Nature's Measuring Tape: A Cognitive Basis for Adaptive Utility

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Abstract

This paper provides a new approach to a utility function, by deriving it from deeper first principles. Using a minimum set of cognitive tools, one can evaluate choice options and their expectations using solely information from one's past and present environments. The adaptive evaluation procedure results in one's utility function being isomorphic to a perceived rank of its magnitude within a reference set, and thus provides a mechanism for one's attitudes towards risk to be shaped entirely by one's experiences, memory, and cognitive imperfections. The proposed parsimonious model links recent developments in economics, psychology, and neuroscience, and shows that environmental context, memory, and cognitive processes may interact in non-trivial and economically relevant ways.

Keywords: Biological basis of behavior, reference-dependent preferences, random utility, procedural approach, memory limitations; perceptual imperfections, Weber-Fechner law, range-frequency theory, decision-by-sampling, overconfidence, well-being.

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1 Introduction

How big is an apple? How high is a salary? How intense is an experience? How young is a person? How frequent is a name? How long is a piece of string? How can one tell?

The concept of a utility function is fundamental to economics and behavioral sciences, yet the process of utility acquisition and its interaction with choice behavior is still not well understood. Among behavioral scientists in adjacent disciplines, there has been a growing interest in the foundations of choice. Kable and Glimcher [2009] summarize the recent findings on the neurobiological mechanism of decision-making as a two-stage procedure consisting of a valuation stage and a choice stage, each occurring in different brain areas. This opens up a possibility that the processes involved at the choice stage might be different from those involved at the valuation stage. Thus, in order to understand final choice behavior, one needs to understand what happens at each stage of decision-making.

While economists have been primarily interested in choice, the present paper takes a step deeper and looks at the process which might occur at the valuation stage. Simply put, before one can construct a utility function $U(x)$, one needs to evaluate, or *measure*, the size of x. I follow a recently advanced evolutionary framework for decision-making. Rather than endowing living organisms with all necessary information, Nature provides the "tools" that enable one to extract information from one's environment and experience (Robson [2001, 2002], Samuelson [2004], Samuelson and Swinkels [2006]). These tools might have been further adapted to deal with more evolutionarily recent tasks (Cosmides and Tooby [1994]). Rayo and Becker [2007] further suggest that Nature may endow humans with happiness as a context-dependent decision tool which allow one to choose among the alternatives.

I thus explore the process and tools by which a given magnitude x can be evaluated. I identify two well-documented domain-independent cognitive tools and, using a parsimonious mathematical model, show that one could evaluate a magnitude of an item entirely using *ordinal comparisons* which, by means of a *frequency* (proportion) tool, are keyed into a universal cardinal scale. This cardinal scale is the interval $[0, 1]$ and is independent of the item's modality - i.e. whether the magnitude is quantity, size, weight, duration, luminosity, and so on. This is done by calculating how frequently a given object "wins" a pairwise ordinal "tournament" against all other objects in the reference set. Such process of evaluation by ordinal rank was advanced by psychologists Stewart, Chater and Brown [2006].[1]

The resultant adaptive evaluation, which is the expected outcome of pairwise comparisons, is a non-decreasing function of a magnitude, and thus rationalizes "relatively more is better" preferences defined on a reference set. Once the reference set changes, so does the adaptive utility. Specifically, a given magnitude is evaluated higher if the reference set is positively rather than negatively skewed, leading to a possibility of non-reflexivity,

[1] Stewart, Chater and Brown [2006] refer to the formal model of magnitude evaluation by the empirical rank developed earlier in Kornienko [2004].

non-transitivity, and preference reversal (as Tversky and Kahneman [1991] pointed out, such "preference anomalies" are a common consequence of a change in a "reference state"). Furthermore, the adaptive evaluation procedure provides a mechanism for one's apparent attitudes towards risk to be determined - at least in part - by one's environmental context.

It is documented that the two cognitive tools are subject to cognitive limitations and thus may lead to informational inaccuracies. To accommodate such perceptual errors, I employ a convolution/scale mixture technique whereby an adaptive evaluation can be expressed as either a sum or a product of a reference magnitude and a pairwise comparison tournament. When the two cognitive tools are perfect, the adaptive magnitude evaluation is equal to the "veridical" (true) rank in the distribution of reference magnitudes. In other words, as long as the two primitive cognitive tools are perfect, the adaptive evaluation is *optimal* in the sense of Robson [2001], minimizing mistakes in binary choices.

Furthermore, I show that a (neoclassical) context-independent utility function may arise when one evaluates a magnitude relatively to a remembered sample and one has a long memory. In contrast, if one's memory is bounded, context effects may arise. One thus would expect that the context effects would be less pronounced when an agent faces familiar type of objects (e.g. orange juice) rather than when one faces relatively unfamiliar objects (e.g. caviar).

Consistently with the principle of efficient use of information, the reference sample can be used two ways - both to construct magnitude evaluation and, similarly to the case-based decision theory of Gilboa and Schmeidler [1995, 2003], to construct probabilities in a frequentist way. When agent relies solely the information from the current environment, an agent may exhibit risk aversion (i.e. choose the expected value of a gamble rather than a gamble) whenever the expected magnitudes exceeds the median (as typically happens in positively skewed distributions), and vice versa. The exposure to other environments may have a spillover effect on one's expected evaluation. For example, one's evaluation of a random environment may decrease (or increase) whenever, in addition, one is exposed to an alternative environment which is stochastically better (worse). As the result, just as expected utility predicts, a first order stochastically higher gamble would be chosen. In general, under assumptions of perfect cognitive tools and memory, the cases when the theory of adaptive magnitude evaluation and the expected utility theory generate different predictions involve empirically less common distributions.

The convolution model allows one to explore of the effects of imperfections in cognitive tools on magnitude evaluation. An imperfect magnitude evaluation is only partially affected by the veridical rank, thus obscuring the empirical relationship between the environment and one's evaluation function. One particularly notable result states that when the reference context is uniform on $(0, 1)$ (which would imply linear evaluation function when cognitive tools are perfect), an *arbitrary* form of a particular ordinal tool imperfection (namely multiplicative discriminability) would result in adaptive evaluation exhibiting non-increasing marginal evaluation. Such distortion of the mental line would lead to apparent risk aversion in an environment where risk neutrality would have been optimal.

The insights generated by the analysis of the valuation stage of decision making are particularly appropriate for understanding human behavior in the situations where researchers are interested in one's evaluations - for example, for the analysis of survey data. Since all economically relevant magnitudes (such as income, consumption, etc.) are attributable to individuals who possess them, the relevant magnitude evaluations necessarily involve interpersonal comparisons. Yet cognitive imperfections may prevent individuals from assessing their possessions accurately. Specifically, when individuals make more inaccurate ordinal comparisons when comparing their possessions to those "above" them, than to those "below" them, their evaluations of own possessions are higher than "true" evaluation, leading to greater satisfaction with their possessions. When individuals have this bias, equality and economic growth may be associated with greater welfare. Instead, if the cognitive tools are perfect, economic processes such as economic growth and income redistribution may surprisingly be welfare neutral. Furthermore, as judgment of own skill relatively to the others involves interpersonal comparisons, the proposed evaluation model allows one to relate individual difficulties in making upward ordinal comparisons, with the observed patterns of overconfidence in relative skill judgment.

This paper is related a large and diverse body of the existing literature in economics, psychology, and neurobiological sciences, only a small portion of which as well as the empirical evidence are surveyed at the end of the paper. In contrast to the existing theoretical works, the present paper (i) adopts a framework whereby choice is a two-stage procedure with evaluation being done in the first stage and choice being done in the second stage; (ii) explicitly models the process which might be involved in an evaluation of a magnitude; (iii) suggests the minimum set of domain-independent cognitive tools which are required for such evaluations; (iv) provides a parsimonious mathematical model of a magnitude evaluation; (v) shows that this evaluation can represent "more is better" preferences; (vi) provides the conditions for the utility function to be equal to an empirical rank; (vii) provides the conditions for the utility function to be independent of the current context; (viii) shows that reference set can be used efficiently to derive expected magnitude evaluation; (ix) shows that expected magnitude evaluation is affected by the exposure to other stochastic environments and by memory; (x) shows that the environments where the theory of adaptive magnitude evaluation and the expected utility theory would generate different predictions are empirically not common; (xi) explicitly incorporates the cognitive imperfections into evaluation; (xii) shows that cognitive imperfections distort the mental line in a predictable way and obscure the empirical relationship between environment and utility; (xiii) shows that particular types of inaccuracies in interpersonal comparisons may increase (decrease) societal welfare; (xiv) shows that overconfidence in relative skill judgment may arise because of cognitive imperfections.

2 Two-Stage Decision Process: Valuation vs. Choice

In their review of the recent findings on the neurobiological mechanism of decision-making in humans and primates, Kable and Glimcher [2009] describe the decision process as a two-stage procedure consisting of a valuation stage (involving ventro-medial prefrontal cortex and striatum) and a choice stage (involving lateral prefrontal and parietal cortices). This leads to a possibility that discrete choice can be decomposed into two distinct processes - the valuation process and choice process.

While economists have been interested in the final, choice, stage, the issue of magnitude evaluation has frequently been overlooked. Yet even for alternatives which differ on a single dimension - like size - one still needs to be able to tell apart the larger object from the smaller object. If one had more-is-better preferences, and there were no constraints on choice, and the cost of choosing a particular object was the same across all objects, one still needs to know which of all alternatives is the largest (and thus most desirable). In contrast, economists assumed that decision-makers have readily available rankings of all alternatives, and concentrated on the final stage of decision making, choice, in the presence of known ranking. Yet, as this paper suggests, the first, valuation, stage, is an important component of decision-making, understanding of which may clarify choice behavior - even when choice alternatives differ only along one dimension. One thus would expect that when the alternatives have different attributes, the situation would be more complicated.

Consider Robinson Crusoe on a desert island, and suppose that he faces a choice between a coconut and a banana, which have magnitudes x_c and x_b respectively. Suppose Crusoe is indifferent between coconuts and bananas, but he prefers a "gigantic" banana to a "tiny" coconut - and vice versa. Thus, before he can make a choice between a coconut and a banana, he needs to be able to evaluate the fruit sizes. Let us suppose that Crusoe's decision making process consists of two stages. In the first, evaluation, stage he evaluates the magnitude x of each fruit, which may then be stored in his memory for a relevant period of time, and in the second, choice stage, conditional on his assessment of fruit sizes, he selects the fruit he finds to be more attractive.

In other words, an individual's decision making process involving a choice between two alternatives can be described as follows.[2]

Two-Stage Binary Choice Process: *Suppose an individual faces a choice between two alternatives $j = 1, 2$, with magnitudes x_j. Then the process of choice between these two alternatives can be described as follows:*

Stage 1: For each of the choice alternatives $j = 1, 2$, an individual evaluates magnitude x_j

[2] The two-stage decision-making process is a simplification which ignores the learning process and the dynamic feedback between the past choice results and future choices.

using magnitude evaluation functions $I_j(x_j)$, and given these magnitude evaluations $I_j = I_j(x_j)$, forms utilities $U_j = U_j(I_j) = U_j(I_j(x_j))$.

Stage 2: Given the utilities U_j of each object $j = 1, 2$, the individual chooses the object 1 over the object 2 with probability $\Omega(U_1, U_2) \in [0, 1]$ as follows:

$$\Omega(U_1, U_2) = \begin{cases} 0 & \textit{whenever} \quad U_1 - U_2 < \nu^- \\ \omega(U_1, U_2) & \textit{whenever} \quad \nu^- \leq U_1 - U_2 \leq \nu^+ \\ 1 & \textit{whenever} \quad U_1 - U_2 > \nu^+ \end{cases} \tag{1}$$

Traditionally, choice theorists made little distinction between these two stages, employing instead a reduced form approach to decision making. Rather than exploring the processes involved at each stage of decision-making, starting with the process of acquisition of magnitude evaluation $I_j(x_j)$ and choice function $\Omega(U_1, U_2)$), the literature instead tends to "compound" these two stages into a model with probability of choice given the attributes of choice alternatives $\Omega(x_1, x_2)$, which is a reduced form of $\Omega(U_1(I_1(x_1)), U_2(I_2(x_2)))$.

In contrast, the above two-stage decision process captures a possibility that there could be different cognitive processes involved at each stage of the the decision making (Kable and Glimcher [2009]). It is thus possible that stochastic decision processes and/or perceptual limitations can happen at each stage of the process, which, in turn, may lead to random choices.[3] It is well documented that individual choices exhibit some randomness. Importantly, this empirically observed variability of choice can arise due to stochastic processes at either or both stages. That is, stochastic choices may occur either because the underlying alternatives are not perfectly discriminable (Thurstone [1927]) or because "utility is not perfectly discriminable" (Luce [1956]), or both. Thus, in order to understand the mechanism of decision-making, one needs to understand the mechanism involved at each stage.

The present paper is agnostic regarding possible processes involved at the second stage of decision making. The above choice rule 1 is consistent with random utility models (started by Thurstone [1927], continued by Luce [1956]). Yet it is agnostic about whether this arises because of the perceptual limitations at the choice stage (similar to Rayo and Becker [2007]), or because of more complex decision process in the brain (for example, as in the drift-diffusion model of Ratcliff [1978] and its extensions - see Fehr and Rangel [2011]).

Instead, the rest of the paper concentrates on the less explored evaluation stage. Thus it is assumed throughout the paper that one has "more is better" preferences, and once the evaluation of alternatives is known, the "larger" objects will be chosen.

Assumption [A1]: Suppose $U_j = u(I_j)$ with $u'(\cdot) > 0$, and $\nu^- = \nu^+ = 0$.

This assumption allows one to concentrate on the features of behavior which could be originated at the valuation stage. Such approach underscores the potential mental effort

[3] There is a possibility that perceptual limitations may lead to "incompleteness of preferences", which can happen at either of the stages of the decision making.

spent evaluating choice alternatives - especially unfamiliar ones. Once the evaluation is done, the choice is trivial. On the other hand, the results of past evaluations could be stored in memory, so that the valuation stage for familiar choice options may be less resource-demanding. The focus on the valuation stage could be particularly relevant in situations where evaluations are of particular importance, for example, for judgment of one's well-being or one's abilities.

3 The Adaptive Model of Magnitude Evaluation

Consider Robinson Crusoe on a desert island. There is a single coconut tree on the island, and Crusoe can see coconuts in a variety of sizes on that tree. One day, a coconut falls off the tree. How could Crusoe evaluate the size of this coconut? Clearly, if the coconut is the smallest one among the ones he can see, it is the least valuable. If instead it is the largest one, it is the most valuable. If it is neither smallest nor largest, its evaluation is in-between. I argue that Crusoe can construct an evaluation function using only two cognitive tools: first, by conducting *pairwise comparison* "tournaments" between the *target* coconut and each of the coconuts which he can see on the tree, and second, by evaluating the target magnitude by the *frequency*, or *proportion* of pairwise comparison tournaments the target coconut "wins".

As the adaptive evaluation procedure is context-dependent, Robinson Crusoe's evaluation is based on the size distribution of coconuts on his island, and will change if he were on another island. For example, Crusoe would be less happier with a 3 inch coconut on an island A where a typical coconut is 5 inch radius rather than on another island B where a typical coconut is 2 inch radius - simply because coconuts smaller than 3 inch are more frequent on island B. Thus it is possible to find a banana such that Crusoe will be happy to trade the 3-inch coconut for this banana on island A, but on island B he would be happy to trade this banana back for the 3-inch coconut - an apparent preference reversal.[4]

More formally, consider an *environment* S consisting of magnitudes of (potentially) observable objects (e.g. sizes of physical goods such as coconuts, houses, cars, incomes, and so on, or magnitudes of a characteristic such as height, beauty, intelligence, etc.), with veridical (true) distribution $F : S \to [0,1]$. An individual observes a *sample* $\breve{S}_{N+1}$ of $N+1$ observations which are independently and identically drawn from set S with distribution F. To evaluate the target magnitude $x \in \breve{S}_{N+1}$ he uses the remaining magnitudes y in the *reference set* $S_N = \breve{S}_{N+1} \setminus \{x\}$ using the following algorithm.

[4]If Robinson Crusoe is destined to have the same coconut (or height, intelligence, and so on) for life, he would choose the island where his coconut wins the pairwise tournaments most frequently. Yet, the adaptive evaluation model is silent regarding his choice over islands when other coconuts (income, consumption) could also be available.

Adaptive Evaluation Algorithm[5] *Suppose an individual faces a reference set S_N and is endowed with ordinal comparison and frequency (proportion) processing tools. Then the evaluation of the target magnitude x can be constructed as follows:*

Step 1 *Using the ordinal comparison tool, compare the "target" magnitude x to every magnitude y in the reference set S_N - that is, judge whether each is larger or smaller than x. Whenever y appears to be similar/equal to x (i.e. $y \sim x$), y is judged to be smaller with a probability p, and larger with probability $1 - p$.*

Step 2 *Using the frequency (proportion) processing tool, estimate how often every $y \in S_N$ is judged to be smaller, or $\frac{N_{y<x}+\hat{N}^{+}_{y\sim x}}{N}$, where $\hat{N}^{+}_{y\sim x}$ is the realized count of cases when a reference magnitude y appears to be similar/equal to x but is judged to be smaller.*

Step 3 *Evaluate the magnitude x relatively to the reference set S_N by the proportion (or frequency of occurrence) of reference magnitudes y which are judged to be smaller:*

$$\hat{I}_{S_N}(x) = \frac{N_{y<x} + \hat{N}^{+}_{y\sim x}}{N} \tag{2}$$

The *realized evaluation* $\hat{I}_{S_N}(x)$ is a *perceived rank* of magnitude x in reference set S_N. It is stochastic whenever the reference set S_N contains elements which appear to have similar magnitudes. If the individual makes many evaluations of magnitude x using the above algorithm, on average the proportion of similar reference magnitudes which are judged to be smaller is p. Let us define the *adaptive evaluation* of magnitude x to be the expected outcome of pairwise comparison, or the expected realized evaluation of this magnitude:[6]

$$I_{S_N}(x) = E\hat{I}_{S_N}(x) = \frac{N_{y<x}}{N} + p\frac{N_{y\sim x}}{N} \tag{3}$$

Note that one can write the realized evaluation as $\hat{I}_{S_N} = I_{S_N} + \epsilon$, where ϵ is some random variable. Following the random utility models pioneered by Thurstone [1927], the rest of the paper will concentrate on the adaptive evaluation I_{S_N} as a candidate for a utility function.

Example 1 *Let $A = \{10, 11, 12, 12, 98, 98, 98, 98, 98, 99, 100\}$ and suppose any $y = x$ has an*

[5] This three-step algorithm is based on counting relative frequencies (proportions). It is easy to modify it into a four-step procedure which is based on counting natural frequencies-of-occurrence instead. This can be done by adding an additional initial step which "normalizes" the range of values for I from 0 to N, and dropping the denominator N from the subsequent steps.

[6] The adaptive evaluation (3) is isomorphic to a *frequency accumulator* representation $I_{S_N}(x) = 1 \cdot \frac{\hat{N}_{y<x}}{N} + p \cdot \frac{\hat{N}_{y\sim x}}{N} + 0 \cdot \frac{\hat{N}_{y>x}}{N}$. where $p \in [0, 1]$ is the value of a similarity "tie".

equal chance to be judged smaller or larger than y. Then

$$\hat{I}_A(10) = \frac{0}{10} \quad \hat{I}_A(11) = \tfrac{1}{10} \quad \hat{I}_A(12) \in \left\{ \frac{2}{10}; \frac{3}{10} \right\}$$
$$\hat{I}_A(98) \in \left\{ \frac{4}{10}; \frac{5}{10}; \frac{6}{10}; \frac{7}{10}; \frac{8}{10} \right\} \quad \hat{I}_A(99) = \tfrac{9}{10} \quad \hat{I}_A(100) = \frac{10}{10} = 1$$

and the adaptive evaluations of each magnitude are

$$I_A(10) = 0; I_A(11) = 0.1; I_A(12) = 0.25; I_A(98) = 0.6; I_A(99) = 0.9; I_A(100) = 1$$

Evidently, the adaptive evaluation $I_{S_N}(x)$ on the reference set S_N is increasing in the magnitude x. Following Tversky and Kahneman [1991], consider complete and transitive "relatively more is better" *preference structure* $\succeq_{S_N}$ where set S_N is a *reference state*. This preference structure $\succeq_{S_N}$ is rationalizable by the adaptive evaluation I_{S_N}. As Tversky and Kahneman [1991] pointed out, "reference shift", or change in a reference state, often results in apparent preference anomalies. Thus, when the environment S changes, so does the reference set S_N, followed by a change in the adaptive evaluation $I_{S_N}(x)$, exhibiting non-reflexivity, non-transitivity and preference reversals when the choices are compared across different reference states.

Example 2 *Let* $B = \{10, 11, 12, 12, 12, 12, 12, 98, 98, 99, 100\}$ *and suppose any* $y = x$ *has an equal chance to be judged smaller or larger than y. Then*

$$\hat{I}_B(10) = \frac{0}{10} \quad \hat{I}_B(11) = \tfrac{1}{10} \quad \hat{I}_B(12) \in \left\{ \frac{2}{10}; \frac{3}{10}; \frac{4}{10}; \frac{5}{10}; \frac{6}{10} \right\}$$
$$\hat{I}_B(98) \in \left\{ \frac{7}{10}; \frac{8}{10} \right\} \quad \hat{I}_B(99) = \tfrac{9}{10} \quad \hat{I}_B(100) = \frac{10}{10}$$

Compare the adaptive evaluations I_B *of each magnitude to* I_A *from Example 1:*

$$I_B(10) = 0 = I_A(10) \quad I_B(11) = 0.1 = I_A(11) \quad I_B(12) = 0.3 > I_A(12)$$
$$I_B(98) = 0.75 < I_A(98) \quad I_B(99) = 0.9 = I_A(99) \quad I_B(100) = 1 = I_A(100)$$

That is, the same absolute magnitude may have different adaptive evaluation in different contexts.

The adaptive evaluation $I_{S_N}(x)$ and its realization $\hat{I}_{S_N}(x)$ depend on the composition of the reference set S_N and on the treatment of similar magnitudes p. The next assumption on the ordinal comparison tool, if holds, results in the realized evaluation $\hat{I}_{S_N}$ and the adaptive evaluation I_{S_N} being equal.

Assumption [A2]: *All reference magnitudes y which are similar to x are judged to be smaller with certainty, i.e. $p = 1$ for all x and y.*

The next assumption on the sample $\breve{S}_{N+1}$ is satisfied if the environment S is a continuum with a continuous veridical distribution $F(x)$.

Assumption [A3]: *Set $\breve{S}_{N+1}$ has no two elements x and y with equal magnitudes, i.e. $Pr(y = x) = 0$ in set $\breve{S}_{N+1}$ for any N.*

If at least one of the above assumptions hold, the following important result arises.

Proposition 1 *Suppose either A2 or A3 or both hold, and that only equal magnitudes are perceived to be similar, i.e. $y \sim x \Leftrightarrow y = x$. Then $\hat{I}_{S_N} = I_{S_N}$, and the adaptive evaluation $I_{S_N}(x)$ is isomorphic to the empirical distribution function $F_N(x)$. As the reference set S_N becomes large, it approaches in the limit to the veridical set $S = \lim_{N\to\infty} S_N$ and the evaluation $I_S(x) = \lim_{N\to\infty} I_{S_N}(x)$ converges uniformly to the veridical (true) distribution $F(x)$. Moreover, $I_S(x)$ is optimal in the sense of Robson [2001], i.e. a chance of mistaken choice of a smaller magnitude in a binary choice is minimized.*

Proof: By definition, the ordinal rank of x in S_N is given by the *empirical distribution function* $F_N(x) = \frac{N_{y\leq x}}{N} = \frac{N_{y<x}+N_{y=x}}{N}$, or, $F_N(x)$ is the frequency of magnitudes in S_N that do not exceed x. If A2 holds, then by assumption $p = 1$ for all x and y whenever $y \sim x \Leftrightarrow y = x$, so that $\hat{N}^{+}_{y\sim x} = N_{y=x}$. If A3 holds, $N_{y\sim x} = N_{y=x} = 0$ for any p. In either case, $I_{S_N}(x) = \hat{I}_{S_N}(x) = \frac{N_{y<x}+N_{y=x}}{N} = \frac{N_{y\leq x}}{N} = F_N(x)$. As all observations in $S_N \subset \breve{S}_{N+1}$ are i.i.d. draws from $F(x)$, the rest follows from Glivenko-Cantelli Theorem (e.g. Durrett [1996]). As Robson [2001] and Netzer [2009] showed, a mistake of choosing a smaller magnitude from a set of two draws from an environment $F(x)$ is minimized when the utility function equals to $F(x)$. ∎

That is, if either the ordinal comparison tool satisfies assumption A2, or if all reference magnitudes in the set S_N are distinct (in line with assumption A3), the adaptive evaluation I_{S_N} is isomorphic to an empirical distribution, or cumulative density function. This isomorphism of a utility function and a cumulative density function was first noticed by Van Praag [1968], and further explored by Kapteyn [1985 and references therein]. Gilboa and Schmeidler [2003] extended the case-based decision theory to represent "at least as likely" binary relation with ranking alternatives by their empirical frequencies. Stewart, Chater and Brown [2006] suggested informally that one can evaluate attribute values by their ordinal rank in a distribution of a sample from memory - whether the attribute is money amounts, time, or probability.

4 Adaptive Magnitude Evaluation on a Continuum

While the logic behind the model presented here works for both discrete and continuous reference distributions, the continuous case is expositionally simpler. Let the environment be S be represented by a veridical distribution F, continuously differentiable on support $(a, b) \subseteq R$, with $f = F' > 0$. Consider a large sample $\breve{S}_{N+1}$ such that $\lim_{N\to\infty} \breve{S}_N = \lim_{N\to\infty} S_N = S$, so that, by Proposition 1, $\lim_{N\to\infty} F_N = F(x)$. It is easy to see that (cumulative density) function $F(x)$ is a *frequency tool* as $F(x) = Pr(y \leq x)$, or equal to the frequency with which y is less or equal than x. To define the *ordinal comparison tool*, let an expected outcome of pairwise comparison of a fixed magnitude x with any magnitude y be represented by the *pairwise comparison function* $\mathcal{D}(x, y) : S \to [0, 1]$, which is non-decreasing in x and non-increasing in y, and is discontinuous at a countable number of points. Then, for any magnitude $x \in S$, the adaptive evaluation $I_S(x)$ of x against reference set S is parsimoniously constructed with two basic cognitive tools as the expected perceived outcome of pairwise comparison tournament of x against every element y in reference set S, or the perceived rank of x in S:[7]

$$I_S(x) = \int_S \underbrace{\mathcal{D}(x, y)}_{ordinal\ tool} \underbrace{dF(y)}_{frequency\ tool} \tag{4}$$

Here, the ordinal comparison tool is the integrand (kernel of integration), and the frequency (or proportion) tool is the variable of integration. The next result follows from the assumptions on the pairwise comparison function $\mathcal{D}$.

Proposition 2 *Suppose an individual is endowed with a frequency (proportion) tool F and a pairwise comparison tool $\mathcal{D}(x, y)$, which is non-decreasing in x, non-increasing in y, and has a finite number of discontinuities. Then*

(i) $I_S(x)$ is a non-decreasing continuous function on set S, and thus rationalizes continuous "more is better" preferences.

(ii) if the reference set A stochastically dominates set B, i.e. $F_A \succeq_{FOSD} F_B$, then $I_B(x) \geq I_A(x)$ for any $x \in X$.

Proof: (i) Since $\mathcal{D}$ is increasing in x and is discontinuous at finitely many points, $I_S(x)$ is non-decreasing and continuous. The preference rationalizability is a standard result.

(ii) By definition of first order stochastic dominance, for any non-decreasing function $U(t)$, $\int_A U(t)dF_A(t) \geq \int_B U(t)dF_B(t)$. But since $\mathcal{D}$ is non-increasing in y, we have that

$$I_A(x) = \int_A \mathcal{D}(x, y)dF_A(y) \leq \int_B \mathcal{D}(x, y)dF_B(y) = I_B(x) \quad \blacksquare$$

[7]Formally, the adaptive evaluation model (8) is closely related to Castagnoli and LiCalzi [1996]'s (expected) probability model.

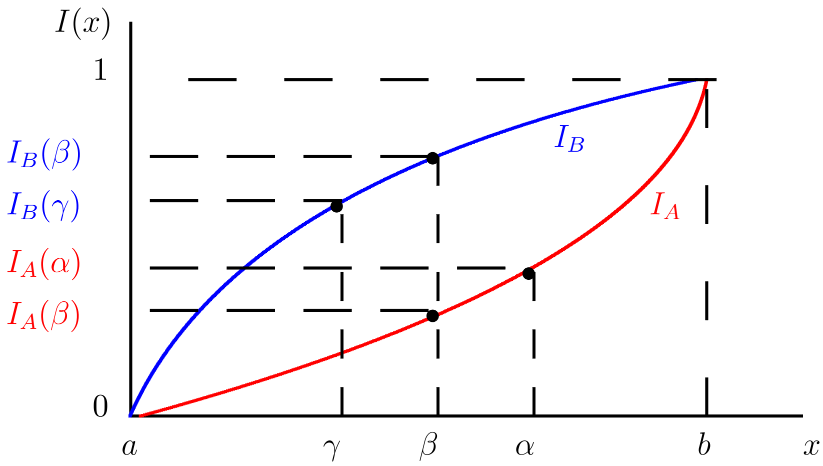


Figure 1: Violations of reflexivity and transitivity across different reference sets for $F_A \succeq_{FOSD} F_B$.

Thus, the adaptive evaluation $I_S(x)$ can rationalize continuous more is better preferences on S. Further, once the context changes, so does the adaptive evaluation. As Figure 1 shows, both reflexivity and transitivity may be violated. That is, one can find some $\beta \in A \cap B$, such that, first, $I_A(\beta) \neq I_B(\beta)$, and, second, it is possible to find $\alpha \in A$ and $\gamma \in B$ such that $I_A(\alpha) > I_A(\beta)$ and $I_B(\beta) > I_B(\gamma)$, but $I_A(\alpha) < I_B(\gamma)$. This has an important consequence. Suppose there exist an "outside" object with exogenously determined valuation $I_Y(\delta)$ where $(A \cup B) \cap Y = \emptyset$ such that $I_A(\beta) < I_Y(\delta) < I_B(\beta)$. Then, an individual would trade β for δ when reference set is A, but once reference set changes to B, he would trade δ back for β - an apparent preference reversal.

This systematic effect of a change in the reference set on subjects' evaluations has been documented in a number of psychological studies, including Parducci [1963, 1965], Stewart [2009], Olivola and Sagara [2009], Wood, Brown, and Maltby [2011], Ungemach, Stewart, and Reimers [2011], Stewart, Reimers, and Harris [2011].

5 Adaptive Evaluation with Perfect Cognitive Tools

Adaptive evaluation with perfect cognitive tools is a basic building block in understanding the cognitive processes involved in adaptive evaluation. It also provides a useful benchmark for understanding the role of cognitive imperfections.

5.1 The Basic Model of Perfect Adaptive Evaluation

Suppose an individual is endowed with a *perfect* frequency (or proportion) tool, so that he correctly perceives the veridical distribution of magnitudes F. Suppose further he is endowed with a *perfect* ordinal comparison tool, i.e. he can tell a bigger from a smaller magnitude even if the magnitudes are only slightly different. In this case, an expected outcome of a pairwise comparison can be written as:[8]

$$\mathcal{D}^P(x,y) = \begin{cases} 0 & \text{whenever} \quad x < y \\ c & \text{whenever} \quad x = y \\ 1 & \text{whenever} \quad x > y \end{cases} \tag{5}$$

where $c \in [0,1]$. This perfect ordinal comparison tool permits two isomorphic representations as a degenerate discriminability z. First, perfect ordinal tool $\mathcal{D}^P$ can be represented in terms of a difference $z = x - y$, degenerate at 0, with $G(z) = H(z) = H(x-y)$, where $H(\cdot)$ is Heaviside (step) function.[9] Thus,

$$\mathcal{D}^P(x,y) = H(x-y) \tag{6}$$

Second, $\mathcal{D}^P$ can be represented in terms of a ratio $z = \frac{x}{y}$, degenerate at 1, with $G(z) = H(z-1) = H\left(\frac{x}{y} - 1\right)$, so that

$$\mathcal{D}^P(x,y) = H\left(\frac{x}{y} - 1\right) \tag{7}$$

Since $H\left(\frac{x}{y} - 1\right) = H(x-y)$, the formulations (6) and (7) represent the same degenerate random variable. This allows us to derive the following benchmark result, which is a continuous-reference-distribution counterpart to the Proposition 1.

Proposition 3 *Suppose an individual is endowed with a perfect ordinal comparison tool (5) and with a perfect frequency (proportion) tool F. Then his adaptive evaluation of magnitude x is given by the cumulative density function of the veridical (true) magnitude distribution $F(x)$:*

$$I_S^P(x) = F(x) \tag{8}$$

Moreover, the perfect adaptive evaluation is optimal in the sense of Robson [2001].

[8] This perfect ordinal tool is similar to the optimal happiness function of Rayo and Becker [2007].

[9] A degenerate random variable z at c has a cumulative distribution equal to Heaviside (step) function $H(z-c)$, with $H(z-c) = 0$ for $z < c$, $H(z-c) = 1$ for $z > c$, and $H(c) \in [0,1]$. It has Dirac delta density function $\delta(x-c)$, and domain $(-\infty, \infty)$. For an arbitrary distribution F, $\int_{-\infty}^{\infty} H(c-z)dF(z) = F(c)$.

Proof: Using representations (6) and (7), get

$$I_S^P(x) = \int_S \mathcal{D}^P(x,y)dF(y) = \int_S H(x-y)\,dF(y) = \int_S H\left(\frac{x}{y}-1\right)dF(y) = F(x)$$

The optimality follows from Robson [2001] and Netzer [2009]. ∎

That is, if both ordinal comparison and frequency (proportion) tools are perfectly accurate, an individual's evaluation $I_S^P(x)$ of magnitude x is isomorphic to the veridical cumulative density function (veridical rank) $F(x)$, and is equal to the frequency of a magnitude x ordinarily "outperforming" other elements in the reference set S. Moreover, as it was shown by Robson [2001] and Netzer [2009], if the adaptive utility is $F(x)$, the probability of mistake in a binary choice in an environment S is minimized.

When cognitive tools are perfect, the shape of the magnitude distribution entirely determines the shape of the utility function. For example, if the veridical magnitude distribution is of the power function form, individual's adaptive evaluation is consistent with a constant relative risk aversion (CRRA) utility function; and if the veridical distribution is exponential, the evaluation exhibits constant absolute risk aversion (CARA).[10]

As the rest of the paper will show, cognitive imperfections result in an adaptive evaluation which is functionally distinct from the veridical distribution F, and thus suboptimal in the sense of Robson [2001].

5.2 Interaction Between Context and Memory

Both Gilboa and Schmeider [1995, 2003] and Stewart, Chater and Brown [2006] assume that an individual uses a database stored in one's memory. It thus may be only partially affected by the current environment, but it can also be affected by the past observations - and thus might be misremembered. Brown and Matthews [2011] extend the decision-by-sampling model to allow for memory to interact with the reference distributions. In other words, Robinson Crusoe's evaluation of a target coconut may be based not only on the set of coconuts which are currently observable to him, but also on his memory of coconut sizes seen previously, or even imagined.

More formally, suppose that at time t_0 the individual's history of observations includes a collection of $T+1$ environments $S_{t_0}, S_{t_0-1}, \ldots, S_{t_0-t}, \ldots, S_{t_0-T}$ with magnitude distributions $F_{t_0}, F_{t_0-1}, \ldots, F_{t_0-t}, \ldots, F_{t_0-T}$. Suppose the individual remembers (samples) observations from period $t_0 - t$ uniformly with the memory rate $\delta_t \in [0,1]$, i.e. any two reference magnitudes $y, y' \in S_{t_0-t}$ have equal probabilities of being included in the reference set S. Using

[10] See Castagnoli and Li Calzi [1996] for these relationships between distribution functions and the shape of utility function.

the general discounting model of Rubinstein [2003] (which permits hyperbolic discounting), write the remembered adaptive evaluation $I_S(x)$ of a magnitude x as a mixture of $T+1$ probability distributions:

$$I_S(x) = \frac{\delta_0}{\Delta} F_{t_0}(x) + \frac{1}{\Delta} \sum_{t=1,\ldots,T} \left(\Pi_{s=1,\ldots,t}\delta_s\right) F_{t_0-t}(x) \tag{9}$$

where $\Delta = \delta_0 + \sum_{t=1,\ldots,T} \left(\Pi_{s=1,\ldots,t}\delta_s\right)$ is a normalization constant.

Since past history may include imaginary environments, the above general formulation permits for saliency both in specific past histories and specific values. Specifically, a salient magnitude value $\tilde{x}$ might enter the reference set S as a past environment S_l containing a degenerate random variable at $\tilde{x}$, i.e. $F_l(x) = H(x - \tilde{x})$; while a salient environment at time k might enter S with $\delta_k > \max\{\delta_0, \ldots, \ldots, \delta_t, \ldots, \delta_T\}$ for all $t \neq k$.

Regardless of the structure of the reference set S, one can decompose the reference set S into a subset S_0 sampled from observations from the current period and a subset S_T sampled from remembered observations in the previous T periods, i.e. $S = S_0 \cup S_T$. Thus, the evaluation (9) can be written as a weighted sum of current F_{t_0} and remembered past $\mathcal{F}_T$ environment:

$$I_S(x) = \frac{\delta_0}{\Delta} F_{t_0}(x) + \left(1 - \frac{\delta_0}{\Delta}\right) F_T(x) \tag{10}$$

The following statement is obvious.

Proposition 4 *Suppose the reference set S includes observations beyond the current environment, i.e. $\frac{\delta_0}{\Delta} < 1$. Then the adaptive evaluation I_S is suboptimal in the sense of Robson [2001].*

That is, when an individual evaluates prospects based on a past memory, his mistakes in a binary choice in the present environment S_0 are not minimized. Furthermore, long memory may make the present environment to have a negligible effect on one's evaluation, leading to context-independent choices.

Proposition 5 *Suppose $\delta_t > 0$ for all t and consider an evaluation with long memory. Then $I(x) = \lim_{T\to\infty} I_S(x)$ is independent of the current context.*

Proof: As $\lim_{T\to\infty} \Delta = \infty$, the weight of the current context is negligible: $\lim_{T\to\infty} \frac{\delta_0}{\Delta} = 0$ and thus $I(x) = \lim_{T\to\infty} I_S(x) = F_T(x)$. ∎

Thus, with long memory, the individual behaves as if he possesses a neoclassical utility function which is determined by the distribution of remembered past observations. Such evaluation, however, is not optimal in the sense of Robson [2001]. Conversely, if one's memory is bounded, one's evaluations will exhibit dependence on the current context. The next result is straightforward.

Proposition 6 *Suppose $\delta_0 > 0$ and $T < \infty$, so that $\frac{\delta_0}{\Delta} > 0$, and S_T fixed. Suppose an individual faces either of the two current environments, A or B with $F_{t_0}^A$ and $F_{t_0}^B$, respectively. Then $I_{S_T \cup A}(x) \lesseqqgtr I_{S_T \cup B}(x)$ whenever $F_{t_0}^A \lesseqqgtr F_{t_0}^B$.*

In other words, when an individual is presented with variable context, her magnitude evaluation will exhibit qualitative, but not quantitative, correspondence with the present context. For example, magnitude judgments of subjects presented with positively skewed distributions will be higher than those presented with negatively skewed distributions as described in Section 3.

5.3 Expected Magnitude Evaluation

The case-based decision theory of Gilboa and Schmeider [1995, 2003] offers an alternative to expected utility theory where agents utilize a database of past experiences to make decisions. Rather than weighting utilities by probabilities, the weights are constructed based on the frequencies of appropriate experiences in the past. As I show here, such database stored in one's memory can be used efficiently as a source of information *both* for constructing an evaluation function and constructing a probability distribution.

Consider again Robinson Crusoe, who faces a coconut tree. Crusoe cannot climb up, but he knows that the tree (randomly) sheds a single coconut per day. Thus, from Crusoe's point of view, his daily coconut consumption is a lottery. How could Crusoe construct an expected magnitude evaluation of that about-to-fall coconut? Note that, as described earlier, the coconut tree serves as a database of coconut magnitudes, which Crusoe uses to evaluate a given coconut. But at the same time, the coconut tree also provides Crusoe with a database for calculating frequencies of coconuts of various sizes, and thus allows Crusoe to calculate a probability with which a coconut of a particular size might fall off the tree.

Consider an environment S with magnitude distribution F_S with support (a, b). The first result shows that, when all observations in the database are relevant both for magnitude evaluation and for probability calculation, the expected magnitude evaluation is independent of the environment.

Proposition 7 *Consider an environment S with magnitude distribution F_S with support (a, b). If the cognitive tools are perfect, the expected magnitude evaluation EI_S is independent of the environment S and is equal to the evaluation of the median magnitude.*

Proof: The proof is simple:

$$EI_S(S) = \int_S I_S(x)dF_S(x) = \int_S F_S(x)dF_S(x) = \frac{1}{2} \quad \blacksquare$$

In other words, if one faces a single stochastic context, one's evaluation of that gamble is equivalent to the evaluation of the median magnitude (which is not the same as expecting to obtain the median magnitude). Thus, when distribution is positively skewed (so that larger magnitudes are less frequent), so that median magnitude is smaller than mean magnitude, the adaptive evaluation of such context is lower than the evaluation of the average magnitude. This can be seen as a form of risk aversion, as the expected value of a gamble is less than the value of its expectation. In contrast, if the distribution is negatively skewed (so that the larger magnitudes are more frequent), the expected value of the gamble is more than the value of its expectations, corresponding to risk loving. We thus have the following straightforward result.

Proposition 8 *Consider an environment S with magnitude distribution F_S with support (a, b), and average magnitude $\mu = E(x)$ and median $m = F_S^{-1}(0.5)$. If the cognitive tools are perfect, then $EI_S(S) > (<) IE(x)$ iff $\mu > (<) m$.*

That is, one's choice between a gamble and its expectation could potentially be governed solely by the properties of the stochastic environment. However, as the subsequent results demonstrate, memory of previously encountered environments could be important for behavior in risky situations.

To see that, suppose there are two coconut trees on the opposite ends of the island. Crusoe needs to be next to the tree to pick up the fallen coconut before wild animals consume it. How would Crusoe decide which tree to choose? He would need to calculate and compare the expected magnitude evaluation for each tree, just as he would do if he were an expected utility maximizer. When Crusoe faces a choice between two trees, his memory plays an important part. Consider first what happens if Crusoe's memory can only hold information about one tree at a time (so that whenever Crusoe walks between the two trees, he forgets everything that he saw previously). Then Proposition 8 would apply. To see that, consider environments A and B, with F_A and F_B on (a, b), so that for any $j = A, B$

$$EI_j(j) = \int_S I_j(x)dF_j(x) = \int_j F_j(x)dF_j(x) = \frac{1}{2}$$

Thus, an amnesiac Crusoe is indifferent regarding which tree to choose.

Instead, suppose Crusoe has perfect memory, so that he remembers the coconuts on each tree. Moreover, suppose his memory allows him to keep track of which coconut in his memory belongs to which tree.[11] For the purposes of magnitude evaluation, he uses the "global" reference set $S_A \cup S_B$ - as the entire universe of all coconuts known to Crusoe is relevant for the magnitude evaluation purposes. But, for the purposes of evaluation of the gambles that each tree provides, only the subset of coconuts associated with a particular tree is relevant for Crusoe's calculation of a probability for a coconut of a particular size falling off that tree.[12] Denote the shares of all coconuts hanging on trees A and B as α and $1-\alpha$, respectively. Then the adaptive evaluation of a magnitude x with respect to the combined reference set $A \cup B$ becomes

$$I_{A\cup B}(x) = \int_S \mathcal{D}^p(y,x)(\alpha f_A(y) + (1-\alpha) f_B(y))dy = \alpha F_A(x) + (1-\alpha)F_B(x)$$

However, the probability distributions associated with each tree are still F_A and F_B. Thus, the expected magnitude evaluations in each environment $j = A, B$ given the combined reference set $A \cup B$ are:

$$\begin{aligned} EI_{A\cup B}(A) &= \int_S (\alpha F_A(x) + (1-\alpha)F_B(x))dF_A(x) = \frac{\alpha}{2} + (1-\alpha)\int_S F_B(x)dF_A(x) \\ EI_{A\cup B}(B) &= \int_S (\alpha F_A(x) + (1-\alpha)F_B(x))dF_B(x) = \frac{1-\alpha}{2} + \alpha\int_S F_A(x)dF_B(x) \end{aligned}$$

In other words, the distribution of coconuts on tree B affects the expected magnitude evaluation of tree A. One thus would expect the the choice of one tree over the other is affected by the interaction between the two contexts, as the following proposition suggests.

Proposition 9 *Consider two environments S_A and S_B with magnitude distributions F_A and F_B with support (a,b). If the agent's cognitive tools are perfect, then $A \succ (\prec) B$ whenever $\int_S F_B(x)dF_A(x) > (<)\frac{1}{2} \Leftrightarrow \int_S F_A(x)dF_B(x) < (>)\frac{1}{2}$. Moreover, unless $F_A = F_B$, the case of indifference is non-generic.*

Proof: The agent's choice depends on the difference between the two expectations:

$$EI_{A\cup B}(A) - EI_{A\cup B}(B) = \frac{\alpha}{2} + \int_S (1-\alpha)F_B(x)dF_A(x) - \frac{1-\alpha}{2} - \int_S \alpha F_A(x)dF_B(x)$$

Note that $\int_S F_A(x)dF_B(x) + \int_S F_B(x)dF_A(x) = 1$, so that

$$EI_{A\cup B}(A) - EI_{A\cup B}(B) = \int_S F_B(x)dF_A(x) - \frac{1}{2} = \frac{1}{2} - \int_S F_A(x)dF_B(x) \ \blacksquare \tag{11}$$

[11] As Mullett and Tunney [2013] find, activations in ventro-medial prefrontal cortex (vmPFC) and the anterior cingulate cortex (ACC) are consistent with encoding "global" ordinal rank, while activations in ventral striatum and thalamus record "local" ordinal rank.

[12] This partitioning the memory set into disjoint subsets for the purpose of probability calculation can be modeled by the similarity function of Gilboa and Schmeider [1995, 2003].

One can understand the interaction of the two environmental contexts better when the contexts are stochastically ordered. As the next result states, if bigger sizes are more common in one context than in the other (in a sense of first order stochastic dominance), the expected evaluation of stochastically bigger (smaller) tree will be higher (lower) than what they would have been worth to the amnesiac Crusoe. In other words, exposing one to information about an alternative context makes one better off if the alternative context is stochastically worse, and vice versa. Furthermore, as one would expect, the stochastically bigger context is more desirable.

Corollary 1 *Iff $F_A \succeq_{FOSD} F_B$, then $EI_{A \cup B}(A) \geq EI_A(A)$, $EI_{A \cup B}(B) \leq EI_B(B)$, and $A \succeq B$.*

Proof: First order stochastic dominance implies that $F_B(x) \geq F_A(x)$ for all x on (a, b), so that

$$\begin{aligned} \int_S F_B(x) dF_A(x) &\geq \int_S F_A(x) dF_A(x) = \frac{1}{2} \\ \int_S F_A(x) dF_B(x) &\leq \int_S F_B(x) dF_B(x) = \frac{1}{2} \end{aligned}$$

Thus,

$$\begin{aligned} EI_{A \cup B}(A) &= \frac{\alpha}{2} + (1 - \alpha) \int_S F_B(x) dF_A(x) \geq \frac{1}{2} = \int_S F_A(x) dF_A(x) = EI_A(A) \\ EI_{A \cup B}(B) &= \frac{1 - \alpha}{2} + \alpha \int_S F_A(x) dF_B(x) \leq \frac{1}{2} = \int_S F_B(x) dF_B(x) = EI_B(B) \end{aligned}$$

and $EI_{S_A \cup S_B}(A) \geq EI_{S_A \cup S_B}(B)$. ∎

The next result assumes that one of the distributions, F_B is concave, so that bigger magnitudes are less frequent than smaller ones. If, in addition, it is second order stochastically dominated by the other context F_A, then F_B is less desirable. This is because both because, from magnitude evaluation point of view, the frequent small magnitudes in B makes magnitudes in A to be more attractive, but also because bigger magnitudes are more frequent in A than in B.

Corollary 2 *If $F_A \succeq_{SOSD} F_B$ and $F_B'' < 0$ on (a, b), then $EI_{A \cup B}(A) \geq EI_A(A)$, and $A \succeq B$.*

Proof: As F_B is increasing and concave, second order stochastic dominance implies that

$$EI_{S_A \cup S_B}(A) - EI_{S_A \cup S_B}(B) = \int_S F_B(x) dF_A(x) - \frac{1}{2} \geq \int_S F_B(x) dF_B(x) - \frac{1}{2} = 0 \ \blacksquare$$

Thus, the expected magnitude evaluation generates the same prediction as expected utility theory in the case of first order stochastic dominance, it is more restrictive in the case of second order stochastic dominance. However, as Stewart, Chater, Brown [2006] point out, many empirically observed distributions exhibit everywhere decreasing density. Thus, from the empirical point of view, the cases when the expected utility theory (under the assumption of risk aversion) and the theory of adaptive magnitude evaluation generate different predictions are not common.

6 Adaptive Evaluation with Ordinal Imperfections

One of the advantages of the proposed mode of adaptive evaluation is that it allows one to clarify the role of cognitive imperfections on magnitude evaluation. Specifically, cognitive imperfections distort one's mental line and lead to magnitude evaluation to be suboptimal in a sense of Robson [2001].

As psychologists discovered in the XIX century, humans tend not to notice the difference between two relatively similar magnitudes. The widely known Weber-Fechner psychophysical law (e.g. Laming [1973]) is the psychologist's counterpart to the law of diminishing marginal utility, and it states that the minimum amount by which stimulus intensity must be changed in order to produce a noticeable variation increases with the stimulus level.

Such findings open up a possibility that the ordinal comparison tool may be imperfect in the sense that any two magnitudes which belong to an interval of *doubt* are *perceived* to be similar. Such ordinal imperfections may lead to smaller magnitudes having higher realized evaluations than larger magnitudes. Thus, due to the stochastic nature of ordinal imperfections, apparent preference anomalies can occur even for the same reference set.

Example 3 *Consider the environments specified in Examples 1 and 2. Suppose the individual perceives* $10 \sim 11 \sim 12$ *and* $98 \sim 99 \sim 100$ *and suppose any* $y = x$ *has an equal chance to be judged smaller or larger than* y. *Then the sample* A *is perceived to be* $A' = \{11, 11, 11, 11, 99, 99, 99, 99, 99, 99, 99\}$ *and thus*

$$\hat{I}_{A'}(x) \in \left\{\frac{0}{10}; \frac{1}{10}; \frac{2}{10}; \frac{3}{10}\right\} \quad \textit{for} \quad x \in \{10, 11, 12\}$$
$$\hat{I}_{A'}(x) \in \left\{\frac{4}{10}; \frac{5}{10}; \frac{6}{10}; \frac{7}{10}; \frac{8}{10}; \frac{9}{10}; \frac{10}{10}\right\} \quad \textit{for} \quad x \in \{98, 99, 100\}$$

Thus the realized evaluation can be non-monotone in magnitude, e.g. $\hat{I}_{A'}(12) < \hat{I}_{A'}(11) < \hat{I}_{A'}(10) < \hat{I}_{A'}(100) = \hat{I}_{A'}(99) = \hat{I}_{A'}(98)$. *However, the adaptive evaluations are non-*

decreasing in magnitude:

$$I_{A'}(10) = I_{A'}(11) = I_{A'}(12) = 0.15$$
$$I_{A'}(98) = I_{A'}(99) = I_{A'}(100) = 0.7$$

Similarly, B is perceived to be $B' = \{11, 11, 11, 11, 11, 11, 11, 99, 99, 99, 99\}$ and thus

$$\hat{I}_{B'}(x) \in \left\{\frac{0}{10}; \frac{1}{10}; \frac{2}{10}; \frac{3}{10}; \frac{4}{10}; \frac{5}{10}; \frac{6}{10}\right\} \quad \textit{for} \quad x \in \{10, 11, 12\}$$
$$\hat{I}_{B'}(x) \in \left\{\frac{7}{10}; \frac{8}{10}; \frac{9}{10}; \frac{10}{10}\right\} \quad \textit{for} \quad x \in \{98, 99, 100\}$$

so that the adaptive evaluations are:

$$I_{B'}(10) = I_{B'}(11) = I_{B'}(12) = 0.3$$
$$I_{B'}(98) = I_{B'}(99) = I_{B'}(100) = 0.85$$

Here, relatively to the "true" evaluation in Examples 1 and 2, the imperfect ordinal comparison tool leads to overvaluing of "small" magnitudes (10 and 11), undervaluing of "large" ones (99 and 100). Since large magnitudes are more frequent in set A' than in set B', all magnitudes are more valuable in set B'.

To understand the complex interactions between ordinal imperfections and the reference set, consider again a large sample from a continuous distribution $F(x)$, and suppose that an individual is endowed with a *perfect* frequency tool, perceiving the distribution of the reference magnitudes to be F. Suppose that the outcome of the pairwise comparison between x and y is determined with certainty whenever x and y are sufficiently far apart, but the ordinal discrimination is stochastic whenever x and y are sufficiently close. Let us suppose that the ordinal discrimination depends the ordinal discriminability variable $z = Z(x, y)$ with $Z_1 > 0, Z_2 < 0$ (e.g. $z = x + y$ or $z = \frac{x}{y}$). Let $G(z)$ be the probability that x is perceived to be no smaller when compared to y whenever z falls into the interval of doubt $K = [k_1, k_2]$. Then, the outcome of the imperfect ordinal comparison tournament is:

$$\mathcal{D}^Z(x, y) = \begin{cases} 0 & \text{whenever} \quad z < k_1 \\ G(z) & \text{whenever} \quad z \in [k_1, k_2] \\ 1 & \text{whenever} \quad z > k_2 \end{cases} \tag{12}$$

and the imperfect adaptive evaluation $I_S(x)$ can be written as the expected outcome of pairwise comparison:

$$I_S^Z(x) = \int_S \mathcal{D}^Z(x, y) dF(y) = \int_S G(Z(x, y)) dF(y) \tag{13}$$

The imperfect adaptive evaluation $I_S^Z(x)$ depends on the form of discriminability $z = Z(x, y)$ and will be explored below.

6.1 Additively Imperfect Ordinal Comparison Tool

Let the ordinal discriminability z be in terms of differences, i.e. $z = x - y \in [k_1, k_2]$, with $k_1 \leq 0 \leq k_2$. For technical simplicity, assume that the interval of doubt is "small", i.e. $k_2 - k_1 < b - a$. Whenever $z > 0$, a reference magnitude y "looms large", with the ordinal tournament assessment of magnitude x being biased downwards. And vice versa, whenever $z < 0$, a reference magnitude y "looms small", and the evaluation of x is boosted upwards. The resulting adaptive evaluation $I_S(x)$ is isomorphic to a *convolution* of a reference variable y and an ordinal discriminability variable z:

$$I_S^A(x) = \int_S \mathcal{D}^A(x, y) dF(y) = \int_S G(x - y) \, dF(y) \tag{14}$$

with the marginal evaluation function being

$$\frac{dI_S^A(x)}{dx} = \int_S g(x - y) \, f(y) dy \quad \text{where} \quad x = z + y$$

Obviously, the perfect comparison tool (6) is a special case of (14). This isomorphism of the additively imperfect adaptive evaluation and a distribution of a variable $x = y + z$ simplifies the subsequent analysis.

Proposition 10 *Suppose the expectation of ordinal discriminability z is zero, i.e. $Ez = 0$. Then, the additively imperfect adaptive evaluation of $I^A(x)$ crosses the veridical adaptive evaluation $I^P(x)$ once, and from above.*

Proof: The reference magnitude Y second order stochastically dominates the additively imperfect reference magnitude $Y + Z$ (or $Y \geq_{SOSD} Y + Z$), and the result is straightforward (see Shaked and Shantikumar [2007], Theorems 3.A.5 and 3.A.34). ∎

In other words, whenever ordinal discriminability z has zero mean, the low values of x are adaptively overvalued, and the high values are undervalued. For example, if the veridical (true) reference distribution is normal $N(\mu_x, \sigma_x^2)$, and the ordinal discriminability variable z is also normal $N(0, \sigma_z^2)$, then the additively imperfect adaptive evaluation of x is the rank in the convolution distribution, which is normal $N(\mu_x, \sigma_x^2 + \sigma_z^2)$, having "fatter" tails than the veridical (true) distribution. This "fatter tail" result holds for most reference distributions.

Proposition 11 *Suppose an individual is endowed with a perfect frequency (proportion) tool F, and an additively imperfect ordinal comparison tool D^A (14) with the ordinal discriminability variable z distributed with G on $[k_1, k_2]$ with $k_1 < 0 < k_2$ and $k_2 - k_1 < b - a$. Then $I_S^A(a) > 0$ and $I_S^A(b) < 1$ as long as $g = G' > 0$ on $[k_1, k_2]$ and $a > -\infty$ and $b < \infty$.*

Proof: For small interval of doubt $k_2 - k_1 < b - a$, an additively imperfect evaluation $I_S^A(x)$ (14) is:

$$I_S^A(x) = \begin{cases} \int_a^x \int_a^{t-k_1} g(t-y)dF(y)dt & \text{if } a \leq x \leq a + k_2 \\ \int_{a+k_2}^x \int_{t-k_2}^{t-k_1} g(t-y)dF(y)dt \quad + I_S^A(a+k_2) & \text{if } a + k_2 < x \leq b + k_1 \\ \int_{b+k_1}^x \int_{t-k_2}^{b} g(t-y)dF(y)dt \quad + I_S^A(b+k_1) & \text{if } b + k_1 < x \leq b \end{cases} \tag{15}$$

Thus, $I_S^A(a) = \int_{a+k_1}^a \int_a^{t-k_1} g(t-y)dF(y)dt$ is strictly positive as long as $a > -\infty$, and $I_S^A(b) = 1 - \int_b^{b+k_2} \int_{t-k_2}^b g(t-y)dF(y)dt$ is strictly less than one as long as $b < \infty$. ∎

Thus, ordinal imperfection alter the shape of magnitude evaluation, obscuring the relationship between context and evaluation, as an imperfect additively imperfect evaluation may not be a distribution function. Ordinal imperfections tend to result in overvaluation of low magnitides and undervaluation of high magnitudes because, when evaluating small objects, the individual incorrectly perceives the existence of objects that are smaller than the smallest object in the set S, and, similarly, he imagines non-existing large objects when evaluating large magnitudes.

Example 4 *Suppose the reference distribution F is uniform on $[\mu-\sigma; \mu+\sigma]$, and distribution of ordinal discriminability $G(z)$ is also uniform on $[k_1, k_2]$ with $k_2 - k_1 < 2\sigma$. Then*

$$I_S^{AU}(x) = \begin{cases} \frac{(\mu-\sigma+k_1-x)^2}{4\sigma(k_2-k_1)} & \text{if } \mu - \sigma \leq x \leq \mu - \sigma + k_2 \\ \frac{x-\mu+\sigma-\frac{k_1+k_2}{2}}{2\sigma} & \text{if } \mu - \sigma + k_2 < x \leq \mu + \sigma + k_1 \\ 1 - \frac{(\mu+\sigma+k_2-x)^2}{4\sigma(k_2-k_1)} & \text{if } \mu + \sigma + k_1 < x \leq \mu + \sigma \end{cases}$$

Thus, in addition to overvaluing small and undervaluing large magnitudes, the evaluation exhibits increasing marginal utility for small magnitudes, decreasing marginal utility for large magnitudes, and constant marginal utility in-between.

6.2 Multiplicatively Imperfect Ordinal Comparison Tool

Let us consider a generalization of Rubinstein's [1988] similarity model (which can be seen as a form of psychophysical Weber-Fechner law). Let $a \geq 0$ and suppose the ordinal discriminability z be in terms of ratios, i.e. $z = \frac{x}{y} \in [k_1, k_2]$, with $0 < k_1 \leq 1 \leq k_2$. That is, the individual can tell apart two distinct magnitudes x and y with certainty only if the ratio of magnitudes $z = \frac{x}{y}$ is outside of the interval of doubt $[k_1, k_2]$, but within this interval an individual can only tell that x is greater than y with some probability $G(z)$. For technical simplicity, assume that the interval of doubt is "small", i.e. $\frac{k_2}{k_1} < \frac{b}{a}$. Whenever $z > 1$, a reference magnitude y "looms large", with the ordinal judgment of magnitude x being biased downwards. And vice versa, whenever $z < 1$, a reference magnitude y "looms small", and the judgment of x is boosted upwards.

The resulting adaptive evaluation $I_S(x)$ is isomorphic to a *scale mixture* of a reference variable y and an ordinal discriminability variable z:

$$I_S^M(x) = \int_S \mathcal{D}^M(x, y) dF(y) = \int_S G\left(\frac{x}{y}\right) dF(y) \tag{16}$$

with the marginal evaluation function being

$$\frac{dI_S^M(x)}{dx} = \int_S \frac{1}{y} g\left(\frac{x}{y}\right) f(y)dy \quad \text{where} \quad x = z \cdot y$$

Obviously, the perfect comparison tool (7) is a special case of (16).

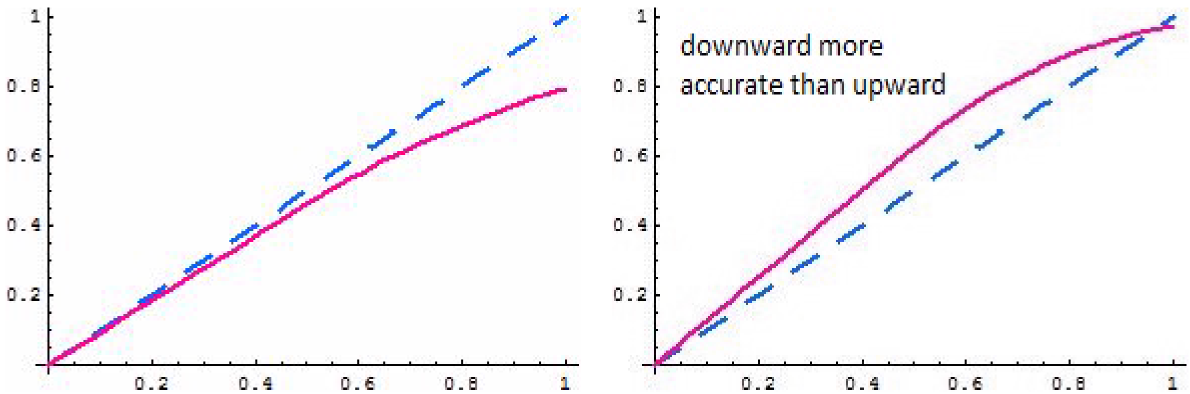
The isomorphism of the multiplicatively imperfect adaptive evaluation and a distribution of a variable $x = y \cdot z$ leads to a number of interesting results, including to an interesting observation that with an *arbitrary* multiplicative discriminability, one's adaptive evaluation based on the uniform reference distribution on $(0, 1)$ will tend to exhibit non-increasing marginal evaluation.

Proposition 12 *Suppose the veridical distribution F is uniform on $(0, 1)$. Then, the multiplicatively imperfect ordinal tool results in a concave magnitude evaluation.*

Proof: This follows from the converse to Khinchine's representation for unimodal distributions and the definition of unimodal distributions (see Dharmadhikari and Joag-dev [1988], Theorem 1.3). ∎

For example, as Figure 2 shows, evaluation relatively to the uniform reference distribution on $(0, 1)$ with a uniform multiplicative tool, will exhibit constant marginal utility for the lower range of veridical magnitudes and decreasing marginal utility for the higher range. The following statement shows that the "fatter tail" result holds for the multiplicatively imperfect ordinal tool as well.

Proposition 13 *Suppose an individual is endowed with a perfect frequency (proportion) tool F, and a multiplicatively imperfect ordinal comparison tool D^M (16) with ordinal discriminability z distributed with G on $[k_1, k_2]$ with $0 < k_1 < 1 < k_2$ and $\frac{k_2}{k_1} < \frac{b}{a}$. Then $I_S^M(a) > 0$ and $I_S^M(b) < 1$ as long as $g = G' > 0$ on $[k_1, k_2]$ and $a > -\infty$ and $b < \infty$.*



a) $x \sim U(0,1)$, $z \sim U[0.5, 2.0]$ b) $x \sim U(0,1)$, $z \sim U[0.5, 1.2]$

Figure 2: Relatively to the uniform veridical distribution on $(0,1)$ (dashed lines) the multiplicatively imperfect evaluation (solid curves) exhibits non-increasing diminishing marginal utility.

Proof: For small interval of doubt $\frac{k_2}{k_1} < \frac{b}{a}$, adaptive evaluation $I_S^M(x)$ is:

$$I_S^M(x) = \begin{cases} \int_{k_1 a}^{x} \int_{a}^{\frac{t}{k_1}} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt & \text{if } a \leq x \leq k_2 a \\ \int_{k_2 a}^{x} \int_{\frac{x}{k_2}}^{\frac{x}{k_1}} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt \quad + I_S^M(k_2 a) & \text{if } k_2 a < x \leq k_1 b \\ \int_{k_1 b}^{x} \int_{\frac{t}{k_2}}^{b} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt \quad + I_S^M(k_1 b) & \text{if } k_1 b < x \leq b \end{cases}$$

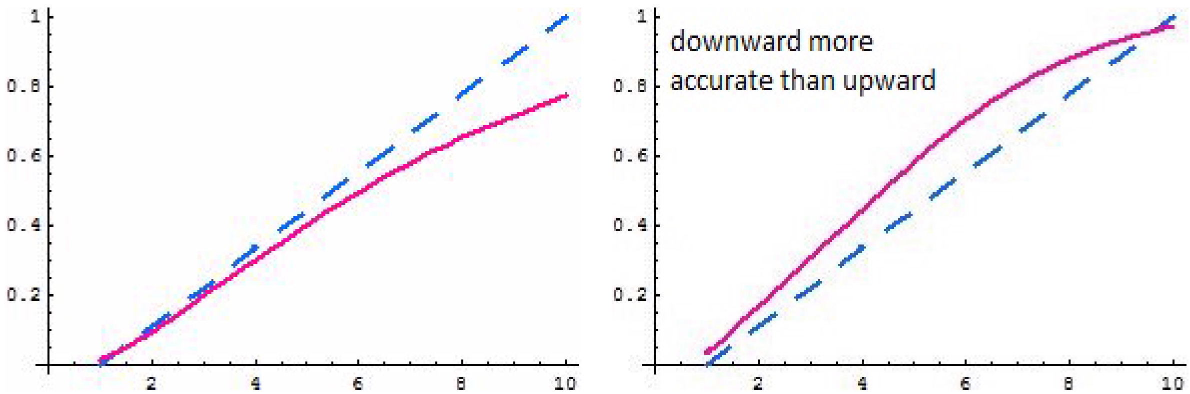
Thus, $I^M(a) = \int_{k_1 a}^{a} \int_{a}^{\frac{t}{k_1}} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt$ is positive as long as $a > 0$, and $I^M(b) = 1 - \int_{b}^{k_2 b} \int_{\frac{t}{k_2}}^{b} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt$ is strictly less than one as long as $b < \infty$. ∎

That is, whether imperfect ordinal tool is additive or multiplicative, in general, the imperfect adaptive evaluation is not a distribution function. This is because the ordinal imperfections lead to the individual incorrectly perceiving the existence of magnitudes which are outside of the reference set, leading to overvaluation of small magnitudes and undervaluation of large ones.

Example 5 *Suppose the reference distribution F is uniform on $[\mu-\sigma; \mu+\sigma]$, and distribution of multiplicative ordinal discriminability $G(z)$ is also uniform on $[k_1, k_2]$ with $\frac{k_2}{k_1} < \frac{\mu+\sigma}{\mu-\sigma}$. Then*

$$I_S^{MU}(x) = \begin{cases} \frac{1}{2\sigma(k_2-k_1)} \left(k_1(\mu-\sigma) - x + x \ln \frac{x}{k_1(\mu-\sigma)}\right) & \textit{if } \mu-\sigma < x < k_2(\mu-\sigma) \\ \frac{1}{2\sigma(k_2-k_1)} \left(-(k_2-k_1)(\mu-\sigma) + x \ln \frac{k_2}{k_1}\right) & \textit{if } k_2(\mu-\sigma) \leq x \leq k_1(\mu+\sigma) \\ 1 - \frac{1}{2\sigma(k_2-k_1)} \left(k_2(\mu+\sigma) - x + x \ln \frac{x}{k_2(\mu+\sigma)}\right) & \textit{if } k_1(\mu+\sigma) < x < \mu+\sigma \end{cases}$$

Thus, in addition to overvaluing small and undervaluing large magnitudes, the evaluation



a) $x \sim U(1,10)$, $z \sim U[0.5, 2.0]$ b) $x \sim U(1,10)$, $z \sim U[0.5, 1.2]$

Figure 3: Relatively to the uniform veridical distribution on $(1, 10)$ (dashed lines) the multiplicatively imperfect evaluation (solid curves) exhibits overvaluation for lower magnitudes and undervaluation for high magnitides.

exhibits increasing marginal utility for small magnitudes, decreasing marginal utility for large magnitudes, and constant marginal utility in-between (see Figure 3).

As Figure 3a shows, if an individual faces similar ordinal imperfections for upward and downward comparisons, it is possible that undervaluation may occur for most of the veridical range. In contrast, as Figure 3b shows, if an individual is better at distinguishing magnitudes which are less rather than those which are greater than the target magnitude, most of the veridical magnitudes might be overvalued.

Multiplicatively and additively imperfect evaluations have some features in common. This is at least in part because one can use a transformation $G\left(\frac{x}{y}\right) = \tilde{G}\left(\ln \frac{x}{y}\right) = \tilde{G}(\tilde{x} - \tilde{y})$, where $\tilde{x} = \ln x$ and $\tilde{y} = \ln y$. While computationally more complex, multiplicative discriminability is more conducive to cognitive non-linearities and has a more solid grounding in cognitive and brain research.

7 Adding Frequency Imperfections

In addition to the well-documented Weber-Fechner law suggesting that human ordinal comparison abilities are imperfect, is it possible that humans may also distort frequencies, albeit the form of frequency imperfections might differ from the form of ordinal imperfections. As Kahneman and Tversky [1979] point out, humans assign non-linear weights to probabilities, overweighing small probabilities and underweighing large ones.[13] Spence [1990] and Hollands

[13] See Hsu, Krajbich, Zhao and Camerer [2009] for a comprehensive list of probability weighting models.

and Dyre [2000] found even a more complex phenomenon for proportion judgments over a variety of stimuli, including continuous sensory modalities such as fullness of the glass, etc. Thus, it is plausible that the frequency tool is also imperfect, further distorting the relationship between the veridical magnitude distribution and adaptive evaluation, and making it more difficult to observe empirically.

Hsu, Krajbich, Zhao and Camerer [2009] suggest that the probability distortions arise at the level of coding in the brain, while Stewart, Chater and Brown [2006] argue that probability distortions align with the empirical frequency distributions. As the research on probability/proportion imperfections is still in infancy, and the cognitive mechanism behind frequency distortions is not clear. Interestingly, in Dehaene, Izard, Spelke, and Pica [2008], subjects appear to lose count, undercounting large numerosities (particularly for sets of dots and sequences of tones). This opens up a possibility that frequency imperfections arise because of *counting errors*, as in large reference sets, the total number of elements N might be undercounted, as well as the number of smaller items $N_{y<x}$ when magnitude x is large (as it "beats" many reference magnitudes). Such imperfections in numerosity judgments may lead to overestimation of cumulative frequency for small magnitudes and underestimation for large ones.[14]

Example 6 *Consider the environments specified in Examples 1 and 2, with any $y = x$ has an equal chance to be judged smaller or larger than y. For illustrative purposes, suppose numerosities greater than* 6 *are undercounted by* 1*. Then for the sample A,*

$$\hat{I}_A^F(10) = \frac{0}{10-1} \quad \hat{I}_A^F(11) = \frac{1}{10-1} \quad \hat{I}_A^F(12) \in \left\{\frac{2}{10-1}; \frac{3}{10-1}\right\}$$
$$\hat{I}_A^F(98) \in \left\{\frac{4}{10-1}; \frac{5}{10-1}; \frac{6}{10-1}; \frac{7-1}{10-1}; \frac{8-1}{10-1}\right\} \quad \hat{I}_A^F(99) = \frac{9-1}{10-1} \quad \hat{I}_A^F(100) = \frac{10-1}{10-1}$$

and the adaptive evaluations are:

$$I_A^F(10) = 0 = I_A(10) \quad I_A^F(11) = 0.11 > I_A(11) \quad I_A^F(12) = 0.28 > I_A(12)$$
$$I_A^F(98) = 0.6 > I_A(98) \quad I_A^F(99) = 0.89 < I_A(99) \quad I_A^F(100) = 1 = I_A(100)$$

For the sample B

$$\hat{I}_B^F(10) = \frac{0}{10-1} \quad \hat{I}_B^F(11) = \frac{1}{10-1} \quad \hat{I}_B^F(12) \in \left\{\frac{2}{10-1}; \frac{3}{10-1}; \frac{4}{10-1}; \frac{5}{10-1}; \frac{6}{10-1}\right\}$$
$$\hat{I}_B^F(98) \in \left\{\frac{7-1}{10-1}; \frac{8-1}{10-1}\right\} \quad \hat{I}_B^F(99) = \frac{9-1}{10-1} \quad \hat{I}_B^F(100) = \frac{10-1}{10-1}$$

[14] Furthermore, in Dehaene, Izard, Spelke, and Pica [2008] subjects from Amazonian indigenous societies also appear to overcount small numerosities, which might further lead to overcounting the number of smaller items $N_{y<x}$ when magnitude x is small (as it "beats" only a few reference magnitudes).

and the adaptive evaluations are:

$$I_B^F(10) = 0 = I_B(10) \quad I_B^F(11) = 0.11 > I_B(11) \quad I_B^F(12) = 0.44 > I_B(12)$$
$$I_B^F(98) = 0.72 > I_B(98) \quad I_B^F(99) = 0.89 < I_B(99) \quad I_B^F(100) = 1 = I_B(100)$$

Combining frequency imperfections with the ordinal imperfections of Example 3 for for the sample A:

$$\hat{I}_{A'}^F(x) \in \left\{\frac{0}{10-1}; \frac{1}{10-1}; \frac{2}{10-1}; \frac{3}{10-1}\right\} \quad \textit{for} \quad x \in \{10, 11, 12\}$$
$$\hat{I}_{A'}^F(x) \in \left\{\frac{4}{10-1}; \frac{5}{10-1}; \frac{6}{10-1}; \frac{7-1}{10-1}; \frac{8-1}{10-1}; \frac{9-1}{10-1}; \frac{10-1}{10-1}\right\} \quad \textit{for} \quad x \in \{98, 99, 100\}$$

and the adaptive evaluations with both imperfections are

$$I_{A'}^F(10) = I_{A'}^F(11) = I_{A'}^F(12) = 0.17$$
$$I_{A'}^F(98) = I_{A'}^F(99) = I_{A'}^F(100) = 0.71$$

For the sample B:

$$\hat{I}_{B'}^F(x) \in \left\{\frac{0}{10-1}; \frac{1}{10-1}; \frac{2}{10-1}; \frac{3}{10-1}; \frac{4}{10-1}; \frac{5}{10-1}; \frac{6}{10-1}\right\} \quad \textit{for} \quad x \in \{10, 11, 12\}$$
$$\hat{I}_{B'}^F(x) \in \left\{\frac{7-1}{10-1}; \frac{8-1}{10-1}; \frac{9-1}{10-1}; \frac{10-1}{10-1}\right\} \quad \textit{for} \quad x \in \{98, 99, 100\}$$

and the adaptive evaluations are:

$$I_{B'}^F(10) = I_{B'}^F(11) = I_{B'}^F(12) = 0.33$$
$$I_{B'}^F(98) = I_{B'}^F(99) = I_{B'}^F(100) = 0.83$$

Here, the ordinal and frequency imperfections combined follow the pattern of ordinal distortion in Example 3.

More formally, let us assume an imperfect frequency tool $w(F)$ as in Quiggin [1982], with $w' > 0$. Then the adaptive evaluation of magnitude x is

$$\hat{I}_S(x) = \int_S G(Z(x, y))dw(F(y)) \tag{17}$$

That is, the adaptive evaluation is determined by three components: by the context of evaluation F, by the ordinal imperfection G and by the frequency (proportion) weighting $w(F)$, resulting in adaptive evaluation very different from the reference distribution. The next result is straightforward.

Proposition 14 *Suppose an individual's magnitude evaluation is given by (17). With a perfect ordinal comparison tool, his adaptive evaluation of magnitude x is entirely determined by its rank in the weighted magnitude distribution:*

$$I_S^{FP}(x) = w(F(x)) \tag{18}$$

so that $I_S^{FP}(x) \lesseqqgtr I_S^P(x)$ whenever $w(F(x)) \lesseqqgtr F(x)$. Moreover, an adaptive evaluation with imperfect frequency tool is suboptimal in the sense of Robson [2001].

Proof: Using representations (6) and (7), get

$$I_S^{FP}(x) = \int_S \mathcal{D}^P(x,y)dw(F(y)) = \int_S H(x-y)\, dw(F(y)) = \int_S H\left(\frac{x}{y}-1\right) dw(F(y)) = w(F(x)) \quad \blacksquare$$

With perfect ordinal tool, if the frequency imperfections follow the pattern found by Kahneman and Tversky [1979], frequency imperfections tend to boost the adaptive evaluations for relatively small magnitudes and depress evaluations of relatively large magnitudes, further aggravating the distortion caused by the ordinal imperfections.

8 Economic Applications

The above model of adaptive evaluation maybe particularly relevant in such non-choice situations, such as surveys. Below I consider how cognitive imperfections may affect subjects' evaluations (and thus responses).

The evaluation procedure maybe particularly relevant in cases when evaluations involve interpersonal comparisons. To evaluate a coconut, Robinson Crusoe compares it against other coconuts present - whether these other coconuts are on a tree, on a store display, or in hands of other people. From the cognitive point of view, Mrs. Brown evaluates her house the same way as Crusoe evaluates a coconut, - using two primitive cognitive tools over a reference set consisting of other houses, - yet these reference houses tend to belong to other people. In a private ownership economy most (if not all) ordinal comparisons are necessarily done interpersonally, particularly for economically relevant variables such as consumption, income, and wealth. A change in Mr. Jones' possessions changes how Mrs. Brown evaluates what she has, appearing as a desire to "keep up with the Joneses".[15] The same cognitive process involves evaluation of individual characteristics such as height, beauty, intelligence, and skill.

[15] While an adaptive evaluation is sentiment-free, in private ownership economies it may lead to a concern with social status discussed by Veblen [1899], Frank [1995], and many others.

Yet cognitive imperfections may prevent individuals from assessing their possessions accurately. Specifically, as the material below demonstrates, when individuals make more inaccurate ordinal comparisons when comparing their possessions to those "above" them, than to those "below" them, their evaluations of own possessions are higher than "true" evaluation, leading to greater satisfaction with their possessions. When individuals have this bias, equality and economic growth may be associated with greater welfare.

8.1 Well-Being and Social Welfare

Consider a private ownership economy with a continuum of individuals, each endowed with an object of magnitude x (e.g. height, intelligence, income, happiness, and so on). Suppose that these magnitudes of individual endowments are observable and thus constitute the reference set S with the veridical magnitude distribution F. Assume that, being atomistic, all individuals have the same reference set S, have perfect frequency tool F, and the same ordinal tool $\mathcal{D}$.

Individuals evaluate the magnitude of their objects via interpersonal encounters with other individuals in the economy. Each encounter between two individuals possessing magnitude x and y, respectively, thus two outcomes of the ordinal comparison tournament. The result of the ordinal comparison tournament for the individual possessing x is given by $\mathcal{D}(x, y)$, while the corresponding result for the individual possessing y is given by $\mathcal{D}(y, x)$. Let us further aggregate the results of each encounter, and define the *surplus* function $\Psi(x \perp y)$ from an encounter between x and y to be the sum of the results of both ordinal comparison tournaments:

$$\Psi(x \perp y) = \mathcal{D}(x, y) + \mathcal{D}(y, x) \tag{19}$$

Clearly, $0 \leq \Psi(x \perp y) \leq 2$. The surplus function $\Psi(x \perp y)$ plays an important role for the societal welfare. As the following Proposition suggests, if the surplus function Ψ has a constant sum property, the distributions of economically important goods can be welfare-neutral.

Proposition 15 *Consider a private ownership economy where all individuals have the same reference set S consisting of magnitudes of objects belonging to all other individuals in the economy with veridical (true) magnitude distribution F. Suppose all individuals have the perfect frequency (proportion) tool F, and identical ordinal comparison tool subject to a constant-sum property $\Psi(x \perp y) = \mathcal{D}(x, y) + \mathcal{D}(y, x) = C \in [0, 2]$ everywhere except at a countable number of encounters x, y. Then the utilitarian measure of the total welfare $W = \int_S I(x) dF(x)$ equals to $\frac{C}{2}$ independently of the magnitude distribution $F(x)$.*

Proof: Rewrite the total welfare as:

$$W = \int_S I_S(x) dF(x) = \int_S \int_S \mathcal{D}(x, y) dF(x) dF(y)$$

Since adaptive evaluation is determined for any x and y, one can also write W as:

$$\begin{aligned} W &= \int_S \int_S \mathcal{D}(y,x)dF(y)dF(x) &= \int_S \int_S (\Psi(x \perp y) - \mathcal{D}(x,y))dF(x)dF(y) = \\ & &= \int_S \int_S \Psi(x \perp y)dF(x)dF(y) - W \end{aligned}$$

Since $\Psi(x \perp y) = C$, we have that

$$W = \frac{1}{2} \int_S \int_S \Psi(x \perp y)dF(x)dF(y) = \frac{C}{2} \quad \blacksquare \tag{20}$$

The economic processes, such as economic growth, and public policies, such as redistributive taxation, tend to change the distribution of individual possessions. As individuals adapt their evaluations of their possessions to the new social environment, the cognitive processes involved in interpersonal comparisons might be important for societal welfare (and thus the success of economic policies). Whenever a pairwise comparison tournament entails opposite outcomes for any two individuals, welfare neutrality of economic processes may arise.[16]

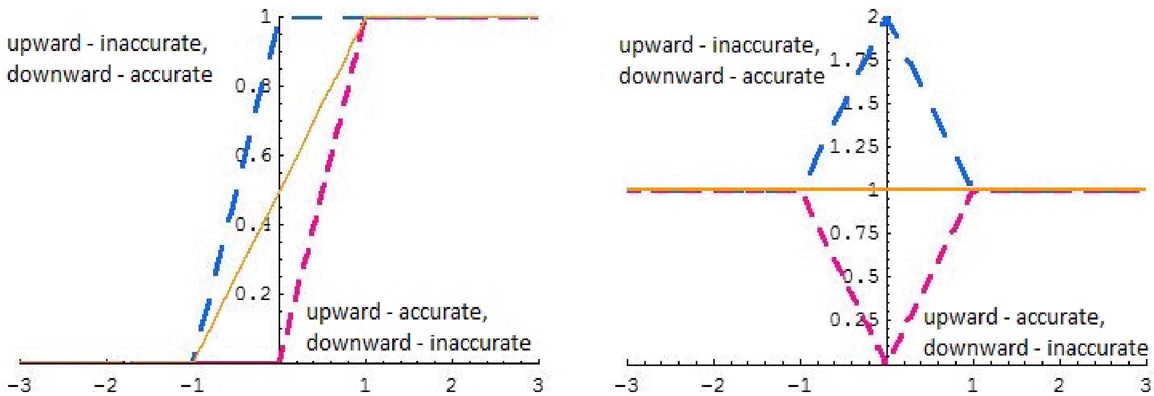
Clearly, with frequency imperfections the welfare neutrality does not hold. With a perfect frequency tool, if the ordinal tool allows for both individuals to "win" the ordinal comparison tournament when the magnitudes x and y are sufficiently similar, it is possible that more equal societal distributions increase welfare. Conversely, equality may decrease welfare if sufficiently similar magnitudes lead to "losses" for both individuals. The next example demonstrates a possibility that different inaccuracies in ordinal comparisons may lead to different social outcomes.

Example 7 *Suppose individuals have additively imperfect evaluations, with the magnitude and ordinal discriminability distributions uniform as in Example 4. Then welfare is given by*

$$W = \frac{2k_1^3 - (k_1 - 2\sigma)^3 + (k_2 - 2\sigma)^3}{24\sigma^2(k_2 - k_1)}$$

Note first that the welfare is independent of mean magnitude μ. Yet if ordinal discriminability $z \sim U[-k,k]$ with $k < \sigma$, the surplus function $\Psi = 1$ for all x and y, and welfare $W = 0.5$ independently of σ. In contrast, as Figure 4 demonstrates, if ordinal tool is subject to "downward comparison inaccuracies", so that ordinal discriminability $z \sim U[-k,0]$ with $k < 2\sigma$, the surplus function Ψ has maximum at $x = y$ and welfare $W = \frac{1}{2} + \frac{k(6\sigma - k)}{24\sigma^2} > 0.5$. Instead, if ordinal tool is subject to "upward comparison inaccuracies", so that ordinal discriminability $z \sim U[0,k]$ with $k < 2\sigma$, the surplus function Ψ has minimum at $x = y$ and welfare $W = \frac{1}{2} - \frac{k(6\sigma - k)}{24\sigma^2} < 0.5$.

[16] Compare this welfare neutrality result to the "happiness paradox" of Easterlin [1974] who pointed out that average self-reported happiness in USA stayed practically unchanged in the post-war period despite real incomes nearly doubled.



a) additive imperfection $z \sim U[k_1, k_2]$ b) surplus function Ψ

Figure 4: When additive ordinal comparisons are symmetrically inaccurate with $z \sim U[-k, k]$, surplus is $\Psi = 1$ for all x, y (brown lines); when ordinal comparisons are "downward inaccurate" with $z \sim U[-k, 0]$, surplus Ψ is maximized at the point of parity $x = y$ (blue lines), and when when ordinal comparisons are "downward inaccurate" with $z \sim U[0, k]$, surplus Ψ is minimized at the point of parity $x = y$ (red lines).

The case of multiplicatively imperfect evaluations is more computationally more complex. To understand the interaction between the ordinal imperfections and inequality, consider first the following example for discrete distributions.

Example 8 *Consider an economy where proportion α of population have income x_L and proportion $1-\alpha$ have incomes $x_H > x_L$. Suppose all agents have perfect ordinal tool with $\mathcal{D}(x,x) = p$, so that $I^p(x_L) = \alpha p$ and $I^P(x_H) = \alpha + p(1-\alpha)$, and welfare $W^P = p + (1-2p)\alpha(1-\alpha)$. Now suppose that agents have a multiplicatively imperfect ordinal tool with the interval of doubt $[\lambda^{-1}, \lambda]$, where $\lambda > 1$. If income distribution F_A is sufficiently unequal with $x_H > \lambda x_L$, the imperfect evaluation $I_A^M = I_A^P$, so that $W_A^M = p + (1-2p)\alpha(1-\alpha)$. If instead, income distribution F_B is sufficiently equal with $x_H \leq \lambda x_L$, the imperfect evaluation $I_B^M(x_L) = I_B^M(x_H) = p$, so that $W_B^M = p$. As $W_A^M - W_B^M = (1-2p)\alpha(1-\alpha)$, welfare is higher in more equal society B as long as encounters with people of similar incomes tend to be seen favorably (i.e. $p > 0.5$). And vice versa, welfare is higher in more unequal society A if, instead, there is bias against encounters with people of similar incomes (i.e. $p < 0.5$). Again, welfare neutrality holds for either case whenever $p = 0.5$.*

The importance of the ordinal tournament between similar magnitudes is further highlighted by the following example.

Example 9 *Suppose individuals have multiplicatively imperfect evaluations, with the magnitude and ordinal discriminability distributions uniform as in Example 5. The resulting welfare W is a complex quadratic function of the ratio $\frac{\mu}{\sigma}$. Again, there is a relationship* *between the behavior of the surplus function* $\Psi(x \perp y) = \frac{(x/y+y/x)-2k_1}{k_2-k_1}$ *and welfare* W*. Specifically, for* $k < \frac{\mu+\sigma}{\mu-\sigma}$*, if ordinal tool is subject to "downward comparison inaccuracy", so that ordinal discriminability* $z \sim U[1/k, 1]$*, the surplus function* Ψ *has maximum at* $x = y$ *and welfare* W *increases with the ratio* $\frac{\mu}{\sigma}$*. That is, for a given level of inequality* σ*, economic growth is socially desirable, while for a given level of aggregate income, inequality is not. Instead, if ordinal tool is subject to "upward comparison inaccuracy", so that ordinal discriminability* $z \sim U[1, k]$*, the surplus function* Ψ *has minimum at* $x = y$ *and welfare* $W =$ *decreases with the ratio* $\frac{\mu}{\sigma}$*.*

Multiplicatively imperfect ordinal tools which satisfy constant sum property are rare, but they also result in welfare neutrality.

Example 10 *Suppose veridical income distribution is uniform on* $[\mu-\sigma, \mu+\sigma]$ *and suppose all agents have multiplicatively imperfect ordinal tool on* $[\lambda^{-1}, \lambda]$*, with* $1 < \lambda^2 < \frac{\mu+\sigma}{\mu-\sigma}$ *with ordinal discriminability* $z = \frac{x}{y}$ *being distributed with* $F^\nu(z) = 0.5 + 0.75\frac{\ln z}{\ln \lambda} - 0.25\left(\frac{\ln z}{\ln \lambda}\right)^3$*. One can verify that as* $F^\nu(z) + F^\nu(z^{-1}) = 1$*, the welfare neutrality holds.*

One can speculate that social aspects of interpersonal comparisons alter the nature of ordinal comparison inaccuracies and thus the shape of the ordinal comparison function. However, regardless of whether the ordinal comparison is entirely cognitive or has any social component to it, the relationship between welfare and income distribution will be determined by the behavior of the social surplus Ψ function, particularly around the point of parity $x = y$. Specifically, if the surplus is higher at parity, equality is likely to be desirable from the point of view of societal welfare, and vice versa.

8.2 Relative Skill Judgment

Svenson (1981) observed that individuals tend to misjudge their skills relatively to the others. The phenomenon of individuals overvaluing their own skills relatively to those of the others has since been widely documented. Let us explore whether cognitive imperfections could contribute to such overconfidence.

First, as Propositions 5 and 7 suggest in the case of the perfect frequency tool, the individuals with the lowest possible skill tend to overvalue their skills relatively to the others, and those at the very top of the distribution will tend to undervalue their skills. As Proposition 8 suggests, this tendency maybe further aggravated by frequency imperfections. Second, as Figure 3b suggests, if individuals are have difficulties in making accurate upward comparisons (i.e. in comparing themselves accurately to those whose skills are better than theirs), the overconfidence in relative skill judgment will arise for a significant fraction of the population.

Burks, Carpenter, Goette and Rustichini [2010] further observe that, on average, the least able tend to overvalue their relative skill, and the most able tend to undervalue themselves, with the judgment gap between the judgment of own skill rank and the true skill rank decreases with the true rank. This phenomenon maybe consistent with possessing some cognitive imperfections in their skill evaluations.

Proposition 16 *For the judgment gap between assessment of own skill $I(x)$ and true rank $F(x)$ to be non-increasing in true skill rank $F(x)$, it is necessary and sufficient that the marginal magnitude evaluation $\frac{dI(x)}{dx}$ (which is the density of convolution/scale mixture of pairwise comparison tournament z and comparison variable y) is no higher than the true density $f(x)$ for all skill levels x.*

Proof: For any skill x, let r stand for the true rank in the distribution $F(\cdot)$, or $r = F(x)$. Then, the evaluation of skill x with true rank r will be given by $I(x) = I(F^{-1}(r))$. Consider how the judgment gap $I(x) - F(x) = I(F^{-1}(r)) - r$ changes with rank r:

$$\begin{aligned}\frac{d(I(x) - F(x))}{dF(x)} &= \frac{d(I(F^{-1}(r)) - r)}{dr} = \frac{d\int_S \mathcal{D}(F^{-1}(r), y)dF(y)}{dr} - 1 = \\ &= \frac{1}{f(x)}\int_S \mathcal{D}'(x, y)dF(y) - 1 = \frac{1}{f(x)}\left(\frac{dI(x)}{dx} - f(x)\right)\end{aligned}$$

For additively imperfect ordinal tool, the condition becomes $\frac{dI^A(x)}{dx} = \int_S g(x-y)f(y)dy \leq f(x)$, and for multiplicatively imperfect tool: $\frac{dI(x)^M}{dx} = \int_S \frac{1}{y} g\left(\frac{x}{y}\right) f(y)dy \leq f(x)$. ∎

Corollary 3 *Suppose skill is distributed uniformly on interval $[a, b]$, and that the ordinal comparison tool z is additively imperfect with density $g(\cdot)$. Then the judgment gap is non-increasing in true rank.*

Proof: Observe that $\frac{dI^A(x)}{dx} = \frac{1}{b-a}\int_S g(x-y)dy \leq \frac{1}{b-a}$. ∎

In other words, as long as individuals have difficulties in making upward ordinal comparisons, and cognitive imperfections result in reduced sensitivity to an increase in own skill (as manifested by a less steep marginal evaluation of skill), the imperfect adaptive evaluation is consistent with overconfidence in relative skill judgment.

9 Related Literature

The parsimonious mathematical model presented here links a number of developments in economics and in adjacent evolutionary, cognitive, and neurobiological sciences.

9.1 Evolutionary Basis of Behavior

The model presented here follows the new evolutionary framework for decision-making, which has been advanced as an alternative to neoclassical utility theory. Robson [2001] advanced an argument in support of a utility function which adapts to the environment.[17] Rather than endowing living organisms with all necessary information, Nature provides the "tools" that enable one to extract information from one's environment and experience (Robson [2001, 2002], Samuelson [2004], Samuelson and Swinkels [2006]), which might have been further adapted to deal with more evolutionarily recent tasks (Cosmides and Tooby [1994], Gigerenzer [1998]). The quest for evolutionary basis of decision-making primitives has been expanded by Robson and Samuelson [2010] to decision and experienced utilities, while Rayo and Becker [2007] further suggest that Nature may endow humans with happiness as a context-dependent measurement tool which allow one to choose among the alternatives. Furthermore, Herold and Netzer [2011] suggested that the non-linear probability weighting function evolved as an evolutionary second-best to complement an adaptive S-shaped value function.

9.2 Domain-Independent Cognitive Tools and Their Imperfections

This paper argues that only two primitive tools are sufficient to "measure" a magnitude against a reference set. One such primitive tool - ordinal comparisons - enables one to tell whether a magnitude is larger, smaller, or equal to another magnitude. Brannon [2002] found that 9-11 months old human infants are able to discriminate between small arrays of objects, while Feigenson, Carey and Spelke [2002] suggested that human infants develop abilities to discriminate continuous variables (such as areas, sizes, densities) even earlier, while Xu [2003] reported that large numerosities are discriminated differently from the smaller ones. Furthermore Walsh [2003] suggested that time, space and numbers involve similar neural mechanisms.

The other primitive tool is frequency. Hasher and Zacks [1979, 1984] suggested that humans update frequencies-of-occurrence automatically and accurately, while Hintzman [1988] proposed a model of retrospective updating of frequency-of-occurrence. Since human children and some animals are equipped with a mental system for counting (Gallistel and Gelman [1992]), natural frequencies can be stored in a numerical format (Jonides and Jones [1992]). Gigerenzer and Hoffrage [1995] and Cosmides and Tooby [1996] further argued that natural frequency format is computationally simple, and thus is evolutionary advantageous way to calculate probabilities (proportions).

The issue of magnitude evaluation and measurement imperfections dates back to the 19th century, and is closely associated with the law of diminishing marginal utility. According to

[17] See a survey on hedonic adaptation by Frederick and Loewenstein [1999].

the Weber-Fechner law of just noticeable differences (see, for example, Link [1992]), humans fail to notice the difference between two magnitudes if this difference is less than a certain threshold, and such minimum noticeable difference is proportional to the stimulus level. For example, most people would notice the difference between having one dollar bill and two dollar bills in one's wallet, but would hardly notice the difference between having 67 and 68 dollar bills. This phenomenon was recently found on the brain level, as Nieder and Miller [2003] found that the representation of numerosities is "compressed" in a monkey brain, prompting Dehaene [2003] to advance an argument in support of a logarithmic mental number line to represent the Weber-Fechner law.[18]

The research on frequency (proportion) imperfections is still in its infancy. As Kahneman and Tversky [1979] pointed out, humans tend to overweight small probabilities and underweight large ones, and this probability distortion may happen at the encoding level in the brain, as Hsu, Krajbich, Zhao and Camerer [2009] found that the neural activity in the striatum is nonlinear in stimulus probabilities. Furthermore, recent research suggests that experienced and stated probabilities may result in opposite pattern of overweighting and underweighting of small and large probabilities (Barron and Erev [2003], Hertwig, Barron, Weber, and Erev [2004]), while Harbaugh, Krause, and Vesterlund [2003, 2010] further report that the implied shape of subjects' probability weighting function is affected by the elicitation procedure and age, with a possibility of the probability weighting function to be linear in adults.

Psychologists have accumulated further evidence that humans tend to evaluate proportions (such as probabilities, proportions of graphical elements, numerosities, an so on) in a non-linear fashion, involving accurate evaluations at the "intermediate" proportions, overestimation of low and underestimate of large proportions (see Spence [1990] and Hollands and Dyre [2000]).

9.3 The Empirical Significance of Stimuli vs. Their Physical Properties

The evaluation mechanism proposed here assumes that the two primitive cognitive tools are used to extract information from one's environment, rather than to rely on the "absolute" physical properties. Robson [2001] proposed that the optimal hedonimeter is an empirical distribution of the environmental stimuli. Suppose one faces a choice between two alternatives, which are non-negative numbers independently drawn from a continuous distribution F. In terms of hedonic utility, the organism can only observe whether an alternative is above or below each of the N threshold values. That is, hedonic values (e.g. "High", "Low")

[18] Dehaene, Izard, Spelke, and Pica [2008] found that education has a "linearizing" effect on the mental number scale, as in Amazonian indigenous cultures the mental number line is found to be logarithmic, while in the Western societies it was found to be closer to the linear.

are assigned following a series of ordinal comparisons to thresholds. When both draws fall within the same region, they are of the same hedonic value. Thus incorrect choices may occur. When both draws fall within the same region, as determined by adjacent threshold values, they are assigned the same hedonic value, despite being distinct. Robson shows that, with no complexity costs, the overall probability of error is minimized when the threshold values c_i satisfy $F(c_i) = \frac{i}{N+1}$ (e.g. if $N = 99$, each threshold level is a percentile). That is, the optimal hedonic utility function is the distribution function F. This result was further examined by Netzer [2009][19] who further pointed out that the hedonic discriminability is particularly important for stimuli which are particularly environmentally relevant - that is, the utility function should be more sensitive for the stimuli which are particularly frequent.

Independently, cognitive scientists Yang and Purves [2004] suggested that it is the empirical significance of stimuli, rather than their physical properties, which is important for survival of a living organism. They note that the same visual aspect of an environment (such as luminance) in different contexts results in different perception of that aspect, and propose that the visual system does not record the physical aspects of stimuli, but instead their statistical relevance as represented by their relative rank in the distribution of stimuli.

9.4 Procedurial Approach to Decision Making

The evaluation algorithm suggested here employs evaluation of a magnitude by its ordinal rank in a comparison set by the frequency of a “winning” outcome in pairwise comparisons. It is thus closely related the decision-by-sampling theory of psychologists Stewart, Chater and Brown [2006], who suggested that, using a series of binary, ordinal comparisons, an alternative’s subjective value is constructed to be its rank in a sample drawn from one’s memory[20] They show that this “decision by sampling” procedure may account for a number of observed regularities involving money, time, and probability, including concave utility functions, losses looming larger than gains, hyperbolic time discounting, and the overestimation of small probabilities and the underestimation of large probabilities. Brown and Matthews [2011] further incorporated memory imperfections into the decision-by-sampling model.

The present paper shares the frequentist procedurial focus on subjective experience with the case-based decision framework pioneered by Gilboa and Schmeidler [1995], which utilizes a database of past experiences to construct decision weights akin probability weights of expected utility. This framework was extended in Gilboa and Schmeidler [2003] to show, as a special case, that one can use empirical frequency distributions to rank an alternative to be “at least as likely” as another.

[19] Netzer [2009], in addition, proposed an alternative fitness-maximizing utility function which also depends on the environment

[20] Compare this to Kornienko [2004].

The evaluation algorithm suggested here involves ranking alternatives by pairwise comparisons. Rubinstein [1980] considered ranking participants in a tournament by counting the number of times a given player beats another player in a tournament (with "ties" not allowed). Landau [1951] used trichotomous outcomes of pairwise tournaments to describe societies' dominance structures.

9.5 Reference-Dependent Models: Theory

The present paper adds to a number of theoretical models explicitly or implicitly employing the isomorphism of a cumulative density function and a utility function. One of the the pioneers of context-dependent evaluation, psychologist Parducci [1963, 1965] observed that the shape of subjects' magnitude evaluations followed the corresponding veridical distributions, albeit with less pronounced curvature. To explain such empirical relationship between magnitude evaluations the and context, he proposed his range-frequency theory whereby the magnitude evaluation is a weighted sum of a veridical rank and a rank in a uniform distribution with the same support. Here, I hypothesize that these qualitative rather than quantitative similarities of evaluations and veridical distributions arise instead because of the imperfections in subjects' cognitive tools.

Simultaneously and independently from Parducci, the isomorphism of a utility function and a cumulative density function was first formally explored by Van Praag [1968], who developed a formal model of consumer behavior where a choice over a large number of consumption goods resulted in the money utility being isomorphic to a normal distribution. This idea was later extended by Kapteyn [1985] (and references therein) who suggested that money utility is c.d.f of an income distribution.

In addition to the above rank-based models, there exist a number of reference-dependent models where alternatives are evaluated, at least in part, against a reference set, or its specific elements. Single-reference point theories developed by psychologists include the adaptation-level theory of Helson [1948] and the prospect theory of Kahneman and Tversky [1979].

With a notable exception of the endogenous reference point theory of Koszegi and Rabin [2006], reference-dependent models developed by economists have been primarily concerned with social comparisons. Pioneered by Duesenberry [1949], "keeping up with the Joneses" models involve comparisons to the average action by the others. The observation that individuals care about their relative position in a social hierarchy led Frank [1985] to develop a model where ordinal rank with respect to the other individuals is a component of one's utility. Harbaugh and Kornienko [2000] consider how the individual reference sample from the common environmental information may affect individual valuations.

Lazear and Rosen [1981], followed by Green and Stokey [1983], suggested that comparisons relatively to peers provide informational improvements. This was further advanced by Samuelson [2004] who highlighted the importance of environmental persistence in an evolutionary context. Rayo and Becker [2007] develop an evolutionary-inspired model with perceptual limitations which allows for hedonic effects of relative success arising due to the increased power of the happiness measurement by using average of peer performance. Wolpert [2010] explores the socially constructed benchmarks using both single point (average) and whole population (rank) models.

9.6 Reference-Dependent Models: Evidence

The empirical evidence suggests that nearly everything that enters one's mental "in-box" is evaluated against a reference set, or its specific elements. There is plenty of evidence that people compare their possessions relatively to what others have (see, for example, Frank [1985], Clark and Oswald [1996], Solnick and Hemenway (1998), and Neumark and Postlewaite [1998]). Nor are moral judgements absolute - as students were found to be appalled by the idea of poisoning a neighbour's dog, yet judged it as a petty crime when poisoning the neighbour herself was on the list (Parducci [1968]). Even evaluation of painful experience is not absolute as it varies with different sequences of pain (Redelmeier and Kahneman [1996]). And so are the judgements of size, weight, or numerousness (Parducci [1963, 1965]), as well as visual perception (Yang and Purves [2004]).

There are a few studies which provide an empirical support for the models based on the isomorphism of evaluation/utility and an empirical rank. Based on the experimental and survey data, Clark, Masclet and Villeval [2006] suggest that one's rank in the income distribution is a strong determinant of effort, while Brown, Gardner, Oswald and Qian [2008] found that one's rank in salary distribution determines job satisfaction and can predict job separation. Using Gallup World Poll data, Barrington-Leigh [2012] reports that, comparing to a specification involving log income, pure ordinal rank is just as successful in explaining the data on life satisfaction, and within country pure rank model provides a better fit.

Olivola and Sagara [2009] argue against context-independent utility functions, and show that empirical distribution of death tolls affects experimental subjects' decisions involving human fatalities in hypothetical situations, and moreover, consistent with cross-country differences in environmental contexts, subjects' choices vary across countries. Wood, Brown and Maltby [2011] found that gratitude is affected by the magnitude distribution of help others provide. Stewart [2009] and Ungemach, Stewart and Reimers [2011] found that evaluations by experimental subjects depend on the relative rank of magnitudes in the reference sets.[21] Stewart, Reimers, and Harris [2011] further argue that since utility, probability weighting and temporal discounting functions are context-dependent, they cannot be stable primitives of economic decision making.

[21] Though Matthews [2012] disputes the results of Ungemach, Stewart and Reimers [2011] in the time domain.

Furthermore, the brain studies suggest that the anticipated reward magnitudes are coded in relative, and not absolute, terms (Seymour and McClure [2008] and references therein). The evidence of rank-dependence on the evaluation stage was recently reported by Mullett and Tunney [2013]. They found that in a human fMRI experiment using monetary values, activations in ventral striatum and thalamus are "locally" context dependent, exhibiting context-dependent rescaling - namely, that £0.30 (the highest value in the "low value" context) resulted in stronger activity than £5 (the lowest value in the "high value" context). They also report that activations in ventro-medial prefrontal cortex (vmPFC) and the anterior cingulate cortex (ACC) are consistent with encoding "global" ordinal rank across two different "local" contexts. For example, an increase in ordinal rank of the stimuli by one unit generated the same increase in activations in ACC and right vmPFC, despite the differences between the adjacently-ranked stimuli varied from £0.10 to £4.30. Importantly, their experimental design involved valuation of monetary values independently of choice. This is in contrast to earlier studies on primate neurons in situations where choice and valuation could potentially be confounding, the activity in the orbitofrontal cortex was reported to be invariant to changes of menu (Padoa-Schioppa and Assad [2008]), yet to be adapting to the range of alternatives (Padoa-Schioppa [2009]).

9.7 Mathematical Apparatus

The mathematical model proposed here is closely related to the mathematical techniques employed in a number of other works.

The model of imperfect ordinal comparisons is closely related to the "expected utility without utility" of Castagnoli and LiCalzi [1996], who suggested to compare lotteries by the expectation of one lottery outperforming the other in the presence of a given prior. The perfect ordinal comparison tool is isomorphic to the optimal happiness function of Rayo and Becker [2007], who model perceptual limitations involved when the expected happiness is compared across different choice options, rather than built into the two cognitive tools considered here. The multiplicatively imperfect ordinal comparison tool is based on the similarity model due to Rubinstein [1988]. The model of imperfect frequency processing uses the rank-dependent formulation of Quiggin [1982] which accounts for cumulative densities.

Here, I utilize the isomorphism of between the expected outcome of imperfect pairwise comparisons and algebra of random variables, and use the convolution and scale mixture techniques to model the additive and multiplicative ordinal imperfections (see, for example, Springer [1979]).

10 Conclusions

This paper provides a conceptual framework which allows one to combine the insights from economics, behavioral, evolutionary and cognitive sciences into one parsimonious mathematical model. It advances a hypothesis that a minimal set of cognitive tools allows one to explore one's environment efficiently, and that utility can be thought as an adaptive measuring tape for choice alternatives.

Undoubtedly, this model has limitations. First, it concentrates on the first stage of the decision process - on magnitude evaluation, rather than on the final stage - choice. The neurobiological processes at the choice stage of decision process is an area of active research (see Fehr and Rangel [2011]), which can be incorporated into the two-stage decision model independently of the processes at the valuation stage. Second, the model here describes the evaluation of objects that differ only in one dimension, while organisms are likely to use additional tools to evaluate multidimensional objects and bundles. It is possible that the similarity function of Gilboa and Schmeidler [1995, 2003] could be a conceptual tool which could allow the analysis of multi-dimensional objects. Third, the model does not explore the interaction between the frequency (rarity) of particular magnitudes and cognitive/memory imperfections. Brown and Matthews [2011] extend decision-by-sampling model to allow for memory distortions which interact with the reference distributions. Fourth, it remains to be determined whether there are additional cognitive tools involved in processing of uncertainty, as Andreoni and Sprenger [2010] document a qualitative difference between certain and uncertain utility.

Yet, despite its shortcomings, the present paper highlights the importance of recent advances in cognitive and brain research for understanding of human economic decision making.

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