Matchings with interval order preferences: stability and Pareto-efficiency

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1 Introduction

We consider two-sided matching markets. Examples of such markets include marriage market, universities-applicants market and others. Pioneering work analysing such kinds of markets is [6]. They considered one-to-one and one-to-many markets, where preferences of individuals on one side over individuals on the other side were considered to be linear orders.

In this paper we analyse a modification of the classical Gale-Shapley admission problem, where preferences of universities are considered to be interval orders. Interval order allows a specific form of indifference in the preference relation. Imagine, each alternative is described with an interval $[l, u]$, and one alternative dominates another if and only if intervals do not overlap and lower bound of the first interval is greater than upper bound of the second interval. Preferences with such property may occur in the cases, when applicants' scoring system (interview, exam or sum of points) is not exactly accurate. In this case if we would construct a weak order directly, according to the scoring results, some students may be undeservedly concerned less preferred than others, when the small scores difference is just the matter of chance.

We show the existence of a stable matching and, moreover, for every stable matching we prove the existence of a linear order extension of universities' preference profile, that does not upset the stability. The second result allows us to say that model with interval orders is, in fact, may be reduced to the model with linear order preferences. Our main result is an extension of the Erdil and Ergin [5] Stable Improvement Cycle Theorem. We provide a criteria that allows us to check whether a stable matching is Pareto-optimal for applicants or not.

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On the other hand, stable mechanism, based on Improvement Cycles, is not strategy-proof. We also provide a strategy-proof applicant-proposing deferred acceptance with tie-breaking, where tie-breaking procedure is organized in a special way. This special tie-breaking allows to lower chances of an applicant-inefficient stable matching (in comparison to that with random-tie breaking). The rest of the paper is organized as follows. In Section 2 the classical Gale-Shapley model is presented and a review of publications is given. In Section 3 a new model is proposed and basic results on stability are obtained. In Section 4 the Stable Improvement Cycle theorem is proven. Section 5 contains an efficiency-oriented tie-breaking procedure and corresponding theorem. Section 6 and 7 include discussion of the linked results and conclusion.

2 The Framework

The following model is considered. Let A be a finite set of applicants, B - a finite set of universities. Each applicant can be admitted to one university, while each university $b \in B$ cannot admit more applicants than its quota q_b.

Definition 1. *The matching is a mapping from $A \cup B$ to the subsets of $A \cup B$ such that:*

- *each applicant $a \in A$ is either admitted to a university $\mu(a) = b$ ($b \in B$) or remains unmatched $\mu(a) = a$,*
- *each university $b \in B$ either admits some subset of applicants $\mu(b) \subseteq A$ or has no students $\mu(b) = b$,*
- *if applicant a is admitted to a university ($\mu(a) = b$), than university b admits this applicant ($a \in \mu(b)$,*
- *numbers of students $|\mu(b)|$, admitted to each university $b \in B$, is less than or equal to its quota q_b).*

This definition was first introduced by Gale and Shapley in [6]. They also assumed, that both applicants and universities have preferences over the opposite side. Preferences of applicants are linear orders over universities; applicant may find some universities unacceptable (worse, than being unmatched). Similarly, preferences of universities are linear orders over individual applicants; again, some applicants may be unacceptable. Preferences of the universities over *subsets* of applicants are considered to be responsive to the their preferences over individuals with qouta restriction.

As both applicants and universities are independent agents and have preferences over each other, the following question arises: does there exist such a matching, than no agent or group would prefer to avoid it? Gale and Shapley call this property 'stability' and introduce the following definition of a stable matching.

Definition 2. *Matching μ is stable if it satisfies the following properties:*

- *individual rationality of applicants*
  no applicant $a \in A$ is matched to an unacceptable university;

- *individual rationality of universities*
  no university $b \in B$ admits an unacceptable applicant;

- *non-wastefullness*
  no pair (a university b and an applicant a) such that the applicant a prefers this university b to her current match $\mu(a)$ and university b finds the applicant acceptable and has an empty seat ($|\mu(b)| < q_b$);

- *pairwise stability*
  no pair (a university b and an applicant a) that a prefers this university b to her current match $\mu(a)$ and b strictly prefers the applicant a to at least one (say, $a' \in \mu(b)$ of its currently admitted applicants.

Gale and Shapley proved that in the case of such preferences the set of stable matchings is non empty. Furthermore, a constructive proof which allows us to find a stable matching was proposed - the so called 'Deferred Acceptance (DA) Procedure'.

Now let us briefly describe the DA algorithm with proposing applicants. In the first step, each applicant applies to her most preferred university. Those universities that got less applications than their quota, 'preliminary admit' all applicants (this rule applies at any step). If the number of applications exceeds university's quota, then university 'preliminary admits' q_b most preferred applicants and rejects the others. At the second step each rejected applicant applies to her second most preferred university. When a university gets second-step application from a candidate, who is preferred to some of the first-step 'preliminary admitted' applicants, it rejects those 'preliminary admitted' applicants and admits (also, preliminary) the new one. After that the next similar step begins. This process lasts until each applicant is either admitted to a university or rejected by all acceptable universities. Last 'preliminary admission' is a resulting stable matching.

The nice property of the described above DA procedure in the Gale-Shapley model is that it always produces a matching, which is weakly Pareto-optimal in the set of stable matchings for the proposing side of the market.

Further research was based on Gale and Shapley's seminal paper. In this paper we are specially interested in the models, allowing indifferences in the preference profiles.

There are many papers that investigate matching problem with indifferences. Problem statement [1] is one of the first that follows a real life example: preferences of municipal schools in Boston and NYC school districts being weak orders. The mechanisms which were originally used in these districts in general produces an unstable matching of applicants and schools. It was in some sense unfair, especially hurting 'naive' children and their parents. Abdulkadiroglu and Sonmez proposed the following new admission procedure: first, ties in schools' preferences are broken randomly; second, deferred acceptance procedure is applied to the admission problem with linear order preferences. Proposed mechanism always constructs a stable matching.

However, it is known, that student-oriented deferred acceptance mechanism does not always produce a student optimal stable matching (see, for example, one of the first paper considering indifferences in preference profile, [9]). In [5] an algorithm, which provides Pareto-efficient (for applicants) stable matching for the matching problem, where preferences of the applicants over universities are linear orders and the universities have weak order preferences over individual applicants, is proposed. This algorithm is based on the so-called Stable Improvement Cycles, a formal definition is given below.

In this paper, we introduce many-to-one matching model, in which, as in previous papers, applicants have linear order preferences over universities, but the universities have the interval order preferences [3] over individual applicants.

Let us give a formal definition of an interval order ([3]). There are several equivalent definitions of an interval order, but we use here one that is most suitable for our purposes.

Definition 3. *Interval order $\succ$ is a partial order on the set X which satisfies the following property: there exists a function I, which assigns a real line interval $I(x) = [l(x), u(x)]$ for each $x \in X$ such that $\forall x, y \in X$ $x \succ y$ iff*

$l(x) > u(y)$.

If the set X is finite, then without loss of generality one can assume that $l(x)$ and $u(x)$ values are integer numbers. Why we think that having interval order preferences over students may be natural for a university? Often universities use some kind of scores to evaluate applicants and form a preference relation. But any scoring method has some error. For example, if some university uses an exam scores with minimum 0 points and maximum 100 points, and gets students' Ann and Bill applications, with scores, say, 78 and 78.2 respectively, it is not clear, whether Bill should have priority to Ann or not. If the exam scoring method has 0.5 standard error, than, probably, Bill and Ann should be considered incomparable in universities preference relation. Moreover, sometimes possible errors are different for different score values.[2]

If for some interval order P_S there exists a function I, such that for all elements of X intervals have the same size, such a binary relation is called a semiorder.

In fact, weak orders are a special case of interval orders. For weak orders there exists an interval function I such that each interval is a single point, that is, $u(a) = l(a)$.

3 The Model

We will use the same notation for the sets of agents and the same definitions of a matching and a stable matching, as in the previous section. Let us formalise our problem statement. Let R denote the preferences profile of applicants. For each $a \in A$ R_a is a linear order over elements of $B \cup \{a\}$.

Let $\succeq$ denote the preference profiles of universities over elements of $A \cup \{b\}$. For each $b \in B$ $\succeq_b$ is an interval order. Furthermore, we assume that each such binary relation satisfies 'no indifference with empty set' property, that is, $\forall b \in B, \forall a \in A$ either $a \succ_b b$ or $a \prec_b b$[3].It means that for each

[2]For example, in Russian university admission system Unified State Exam scores are used for ranking purposes. Unified State exam scores are obtained in the following non-linear way. If Ann obtained 65 and Bill obtained 70 points, it means that he have solved 3 more tasks in the exam, but if Paul obtained 25 and Kate obtained 30 points, it means, that Kate have solved just one more task. In the current system these considerations are not taken into account, and even 1 point excess is enough to be preferred by university.

[3]Comparison with b means comparison with having an empty seat

university applicants are clearly divided into two groups: acceptable and unacceptable ones.

As we consider many-to-one matching problem, we have to define preferences of the universities over the *sets* of applicants. We do not define these preference relation explicitly, but we consider that the preference relation of each university over the sets of applicants is responsive to the preference relation over individuals: $\forall b \in B$, $\forall A' \subseteq A$ such that $|A'| < q_b$, $a_1, a_2 \in A \setminus A'$ if $a_1 \succ_b a_2$ then $A' \cup \{a_1\} \succ_b A' \cup \{a_2\}$.

3.1 Existence of stable matching

The definition of stable matching has already been given in the previous section. Gale and Shapley [6] have shown that in the case of linear order preferences on both sides of the market stable matching always exists.

It is easy to show the existence of a stable matching in case when universities' preferences are the interval orders.

Theorem 1. *If applicants have linear orders over universities, and universities have interval over individual applicants, then stable matching always exists.*

Proof. Indifferences in an interval order preference relation might be resolved so that each university's preference relation will be transformed into linear order. Naturally, if $x \approx_b y$ under true preference relation of the university b, it means, that intervals $I_b(x)$ and $I_b(y)$ are overlapped. Let us construct new preference relation $\succeq'_b$ such that $x \succ'_b y$ iff $\frac{l_b(x)+u_b(x)}{2} > \frac{l_b(y)+u_b(y)}{2}$. Incentives to form a blocking pair are wider for universities with $\succeq'$ preferences, than for universities with $\succeq$ preferences. So a matching which is stable under these transformed preference profile $\succeq' = \{\succ_b\}_{,b \in B}$ is also stable under original interval order preference profile $\succeq$. As a transformed preference profile is a profile of linear orders, the Gale-Shapley theorem allows us to conclude, that such stable matching always exists. $\square$

This result, in fact, follows from the more general statement: stable matching exists for any partial orders preference profile.

The next natural question is the following: is it possible to find all stable matchings in the discussed model, if we consider all possible transformed profiles $\succeq'$ and all matchings, which are stable under these transformed preference? We find out that this is true.

Theorem 2. *For each stable matching μ in the model with interval order preferences there exists such transformed universities' preference profile $\succeq'$, which consists of linear orders and does not contradict the original preferences $\succeq$, such that matching μ is also stable under this linear order preference profile.*

Proof. Consider some matching μ which is stable under original preference profile $(\succeq, R)$. Let us construct a linear order preference profile $\succeq'$ which does not upset the original preference relations in profile $\succeq$ and the stability of μ.

New preference profile $\succeq'$ must satisfy two properties:

- no contradiction with original preferences: if $a \succ_b a'$, than $a \succ'_b a'$,
- new blocking pair should not occur: if a' prefers university b to her current match, than under $\succ'_b$ university b must strictly prefer any of its current students to a.

As blocking pairs include one university and one applicant, linear preference relation extension may be constructed independently for each university.

Consider university b and the set of applicants A'_b, which include all applicants, incomparable to or dominated by any applicant, admitted to university b. Formally, A'_b=$\{a' \in A \setminus \mu(b) | \forall a \in \mu(b) a \succeq_b a'\}$. Moreover, let us define the set A''_b=$A \setminus (A'_b \bigcup \mu(b))$. Now we have divided the set of applicants into three disjoint sets: $A''_b, \mu(b), A'_b$.

Any applicant a' in the set A'_b may be (potentially) interested in a seat at the university b more than at her current match $\mu(a')$. So, to prevent forming a blocking pair (a', b) any applicant in A'_b must be dominated by any applicant in $\mu(b)$ under modified linear order preference relation $\succeq'_b$.

Note, that each applicant in set A''_b must be admitted to a university, that she prefers to university b. If this would not be true, university b and those applicant would form a blocking pair.

Now let us construct transformed interval function $I*$ such that under corresponding preference relation $\succ *_b$ any applicant in A' will be dominated by any applicants in $\mu(b)$ without contradiction to the original preference relation.

For all applicants in $\mu(b)$ and $A''(b)$ $u*(a)$ and $l*(a)$ values, assigned by $I*$ function remain the same as those assigned by I function. On the other hand, boundaries for applicants in A' will be changed.

Let us define two subsidiary variables: minimum lower bound among all applicants currently admitted to b: $\underline{l_b} = \min_{a\in\mu(b)}(l_b(a))$; and maximum upper bound among all applicants who are not admitted, but potentially desire a seat at b: $\overline{u_b} = max_{a\in A'_b}(u_b(a))$.

If $\overline{u_b} < \underline{l_b}$, than no transformation is needed and $I* = I$ since any applicant in $\mu(b)$ is preferred to any applicant in A' under original preferences of the university b. Otherwise for each applicant a' in A'_b the following new bounds are assigned: $u*(a') = u(a') - (\overline{u_b} - \underline{l_b}) - 1$, $l*(a') = u(a') - (\overline{u_b} - \underline{l_b}) - 1$. We just move all intervals of applicants in A'_b below the lowest bound of the admitted applicants.

Now let us show that an interval order $\succeq *_b$, based on interval function $I*$ does not contradict with original preference relation $\succeq_b$.

1. Any preference relation between pairs of applicants within each of sets A'',A',$\mu(b)$ clearly remains unchanged.

2. For pair of applicants $a' \in A'_b$ and $a \in \mu(b)$:

   - if $a \succeq_b a'$, then $l_b(a) > u_b(a')$ and, naturally, $l_b(a) > u_b(a') - (\overline{u_b} - unverlinel_b) - 1$,
   - if $a \approx_b a'$, then simultaneously $u_b(a') \geq l_b(a)$ and $u_b(a) \leq l_b(a')$. By construction of $I*$ $u*_b(a') < l_b(a)$, as we distract from $u_b(a')$ the number, which is higher than maximal $u_b(a') - l_b(a)$ difference among all possible pairs (a, a'),
   - it is impossible, that $a' \succeq_b a$ by construction of $A'(b)$.

3. For any two applicants $a'' \in A''_b$ and $a \in \mu(b)$ nothing changes under interval function $I*$ in comparison to I.

4. For pair of applicants $a' \in A'_b$ and $a'' \in A''_b$:

   - if $a'' \succeq_b a'$, then $l_b(a'') > u_b(a')$ and, naturally, $l_b(a'') > u_b(a') - (\overline{u_b} - unverlinel_b) - 1$,
   - if $a'' \approx_b a'$, then simultaneously $u_b(a') \geq l_b(a'')$ and $u_b(a'') \leq l_b(a')$. Under new interval $[l*_b(a''), u8_b(a'')]$ applicants either remain incomparable, or a'' is being preferred to a'.
   - it is impossible that $a'' \succeq_b a'$, because, by construction, $\exists a$ such that $l_b(a'') > u_b(a)$, but $\forall a$ $u_b(a) \geq l_b(a')$. It means that $l_b(a'') >$

$l_b(a')$. Using the fact that $u_b(a'') \geq l_b(a'')$ we get that $u_b(a'') > l_b(a')$, so a'' can never be dominated by a'.

We have constructed new interval function $I*$, such that corresponding interval order $\succ *_b$ does not contradict original preference relation and, moreover, all applicants, who could possibly form a blocking pair with university b, are now strictly dominated by applicants in $\mu(b)$. Now we can resolve remaining indifferences in an arbitrary way, as they do not affect the stability of matching mu.

In our model we assume that each university's preference relation $\succeq_b$ satisfies 'no indifference with empty set' condition, so no breaking ties with empty set is needed. In addition, as definition of stable matching uses only preference of the universities over individuals, we do not need to implicitly construct preference relation over the sets of applicants. $\square$

Now we have proven that stable matching always exists and, furthermore, for each stable matching μ some linear order profile always exists which does not contradict original preference profile and stability of μ.

3.2 Pareto-dominated stable matching

However, the next natural question arises. Are all of these stable matchings Pareto-efficient? In this paper we consider only applicants' preferences for Pareto-efficiency analysis. In practice applicants are usually an active part of the market while universities (schools, etc.) are often only public service providers, so their preferences are just defined by law.

Let us discuss the following simple example with 3 applicants a_1, a_2, a_3 and 3 universities b_1, b_2, b_3, with each quota q_{b_i} equal to one. Preferences of the applicants are

$a_1: \quad b_1 P b_2 P b_3$

$a_2: \quad b_2 P b_1 P b_3$

$a_3: \quad b_2 P b_3 P b_1$

Preferences of the universities are

$b_1: \quad a_3 \approx a_2 \quad a_2 \approx a_1 \quad a_3 \succ a_1$

$b_2: \quad a_1 \approx a_3 \quad a_3 \approx a_2 \quad a_1 \succ a_2$

$b_3: \quad a_1 \approx a_2 \quad a_2 \approx a_3 \quad a_1 \succ a_3$

Preference relation of each university in this example is a simplest semiorder [2], a special case of an interval order.

Let us construct two different linear order preference profiles $\succ'$ and $\succ''$, such that both do not upset original preferences $\succeq$

Under $\succ'$ preference relation of b_2 is $a_1 \succ'_{b_2} a_2 \succ'_{b_2} a_3$. At the first step of the deferred acceptance applicant-proposing procedure a_1 applies to b_1, while both a_2 a_3 apply to b_2. University b_2, gets two application for only one seat and rejects a_3 (according to $\succ'_{b_2}$. Then a_3 applies to b_3 and the DA procedure stops. Stable matching $\mu = (\begin{smallmatrix} b_1 b_2 b_3 \\ a_1 a_2 a_3 \end{smallmatrix})$ is constructed.

Under $\succ''$ preference relation of b_2 is $a_1 \succ''_{b_2} a_3 \succ''_{b_2} a_2$ and preference relation of b_1 is $a_3 \succ''_{b_1} a_2 \succ''_{b_1} a_1$. The beginning of the procedure is the same as above: a_1 applies to b_1 and both a_2 and a_3 apply to b_2. University b_2, according to $\succ''_{b_2}$ must reject a_2. At the second step rejected a_2 applies to b_1. University b_1, in its turn, rejects a_1. Now (third step) a_1 applies to b_2. University b_2 again has too much applications and rejects a_3. Finally, a_3 applies to b_3 and the mechanism stops here. Another stable matching is found $\nu = (\begin{smallmatrix} b_1 b_2 b_3 \\ a_2 a_1 a_3 \end{smallmatrix})$.

It is easy to show that matching μ is weakly better for applicants than ν. Indeed, applicant a_1 prefers b_1 to b_2, and applicant a_2 likes b_2 more than b_1. Third applicant is indifferent between these two matchings, as she is admitted to the same university under both of them.

So, matching ν is obtained via DA applicant-oriented procedure, and it is stable, but inefficient in terms of applicants' preferences. In the next section we will prove a theorem that allows us to check, whether some particular stable matching is applicant-side Pareto-efficient, and, if necessary, transform it into a Pareto-efficient one.

4 Stable Improvement Cycle. Main Result.

Ergin and Erdil [5] first introduced definition of a Stable Improvement Cycle.

Let $C(b, \mu) = \{a \in A | b R_a \mu(a)\}$. In addition, let $D(b, \mu) = \{a \in C | \forall a' \in C \ a \succeq_b a'\}$.

Definition 4. *A Stable Improvement Cycle consists of distinct applicants $a_1, ..., a_n \equiv a_0$ $(n \geq 2)$ such that*

- $\mu(a_i) \in B$ *(each applicant in a cycle is assigned to a university),*
- $\forall a_i \ \mu(a_{i+1}) R_{a_i} \mu(a_i)$
- $\forall a_i \ a_i \epsilon D(\mu(a_{i+1}), \mu)$

Let $SIC(a_i) = b_{i+1}$, $SIC(b_i) = a_{i-1}$

To prove our main Theorem below, we first need the following lemma [5]

Lemma 1. *Fix* $\succeq$ *and* P. *Assume that* μ *is a stable matching that is applicant-side Pareto dominated by another matching* ν. *Let* A' *denote the set of applicants who prefer their university under* ν *to their university under* μ. *Let* $B' = \mu(A')$. B' *is the set of universities, to which applicants in* A' *are assigned under* μ. *The following is true:*

1. *Applicants in* $A \setminus A'$ *have the same match under* μ *and* ν.
2. *For any university* b, $|\mu(b)| = |\nu(b)|$.
3. *Each applicant in* A' *is assigned to a university both under* μ *and* ν.

Original proof of the lemma, provided by [5] is in fact the same even for the case of partial order preferences of the universities.

Now we can formulate and prove our main result.

Theorem 3. *Fix* $\succeq$ *and* R, *and let* μ *be a stable matching. If* μ *is Pareto-dominated by another stable matching nu, then it admits a Stable Improvement Cycle.*

Proof. Let, again, A' denote the set of applicants who prefer their university under ν to their school under μ. Let $B' = \mu(A')$, that is, the set of universities, to which applicants in A' are assigned under μ.

Let $C'(b, \mu)$ denote a subset of the set A', where each applicant desires to be admitted to the university b instead of her current university under μ. Formally, $C'(b, \mu) = \{a \in A' | bR_a\mu(a)\}$. In addition, let $D'(b, \mu)$ denote a subset of $C'(b, \mu)$, which includes only applicants with the maximum upper bound of the interval $I(a)$. Formally, $D'(b, \mu) = \{a \in C'(b, \mu) | u(a) = max_{x \in C'(b,\mu)}(u(x))\}$.

Now we can construct $G(V, E)$ - an oriented graph, where the set of vertices $V = B'$ and an edge $e(b_1, b_2) \in E$ iff $\exists a \in \mu(b_1)$ such that $a \in D'(b_2, \mu)$. In other words, edge $e(b_1, b_2)$ is in graph G if there exist an applicant such that

- she is assigned to the university b_1 under μ, but prefers university b_2 to university b_1,

- among all applicants, who prefer university b_2 to their current university under μ, this applicant a belongs to the group of the applicants with the maximum interval upper bound.

We will attach a label 'a' to this edge (b_1, b_2)

Graph G always contains a cycle, as each university in B' is preferred by at least one applicant to her university under μ (here we use Lemma 1). Let us take any such cycle (in particular, take applicants, who label edges in the cycle) and prove that it is a Stable Improvement Cycle.

First and second properties of the Stable Improvement Cycle hold by construction. Let us show that the third property also holds. Applicant a is dominated under b's preferences by some applicant a' iff $l_b(a') > u_b(a)$. By construction of the cycle, $\forall b$ it is true that a=SIC(b) has the maximal $u_b(a)$ among all applicants in $C'(b, \mu)$, that is, all applicants in A' who desire to be admitted to b. So no applicant from $C'(b, \mu)$ can dominate applicant a.

It remains to show, that applicant a=SIC(b) is **not** preferred by university b to some applicant in $A \setminus A'$. Let us prove it by contradiction.

Suppose there $\exists$ an applicant $x \in A \setminus A'$ and some university $b \in B'$ such that $b \succ_x \mu(x)$ and $x \succ_b$ SIC(b) ($l_b(x) > u_b$(SIC(b)) in terms of interval function). In this case applicant x and university b would form a blocking pair and a matching, constructed after applying SIC, will not be stable.

Stability of ν implies that for all $y \in \nu(b)$ it is true that $y \succeq_b x$ (here we use Lemma 1). In terms of interval boundaries it means that $u_b(y) \geq l_b(x)$.On the other hand, SIC(b) and any applicant in $\nu(b)$ belong to $C'(b, \mu)$ and SIC(b)$\in$ $D'(b, \mu)$ so, by construction of set D', u_b(SIC(b)) is maximal, so $\forall y \in \nu(b)$ u_b(SIC(b)) $\geq u_b(y)$.

From the two statements above we get u_b(SIC(b)) $\geq u_b(y) \geq l_x$, which means, that x does not dominate SIC(b) according to b's preferences. We can conclude that x and b will never form a blocking pair. Now we have proved that for constructed cycle all properties of Stable Improvement Cycle hold.

□

This theorem provides a criteria, which allows to check, whether some particular stable matching is applicant-efficient. Furthermore, we can think about stable and efficient mechanism. First, indifferences in universities' preferences are broken arbitrary. Second, an applicant proposing DA procedure is applied and some stable matching is obtained. Third, Stable Improvement Cycle is constructed, if exists, and matching is improved. The last step is

repeated as many times as necessary. Such a mechanism will always produce a stable matching.
Unfortunately, such mechanism is not strategy-proof. Erdil and Ergin [5] show this for the case of weak orders, so it is also true in our more general setting.

5 Strategy-proof tie-breaking and efficient outcome?

In this section we introduce a specific form of tie-breaking, which allows us to reduce chances of an inefficient outcome. Consider the following: preference relation of each applicant (university) is independently randomly chosen from the set of all possible linear (interval) orders. Our question is how to break ties in particular university's in order to reduce chance of inefficient outcome without taking into account preferences of other universities and applicants? When we deal with weak orders, this question has no answer, but with interval orders this is not the same. In the interval orders some ties are different from the others in terms of possible efficiency losses.
In the previous section it was shown that any inefficient stable matching admits Stable Improvement Cycle. If we consider each university separately

5.1 Example

Let $A = a_1, a_2, a_3$ be a set of applicants and let $\succeq_b$: $a_1 \succ_b a_3, a_1 \approx_b a_2, a_2 \approx_b a_3$, while $q_b = 1$.
We are mostly interested in situation where under deferred acceptance procedure university b receives proposals from each of the three applicants. Now consider three possible linear extensions of this interval order:

- $a_1 \succ'_b a_2 \succ'_b a_3$. In this case a_1 will be admitted to the university b. Two other applicants will be rejected and, therefore, admitted to some less preferred university. Both a_2 and a_3 will create edges pointing to university b in the Improvement Graph.

- $a_1 \succ'_b a_3 \succ'_b a_2$. Again, a_1 will be admitted to the university b and both a_2 and a_3 will create to edges in Improvement Graph.

- $\mathbf{a_2} \succ'_b a_1 \succ'_b a_1$. In this case, on the contrary, a_2 will be admitted to the university b and only a_1 will create edge in Improvement Graph (as a_1 is strictly better then a_3 according to original preferences, a_3 will not be able to point to b in the Improvement Graph).

In the latter case we obtain only one edge, while in the former cases we obtain two edges. So in the latter case chances of obtaining Stable Improvement Cycle are lower (all other things being equal).

The following procedure is based on a simple idea, illustrated by the above example

- Step 1. Consider original preference relation $\succeq_b$ and 'best' (undominated) antichain A_1=$\{a_1, ..., a_k\}$. Let $a_i \succ'_b a_j$ if $a_i P \subset a_j P$, that is, a_i dominates a_j in the new transformed preference relation $\succ -_b$ if a_i dominates strictly smaller set of alternatives than a_j. All remaining ties among elements in the antichain are broken randomly.

- Step $t \in [2, T]$. Consider the 'best' antichain A_t on $A \backslash (\cup A_{t-1})$. Repeate the step 1 for the A_t. Repeat steps until no elements are remaining in $A \setminus (cup A_t)$.

- Step $T + 1$. For any elements $a \in A_n$, $a' \in A_m$ $a \succ'_b a'$ iff $n < m$.

We can now state the following general result.

Theorem 4. *Let all preference relations of the universities be interval orders of special form, where each maximal antichain has the same length. If we break the ties of each university according to the procedure, described above, then under deferred acceptance procedure with proposing applicants we have the lowest (among all possible 'independent' tie-breakings) chances of constructing an applicant-inefficient stable matching.*

In fact this theorem allows us to construct a new version of deferred acceptance with tie-breaking. First, break ties according to the procedure above. Second, apply deferred acceptance procedure with applicants proposing. Such procedure is strategy-proof for applicants (as first step tie-breaking does not consider there preferences, while the second step is just classical Gale-Shapley procedure). At the same time, probability of obtaining an inefficient stable matching with such procedure is lower then with random tie-breaking procedure.

6 Discussion

Interval orders are, in fact, a special case of the partial orders. So the results, obtained for matching with preferences, being partial orders, are directly applicable to our problem statement. Although there are plenty of papers, analysing problem setting with partial orders preferences, none of them solves the same task as ours. Some of the papers (see, for example, [8]) consider more restrictive definitions of stability: super-stability or strict stability. In our setting this two concepts are, in fact, the same. Super-stability, as the original Gale-Shapley stability, requires no blocking pair property, but uses a different definition of a blocking pair. An and a university are said to be a blocking pair if the applicant weakly prefers the university to her current match, and university weakly prefers applicant to any of its current students. It is obvious that super-stable matching may not exist. If the set of super-stable matchings is non-empty, it forms a distributive lattice, so there exists a unique applicant-optimal matching.

Novel paper [4] assumes the firm-worker model where firms does not have sufficient information to rank potential employees and at the beginning has partial order preferences. Additional information may be obtained via interviews. Authors construct a mechanism, which allows to find firms-efficient (or employee-efficient) stable matching and minimize number of interviews. The main difference with our setting is that firms are assumed to have unknown to themselves strict preferences over employees, while in our setting universities are assumed to be truly indifferent.

Paper [7] is one of the closest to our setting. Authors use the same definition of stability and construct a mechanism, which finds a men-efficient stable matching. Their result has two main differences with ours. First, their mechanism may not find an efficient stable matching, even it is clear, that efficient stable matchings always exist. Second, they do not provide criteria of one-side efficiency of some particular matching.

7 Conclusion

In this paper we analyse an extension of the classical university-applicant Gale-Shapley model, where we allow preferences of the universities be interval orders. This extension covers admission systems, where scoring is used for ranking applicants, but scoring method may have some error. We found out that stable matching always exists and can be found via deferred acceptance procedure. Unfortunately, applicant-proposing deferred acceptance procedure may produce an applicant-inefficient stable matching. Our first result is a criteria, which allows to determine, whether some particular stable matching is applicant-efficient and improve a matching, if necessary. Our results has direct practical implications in centralized admission mechanisms.

Unfortunately, the mechanism, based on application of this criteria, is not strategy-proof. So, we propose another mechanism, which is strategy-proof and have reduced chances of obtaining inefficient stable matching under DA with applicants proposing.

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