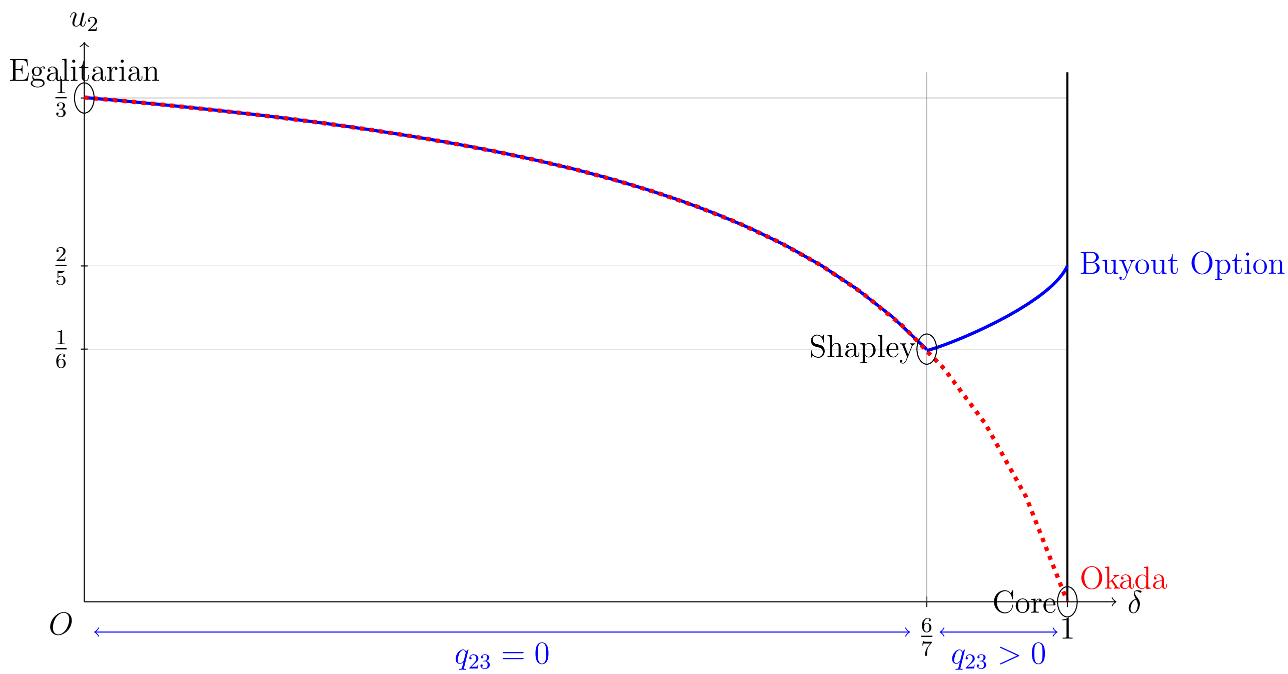

Figure 1: Patience causes Inefficiency

Simultaneously solving the three equations (33), (34), and (35) completes the proof of the second part. □

If $\delta = 1$, then the equilibrium expected payoff vector is $\left(\frac{5}{9}, \frac{2}{9}, \frac{2}{9}\right)$ and each worker form a labor union with probability $\frac{1}{2}$. If $\delta = \frac{6}{7}$, then the equilibrium expected payoff vector is $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$, which coincides to the Shapley value of the underlying characteristic function form game. Note that the unique core allocation is $(1, 0, 0)$ and this allocation. If the agents have no buyout option as Okada (2011), then the second type of equilibria (Weak Solidarity) is impossible. Hence, without buyout options, for all $0 \leq \delta \leq 1$, the equilibrium expected payoff vector must be $\left(\frac{2-\delta}{6-5\delta}, \frac{2-2\delta}{6-5\delta}, \frac{2-2\delta}{6-5\delta}\right)$ and this converges to the core allocation. However, allowing buyout options to each agent, non-core allocations can be obtained as an equilibrium. See Figure 1. Considering that most of the power indexes do not belong to core allocations, buyout options can justify such power indexes.

Remark. When the workers do not make an offer each other, the winning coalition will be formed immediately and hence the sum of equilibrium expected payoffs must be 1, no matter what δ is, as long as $\delta \leq \frac{6}{7}$. However, if $\frac{6}{7} < \delta < 1$, there must be efficiency loss, that is, the sum of equilibrium expected payoffs is strictly less than 1.

7.2 The effect of workers' recognition probability

In this subsection, we assume that the recognition probability is $(1-2p, p, p)$, that is each worker can be selected as a proposer with probability $0 < p < \frac{1}{2}$. For ease of exposition, fix $a = 1$ and $\delta = 1$. In a symmetric equilibrium (x, q), the excess surplus for each coalition is:

- $e(\{1,2\}, x) = e(\{1,3\}, x) = 1 - (x_1 + x_2)$;
- $e(\{2,3\}, x) = x_2^{\pi^\circ(2,\{2,3\})} - 2x_2 = 2p\delta - 2x_2 = 2p - 2x_2$; and
- $e(N, x) = 1 - (x_1 - 2x_2)$.

Since $\delta = 1$, note that $x_i = u_i$ for each $i \in N$ and $x_N = u_N = 1$. Again, forming N is dominated and hence $q_{12} = q_{13} = \frac{1}{2}$. At $\delta = 1$, for any positive workers' recognition probability, each worker form a labor union with positive probability, due to Theorem 3. Now we show that if the workers' recognition probability is lower than a certain level, then they form a union for sure whenever they are supposed to propose.

Proposition 10. *There are two types of cutoff strategy symmetric SSPE depend on p.*

i. (Weak Solidarity.) If $\frac{1}{4} < p < \frac{1}{2}$, then each worker makes an offer to each other with probability $q_{23} = q_{32} = \frac{1-2p}{2p}$ and the equilibrium expected payoff is

$$\left(1 - \frac{4}{3}p, \frac{2}{3}p, \frac{2}{3}p\right).$$

ii. (Strong Solidarity.) If $p \leq \frac{1}{4}$, then each worker makes an offer to each other with probability 1 and the equilibrium expected payoff is

$$\left(1 - \frac{8p^2}{1+2p}, \frac{4p^2}{1+2p}, \frac{4p^2}{1+2p}\right).$$

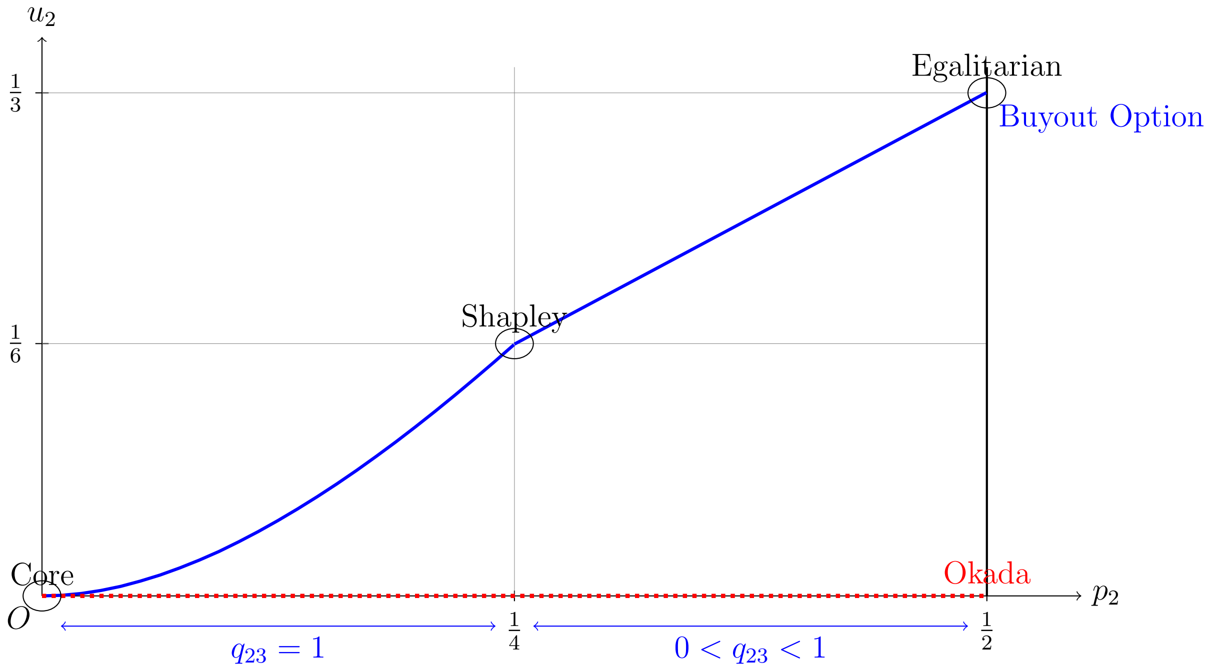

Figure 2: The less likely to be recognized, the more likely to form a union

Proof. **Cases 1: Weak Solidarity.** Suppose that $0 < q_{23} < 1$. It must be $e(23, x) = e(12, x)$, which implies that $x_2 = \frac{2}{3}p$. Each worker's equilibrium expected payoff is

$$x_2 = p(2p - x_2) + pq_{23}x_2 + (1 - 2p)q_{12}x_2, \tag{36}$$

which implies $x_2 = \frac{4p^2}{1+4p-2pq_{23}}$. It follows, with the condition $e(23, x) = e(12, x)$, that $x_2 = \frac{4p^2}{1+4p-2pq_{23}} = \frac{2}{3}p$, or equivalently, $q_{23} = \frac{1-2p}{2p}$. Plugging $q = \frac{1-2p}{2p}$, we have $x_2 = \frac{2}{3}p$ and $x_1 = 1 - 2x_2 = 1 - \frac{4}{3}p$. Since $q_{23} = \frac{1-2p}{2p}$ is assumed between 0 and 1, it must be $0 < \frac{1-2p}{2p} < 1$. This condition requires that $\frac{1}{4} < p < \frac{1}{2}$, which completes the proof of the first part.

Cases 2: Strong Solidarity. Suppose that $q_{23} = 1$. It must be $e(23, x) \geq e(12, x)$, which implies that $x_2 \leq \frac{2}{3}p$. Each worker's equilibrium expected payoff is

$$x_2 = p(2p - x_2) + px_2 + (1 - 2p)q_{12}x_2, \tag{37}$$

which implies $x_2 = \frac{4p^2}{1+2p}$ and $x_1 = 1 - 2x_2 = 1 - \frac{8p^2}{1+2p}$. Plugging x_2 into the condition $e(23, x) \geq e(12, x)$, it must be $\frac{4p^2}{1+2p} \leq \frac{2}{3}p$, or equivalently, $p \leq \frac{1}{4}$. This completes the proof of the second part. □

If $p = \frac{1}{4}$, each worker form a union with probability 1 and its equilibrium expected payoff coincides to the Shapley value of the underlying characteristic function form game. As $p \to \frac{1}{2}$, that is the firm has little chance to propose, the equilibrium expected payoff converges to the eqalitarian solution, in which all the agents split the surplus equally. As $p \to 0$, that is workers has little chance to propose, the equilibria expected payoff converges to the core allocation.

Figure 2 illustrates the effect of buyout option comparing the result with standard models which have no buyout option. In random-proposer models without buyout option, as $\delta \to 1$, the equilibrium payoff must be in the core as long as the core is nonempty. More specifically in Okada (2011), if $\delta = 1$, workers' payoff is always zero no matter what workers recognition probability. However, as the result of buyout option, workers can form a union and increase their bargaining power by unifying their negotiation channel, and hence they can get a wage more than their marginal product.

Appendix: Existence of SSPE

In this section, we present an outline for a proof of the existence of a cutoff strategy SSPE. This is based on Eraslan (2002). For all $\pi \in \Pi$, if $n(\pi) = 2$ then x^π and q^π are uniquely defined by, for all $i \in N(\pi)$,

i) $x_i^\pi = (1-\delta)v(\pi_i) + \delta p_i^\pi v(N)$; and

ii) $q_i(N(\pi)) = 1$.

Now we take an arbitrary $\pi \in \Pi$ and suppose that there exist $\mathbf{x}^\pi = \{\{x_i^{\pi'}\}_{i \in N(\pi')}\}_{\pi' \in \Pi^\pi}$ and $\mathbf{q}^\pi = \{\{q_i^{\pi'}\}_{i \in N(\pi')}\}_{\pi' \in \Pi^\pi}$, such that for all $\pi' \in \Pi^\pi$ and all $i \in N(\pi')$

i) $x_i^{\pi'} = (1-\delta)v(\pi'_i) + \delta u_i^{\pi'}(\mathbf{x}^\pi, \mathbf{q}^\pi)$; and

ii) $q_i^{\pi'} \in \Delta(D_i^{\pi'}(\mathbf{x}^\pi))$.

Now we show that there exist $\{x_i^\pi\}_{i \in N(\pi)}$ and $\{q_i^\pi\}_{i \in N(\pi)}$ which satisfies the two conditions in Proposition 3. Define $X^\pi = \{x \in \mathbb{R}^{n(\pi)} \mid \sum_{i \in N(\pi)} x_i \leq v(N)\}$, $\Sigma_i^\pi = \Delta(\mathcal{P}_i(N(\pi))$, and

$\Sigma^\pi = \times_{i\in N(\pi)}\Sigma_i^\pi$. Given $q \in \Sigma^\pi$, define $V(\cdot;q) : X^\pi \longrightarrow X^\pi$ such that for all $i \in N(\pi)$:

$$\begin{aligned} V_i(x,q) &= p_i^\pi \sum_{S\in\mathcal{P}_i(N(\pi))} q_i(S)\left[x_i^{\pi(i,S)} - \sum_{j\in S} x_j\right] \\ &\quad + \sum_{j\in N} p_j^\pi \sum_{S\in\mathcal{P}_j(N(\pi))} q_j(S)\left[x_i \mathbb{1}(i\in S) + x_i^{\pi(j,S)}\mathbb{1}(i\notin S)\right]. \end{aligned}$$

Lemma 19. *Given $q \in \Sigma^\pi$, $V(\cdot,q)$ is a contraction mapping.*

Since $V(\cdot;q)$ is a contraction mapping, for each $q \in \Sigma^\pi$, there exists a unique fixed point $\xi : \Sigma^\pi \longrightarrow X^\pi$ such that

$$\xi(q) = V(\xi(q),q).$$

Lemma 20. *ξ is a continuous function.*

Now we define $Q_i : X^\pi \rightrightarrows \Sigma_i^\pi$ for each $i \in N(\pi)$:

$$Q_i(x) = \underset{q\in\Sigma_i^\pi}{\operatorname{argmax}} \sum_{S\in\mathcal{P}_i(N(\pi))} q_i(S)\left[x_i^{\pi(i,S)} - \sum_{j\in S} x_j\right],$$

and $Q = \times_{i\in N(\pi)} Q_i$.

Lemma 21. *Q is nonempty, compact-valued, convex-valued, and upper-hemicontinuous.*

Define $R : \Sigma^\pi \rightrightarrows \Sigma^\pi$ such that

$$R(q) = \{q' \in \Sigma^\pi \mid q' \in Q(\xi(q))\}.$$

Lemma 22. *R is nonempty, compact-valued, convex-valued, and upper-hemicontinuous.*

By Kakutani fixed point theorem, there exists $q^\pi \in \Sigma^\pi$ such that $q^\pi \in R(q^\pi)$. Let $x^\pi = \xi(q^\pi)$. Define $\mathbf{x} = \mathbf{x}^\pi \cup x^\pi$ and $\mathbf{q} = \mathbf{q}^\pi \cup q^\pi$. Then for all $i \in N(\pi)$, x_i^π and q_i^π satisfies the two conditions in Theorem 1. Therefore, $(\mathbf{x},\mathbf{q})$ consists of a cutoff strategy SSPE of the subgame with the arbitrary chosen coalitional state π. By induction argument, the game with the initial state π° has a cutoff strategy SSPE and the existence follows.

Theorem 4. *For any bargaining game $\Gamma = (N,v,p,\delta)$, there exists a cutoff strategy SSPE.*

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