Shared Risk in n-player Games

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Abstract

We analyze a game where the world is either in a good state or a state of disaster. We assume each player chooses an action in which her payoff in the good state of the world is increasing, but so is the probability of the bad state. After describing the conditions under which all pure strategy Nash Equilibria (PSNE) are symmetric and, alternatively, at most one symmetric PSNE exists, we show that if multiple PSNE exist (symmetric or otherwise), the least risky strategy profile is optimal among that set. However, no interior PSNE is socially optimal among all strategy profiles, as all interior PSNE are too risky.

1 Introduction

Consider a driver on a highway. The driver may choose to drive faster and more riskily and, in the absence of an accident, arrive at his destination more quickly. But driving riskily will, of course, increase the probability of an accident. In the absence of other drivers, he would simply conduct his own cost benefit analysis and find his optimal level of risk. With many drivers, however, how each drives will influence the others, and the system becomes more complicated. This basic problem describes many scenarios, such as airlines choosing security procedures, lenders deciding how many high-risk, high-interest loans to award, and apartment owners in an earthquake-zone determining how much to contribute towards the retrofit of their building. We investigate the characteristics of the pure strategy Nash Equilibria (PSNE) in these types of systems, and how the PSNE outcomes compare to those under the socially optimal strategy profile.

More specifically, we analyze a game where there are two states of the world: a good state and a state of disaster. We assume each of a finite set of players chooses an action that

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is increasing in both risk and potential reward. In the good state of the world, the players receive a payoff that is increasing in the action they choose, but the probability of the bad state, in which each player receives a minimum payoff, is also increasing in the actions of all the players.[1]

We first focus on symmetric pure strategy Nash equilibria (PSNE) and show conditions related to the probability of disaster that guarantee that all PSNE are symmetric or, alternatively, that at most one PSNE is symmetric. Under the latter condition we also show that, if we define the risk level of a particular strategy profile more strictly, then only PSNE of similar riskiness exist.

Before turning to applications, we discuss conditions under which a PSNE is guaranteed to exist, and show that for a continuous strategy space and differentiable payoff function any interior equilibrium will have a higher than optimal risk-level (as defined by the probability of disaster). In an interior PSNE, each player who plays an interior strategy has balanced the reward he receives with the risk he is taking on, but he does not account for the risk that he imposes on the other players in the game. The presence of this externality ensures that each player will choose an overly-risky action (relative to the social optimum) in equilibrium. This point is particularly relevant when we consider games with multiple Pareto-ranked equilibria or, in other words, coordination games.[2] Even if we are able to somehow select the socially optimal PSNE for the game we analyze, we would only have a second-best solution unless, in equilibrium, all players play the minimum strategy or, in some cases, the maximum one.

Our work is most closely related to that of Geoffrey Heal and Howard Kunreuther on coordination and risk in what they term games of interdependent security (IDS).[3] They study scenarios where each player chooses a strategy from a binary set and faces a binary set of possible outcomes, where the choices of all the players in the system may influence the probability of reaching a particular outcome. For example, in an application we discuss in more detail in Section 4, they consider the problem an airline faces when deciding whether

[1]In our model, it makes no difference if the bad state affects the entire system with some probability or visits player i with some probability: the optimizing player will behave the same.

[2]A large literature focuses on analyzing ways that we may end up at the socially optimal equilibrium of a coordination game. See, for example, Crawford and Haller (1990). Relatedly, much experimental literature investigates which equilibrium we actually do end up choosing. For an example of this, see van Huyck et al. (1990).

[3]See, for example, Kunreuther and Heal (2003); Heal and Kunreuther (2005, 2006, 2007).

to invest in a baggage x-ray machine to detect bombs. Since it is assumed that the airline will only have time to x-ray baggage on the first leg of travel, those passengers who transfer from other airlines will not have their baggage checked unless the airline that provided their first leg has an x-ray machine of its own.

When all airlines are symmetric in investment cost all PSNE are symmetric as well and one of three scenarios will occur depending on the parametrization: in the unique PSNE no airline buys an x-ray machine, in the unique PSNE all airlines buy an x-ray machine, or both equilibria exist. Our model extends this game by allowing airlines to choose the level of investment in security. Specifically, our model is akin to letting each airline choose the proportion of bombs that its security detects (with the investment cost rising in the effectiveness). We show that all PSNE in this particular game are symmetric and that any non-corner PSNE results in underinvestment.

The model we consider is (almost) contained in the model used in Cooper and John (1988), which emphasizes the role of spillovers and strategic complementarities.[4] Spillovers refer to the payoff of an agent being impacted by the actions of others, while strategic complementarity denotes a scenario where one (or a set of) agent(s) playing a higher (or lower) strategy induces other agents to do the same.[5] While our emphasis is more specific and slightly different, when appropriate we try to connect the observations from this literature, as well as those from the literature on supermodularity, with our results.

2 Model

Player $i \in \{1, 2, 3, ..., N\}$ chooses action $a_i \in \mathbb{A} \subset \mathbb{R}$. The action of each player, in addition to impacting the player's expected payoff, also impacts the riskiness of the system. Let $p : \mathbb{A}^N \to (0, 1)$ and $x : \mathbb{A} \to (0, \infty)$ be strictly increasing functions. $p(a_1, a_2, ..., a_N)$ represents the probability that the system encounters a disaster, in which case each player receives a payoff of 0.[6] Otherwise, player i receives a payoff of $x(a_i)$.

[4]For some of our results we do not assume the payoff functions are twice differentiable, as they do throughout.

[5]See also Bulow et al. (1985) for discussion of strategic complements and substitutes.

[6]Notice that we assume that with probability greater than 0 and less than 1 that a disaster will occur. Further, though we model the disaster to impact all players, our analysis does not change if the disaster impacts only one or a subset of players at a given time. In other words, if we have $p_i(a_1, a_2, ..., a_N)$ impact only player i, our results still hold.

A strategy profile in this game is given as $\vec{a} = (a_1, a_2, ..., a_N)$. We also make use of the notation $\vec{a} = \langle \vec{a}_{-i}, a_i \rangle$ to highlight player i's action. In addition to the functional assumptions above, we also assume

Assumption 1. *The probability of disaster is symmetric in the players' strategies. For example,* $p(a_1, a_2, ..., a_N) = p(a_2, a_1, ..., a_N)$.

To facilitate economic meaning in our model, we are interested in discussing the systemic riskiness of a given strategy profile. To that end,

Definition 1. *Strategy profile $\vec{a}$ is riskier than strategy profile $\vec{b}$ if $p(\vec{a}) > p(\vec{b})$.*

Each risk neutral player's objective is to maximize her return. For notational simplicity let $q(\vec{a}) = 1 - p(\vec{a})$. A strategy profile $\vec{a}$ is a pure strategy Nash Equilibrium (PSNE) iff $\forall i \in \{1, 2, 3, ..., N\}$ and $\forall b \in \mathbb{A}$

$$x(a_i)q(\vec{a}) \geq x(b)q(\langle \vec{a}_{-i}, b \rangle). \tag{1}$$

Define $Q(\vec{a}, a_i, b) = \frac{q(\langle \vec{a}_{-i}, b \rangle)}{q(\vec{a})}$. In other words, $Q(\vec{a}, a_i, b)$ is the ratio of the probability of not incurring disaster under strategy profile $\langle \vec{a}_{-i}, b \rangle$ to the probability of not incurring a disaster under strategy profile $\vec{a}$. We also, less precisely, think of this as the ratio of the new probability of no disaster to the old if i switches from a_i to b. For example, $Q(\vec{a}, a_i, b) = 2$ indicates the system is twice as likely to not incur disaster when player i plays b rather than a_i.

We rewrite (1) as

$$\frac{x(a_i)}{x(b)} \geq Q(\vec{a}, a_i, b), \tag{2}$$

to emphasize the risk-reward calculation that the potentially deviating player makes. If the ratio of the returns is greater than the ratio of "no disaster" probabilities, then the deviation is not a fruitful one.

Several results below depend on $Q(\vec{a}, a_i, b)$ being monotone in the risk level of $\vec{a}$ when $b > a_i$.[7] If $Q(\vec{a}, a_i, b)$ is strictly increasing in the risk level of $\vec{a}$ when $b > a_i$, then if player

[7]Notice that this equivalent to saying $Q(\vec{a}, a_i, b)$ is monotone in the risk level of $\vec{a}$ when $a_i < b$, but due to the reciprocal nature of $Q(\vec{a}, a_i, b)$, if $Q(\vec{a}, a_i, b)$ is increasing in the risk level of $\vec{a}$ when $b > a_i$, then $Q(\vec{a}, a_i, b)$ will be decreasing in the risk level of $\vec{a}$ when $b < a_i$.

i deviates from a_i to b, the probability of "no disaster" will decrease proportionately less when the system's overall risk level is greater. Consider a system consisting of drivers on a highway. If $Q(\vec{a}, a_i, b)$ is strictly increasing, this corresponds to one (or a given number of) additional driver(s) choosing to begin to drive recklessly lowering the probability of having an accident-free day *proportionately* more when fewer drivers in the system are initially driving recklessly.

Alternatively, if $Q(\vec{a}, a_i, b)$ is strictly decreasing in the risk level of $\vec{a}$ when $b > a_i$ the change to the probability of "no disaster" will decrease proportionately more at higher risk levels when i switches from a_i to b. In the driving analogy, this corresponds to one (or a given number of) additional driver(s) choosing to begin to drive recklessly lowering the probability of having an accident-free day more when more drivers in the system are initially driving recklessly.

3 Characteristics of the Pure Strategy Nash Equilibria

We first examine the implications for symmetric PSNE in our model. A strategy profile $\vec{a} = (a, a, ..., a)$ is a symmetric PSNE, iff $\forall b \in \mathbb{A}$

$$x(a)q(\vec{a}) \geq x(b)q(\langle \vec{a}_{-i}, b \rangle). \tag{3}$$

By restricting $Q(\vec{a}, a_i, b)$ to be monotone in the riskiness of $\vec{a}$ (conditional on $a_i < (>)b$), we are able to discuss the implications for symmetric PSNE in this model. First,

Proposition 1. *Suppose that for all $a_i, b \in \mathbb{A}$ such that $b > a_i$, $Q(\vec{a}, a_i, b)$ is strictly increasing in the riskiness of $\vec{a}$. Then every PSNE is symmetric.*

Proof. Let us proceed by contradiction. Suppose $\vec{c} = (c_1, c_2, ..., c_N)$ is an asymmetric PSNE and $Q(\vec{a}, a_i, b)$ is strictly increasing in the riskiness of $\vec{a}$ for all $a_i < b$. Without loss of generality, assume $c_1 < c_2$. Since $\vec{c}$ is a PSNE, player 1 cannot profitably deviate from c_1 to c_2, implying $q(\vec{c})x(c_1) \geq q(c_2, c_2, c_3, ..., c_N)x(c_2)$ and $\frac{x(c_1)}{x(c_2)} \geq \frac{q(c_2,c_2,c_3,...,c_N)}{q(\vec{c})} = Q(\vec{c}, c_1, c_2))$.

Likewise, player 2 cannot profitably deviate from c_2 to c_1. So, $q(\vec{c})x(c_2) \geq q(c_1, c_1, c_3, ..., c_N)x(c_1)$, which implies $\frac{x(c_1)}{x(c_2)} \leq \frac{q(\vec{c})}{q(c_1,c_1,c_3,...,c_N)} = Q((c_1, c_1, c_3, ..., c_N), c_1, c_2)$. But $Q(\vec{c}, c_1, c_2) \leq \frac{x(c_1)}{x(c_2)} \leq$

$Q((c_1, c_1, c_3, ..., c_N), c_1, c_2)$ implies $Q(\vec{a}, c_1, c_2)$ is not strictly increasing in the risk level of $\vec{a}$, since $p(\vec{c}) > p(c_1, c_1, c_3, ..., c_N)$. Contradiction. $\square$

$Q(\vec{a}, a_i, b)$ strictly increasing is necessary but not sufficient for strategic complementarity (or, equivalently, for the game to be supermodular). Intuitively, $Q(\vec{a}, a_i, b)$ strictly increasing implies that *if* one player finds that playing a higher strategy leads to a higher payoff, then all the other players will also find it optimal to increase their strategies (leading to only symmetric PSNE). Strategic complementary, on the other hand, implies that *any* increase in one player's strategy (whether it improves her payoff or not) leads others to increase their strategies as well.

More formally, the relevant condition for supermodularity would be for the payoffs to be increasing in first differences. So, for $b > a_i$, $x(b)q(\langle a_{-i}, b\rangle) - x(a_i)q(\langle a_{-i}, a_i\rangle)$ is increasing in $a_j \neq a_i$. If we assume that $q(\cdot)$ and $x(\cdot)$ are differentiable and $\mathbb{A}$ is compact and convex (as we do below in Assumption 2), then this is equivalent to $x(b)\frac{\partial q(\langle a_{-i}, b\rangle)}{\partial a_j} - x(a_i)\frac{\partial q(\langle a_{-i}, a_i\rangle}{\partial a_j} > 0$. $Q(\cdot)$ increasing implies that $\frac{\partial q(\langle a_{-i}, b\rangle)}{\partial a_j} > \frac{\partial q(\langle a_{-i}, a_i\rangle}{\partial a_j}$, but without further assumptions regarding $x(\cdot)$, we cannot conclude that the game is necessarily supermodular ($x(b) > x(a_i)$ and $\frac{\partial q}{\partial a_j} < 0$).[8]

Alternatively, if $Q(\vec{a}, a_i, b)$ is decreasing in the risk level of $\vec{a}$ (conditional on $a_i < b$), we can rule out coordination problems among symmetric PSNE.

Proposition 2. *Suppose that for all $b > a_i$, $Q(\vec{a}, a_i, b)$ is decreasing in the riskiness of $\vec{a}$. Then there is at most one symmetric PSNE.*

Proof. Let us proceed by contradiction. Suppose $\vec{c} = (c, c, ...c)$ and $\vec{d} = (d, d, ...d)$ are both symmetric PSNE and $Q(\vec{a}, a_i, b)$ is decreasing in the risk level of $\vec{a}$ for all $a_i < b$.Without loss of generality, assume $c < d$. That $\vec{c}$ is a PSNE implies $x(c)q(\vec{c}) \geq x(d)q(\langle \vec{c}_{-i}, d\rangle)$. We can rewrite this condition as $\frac{x(c)}{x(d)} \geq Q(\vec{c}, c, d)$.

That $\vec{d}$ is a PSNE implies $x(d)q(\vec{d}) \geq x(c)q(\langle \vec{d}_{-i}, c\rangle)$. We can rewrite this condition as $\frac{x(c)}{x(d)} \leq Q(\langle \vec{d}_{-i}, c\rangle, c, d)$. Combining the two constraints, $Q(\vec{c}, c, d) \leq \frac{x(c)}{x(d)} \leq Q(\langle \vec{d}_{-i}, c\rangle, c, d)$,

[8]Strategic complementarity is necessary for multiple symmetric PSNE to exist when the payoff function is differentiable (Cooper and John, 1988, Proposition 1). Additionally, supermodularity guarantees the existence of a PSNE (Topkis, 1979, Theorem 3.1) and further, that there exist largest and smallest serially undominated strategies $\underline{a_i}$ and $\bar{a}_i$, respectively, that the symmetric strategy profiles where all players play these strategies are PSNE, and that all PSNE fall between these two in terms of risk profile (Milgrom and Roberts, 1990, Theorem 5). Of course, $\underline{a_i}$ may equal $\bar{a}_i$ in which case the PSNE is unique.

implying $Q(\vec{a}, c, d)$ is not strictly decreasing in the risk level of $\vec{a}$, since $p(\vec{c}) < p(\langle \vec{d}_{-i}, c\rangle)$. Contradiction. $\square$

Essentially, if $Q(\vec{a}, a_i, b)$ is decreasing in the risk level of $\vec{a}$, then the game exhibits strategic substitutability. As discussed by Cooper and John (1988), strategic substitutability means that one player increasing her strategy decreases the optimal strategy for the other players.[9] When the payoff function is differentiable, this further implies that a unique symmetric PSNE exists (Cooper and John, 1988, Proposition 1).[10]

If we more strictly define what it means for a particular strategy profile to be riskier than another, then we are able to strengthen Proposition 2.

Definition 2. *Define $\vec{b}$ to be strictly riskier than $\vec{a}$ if $\forall b_i \in \vec{b}$, $p(\langle \vec{a}_{-i}, b_i\rangle) < p(\vec{b})$.*

Essentially, this definition says that strategy profile $\vec{b}$ is strictly riskier than $\vec{a}$ if the risk level of $\vec{b}$ is still greater after switching in any one b_i into $\vec{a}$ for a_i. Then,

Proposition 3. *Suppose that for all $b > a_i$, $Q(\vec{a}, a_i, b)$ is decreasing in the riskiness of $\vec{a}$. Then no PSNE will be strictly riskier than any other.*

Proof. Let us proceed by contradiction. Suppose $\vec{c}$ and $\vec{d}$ are both PSNE and $Q(\vec{a}, a_i, b)$ is decreasing in the risk level of $\vec{a}$ for all $b > a_i$. Without loss of generality, assume $\vec{d}$ is strictly riskier than $\vec{c}$ and $c_1 < d_1$. That $\vec{c}$ is a PSNE implies that $x(c_1)q(\vec{c}) \geq x(d_1)q(\langle \vec{c}_{-1}, d_1\rangle)$. We can rewrite this condition as $\frac{x(c_1)}{x(d_1)} \geq Q(\vec{c}, c_1, d_1)$.

That $\vec{d}$ is a PSNE implies $x(d_1)q(\vec{d}) \geq x(c_1)q(\langle \vec{d}_{-1}, c_1\rangle)$. We can rewrite this condition as $\frac{x(c_1)}{x(d_1)} \leq Q(\langle \vec{d}_{-1}, c_1\rangle, c_1, d_1)$. Combining the two constraints, $Q(\vec{c}, c_1, d_1) \leq \frac{x(c_1)}{x(d_1)} \leq Q(\langle \vec{d}_{-1}, c_1\rangle, c_1, d_1)$, implying $Q(\vec{a}, c_1, d_1)$ is not strictly decreasing in the risk level of $\vec{a}$, since $p(\vec{c}) < p(\langle \vec{d}_{-1}, c_1\rangle)$ ($\vec{b}$ is strictly riskier than $\vec{a}$). Contradiction. $\square$

Of course, we would like to be able to rank the PSNE when multiple equilibria do exist. Since the players are assumed to be risk neutral, we simply define the social welfare as the sum of the expected payoffs.

[9]To see the connection to the definition of strategic substitutability given in Cooper and John (1988), assume $q(\vec{a})$ is twice differentiable and $x(a_i)$ is differentiable. The expected payoff for player i is then $\pi_i(\vec{a}) = x(a_i)q(\vec{a})$. If $Q(\vec{a}, a_i, b)$ is decreasing in the risk level of $\vec{a}$, then $\frac{\partial^2 \pi_i}{\partial a_i \partial a_j} < 0$.

[10]See also Cooper (1999, p. 21) for further discussion.

Definition 3. *$\vec{a}$ is socially preferable to $\vec{b}$ if $\sum_{i=1}^{N} x(a_i)q(\vec{a}) > \sum_{i=1}^{N} x(b_i)q(\vec{b})$. $\vec{a}$ is a socially optimal strategy profile if $\forall \vec{b} \in \mathbb{A}^N$, $\sum_{i=1}^{N} x(a_i)q(\vec{a}) \geq \sum_{i=1}^{N} x(b_i)q(\vec{b})$.*

Then, in cases where multiple PSNE exist,

Proposition 4. *If $\vec{a}$ and $\vec{b}$ are PSNE with $\vec{b}$ strictly riskier than $\vec{a}$, then $\vec{a}$ is socially preferable.*

Proof. Since $\vec{a}$ is a PSNE, $x(a_i)q(\vec{a}) \geq x(b_i)q(\langle \vec{a}_{-i}, b_i \rangle)$. Since $\vec{b}$ is strictly riskier than $\vec{a}$, $q(\langle \vec{a}_{-i}, b_i \rangle) > q(\vec{b})$. Therefore, $x(b_i)q(\langle \vec{a}_{-i}, b_i \rangle) > x(b_i)q(\vec{b})$ and subsequently, $x(a_i)q(\vec{a}) > x(b_i)q(\vec{b})$, implying $\sum_{i=1}^{N} x(a_i)q(\vec{a}) \geq \sum_{i=1}^{N} x(b_i)q(\vec{b})$. $\square$

Essentially, the (strictly) safest PSNE will be socially optimal among the set of equilibria. Then, in the case of symmetric PSNE,

Corollary 1. *If $\vec{a}$ and $\vec{b}$ are symmetric PSNE with $b > a$, then $\vec{a}$ is socially preferable.*

Corollary 1 follows directly from the observation that if $\vec{a}$ and $\vec{b}$ are symmetric equilibria with $\vec{b}$ riskier than $\vec{a}$, then $\vec{b}$ is strictly riskier than $\vec{a}$ as well. But even when an equilibrium is socially optimal *among the set of equilibria*, the equilibrium is unlikely to be socially optimal among the set of all strategy profiles. To see this, further assume,

Assumption 2. *(i) $\mathbb{A}$ is compact and convex, i.e. $\mathbb{A} = [\underline{a}, \bar{a}]$.*
(ii) $q : \mathbb{A}^N \to (0,1)$ and $x : \mathbb{A} \to (0,\infty)$ are twice differentiable.

Then,

Proposition 5. *Given Assumption 2, if a strategy profile $\vec{a}$ is socially optimal* and *a PSNE, then all players either play the minimum (safest) strategy or the maximum (riskiest) strategy.*

Proof. Proceed by contradiction. Assume $\vec{a}$ is a socially optimal strategy profile and an interior PSNE (one in which at least one player plays an interior strategy (i.e. plays in the open interval $(\underline{a}, \bar{a})$)). To be an interior PSNE, the necessary first order condition is $\forall i \in \{1, 2, ..., N\}$ who play interior strategies

$$x'(a_i)q(\vec{a}) + x(a_i)\frac{\partial q}{\partial a_i} = 0.$$

The socially optimal strategy profile maximizes $\sum_{i=1}^{N} x(a_i)q(\vec{a})$. The necessary first order condition is then $\forall i \in \{1, 2, ..., N\}$ who play interior strategies

$$x'(a_i)q(\vec{a}) + x(a_i)\frac{\partial q}{\partial a_i} + \sum_{j \neq i} x(a_j)\frac{\partial q}{\partial a_i} = 0.$$

Subtracting the first condition from the second, we see that $\sum_{j\neq i} x(a_j)\frac{\partial q}{\partial a_i} = 0$. But since $x(a_i) > 0$ and $\frac{\partial q}{\partial a_i} < 0$, then $\sum_{j\neq i} x(a_j)\frac{\partial q}{\partial a_i} < 0$. Contradiction. $\square$

Corollary 2. *Under Assumption 2, in any interior PSNE, social welfare can always be improved by at least one agent choosing a lower (less risky) action.*

To see this, notice that differentiating the social welfare (sum of agents expected payoffs) with respect to a_i yields $x'(a_i)q(\vec{a}) + x(a_i)\frac{\partial q}{\partial a_i} + \sum_{j\neq i} x(a_j)\frac{\partial q}{\partial a_i}$. However, when the strategy profile is an interior PSNE, the first two terms sum to 0, and the third term, $\sum_{j\neq i} x(a_j)\frac{\partial q}{\partial a_i} < 0$. This implies that the social welfare can always be improved by any interior a_i decreasing when $\vec{a}$ is a PSNE.

Before discussing the role of shared risk in a more general sense, let us note conditions to guarantee the existence of a PSNE. Above, we mention a few special cases (supermodularity and $Q(\vec{a}, a_i, b)$ decreasing) that imply that a PSNE exists. More generally, if we assume

Assumption 3. *Define $\pi_i(\vec{a}) = x(a_i)q(\vec{a})$. Assume that for all i, $\pi_i(\vec{a})$ is quasiconcave.*

Then,

Proposition 6. *Under Assumptions 2 and 3, a PSNE exists.*

Proof. Follows directly from the Debreu-Glicksberg-Fan theorem (Debreu, 1952; Glicksberg, 1952; Fan, 1952): (1) The strategy space is nonempty, convex, and compact, (2) the payoff function is continuous, and (3) the payoff function is quasiconcave in the strategy profile. $\square$

Above we assume that choosing an action a_i either has payoff $x(a_i)$ or 0, with each associated probability depending on the overall risk level in the system. We can more generally assume that the payoff for i follows some probability distribution that depends on the system's actions. Let $x_i \sim F(\vec{a})$ be an absolutely continuous random variable that denotes the payoff to the ith agent, where $F(\cdot)$ has corresponding density $f(\cdot)$. How each agent evaluates the distribution of his payoff is, of course, subjective, but assume utility depends only on the mean and variance of the payoff distribution.

Analogous to our assumptions above,

Assumption 4. *Assume $\mathbb{A} = [\underline{a}, \bar{a}]$ where $\underline{a}, \bar{a} \in \mathbb{R}$. Let $\bar{x}_i : \mathbb{A}^N \to \mathbb{R}$ and $\sigma_i^2 : \mathbb{A}^N \to [0, \infty)$ be differentiable in all arguments. Assume the expectation and variance of the ith player's payoff are defined and let them be denoted by $\bar{x}_i(\vec{a})$ and $\sigma_i^2(\vec{a})$, respectively. Assume that $\frac{\partial \sigma_i^2}{\partial a_i} > 0$, $\frac{\partial \bar{x}_i}{\partial a_i} > 0$ and, for $i \neq j$, (i) $\frac{\partial \sigma_i^2}{\partial a_j} > 0$ and (ii) $\frac{\partial \bar{x}_i}{\partial a_j} \leq 0$.*

Essentially, (i) ensures that the risk level faced by the jth agent is increasing in the actions of the ith agent, while (ii) ensures that the mean return is non-increasing. Above, these assumptions are implicit in $q(\vec{a})$ decreasing in each a_i with an expected return of $x(a_i)q(\vec{a})$. Further, we assume that the agents are risk averse. Formally,

Assumption 5. *Let $u_i : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be increasing in the first argument and decreasing in the second. Let $u_i(x_i(\vec{a}), \sigma_i^2(\vec{a}))$ denote the utility of the ith agent when the strategy profile $\vec{a}$ is played. Assume, further, that $u_i(\cdot)$ is quasiconcave in $\vec{a}$.*

If we define the social welfare implied by a strategy profile $\vec{a}$ as the sum of the agents' utilities under $\vec{a}$, then

Proposition 7. *Under Assumptions 4 and 5, (i) a PSNE exists, (ii) any interior PSNE will not be socially optimal, and (iii) at any interior PSNE, social welfare can always be improved by at least one agent choosing a lower (less risky) action.*

Proof. A PSNE is guaranteed to exist again by Debreu-Glicksberg-Fan.

Let us proceed by contradiction for (ii). Suppose otherwise, that an interior PSNE is socially optimal. The necessary first order condition is that $\forall i$ that play interior strategies $\frac{\partial u_i}{\partial a_i} = 0$. The social welfare is given by $\sum_{i=1}^N u_i(\bar{x}_i(\vec{a}), \sigma_i^2(\vec{a}))$. The necessary first order condition for maximizing the social welfare is then $\frac{\partial u_i}{\partial a_i} + \sum_{j \neq i} \frac{\partial u_j}{\partial a_i} = 0$. Since $\vec{a}$ is an interior PSNE, $\exists i$ such that $\frac{\partial u_i}{\partial a_i} = 0$. But, $\frac{\partial u_j}{\partial a_i} = \frac{\partial u_j}{\partial \bar{x}_j}\frac{\partial \bar{x}_j}{\partial a_i} + \frac{\partial u_j}{\partial \sigma_j^2}\frac{\partial \sigma_j^2}{\partial a_i} < 0$. This implies $\frac{\partial u_i}{\partial a_i} + \sum_{j \neq i} \frac{\partial u_j}{\partial a_i} < 0$. Contradiction.

(iii) follows directly from $\frac{\partial u_i}{\partial a_i} + \sum_{j \neq i} \frac{\partial u_j}{\partial a_i} < 0$ at any interior PSNE. □

Propositions 5 and 6 get at the heart of the paper. In these games, each player imposes a negative externality on the others in the system by increasing the risk level each faces (and possibly decreasing others mean returns). Therefore, in any interior PSNE, though the players have all balanced their own personal risk reward calculation, they do not account for a further negative impact on the rest of the system, and as a result, they collectively pursue a strategy profile that is riskier than socially optimal.

4 Applications

We analyze three applications. The first, airline security, is a problem introduced by Kunreuther and Heal (2003). In the second application, we examine the problem of neighbors in an apartment building determining the optimal investment to retrofit the building for earthquake safety. Last, we focus on a model of lenders determining what proportion of their portfolio to make high-risk, high reward.

Airline security. Kunreuther and Heal (2003) introduce a game where N airlines choose whether to invest in an x-ray machine that checks baggage for bombs. The machine, however, would only be used to check baggage that originates with the airline, so passengers that transfer from another airline would still be able to smuggle a bomb aboard as long as their airline of origin does not invest in a machine of its own. Assume, for tractability in our framework, that even if no airline invests in the x-ray machine, the probability that disaster occurs is less than 1. Using our notation $\mathbb{A} = \{0, 1\}$, where 1 indicates investing in security and 0 is not doing so.

Rather than the disaster impacting the entire network, as we assume above, the disaster here impacts only the airline where the bomb goes off. To adjust our model accordingly, let $p_i(\vec{a})$ be the probability that a bomb goes off on airline i, where $p_i : \mathbb{A}^N \to (0, 1)$ is increasing in each argument. The payoff to each airline when a bomb does not go off is, again using our notation,

$$x(a_i) = \begin{cases} Y & \text{if } a_i = 0 \\ Y - c & \text{if } a_i = 1. \end{cases}$$

Let $x(a_i) = 0$ when a bomb goes off on airline i. Each airline i chooses $a_i \in \mathbb{A}$ to maximize $x(a_i)q_i(\vec{a})$ where $q_i(\vec{a}) = 1 - p_i(\vec{a})$. A strategy profile $\vec{a}$ is a PSNE iff $\forall i \in \{1, 2, ...N\}$ and $\forall b \in \mathbb{A}$

$$x(a_i)q_i(\vec{a}) \geq x(b)q_i(\langle \vec{a}_{-i}, b\rangle),$$

which, as above, we can rewrite as

$$\frac{x(a_i)}{x(b)} \geq Q_i(\vec{a}, a_i, b),$$

where $Q_i(\vec{a}, a_i, b)$ is the ratio of the probability of "no disaster" when i switches to b to the probability when i plays a_i.

Notice that since investing in an x-ray machine stops all bombs from customers beginning their travel on your airline, and only those bombs, one might conclude that choosing to invest in an x-ray machine will lower the probability of disaster for your airline by the same amount regardless of the other airlines investment decision. However, since two bombs going off on a plane is no worse than one going off, the impacts of a bomb going off from an originating traveler and from a transfer are non-additive. In other words, choosing to invest increases the probability of "no disaster" less when the probability of a transferring passenger having a bomb is greater.

Let α be the probability of a passenger attempting to smuggle a bomb aboard your airline and $\beta(\vec{a}_{-i})$ be the probability of a bomb being transferred onto your flight. Then,

$$p(\vec{a}) = \begin{cases} \alpha + \beta(\vec{a}_{-i}) - \alpha\beta(\vec{a}_{-i}) & \text{if } a_i = 0 \\ \beta(\vec{a}_{-i}) & \text{if } a_i = 1. \end{cases}$$

But for our results on symmetric PSNE, we care about the change in the ratio of the probabilities of no disaster when i changes his action. To that end, when moving from a less risky to a riskier strategy (from 1 to 0),

$$Q_i(\vec{a}, 1, 0) = \frac{1 - \alpha - \beta(\vec{a}_{-i}) + \alpha\beta(\vec{a}_{-i})}{1 - \beta(\vec{a_{-i}})} = 1 - \alpha.$$

$Q_i(\vec{a}, a_i, b)$ does not depend on the spillover risk (or on the actions of the other agents), only on the direct risk. $Q_i(\vec{a}, a_i, b)$ is also independent of a_i: since $|\mathbb{A}| = 2$, $Q_i(\vec{a}, 1, 0)$ is the only configuration where a_i is less risky than b. In other words, $Q_i(\vec{a}, a_i, b)$ is independent of the risk profile of $\vec{a}$.

Since $Q_i(\vec{a}, a_i, b)$ is neither strictly increasing nor decreasing in the risk profile of $\vec{a}$, we are unable to apply either of our results concerning symmetric PSNE. Fortunately, using a different technique Kunreuther and Heal (2003) show that all equilibria are symmetric. Further, depending on the precise parameterization, the set of PSNE may contain both symmetric equilibria or only one of them (either PSNE is possible under the latter scenario). We can say, as also noted in Kunreuther and Heal (2003), that when both equilibria exist, all airlines investing in an x-ray machine is socially preferable (Proposition 3).

Suppose we change the problem slightly so that each airline can choose not just whether to invest in bomb-detection or not, but rather, *how much* to invest. Specifically, suppose that each airline chooses $a_i \in \mathbb{A} = [\epsilon, 1-\epsilon]$, where $1-a_i$ is the proportion of the bombs in the baggage of passengers who originate with your airline that are detected.[11] This specification ensures that the risk level of $\vec{a}$ is increasing in a_i.

Note that $Q_i(\vec{a}, a_i, b)$ changes slightly now. The probability of a bomb getting through airline i's screening is now $a_i\alpha$. So, defining α and $\beta(\vec{a}_{-i})$ as above,

$$Q_i(\vec{a}, a_i, b) = \frac{1 - (b\alpha + \beta(\vec{a}_{-i}) - b\alpha\beta(\vec{a}_{-i}))}{1 - (a_i\alpha + \beta(\vec{a}_{-i}) - a_i\alpha\beta(\vec{a}_{-i}))} = \frac{1 - b\alpha}{1 - a_i\alpha}.$$

So $Q_i(\vec{a}, a_i, b)$ is still independent of $\vec{a}_{-i}$, but now depends on both a_i and α. If we assume that $Q_i(\vec{a}, a_i, b)$ is differentiable, then,

$$\frac{\partial Q_i}{\partial a_i} = \frac{a_i(1 - b\alpha)}{(1 - a_i\alpha)^2} > 0.$$

Therefore, we can invoke Proposition 1 to note that even when airlines are able to choose their level of investment, all PSNE are symmetric. Further, other than PSNE in which all airlines invest in the maximum level or (in some circumstances) PSNE in which all airlines invest in the minimum level, all PSNE exhibit underinvestment relative to the social optimum

[11] And, of course, also let the cost of security $c(1 - a_i)$ be increasing in $1 - a_i$, to maintain the appropriate framework.

(Corollary 2, Proposition 5). This example starkly illustrates the externality present in this type of scenario. Each airline chooses whether to invest, but the impact of that choice does not depend on the other airlines' actions. So, of course, since each airline's choice impacts the risk others face, and the potential benefit is unaccounted for, underinvestment is likely.

Earthquake retrofitting. N neighbors own apartments in a building in a city where earthquakes are common. Each neighbor i is asked to voluntarily contribute $a_i \in \mathbb{A} = [0, \bar{a}]$ dollars to retrofit the building: reinforcing the building to minimize damage from future earthquakes. Define the total investment level $K = \sum_{i=1}^{N} a_i$. Let $p : [0, N\bar{a}] \to (0,1)$ be a decreasing function. With probability $p(K)$, an earthquake will destroy the building and all residents will receive a payoff normalized to 0. As usual, we assume disaster cannot completely be eliminated nor will one occur with certainty. In the absence of disaster, assume each player receives $x(a_i) = \bar{a} - a_i + \epsilon$. Notice that investing fully and not having the building destroyed gives a higher payoff than disaster.

Consider $Q(\vec{a}, a_i, b) = \frac{q(K-(a_i-b))}{q(K)}$, where $b < a_i$ implies that $\langle \vec{a}_{-i}, b \rangle$ is riskier than $\vec{a}$. Assume $q(K)$ is differentiable and define $\delta = a_i - b$. Then,

$$\frac{\partial Q}{\partial K} = \frac{q(K)q'(K-\delta) - q'(K)q(K-\delta)}{[q(K)]^2},$$

implying that $Q(\vec{a}, a_i, b)$ is increasing (decreasing) in K if

$$\frac{q(K)}{q'(K)} > (<) \frac{q(K-\delta)}{q'(K-\delta)}.$$

Notice that the risk level of $\vec{a}$ is strictly decreasing in K. Thus if $Q(\vec{a}, a_i, b)$ is increasing (decreasing) in K, $Q(\vec{a}, a_i, b)$ is decreasing (increasing) in the risk level.

Since $q(K) > q(K-\delta)$, it is sufficient that $q'(K)$ be decreasing to ensure that $Q(\cdot)$ is increasing in K and subsequently $Q(\cdot)$ is decreasing in the risk level. This condition can be described, essentially, as the marginal product of investment diminishing (an additional dollar investment increases the probability of no disaster more at lower levels of total investment), and seems plausible.

For $Q(\vec{a}, a_i, b)$ decreasing in the risk level of $\vec{a}$ we can invoke Propositions 2 and 3, stating that there will be at most one symmetric PSNE, and if more than one PSNE exists, no PSNE will be strictly riskier than another. And, of course, at any interior PSNE, the players will underinvest (be too risky) compared to the social optimum.

Lenders. Lender $i \in \{1, 2, ..., N\}$ gives two types of loans: high-risk, high-interest loans and low-risk, low-interest loans. Assume that in good economic times the expected payoff to lender i is higher for high-risk loans, but the higher the proportion of high-risk loans, the more instability in the economy. If the economy goes into crisis, all of the lenders fail with payoffs normalized to 0.

The lender's problem then is to choose the proportion of his capital to devote to high-risk loans: $a_i \in \mathbb{A} = [0, 1]$. Assume each lender receives return $x(a_i)$ if no crisis occurs, where $x : [0, 1] \to (0, \infty)$ is strictly increasing and differentiable. Suppose the risk level in the economy can be thought of as the sum of the institutions' proportions of high-risk loans weighted by the size of each institution. In other words, let $R = \sum_{i=1}^{N} a_i M_i$ where $M_i \in (0, 1]$ represents the size of each institution relative to the size of the loan market. Even though the risk is not symmetric in the institution actions (i.e. $R(a_1, a_2, ..., a_N)$ may not equal $R(a_2, a_1, ..., a_N)$), we will not invoke Proposition 1, which relies on that assumption.

Let the probability of crisis be given by $p(R)$, where $p : [0, 1] \to (0, 1)$ is strictly increasing in each argument and differentiable, and let $q(R) = 1 - p(R)$. Assume $b > a_i$, then $Q(\vec{a}, a_i, b) = \frac{q(R+(b-a_i)M_i)}{q(R)}$, and differentiating with respect to R, defining $\delta = b - a_i$ as before,

$$\frac{\partial Q}{\partial R} = \frac{q(R)q'(R + \delta M_i) - q'(R)q(R + \delta M_i)}{[q(R)]^2}.$$

Recall that $q'(R) < 0$, implying that $Q(\vec{a}, a_i, b)$ is increasing (decreasing) in R if $\forall i$

$$\frac{q(R)}{q'(R)} > (<) \frac{q(R + \delta M_i)}{q'(R + \delta M_i)}.$$

Therefore it is sufficient for $|q'(R)| < |q'(R + \delta M_i)|$, for $Q(\vec{a}, a_i, b)$ to be decreasing in the risk level of $\vec{a}$. In other words, if the marginal decrease in the probability of no disaster is lower in magnitude at lower levels of risk (fewer high risk loans), then $Q(\vec{a}, a_i, b)$ is decreasing in the risk level of the system, implying that at most one symmetric PSNE exists (Proposition

2).

Of course, regardless of whether $Q(\vec{a}, a_i, b)$ is decreasing in the risk level of the system, the results concerning social optimality are still valid. If an interior equilibrium exists, then this society is oversupplying high-risk loans.

5 Conclusion

We analyze a game with N players, each who chooses an action $a_i \in \mathbb{A}$. With probability that is increasing in all players' actions, the system undergoes a disaster and everyone receives a minimum payoff. Otherwise, each player receives a payoff increasing in his action. While we do attempt to characterize the PSNE and determine when coordination may be a problem, perhaps more interestingly, we show that no interior PSNE will be socially optimal. So even when coordination problems exist, the optimal PSNE still may not achieve the best result for the society.

We view this paper as a partial bridge between Kunreuther and Heal (2003) and Cooper and John (1988). We somewhat generalize and extend Kunreuther and Heal (2003), but our focus is more applied than that of Cooper and John (1988), as we assume greater structure in our model to fit the particular class of problems we have in mind. Hopefully, the balance we strike is useful and we complement both rather than neither.

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