Bargaining and Power

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Abstract. We introduce a new power configuration which takes into account bargaining between players within in coalitions. We show that under very weak conditions on a bargaining solution there is a power configuration which is stable with respect to renegotiations. We further show that given this power configuration there is a coalition which is both internally and individually stable. Finally we consider two different bargaining solutions on apex games and show under which conditions there are core stable coalitions.

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1 Introduction

Consider a committee or a parliament which has to make decisions. Usually, players of parties with similar interests form coalitions which are able to enforce their will. This is the case particularly, if this committee will gather on a regular base. Based on possible coalitions a player might join, one can make statements about his power in the committee or in a coalition he is member of.

The measurement of a member's power in a committee has been the subject of many articles. Famous Examples are the Shapley-Shubik power index (SSPI, Shapley and Shubik, 1954) and various versions of the Banzhaf-Coleman power index (BCPI, Banzhaf, 1965). However, these indices per se do not consider players' power in coalitions apart from the whole player set.

Shenoy (1979) introduced the power of players in each coalition based the Shapley-Shubik index. In particular, the author considered a coalition formation game where players' preferences over coalitions depend on their power in coalitions. This idea has been further developed and generalized by Dimitrov and Haake (2006). However, these idea of power within a coalition did not take into account anything outside of this coalition.

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The Owen value (OV, Owen, 1977) and the Casajus value (CV Casajus, 2009) are adaptations of the Shapley-Shubik index which take into account the partition of the player set into coalitions. The first one has been used by **?**Hart and Kurz (1984) to introduce a similar coalition formation game as Shenoy (1979), but under consideration of the behaviour of players outside of a fixed coalition. Although the power of a player therefore depends on other coalitions as well, the power a player receives in two different coalitions are completely independent.

We interpret power as a payoff of players, for instance in a parliament where a government of several parties has to agree on the allocation of cabinet seats among parties. In this case, a player can use the power in one coalition to enforce a certain power in another coalition. In other words: Power in one coalition can be used to bargain about power in another coalition. We will illustrate this idea in the following example.

Example 1.1. The German Bundestag currently consists of five parliamentary parties, CDU/CSU (1), FDP (2), SPD (3), Linke (4), and B90/Grne (5). A coalition has the absolute majority if and only if it contains at least one of the following coalitions: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, or $\{2,3,4\}$. In the model of Shenoy (1979) there is no stable outcome of this game; in the model of Hart and Kurz (1984) each of these four coalitions is stable.

The government consists of CDU/CSU and FDP. Under the assumption that parties in the opposition do not collaborate, the above mentioned power indices deliver the following values for the governmental parties.

	SSPI	BCPI	OV	CV
CDU	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{2}{3}$
FDP	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{3}$

The idea that these two parties are equally powerful as SSPI and BCPI suggest is not very convincing, given that there are two other parties outside of the government which each have the absolute majority together with CDU/CSU. OV distinguishes between the two parties, but if we assume that the remaining parties work together then we have $OV_{CDU} = OV_{FDP} = \frac{1}{2}$ although CDU has much better chances to find a different party for a government coalition than FDP. The cabinet consists of 16 ministers of which 11 are members of CDU/CSU and 5 belong to FDP. Hence, in this example the outside option value is closest to the actual distribution of power between parties.

The previous models always made the assumption that the power of players in coalitions in specified ex ante and leads to a coalition formation game. Nevertheless, in reality players bargain over the distribution of power in coalitions, they check their options in other coalitions, and they renegotiate. Hence, a separation of the coalition formation process from the power distribution is not convincing.

Our model brings together these two concepts; the power of a player within a coalition depends on two things:

1. His marginal contribution: A player who is needed in a winning coalition is more powerful than a player who could leave the coalition without effect.
2. His outside option: A player who is very powerful in another coalition is more powerful than a player who has no other options.

We assume that the allocation of power in coalitions follows a fixed bargaining rule (Nash, 1950) and depends on these two variables. After any renegotiation, the outside options of players may have changed and lead to a new renegotiation. We are not focussing on this dynamic process, but on the question whether we can find an allocation which is stable with respect to renegotiation. In this case an application of the bargaining rule would not change the result. We will show in Section 2 that under very weak conditions on a bargaining rule such an allocation exists. In Section 3 we consider a special bargaining rule and show that under some restrictions this stable allocation is even unique.

We can interpret such an allocation as the result of exploratory talks between all groups of parties. As this allocation will not be renegotiated, the preferences of players over coalitions based on this allocation are very robust. In Section 4 we give conditions such that a coalition exists which is both internally stable (i.e. no group of player would leave it to stay alone) and individually stable (no player would leave the coalition to join another one).

Section 5 is devoted to a special class of games, namely apex games. Karos (2012) considered the coalition formation game after application of the Shapley-Shubik index or the normalized Banzhaf-Coleman index. It has been shown that for each coalition there is a group of players within this coalition which can improve by leaving and joining the remaining players. We show that in our model there are some conditions depending on the bargaining rule which guarantee the existence of a coalition which will not be left by any players.

In Section 6 we further extent our model. Especially in parliaments not all coalitions which can reach the absolute majority are equally probable. There are parties which will never collaborate due to there political interests. As the stable allocation in our model depends on outside options, we have to ensure that the allocation in an impossible coalition does not affect the allocation in any other coalition. We show that if each player can chose to stay alone, a stable allocation still exists. In particular, if there is at least one possible winning coalition, then we can find a coalition which is stable as before.

In Section 7 we give some concluding remarks and possible further developments of our model.

2 The Model

Throughout the paper let N be a finite set of players. A *coalition* is a subset $S \subseteq N$ and the set $\mathcal{P} = \mathcal{P}(N)$ is the collection of all coalitions. For $i \in N$ we denote by $\mathcal{P}_i$ the collection of all coalitions containing i. A *simple game* is a function $v : \mathcal{P}(N) \to \{0, 1\}$ with $v(\emptyset) = 0$. A simple game v is called *proper* if $v(S) + v(N \setminus S) \leq 1$ for all $S \subseteq N$ and *monotonic* if $v(S) \leq v(T)$

for all S, T with $S \subseteq T$. A coalition $S \subseteq N$ with $v(S) = 1$ is called *winning*. A winning coalition S which does not contain any proper subcoalition which is also winning is called *minimal winning*. If S is a winning coalition and $i \in S$ is such that $v(S \setminus \{i\}) = 0$ then i is called *pivotal* in S with respect to v. Two players $i, j \in N$ are called *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

For $i \in N$ and $S \subseteq N$ we define

$$d_i^m(S) = v(S) - v(S \setminus \{i\}).$$

as the *marginal contribution* of player i to S in v. Note, that $d_i^m(S)$ depends on v; we skip v in the notation for comfort reasons, though.

Definition 2.1. Let v be a simple game. A *power configuration* $x = (x_i)_{i \in N}$ for v is a vector of maps $x_i : \mathcal{P}_i \to \mathbb{R}$ such that $x_i(S) = 0$ for all $i \in S$ if $v(S) = 0$ and $\sum_{i \in S} x_i(S) \leq v(S)$ for all $S \subseteq N$. A power configuration x is called *efficient* if $\sum_{i \in S} x_i(S) = v(S)$ for all $S \subseteq N$, and called *individual rational* if $x_i(S) \geq 0$ for all $S \in \mathcal{P}_i$. The collection of all power configurations for v is denoted by $\Delta(v)$ and the collection of all individual rational power configuration is denoted by $\Delta_{ir}(v)$.

We can think of a power configuration as a set of agreements which clarify in each coalition S how power is distributed in S. Let x be a power configuration. The condition $x_i(S) = 0$ for loosing coalitions and all $i \in S$ reflects that a player should not have any power if he is member of a loosing coalition. Let S be a winning coalition and let $i \in S$. Player i is contained in many other coalitions (for instance $N \setminus S \cup \{i\}$), in particular, each coalition $T \subseteq N \setminus S \cup \{i\}$ with $i \in T$ ensures him power $x_i(T)$. We define player i's *outside option in S* as

$$d_i^o(S, x) = \max_{T \subseteq N \setminus S} x_i(T \cup \{i\}).$$

When the members of S are negotiating on how power within S should be shared, the two values d_i^m and d_i^o are crucial for the bargaining position of i. The next definition specifies what we mean by bargaining in our context.

Definition 2.2. A *disagreement point* for S is a vector $d = d(S) \in \mathbb{R}^S$. A *bargaining solution* is a map F such that $F(S, v(S), d) \in \mathbb{R}^S$, and

$$\sum_{i \in S} F_i(S, v(S), d) \leq v(S)$$

for each proper monotonic simple game v, each coalition $S \subseteq N$, and each disagreement point d for S.

The triple $(S, v(S), d)$ is called a *bargaining problem*. It describes exactly the previous situation: The players $i \in S$ negotiate about a worth $v(S)$ where the disagreement point d represents their bargaining positions. We have pointed out that $d_i(S)$ depends on two things, namely the marginal contributions $d_i^m(S)$

and the outside option $d_i^o(S,x)$ given a power configuration x. In our model we assume that disagreement points are a convex combination of d_i^m and d_i^o. Henceforth, let

$$d_i(S,x) = \alpha d_i^m(S) + (1-\alpha)\, d_i^o(S,x)$$

be the disagreement point in the power bargaining problem within coalition S. It is clear that the outside option $d_i^o(S,x)$ of a player $i \in S$ can be positive only if $(N \setminus S) \cup \{i\}$ is winning. Because of properness of v this can be the case only if $S \setminus \{i\}$ is losing. Hence, a player $i \in S$ can only have a positive outside option if he is pivotal in S. The parameter α specifies how this outside option shall be weighted.

We are no facing the following problem: Given any power configuration x, we have a set of bargaining problems with disagreement points depending on x, that is players renegotiate their power. After application of a bargaining solution F, we end up with a new power configuration. We are looking for a power configuration which is stable with respect to renegotiation. The next definition formalizes this idea.

Definition 2.3. Let F be a bargaining solution and v be a simple game. A power configuration $x \in \Delta(v)$ is called *stable with respect to* F if for all winning coalitions $S \subseteq N$ and all $i \in S$ the following holds.

$$\begin{array}{rcl} x_i(S) & = & F_i(S, v(S), d(S,x)) \\ d_i(S,x) & = & \alpha d_i^m(S) + (1-\alpha)\, d_i^o(S,x) \\ d_i^o(S,x) & = & \max_{T \subseteq N \setminus S} x_i(T \cup \{i\}). \end{array} \tag{1}$$

For general bargaining solutions F we cannot assume that such a stable payoff configuration exists for all proper monotonic simple games. Hence, the aim of the remainder of this section is to find sufficient conditions on F such that stable power configuration exist.

In classical bargaining theory we have that $\sum_{i\in S} d_i(S) \leq v(S)$ for each bargaining problem. We do not restrict our attention to this case. If the disagreement point is such that it cannot be reached by any allocation of $v(S)$, one usually talks about a *bankruptcy problem* (see for instance Curiel et al., 1987). We will not distinguish between bargaining problems and bankruptcy problems. The following properties a bargaining solution might satisfy account for this and are therefore slightly different from definitions which can be found in literature on bargaining problems.

Definition 2.4. A bargaining solution F is called

1. *individual rational* if $F_i(S, v(S), d) \geq 0$ for all proper monotonic simple games v, all coalitions $S \subseteq N$, all $i \in S$, and all $d \in \mathbb{R}^S$.
2. *efficient* if $\sum_{i\in S} F_i(S, v(S), d) = v(S)$ for all proper monotonic simple games v, all coalitions $S \subseteq N$, all $i \in S$, and all $d \in \mathbb{R}^S$.
3. *continuous* if $F(S, v(S), .)$ is continuous for all coalitions $S \subseteq N$ and all proper monotonic simple games v.

4. *fair* if for all proper monotonic simple games v and all coalitions $S \subseteq N$ there is $i \in S$ with $F_i(S, v(S), d) \geq d_i(S)$ if and only if $F_j(S, v(S), d) \geq d_j$ for all $j \in S$.

Individual rationality does not guarantee that all players are satisfied by their power in the sense that $F_i(S, v(S), d) \geq d_i$. It rather says that no player should pay (i.e. have negative power) for joining a coalition. Since we do not assume that each player i can receive at least d_i, fairness ensures that all players are on the same side of d: A player i cannot get more than d_i if in the same coalition another player j receives less than d_i. Efficiency is standard, it can be understood as a normalization such that the distributed power in each winning coalition sums up to 1. Continuity ensures that a small change in disagreement points cannot change arbitrarily large change in the bargaining outcome

Example 2.5. 1. The *egalitarian bargaining solution* is defined as

$$E_i(S, v(S), d(S)) = d_i(S) + \frac{1}{|S|}\left(v(S) - \sum_{j \in S} d_j(S)\right).$$

Clearly, E is efficient, fair, and continuous. However, E is not individual rational, as $\sum_{j \in S} d_j(S)$ might be very large.

2. The *constrained egalitarian bargaining solution* (see for instance Curiel et al., 1987) is defined as

$$\tilde{E}_i(S, v(S), d(S)) = \max\{d_i - \lambda, 0\}$$

where λ is such that $\sum_{i \in S} E_i(S, v(S), d(S)) = v(S)$. $\tilde{E}$ is individual rational, efficient, and continuous. However, $\tilde{E}$ is not fair.

3. The *proportional bargaining solution*[1] is defined as

$$P_i(S, v(S), d(S)) = \begin{cases} \frac{d_i(S)}{\sum_{i \in S} d_i(S)} v(S), & \text{if } \sum_{i \in S} d_i(S) \neq 0, \\ \frac{1}{|S|} v(S), & \text{if } \sum_{i \in S} d_i(S) = 0. \end{cases}$$

We see on the first sight that P is individual rational, efficient, and fair. But, P is not continuous in $d = 0$.

The following theorem focuses on continuous bargaining solutions and gives sufficient conditions for the existence of a stable power configuration. The proportional bargaining solution, which is not continuous, is considered in the next section.

Theorem 2.6. *Let F be a continuous bargaining solution and v be a proper monotonic simple game.*

1. *If F is individual rational then there is a stable $x \in \Delta(v)$.*

[1] This is a special case of the proportional bargaining solution introduced in Kalai (1977)

2. *If F is fair and efficient then there is a stable $x \in \Delta(v)$.*

Proof. Let F be a bargaining solution and v be a proper monotonic simple game. We define the map $\hat{F} : \Delta(v) \to \Delta(v)$ as

$$\hat{F}_{i,S}(x) = F_i(S, v(S), d(S, x)).$$

Then a power configuration $x \in \Delta(v)$ is stable with respect to F if and only if $\hat{F}(x) = x$. Hence, we have to show that $\hat{F}$ has a fixed point. First we show that if F is continuous then $\hat{F}$ is continuous as well. For this purpose we need the following construction. Let $x \in \Delta(v)$, $S \subseteq N$ and $i \in S$. Then let $\mathcal{T}_i^S(x) \subseteq \mathcal{P}(N \setminus S)$ be such that

$$x_i(T \cup \{i\}) \geq x_i(T' \cup \{i\})$$

for all $T \in \mathcal{T}_i^S(x)$ and all $T' \subseteq N \setminus S$. That is, given the power configuration x, $\mathcal{T}_i^S(x)$ is the collection of optimal coalitions for player i outside of S. In particular, we have $d_i(S, x) = x_i(T)$ for all $T \in \mathcal{T}_i^S(x)$. Let now

$$[x] = \left\{ y \in \Delta(v);\ \mathcal{T}_i^S(x) \cap \mathcal{T}_i^S(y) \neq \emptyset \text{ for all } S \subseteq N, i \in S \right\}.$$

If $y \in [x]$, we can find for each player i in each coalition S a coalition T such that $d_i(S, x) = x_i(T)$ and $d_i(S, y) = y_i(T)$; in particular, $T \in \mathcal{T}_i^S(x) \cap \mathcal{T}_i^S(y)$. Note that

$$\hat{F}_{i,S}(x) = F_i(S, v(S), x_i(T))$$

for all $T \in \mathcal{T}_i^S(x)$. Hence, $\hat{F}$ is continuous on $[x]$ for all $x \in \Delta(v)$ as F is continuous. By definition of $\mathcal{T}_i^S(x)$ it is straightforward that $[x]$ is closed for all $x \in \Delta(v)$. As further N and $\mathcal{P}(N)$ are finite, there can only be a finite number of sets of this type, i.e. there are $x_1, \ldots, x_n$ such that

$$\Delta(v) = \bigcup_{k=1}^{n} [x_k].$$

As $\hat{F}$ is continuous on $[x_k]$ and $[x_k]$ is closed for all $k = 1, \ldots, n$, $\hat{F}$ is continuous on $\Delta(v)$.

1. Let now F be individual rational. Then we have $F(\Delta_{ir}(v)) \subseteq \Delta_{ir}(v)$. Hence, by Brouwer's fixed point theorem, there is a fix point of $\hat{F}$ in $\Delta_{ir}(v)$.

2. Let F be fair and efficient. Since $F_i(\{i\}, v(\{i\}), d) = v(\{i\}) \geq 0$, we have $d_i(S) \geq 0$ for all $S \subseteq N$. Let now

   $$Q = \{x \in \Delta(v);\ -(|N| - 1) \leq x_i(S) \leq 1 \text{ for all } S \subseteq N, i \in S\}.$$

   We show that $\hat{F}(Q) \subseteq Q$. Let therefore $x \in Q$ and $S \subseteq N$. We consider two cases

(a) Let $F_i(S, v(S), d(S,x)) \geq d_i(S,x) \geq 0$ for all $i \in S$. Since

$$\sum_{i \in S} F_i(S, v(S), d(S,x)) \leq v(S) \leq 1,$$

we have that $0 \leq F_i(S, v(S), d(S,x)) \leq 1$.

(b) Let $F_i(S, v(S), d(S,x)) \leq d_i(S,x)$. Then

$$\begin{aligned} F_i(S, v(S), d(S,x)) &= v(S) - \sum_{j \in S \setminus \{i\}} F_j(S, v(S), d(S,x)) \\ &\geq -\sum_{j \in S \setminus \{i\}} d_j(S,x) \\ &\geq -\sum_{j \in S \setminus \{i\}} \max_{x \in Q, T \subseteq N \setminus S} x_j(T \cup \{i\}) \\ &\geq -(|N|-1). \end{aligned}$$

Hence,

$$-(|N|-1) \leq F_i(S, v(S), d(S,x)) \leq d_i(S,x) \leq \max_{i,S} x_i(S) \leq 1.$$

So, we have that $\hat{F}(x) \in Q$. Again, by Brouwer's fixed point theorem, there is a fixed point of $\hat{F}$.

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Theorem 2.6 ensures the existence of stable power configurations for continuous bargaining solutions under very weak conditions. Together with Example 2.5 it implies the following corollary immediately.

Corollary 2.7. *1. For each proper monotonic simple game v there is a stable payoff configuration $x \in \Delta(v)$ with respect to $\tilde{E}$.*

2. For each proper monotonic simple game v there is a stable payoff configuration $x \in \Delta(v)$ with respect to E.

Although the existence of a stable power configuration for all proper monotonic simple games is a strong result, Theorem 2.6 does not guarantee uniqueness of the stable power configuration. The next example shows that in general we cannot expect uniqueness, even under the conditions of Theorem 2.6.

Example 2.8. Let $\alpha = 0$, that is $d(S,x) = d^o(S,x)$, and let v be the proper monotonic simple game on $N = \{1,2,3,4,5,6\}$ with minimal winning coalitions $\{1,2,3\}$, $\{1,4,5\}$, $\{2,4,6\}$, and $\{3,5,6\}$. A stable power configuration with respect to P, E, and $\tilde{E}$ is for instance given by $x_i(S) = \frac{v(S)}{|S|}$ for all $S \subseteq N$ and all $i \in S$. However, this is not the only stable power configuration. Let y be defined as follows:

$$y_i(S) = \begin{cases} 0, & \text{if } v(S) = 0 \text{ or } i \notin S \\ \frac{1}{|S|}, & \text{if } v(S) = 1 \text{ and } |S| \geq 5, \\ 1, & \text{if } (S = \{1,2,3,6\} \text{ or } S = \{1,4,5,6\}) \text{ and } i = 1, \\ 1, & \text{if } (S = \{1,2,4,6\} \text{ or } S = \{1,3,5,6\}) \text{ and } i = 6, \\ 1, & \text{if } v(S) = 1, |S| = 3, i = 1, \text{ and } i \in S, \\ 0, & \text{if } v(S) = 0, |S| = 3, i \neq 1, \text{ and } 1 \in S, \\ 1, & \text{if } v(S) = 1, |S| = 3, i = 6, \text{ and } i \in S, \\ 0, & \text{if } v(S) = 0, |S| = 3, i \neq 6, \text{ and } 6 \in S. \end{cases}$$

Then we have $\hat{P}(x) = \hat{E}(x) = \hat{\tilde{E}}(x) = x$, that is x is stable with respect to P, E, and $\tilde{E}$, too.

We close this section with some properties of stable power configurations. The first two parts of the following lemma are obvious and their proof are omitted.

Lemma 2.9. *Let v be a proper monotonic simple game, let F be a bargaining solution, and let $x \in \Delta(v)$ be stable with respect to F.*

1. *If F is efficient then x is efficient.*
2. *If F is individual rational then x is individual rational.*
3. *If F is symmetric and i, j are symmetric with respect to v then x' defined as*

$$x'_i(S) = \begin{cases} x_j(S), & \text{if } i, j \in S, \\ x_j(S \setminus \{i\} \cup \{j\}), & \text{if } i \in S \text{ and } j \notin S, \end{cases}$$

and

$$x'_j(S) = \begin{cases} x_i(S), & \text{if } i, j \in S, \\ x_i(S \setminus \{j\} \cup \{i\}), & \text{if } j \in S \text{ and } i \notin S, \end{cases}$$

is stable with respect to F as well.

Proof. We prove only the last part. Let $x \in \Delta(v)$ be stable with respect to F and let $i, j \in N$ by symmetric with respect to v. Let $p^{i,j} : N \to N$ be the permutation defined as

$$p^{i,j}(k) = \begin{cases} i, & \text{if } k = j, \\ j, & \text{if } k = i, \\ k, & \text{if } k \neq i, j. \end{cases}$$

Then $T \in \mathcal{T}_k^S(x)$ if and only if $p^{i,j}(T) \in \mathcal{T}_{p^{i,j}(k)}^{p^{i,j}(S)}(x')$. Hence,

$$\begin{aligned}
F_{p^{i,j}(k)}\left(p^{i,j}(S), v\left(p^{i,j}(S)\right), d\left(p^{i,j}(S), x'\right)\right) &= F_k(S, v(S), x(T)) \\
&= x_k(S) \\
&= x'_{p^{i,j}(k)}(S).
\end{aligned}$$

■

The first two parts of the Lemma need no further explanation. For the last part one has to keep in mind that a stable power configuration need not to be unique. In particular, not each stable power configuration must be symmetric. Lemma 2.9 guarantees that the set of all stable power configuration is symmetric. An easy consequence is the following corollary.

Corollary 2.10. *Let v be a proper monotonic simple game, let F be a symmetric bargaining solution such that there is a unique $x \in \Delta(v)$ which is stable with respect to F. Then $x_i(S) = x_j(S)$ for all $S \subseteq N$ with $i, j \in S$ and $x_i(S \cup \{i\}) = x_j(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.*

3 The Proportional Bargaining Solution

We have already mentioned that P is not continuous in $d = 0$, such that we cannot apply Theorem 2.6. Nevertheless, we can already make some statements about stable power configurations, if they exist. The proofs are straightforward and therefore omitted.

Lemma 3.1. *Let v be a proper monotonic simple game let $x \in \Delta(v)$ be stable with respect to P.*

1. *If $S \subseteq N$ is such that no $i \in S$ is pivotal in S with respect to v then $x_i(S) = \frac{v(S)}{|S|}$ for all $i \in S$.*
2. *If $\alpha > 0$ and if $S \subseteq N$ is such that there is only one player $i \in S$ who is pivotal in S with respect to v then $x_i(S) = 1$ and $x_j(S) = 0$ for all $j \in S \setminus \{i\}$.*
3. *If $\alpha > 0$ and if $S \subseteq N$ is such that there is at least one player in S who is pivotal in S with respect to v then $x_j(S) = 0$ for all $j \in S$ which are not pivotal in S.*

Note that the last two results in Lemma 3.1 depend on the parameter α. For $\alpha = 1$ the stable power would be unique and very easy to find: For any winning coalition S let $S' \subseteq S$ be the set of players which is pivotal in S. Then

$$x_i(S) = \begin{cases} \frac{1}{|S'|}, & \text{if } i \in S', \\ 0, \text{else.} \end{cases} \tag{2}$$

On the other side, if $\alpha = 0$, a player $i \in S$ which is pivotal in S and does not have any positive outside option (for instance because $(N \setminus S) \cup \{i\}$ is not winning) has power 0 in S.

So far, we do not have empirical evidence how α should be chosen. In the next theorem we show that for strictly positive α a stable power configuration with respect to P always exists. Moreover, we give a lower bound for α such that this stable power configuration is unique.

Theorem 3.2. *1. Let $\alpha > 0$. Then there is $x \in \Delta(v)$ which is stable with respect to P.*

2. Let $\alpha > \frac{|N|-2}{|N|}$. Then there is a unique $x \in \Delta(v)$ which is stable with respect to P.

Proof. By Lemma 3.1 $\hat{P}_{i,S}$ is constant and therefore continuous for all coalitions $S \subseteq N$ which do not contain at least two pivotal players.

1. Let $\alpha > 0$. We have to show that $\hat{P}_{i,S}$ is continuous for all coalitions $S \subseteq N$ which contain at least two pivotal players. We see that

$$\hat{P}_{i,S}(x) = \frac{d_i(S)}{\sum_{j\in S} d_j(S)} v(S) = \frac{d_i(S)}{\alpha \sum_{j\in S} d_j^m(S) + (1-\alpha)\sum_{j\in S} d_j^o(S)} v(S).$$

As $\sum_{j\in S} d_j^m(S) \geq 2$ and $\alpha > 0$, we have that $\hat{P}$ is continuous for all $x \in \Delta(v)$. Since $\hat{P}(\Delta_{ir}(v)) \subseteq \Delta_{ir}(v)$, there must be a fixed point of $\hat{P}$ in $\Delta_{ir}(v)$.

2. Let $\alpha \geq \frac{|N|-2}{|N|}$. If v is a proper monotonic simple game such that there is $i \in N$ with $v(\{i\}) = 1$ then the only stable power configuration with respect to P is

$$x_k(S) = \begin{cases} 1, & \text{if } k = i \text{ and } i \in S, \\ 0, & \text{else} \end{cases}$$

by Lemma 3.1. So let v be such that $v(\{i\}) = 0$ for all $i \in N$. We show that $\hat{P}$ is a contraction on $\Delta_{ir}(v)$. For this purpose, note that for the partial derivatives of $\hat{P}$ we have

$$\frac{\partial \hat{P}_{i,S}}{\partial x_j(T)} = 0$$

for all $S \subseteq N$ which do not contain at least two pivotal players and for all $S \subseteq N$ and all $i \in S$ which are not pivotal in S, for all $T \subseteq N$ and all $j \in T$. We also have $\frac{\partial \hat{P}_{i,N}}{\partial x_j(T)} = 0$ for all $i \in N$, all $T \subseteq N$ and all $j \in T$ as N does not contain any player with a positive outside option. Let therefore $S \subsetneq N$ contain at least two pivotal players and let $i \in S$ be pivotal. Let further $T_j^S(x) \in \mathcal{T}_j^S(x)$ for all $j \in S$. Then we have

$$\frac{\partial \hat{P}_{i,S}}{\partial x_j(T)} = \begin{cases} \frac{(1-\alpha)\left(\alpha \sum_{k\neq i} d_k^m(S)+(1-\alpha)\sum_{k\neq i} d_k^o(S)\right)}{\left(\alpha \sum_{k\in S} d_k^m(S)+(1-\alpha)\sum_{k\in S} d_k^o(S)\right)^2}, & \text{if } i=j \text{ and } T = T_i^S(x), \\ -\frac{(1-\alpha)(\alpha d_i^m(S)+(1-\alpha)d_i^o(S))}{\left(\alpha \sum_{k\in S} d_k^m(S)+(1-\alpha)\sum_{k\in S} d_k^o(S)\right)^2}, & \text{if } i\neq j \text{ and } T = T_j^S(x), \\ 0, & \text{else.} \end{cases}$$

Hence,

$$\begin{aligned} \sum_{j\in N, T\subseteq N} \left|\frac{\partial \hat{P}_{i,S}}{\partial x_j(T)}\right| &= \frac{1-\alpha}{\alpha \sum_{k\in S} d_k^m(S) + (1-\alpha)\sum_{k\in S} d_k^o(S)} \\ &\quad + \frac{(1-\alpha)(|S|-2)(\alpha d_i^m(S)+(1-\alpha)d_i^o(S))}{\left(\alpha \sum_{k\in S} d_k^m(S) + (1-\alpha)\sum_{k\in S} d_k^o(S)\right)^2} \\ &\leq \frac{(1-\alpha)(|S|-1)}{\alpha \sum_{k\in S} d_k^m(S) + (1-\alpha)\sum_{k\in S} d_k^o(S)} \\ &\quad - \frac{\alpha}{\left(\alpha \sum_{k\in S} d_k^m(S) + (1-\alpha)\sum_{k\in S} d_k^o(S)\right)^2} \\ &\leq \frac{(1-\alpha)(|S|-1)}{2\alpha} - \frac{\alpha}{|S|^2} \end{aligned}$$

For the Jacobian matrix $D_{\hat{P}}$ we therefore find

$$\begin{aligned} \left\|D_{\hat{P}}\right\| &= \max_{i\in N, S\subsetneq N} \sum_{j\in N, T\subseteq N} \left|\frac{\partial \hat{P}_{i,S}}{\partial x_j(T)}\right| \\ &\leq \frac{(1-\alpha)(|N|-2)}{2\alpha} - \frac{\alpha}{(|N|-1)^2}. \end{aligned}$$

Since this bound is decreasing with increasing α and since $\alpha \geq \frac{|N|-1}{|N|}$ we find

$$\left\|D_{\hat{P}}\right\| \leq 1 - \frac{|N|-2}{|N|(|N|-1)^2}.$$

Hence, $\hat{P}$ is a contraction and has therefore a unique fixed point by Banach's fixed point theorem.

$\blacksquare$

4 Coalition Formation

A *hedonic coalition formation game* (Drèze and Greenberg, 1980) is a set N together with a profile of preferences $(\succeq_i)_{i\in N}$[2], where $\succeq_i$ is defined on $\mathcal{P}_i$ for all $i \in N$.

[2]A preference relation is a complete, transitive, and reflexive binary relation.

It has been pointed out by Dimitrov and Haake (2006) (see also Shenoy, 1979) that a proper monotonic simple game v, together with a power configuration $x \in \Delta(v)$, induces a hedonic game as follows. For $i \in N$ and $S, T \in \mathcal{P}_i$ let $\succeq_i$ be defined as

$$S \succeq_i T \qquad \text{if and only if} \qquad x_i(S) \geq x_i(T).$$

The outcome of a general hedonic coalition formation game is a partition of the player set which is stable in some sense. In our case we are interested in stable coalitions rather than partitions. Therefore we slightly adapt the classical definitions.

Definition 4.1. Let v be a proper monotonic simple game, let $x \in \Delta(v)$, and let $S \subseteq N$ be winning. S is called *individually stable* if for each $i \in S$ and each $T \subseteq N \setminus S$ it holds that $x_i(T \cup \{i\}) \leq x_i(S)$.

Roughly speaking, a winning coalition S is individually stable if no player $i \in S$ has an incentive to leave S and join any coalition $T \subseteq N \setminus S$.

Let $x \in \Delta_{ir}(v)$ be efficient. It is easy to see that a coalition S which does not contain any pivotal player must be individual stable as no player can improve by moving to a loosing coalition. However, these coalitions do not always seem *credible* in the following sense: Although no player can improve by leave the coalition, there might still be a group of players T inside S which could improve by excluding the remaining players. We therefore call a coalition $S \subseteq N$ *internally stable* if for each $T \subseteq S$ there is $i \in T$ such that $x_i(S) \geq x_i(T)$. The question is now: Can we find a coalition which is both individually stable and internally stable? The answer is yes, in the following set up.

Theorem 4.2. *Let v be a proper monotonic simple game, let F be an individual rational and fair bargaining solution, let $\alpha = 0$, and let $x \in \Delta(v)$ be stable with respect to F. Then there is a coalition $S \subseteq N$ which is both individually stable and internally stable.*

Proof. First we show that there is an internally stable winning coalition S. For this reason note that $x_i(N) \geq d_i^o(N) = 0$ for all $i \in N$ or $x_i(N) < d_i^o(N) = 0$ for all $i \in N$ by fairness. Hence, individual rationality implies that $x_i(N) \geq 0$ for all $i \in N$. Let now $S_0 = N$ and $S_k \subsetneq S_{k-1}$ such that $x_i(S_k) > x_i(S_{k-1})$ for all $i \in S_k$. If k is such that there is no S_{k+1} then $S = S_k$ is internally stable.

We show that there is an individually and internally stable coalition. For this reason let S be internally stable. Since F is fair and $\alpha = 0$, we have either $x_i(S) \geq d_i^o(S, x)$ for all $i \in S$ or $x_i(S) < d_i^o(S)$ for all $i \in S$. In the first case this means

$$x_i(S) \geq \max_{T \subseteq N \setminus S} x_i(T \cup \{i\}),$$

hence, S is individually stable. So, let $i \in S$ and let $x_i(S) < d_i^o(S)$. Let $T_1 \in \mathcal{T}_i^S(x)$. Since $d_i^o(S) > x_i(S) \geq 0$, T_1 must be a winning coalition. Because of individual rationality there is no loosing $T' \subsetneq T_1 \cup \{i\}$ with $x_j(T') > x_j(T_1 \cup \{i\})$ for all $j \in T'$. As i is pivotal in T_1, i is contained in each winning

$T' \subseteq T_1 \cup \{i\}$. Since $T_1 \in \mathcal{T}_i^S(x)$, we have that $x_i(T_1 \cup \{i\}) \geq x_i(T')$. Thus, T_1 is internally stable. Now, either $x_i(T_1) \geq d_i^o(T_1, x)$ or $x_i(T_1) \leq d_i^o(T_1, x)$. In the first case T_1 is individually stable. In the latter case we define

$$T_k \in \mathcal{T}_i^{T_{k-1}}(x)$$

for all $k \geq 1$. Then all T_k are internally stable and T_k is individually stable if and only if $x_i(T_k) \geq x_i(T_{k+1})$. As N and therefore $\mathcal{P}(N)$ are finite, there is k such that $T_{k+1} = T_l$ for some $l \leq k$. Let k^* be the first such k^*. Then

$$x_i(T_{k^*}) \geq x_i(T_l) = x_i(T_{k^*+1}) = d_i^{T_{k^*}}(S)$$

and we see that T_{k^*} is individually stable. ∎

5 Application: Apex Games

An *apex game* a_{iJ} on a player set $N = \{i\} \cup J$, where $|J| \geq 3$ is defined by

$$a_{iJ}(S) = \begin{cases} 1, & \text{if } (i \in S \text{ and } S \cap J \neq \emptyset) \text{ or } J \subseteq S, \\ 0, & \text{else.} \end{cases}$$

Player i is called *apex player* and $j \in J$ is called minor player.

In this section we show that for the bargaining solutions E, $\tilde{E}$, and P there is always a unique stable power configuration for each apex game. We also investigate the induced hedonic coalition formation game. We already now that we can find internally and individually stable coalitions in case of $\alpha = 0$. Now, we consider arbitrary $\alpha \in [0, 1]$. We show under which conditions we can find coalitions which satisfy the following, stronger notion of stability.

Definition 5.1. Let v be a proper monotonic simple game, let $x \in \Delta(v)$, and let $S \subseteq N$ be winning. A *deviation* of S is a coalition T such that $x_i(T) > x_i(S)$ for all $i \in S \cap T$ and $x_i(T) > 0$ for each $i \in T \setminus S$. S is called *core stable* if there is no deviation of S.

The difference between individually stability and core stability is crucial. A coalition is individual stable if no player has incentive to leave it. A coalition is core stable if no group of players has incentive to leave it. In particular, core stability implies both individual and internal stability.

The Proportional Solution We already know that there is a stable power configuration with respect to the proportional solution if $\alpha > 0$. In case of apex games such a power configuration exists also for $\alpha = 0$. Moreover this power configuration is even unique for arbitrary α as the following theorem shows.

Theorem 5.2. *The unique $x \in \Delta(a_{iJ})$ which is stable with respect to P is given by*

$$x_i(S) = \begin{cases} \frac{1}{|N|}, & \text{if } S = N, \\ \frac{|J|}{(1+\alpha)|J|+1-\alpha}, & \text{if } |S \cap J| = 1, \\ 1, & \text{if } 2 \leq |S \cap J| \leq |J| - 1 \end{cases}$$
$$x_j(S) = \begin{cases} \frac{1}{|N|}, & \text{if } S = N, \\ \frac{1}{|J|}, & \text{if } S = J, \\ \frac{\alpha|J|+1-\alpha}{(1+\alpha)|J|+1-\alpha}, & S \cap J = \{j\}, \\ 0, & \text{if } 2 \leq |S \cap J| \leq |J| - 1 \end{cases}$$

for all winning coalitions $S \subseteq N$ and all minor players $j \in S$.

Proof. It can easily be shown that x is stable with respect to P. We show that x is the unique stable power configuration. Let therefore $y \in \Delta(v)$ be stable with respect as well. By Lemma 3.1 we have $y_i(S) = 1$ for all winning $S \subseteq N$ with $2 \leq |S \cap J| \leq |J| - 1$. Consequently, $d_i(\{i,j\}, y) = 1$ for all $j \in J$. Hence,

$$y_j(\{i,j\}) = \frac{\alpha + (1-\alpha) y_j(J)}{1 + \alpha + (1-\alpha) y_j(J)} = 1 - \frac{1}{1 + \alpha + (1-\alpha) y_j(J)}$$

We also have that

$$y_j(J) = \frac{\alpha + (1-\alpha) y_j(\{i,j\})}{\alpha |J| + (1-\alpha) \sum_{k \in J} y_k(\{i,k\})}.$$

Let $Y = \sum_{k \in J} y_k(\{i,k\})$. Then we see

$$y_j(J) = \frac{1 - \frac{1-\alpha}{1+\alpha+(1-\alpha)y_j(J)}}{|J| - (1-\alpha) Y}$$

for all $j \in J$. Hence,

$$|J| - (1-\alpha) Y = \frac{2\alpha + (1-\alpha) y_j(J)}{y_j(J)(1 + \alpha + (1-\alpha) y_j(J))}.$$

As the left hand side of this equation does not depend on j, we must have $y_j(J) = y_k(J)$ for all $j, k \in J$. Hence, by efficiency of P, $y_j(J) = \frac{1}{|J|}$. For the remaining coalitions S it can now easily be shown that $x_k(S) = y_k(S)$ for all $k \in S$. ■

We see immediately that a coalition S which contains i and at least two minor players can neither be core stable nor be a deviation of any coalition. The consequence from this observation is the following corollary.

Corollary 5.3. *Let a_{iJ} be an apex game on N and let $x \in \Delta(a_{iJ})$ be stable with respect to P. In the induced hedonic game there is a stable coalition. In particular,*

$$\begin{aligned} J \text{ is core stable} \quad & \text{iff} \quad |J| \leq \sqrt{\tfrac{1}{\alpha}} + 1, \\ \{i, j\} \text{ is core stable for all } j \in J \quad & \text{iff} \quad |J| \geq \sqrt{\tfrac{1}{\alpha}} + 1, \end{aligned}$$

and there are not further core stable coalitions.

The existence of a core stable coalition for each apex game is a very nice feature of the power configuration x. In fact, both for the Shapley-Shubik index and the Banzhaf-Coleman index core stable coalitions do not exist in any apex game (see Karos, 2012)

The Egalitarian Bargaining Solution We already know that there is a stable power configuration with respect to E for each proper monotonic simple game. From the definition of the constrained bargaining solution $\tilde{E}$ it is clear that in a stable power configuration x we have $x_i(S) \leq 1$ for all $S \subseteq N$ and all $i \in S$. The next Lemma shows that this bound holds also for E.

Lemma 5.4. *Let v be a proper monotonic simple game and let $x \in \Delta(v)$ be stable with respect to E. Then $x_i(S) \leq 1$ for all $S \subseteq N$ and all $i \in S$.*

Proof. Assume that there is $S \subseteq N$ and $i \in S$ such that $x_i(S) > 1$ and let $\varepsilon = x_i(S) - 1$. Since

$$\begin{aligned} 1 + \varepsilon &= x_i(S) \\ &= \alpha + (1-\alpha)\, d_i^o(S) + \frac{1}{|S|}\left(1 - |S|\,\alpha - (1-\alpha)\sum_{j \in S} d_j^o(S)\right) \\ &\leq \frac{|S|-1}{|S|}(1-\alpha)\, d_i^o(S) + \frac{1}{|S|} \end{aligned}$$

we find

$$d_i^o(S_1) \geq \frac{1 + \frac{|S|}{|S|-1}\varepsilon}{1-\alpha} \geq 1 + \frac{|S|}{|S|-1}\varepsilon$$

Let $T_1 \in \mathcal{T}_i^S(x)$, i.e. $x_i(T_1) \geq 1 + \frac{|S|}{|S|-1}\varepsilon > x_i(S)$. Then we find for the same reasons as before $d_i^o(T_1) > x_i(T_1)$. Let now $T_{k+1} \in \mathcal{T}_i^{T_k}(x)$ for all $k \geq 1$. With the same arguments we have $x_i(T_{k+1}) > x_i(T_k)$ for all $k \geq 1$. But this is impossible since there is only a finite number of coalitions. Hence, $\varepsilon = 0$, that is $x_i(S_1) \leq 1$. ■

We show now that the stable power configuration is unique.

Theorem 5.5. *Let a_{iJ} be an apex game on N.*

1. *If $|J| = 3$, the unique $x \in \Delta(a_{iJ})$ which is stable with respect to E is given by*

$$\begin{aligned}
x_i(S) &= \begin{cases} \frac{1}{4}, & \text{if } S = N, \\ \frac{1}{2} + \frac{\alpha^2 - 1}{2\alpha^2 - 4\alpha - 4}, & \text{if } |S \cap J| = 1, \\ \frac{1}{3} + \frac{1+\alpha}{2+2\alpha-\alpha^2}, & \text{if } |S \cap J| = 2, \end{cases} \\
x_j(S) &= \begin{cases} \frac{1}{4}, & \text{if } S = N, \\ \frac{1}{3}, & \text{if } S = J, \\ \frac{1}{2} - \frac{\alpha^2 - 1}{2\alpha^2 - 4\alpha - 4}, & \text{if } S \cap J = \{j\}, \\ \frac{2}{3} - \frac{1+\alpha}{2+2\alpha-\alpha^2}, & \text{if } |S \cap J| = 2 \end{cases}
\end{aligned} \tag{3}$$

for all winning coalitions $S \subseteq N$ and all minor players $j \in S$.

2. *If $|J| \geq 4$, the unique $x \in \Delta(a_{iJ})$ which is stable with respect to E is given by*

$$\begin{aligned}
x_i(S) &= \begin{cases} \frac{1}{|N|}, & \text{if } S = N, \\ 1 - \frac{\alpha}{2} - \frac{1-\alpha}{2|J|}, & \text{if } |S \cap J| = 1, \\ 1, & \text{if } 2 \leq |S \cap J| \leq |J| - 2, \\ 1 - \frac{|J|-1}{|J|}\left(\frac{\alpha(1-\alpha)}{2} + \frac{(1-\alpha)^2}{2|J|}\right), & \text{if } |S \cap J| = |J| - 1 \end{cases} \\
x_j(S) &= \begin{cases} \frac{1}{|N|}, & \text{if } S = N, \\ \frac{1}{|J|}, & \text{if } S = J, \\ \frac{\alpha}{2} + \frac{1-\alpha}{2|J|}, & \text{if } |S \cap J| = 1, \\ 0, & \text{if } 2 \leq |S \cap J| \leq |J| - 2, \\ \frac{|J|-1}{|J|}\left(\frac{\alpha(1-\alpha)}{2} - \frac{(1-\alpha)^2}{2|J|}\right), & \text{if } |S \cap J| = |J| - 1 \end{cases}
\end{aligned} \tag{4}$$

for all winning coalitions $S \subseteq N$ and all minor players $j \in S$.

Proof. In both cases it is easy to verify that x is stable with respect to E. We show that in both cases x is unique.

1. Since $x_i(\{i,j\}) \leq 1$ for all $j \in J$, we have

$$\begin{aligned}
x_i(\{i\} \cup J \setminus \{j\}) &= \frac{1}{3} + \frac{2}{3}\alpha + \frac{2}{3}(1-\alpha)\, x_i(\{i,j\}) \\
&\geq x_i(\{i,j\}).
\end{aligned}$$

We show that there is no $j \in J$ such that $x_i(\{i,j\}) > x_i(\{i,j,k\})$ for all $k \in J \setminus \{j\}$. Assume that there is such $j \in J$. Then a stable power configuration must solve the following equation system.

$$\begin{array}{rcl}
x_i(\{i,j\}) & = & \frac{1}{2}+\frac{1-\alpha}{2}x_i(\{i,k,l\})-\frac{1-\alpha}{2}x_j(J)\\
x_j(\{i,j\}) & = & \frac{1}{2}-\frac{1-\alpha}{2}x_i(\{i,k,l\})+\frac{1-\alpha}{2}x_j(J)\\
x_i(\{i,k\}) & = & \frac{1}{2}+\frac{1-\alpha}{2}x_i(\{i,j\})-\frac{1-\alpha}{2}x_k(J)\\
x_k(\{i,k\}) & = & \frac{1}{2}-\frac{1-\alpha}{2}x_i(\{i,j\})+\frac{1-\alpha}{2}x_k(J)\\
x_i(\{i,l\}) & = & \frac{1}{2}+\frac{1-\alpha}{2}x_i(\{i,j\})-\frac{1-\alpha}{2}x_l(J)\\
x_l(\{i,l\}) & = & \frac{1}{2}-\frac{1-\alpha}{2}x_i(\{i,j\})+\frac{1-\alpha}{2}x_l(J)\\
x_i(\{i,j,k\}) & = & \frac{1}{3}+\frac{2\alpha}{3}+\frac{2(1-\alpha)}{3}x_i(\{i,l\})\\
x_i(\{i,j,l\}) & = & \frac{1}{3}+\frac{2\alpha}{3}+\frac{2(1-\alpha)}{3}x_i(\{i,k\})\\
x_i(\{i,k,l\}) & = & \frac{1}{3}+\frac{2\alpha}{3}+\frac{2(1-\alpha)}{3}x_i(\{i,j\})\\
x_j(J) & = & \frac{1}{3}+\frac{2(1-\alpha)}{3}x_j(\{i,j\})-\frac{1-\alpha}{3}x_k(\{i,k\})-\frac{1-\alpha}{3}x_l(\{i,l\})\\
x_k(J) & = & \frac{1}{3}-\frac{1-\alpha}{3}x_j(\{i,j\})+\frac{2(1-\alpha)}{3}x_k(\{i,k\})-\frac{1-\alpha}{3}x_l(\{i,l\})\\
x_l(J) & = & \frac{1}{3}-\frac{1-\alpha}{3}x_j(\{i,j\})-\frac{1-\alpha}{3}x_k(\{i,k\})+\frac{2(1-\alpha)}{3}x_l(\{i,l\})
\end{array}$$

The unique solution of this system delivers

$$\begin{array}{rcl}
x_i(\{i,j\}) & = & \dfrac{2\alpha^4-8\alpha^3-9\alpha^2+22\alpha+11}{2\alpha^4-14\alpha^3+34\alpha+14}\\
x_i(\{i,j,k\}) & = & \dfrac{7\alpha^4+2\alpha^3-6\alpha^2-40\alpha-17}{3\alpha^4-21\alpha^3+51\alpha+21}.
\end{array}$$

It can now be shown that $x_i(\{i,j,k\})>x_i(\{i,j\})$ for all $\alpha\in[0,1]$.

Assume now that there is $j\in J$ such that $x_i(\{i,j,l\})\geq x_i(\{i,j\})>x_i(\{i,j,k\})$ for $j,l\in J\setminus\{j\}$. Then a stable power configuration must solve

$$\begin{array}{rcl}
x_i(\{i,j\}) & = & \frac{1}{2}+\frac{1-\alpha}{2}x_i(\{i,k,l\})-\frac{1-\alpha}{2}x_j(J)\\
x_j(\{i,j\}) & = & \frac{1}{2}-\frac{1-\alpha}{2}x_i(\{i,k,l\})+\frac{1-\alpha}{2}x_j(J)\\
x_i(\{i,k\}) & = & \frac{1}{2}+\frac{1-\alpha}{2}x_i(\{i,j,l\})-\frac{1-\alpha}{2}x_k(J)\\
x_k(\{i,k\}) & = & \frac{1}{2}-\frac{1-\alpha}{2}x_i(\{i,j,l\})+\frac{1-\alpha}{2}x_k(J)\\
x_i(\{i,l\}) & = & \frac{1}{2}+\frac{1-\alpha}{2}x_i(\{i,j\})-\frac{1-\alpha}{2}x_l(J)\\
x_l(\{i,l\}) & = & \frac{1}{2}-\frac{1-\alpha}{2}x_i(\{i,j\})+\frac{1-\alpha}{2}x_l(J)\\
x_i(\{i,j,k\}) & = & \frac{1}{3}+\frac{2\alpha}{3}+\frac{2(1-\alpha)}{3}x_i(\{i,l\})\\
x_i(\{i,j,l\}) & = & \frac{1}{3}+\frac{2\alpha}{3}+\frac{2(1-\alpha)}{3}x_i(\{i,k\})\\
x_i(\{i,k,l\}) & = & \frac{1}{3}+\frac{2\alpha}{3}+\frac{2(1-\alpha)}{3}x_i(\{i,j\})\\
x_j(J) & = & \frac{1}{3}+\frac{2(1-\alpha)}{3}x_j(\{i,j\})-\frac{1-\alpha}{3}x_k(\{i,k\})-\frac{1-\alpha}{3}x_l(\{i,l\})\\
x_k(J) & = & \frac{1}{3}-\frac{1-\alpha}{3}x_j(\{i,j\})+\frac{2(1-\alpha)}{3}x_k(\{i,k\})-\frac{1-\alpha}{3}x_l(\{i,l\})\\
x_l(J) & = & \frac{1}{3}-\frac{1-\alpha}{3}x_j(\{i,j\})-\frac{1-\alpha}{3}x_k(\{i,k\})+\frac{2(1-\alpha)}{3}x_l(\{i,l\}).
\end{array}$$

In this case there is again for each $\alpha\in[0,1]$ a unique solution, in particular we have

$$\begin{array}{rcl}
x_i(\{i,j\}) & = & \dfrac{4\alpha^4-13\alpha^3-6\alpha^2+23\alpha+10}{4\alpha^4-19\alpha^3+3\alpha^2+35\alpha+13}\\
x_i(\{i,j,k\}) & = & \dfrac{8\alpha^4+19\alpha^3-21\alpha^2-83\alpha-31}{12\alpha^4-57\alpha^3+9\alpha^2+105\alpha+39}.
\end{array}$$

It can now be shown that $x_i(\{i,j,k\}) > x_i(\{i,j\})$ for all $\alpha \in [0,1]$.

After we ruled out the previous two possibilities, it must now be the case that $d_i^o(\{i,j\}) = x_i(\{i\} \cup J \setminus \{j\})$ for all $j \in J$. Hence, a stable power configuration must solve

$$\begin{array}{rcl}
x_i(\{i,j\}) & = & \frac{1}{2} + \frac{1-\alpha}{2} x_i(\{i,k,l\}) - \frac{1-\alpha}{2} x_j(J) \\
x_j(\{i,j\}) & = & \frac{1}{2} - \frac{1-\alpha}{2} x_i(\{i,k,l\}) + \frac{1-\alpha}{2} x_j(J) \\
x_i(\{i,k\}) & = & \frac{1}{2} + \frac{1-\alpha}{2} x_i(\{i,j,l\}) - \frac{1-\alpha}{2} x_k(J) \\
x_k(\{i,k\}) & = & \frac{1}{2} - \frac{1-\alpha}{2} x_i(\{i,j,l\}) + \frac{1-\alpha}{2} x_k(J) \\
x_i(\{i,l\}) & = & \frac{1}{2} + \frac{1-\alpha}{2} x_i(\{i,j,k\}) - \frac{1-\alpha}{2} x_l(J) \\
x_l(\{i,l\}) & = & \frac{1}{2} - \frac{1-\alpha}{2} x_i(\{i,j,k\}) + \frac{1-\alpha}{2} x_l(J) \\
x_i(\{i,j,k\}) & = & \frac{1}{3} + \frac{2\alpha}{3} + \frac{2(1-\alpha)}{3} x_i(\{i,l\}) \\
x_i(\{i,j,l\}) & = & \frac{1}{3} + \frac{2\alpha}{3} + \frac{2(1-\alpha)}{3} x_i(\{i,k\}) \\
x_i(\{i,k,l\}) & = & \frac{1}{3} + \frac{2\alpha}{3} + \frac{2(1-\alpha)}{3} x_i(\{i,j\}) \\
x_j(J) & = & \frac{1}{3} + \frac{2(1-\alpha)}{3} x_j(\{i,j\}) - \frac{1-\alpha}{3} x_k(\{i,k\}) - \frac{1-\alpha}{3} x_l(\{i,l\}) \\
x_k(J) & = & \frac{1}{3} - \frac{1-\alpha}{3} x_j(\{i,j\}) + \frac{2(1-\alpha)}{3} x_k(\{i,k\}) - \frac{1-\alpha}{3} x_l(\{i,l\}) \\
x_l(J) & = & \frac{1}{3} - \frac{1-\alpha}{3} x_j(\{i,j\}) - \frac{1-\alpha}{3} x_k(\{i,k\}) + \frac{2(1-\alpha)}{3} x_l(\{i,l\}).
\end{array}$$

We find that in this case the unique solution is given in 3.

2. Let $x \in \Delta(a_{iJ})$ be stable with respect to E.

$$\mathcal{S} = \{S \subseteq N;\ i \in S, |S \cap J| = 2\}.$$

Let $S_1 \in \mathcal{S}$ such that $x_i(S_1) \geq x_i(S)$ for all $S \in \mathcal{S}$ and let $S_2 \in \mathcal{S}$ such that $S_1 \cap S_2 = \{i\}$ and $x_1(S_2) \geq x_1(S)$ for all $S \in \mathcal{S}$ with $S \cap S_1 = \{i\}$. Then $d_j(S_k) = 0$ for $k = 1, 2$ all $j \in S_k \cap J$. Hence,

$$\begin{array}{rcl}
x_i(S_1) & \geq & \alpha + (1-\alpha) x_i(S_2) + \dfrac{1}{3}(1 - \alpha - (1-\alpha) x_i(S_2)) \\
& = & \dfrac{1+2\alpha}{3} + \dfrac{2-2\alpha}{3} x_i(S_2).
\end{array}$$

We also see for the same reasons that $x_i(S_2) \geq \frac{1+2\alpha}{3} + \frac{2-2\alpha}{3} x_i(S_1)$ such that

$$\begin{array}{rcl}
x_i(S_1) & \geq & \dfrac{1+2\alpha}{3} + \dfrac{2-2\alpha}{3}\left(\dfrac{1+2\alpha}{3} + \dfrac{2-2\alpha}{3} x_i(S_1)\right) \\
& = & \dfrac{5+8\alpha-4\alpha^2}{9} + \dfrac{4-8\alpha+4\alpha^2}{9} x_i(S_1).
\end{array}$$

Hence, $x_i(S_1) \geq 1$ and for the same reasons $x_i(S_2) \geq 1$. By Lemma 5.4 we have $x_i(S_1) = 1$.

If S is such that $2 \leq |S \cap J| \leq |J| - 2$ then $d_i(S) = 1$, $d_j(S) = 0$ for all $j \in S \cap J$, and hence, $x_i(S) = 1$ and $x_j(S) = 0$. Similar to the proof of

Theorem 5.2 we show that $x_j(J) = x_k(J)$ for all $j, k \in J$ and conclude $x_j(J) = \frac{1}{|J|}$. Hence, we have

$$x_i(\{i,j\}) = 1 + \frac{1}{2}\left(1 - 1 - \alpha - (1-\alpha)\frac{1}{|J|}\right) = 1 - \frac{\alpha}{2} - \frac{1-\alpha}{2|J|}$$

for all $j \in J$. Hence, $x_j(\{i,j\}) = \frac{\alpha}{2} + \frac{1-\alpha}{2|J|}$. Finally,

$$\begin{aligned} x_i(\{i\} \cup J \setminus \{j\}) &= \frac{1}{|J|} + \frac{|J|-1}{|J|}\alpha + \frac{|J|-1}{|J|}(1-\alpha)\left(1 - \frac{\alpha}{2} - \frac{1-\alpha}{2|J|}\right) \\ &= 1 - (1-\alpha)\frac{|J|-1}{|J|}\left(\frac{\alpha}{2} + \frac{1-\alpha}{2|J|}\right) \end{aligned}$$

and therefore $x_k(\{i\} \cup J \setminus \{j\}) = (1-\alpha)\frac{|J|-1}{|J|}\left(\frac{\alpha}{2} + \frac{1-\alpha}{2|J|}\right)$ for all $k \in J \setminus \{j\}$.

■

With Theorem 5.5 the next corollary is easy to prove.

Corollary 5.6. *Let a_{iJ} be an apex game on $N = \{i\} \cup J$.*

1. *Let $|J| = 3$. Then J is core stable if and only if $\alpha \leq \frac{\sqrt{3}-1}{2}$. In this case J is the only core stable coalition. Further $\{i,j\}$ is core stable for each $j \in J$ if and only if $\alpha = 1$. In this case there are no further core stable coalitions. If $\frac{\sqrt{3}-1}{2} < \alpha < 1$ there are no core stable coalitions.*

2. *Let $|J| \geq 4$. Then J is core stable if and only if $|J| \leq \frac{1+\alpha}{\alpha}$. In this case J is the only core stable coalition. Further $\{i,j\}$ is core stable for each $j \in J$ if and only if $\alpha = 1$. In this case there are no further core stable coalitions. If $\frac{1}{|J|-1} < \alpha < 1$ then there are no core stable coalitions.*

Proof.

1. Note that $x_j(\{i,j\}) \leq \frac{1}{3} = x_j(J)$ if and only if $\alpha \leq \frac{\sqrt{3}-1}{2}$. Since $x_j(\{i,j,k\}) \leq x_j(\{i,j\})$, we have that J is core stable if and only if $\alpha \leq \frac{\sqrt{3}-1}{2}$. In this case each coalition of type $\{i,j,k\}$ is blocked by J and each coalition of type $\{i,j\}$ is blocked by $\{i\} \cup J \setminus \{j\}$.

   If $1 > \alpha > \frac{\sqrt{3}-1}{2}$ then J is blocked by $\{i,j\}$, $\{i,j\}$ is blocked by $\{i\} \cup J \setminus \{j\}$, and $\{i\} \cup J \setminus \{j\}$ is blocked by J.

   If $\alpha = 1$ then J is blocked by $\{i,j\}$ and $\{i\} \cup J \setminus \{j\}$ is blocked by J. However, $\{i,j\}$ is not blocked by $\{i\} \cup J \setminus \{j\}$, since $x_k(\{i\} \cup J \setminus \{j\}) = 0$ for all $k \in J \setminus \{j\}$.

2. Note that $x_j(\{i,j\}) \leq \frac{1}{|J|} = x_j(J)$ if and only if $\alpha \leq \frac{1}{|J|-1}$. Since $x_j(\{i\} \cup J \setminus \{k\}) \leq x_j(\{i,j\})$ for all $k \in J \setminus \{j\}$, we have that J is core stable if and only if $\alpha \leq \frac{1}{|J|-1}$ or equivalently $|J| \leq \frac{1+\alpha}{\alpha}$. In this case

each winning coalition which contains i and at least two minor players is blocked by J and $\{i,j\}$ is blocked by $\{i\} \cup J \setminus \{j\}$.

If $\alpha = 1$ then J is blocked by $\{i,j\}$ and $\{i\} \cup J \setminus \{j\}$ is blocked by J. However, $\{i,j\}$ is not blocked by $\{i\} \cup J \setminus \{j\}$, since $x_k (\{i\} \cup J \setminus \{j\}) = 0$ for all $k \in J \setminus \{j\}$.

If $\frac{1}{|J|-1} < \alpha < 1$ then J is blocked by $\{i,j\}$, $\{i,j\}$ is blocked by $\{i\} \cup J \setminus \{j\}$, and each coalition which contains i and at least two minor players is blocked by J.

■

Remark 5.7. In case of $\tilde{E}$ we have that $d_k (S) \leq 1$ for all winning coalitions S and all $k \in S$. In particular, for each winning coalition S except J we find that $d_k (S) \geq -\frac{1}{|S|} \left(1 - \sum_{l \in S} d_l (S)\right)$ for all $k \in S$. Hence, we have that for $x \in \Delta (v)$ which is stable with respect to $\tilde{E}$ it holds true that $x_k (S) = E_i (S, v (S), d (S))$. With similar arguments as in the proof of Theorem 5.5 it can be shown that $x_j (J) = \frac{1}{|J|}$ for all $j \in J$. Hence, stable payoff configurations with respect to E and $\tilde{E}$ coincide on all apex games.

6 Impossible Coalitions

In many applications of simple games the formation of certain coalitions is impossible. This might be because of legal issues (such as antitrust legislation) or simply because some political parties have so different interests that they cannot work together. So far, we ignored such restriction. However, as the disagreement points of players depend on their outside options, we should guarantee that a player cannot use his hypothetical power in a coalition which will never form.

We say that $\mathcal{R} \subseteq \mathcal{P}$ is a *coalition restriction* if $\{i\} \in \mathcal{R}$ for all $i \in N$. This condition simply says that each player can stay alone, in particular, each player has the outside option to stay alone.

Definition 6.1. Let F be a bargaining solution, $\mathcal{R}$ be a coalition restriction, and v be a simple game. A power configuration $x \in \Delta (v)$ is called *stable with respect to F under restriction* $\mathcal{R}$ if for all winning coalitions $S \in \mathcal{R}$ and all $i \in S$ the following holds.

$$\begin{array}{rcl} x_i (S) & = & F_i (S, v (S), d (S, x)) \\ d_i (S, x) & = & \alpha d_i^m (S) + (1 - \alpha) d_i^o (S, x) \\ d_i^o (S, x) & = & \max_{T \in \mathcal{R}, T \subseteq N \setminus S} x_i (T \cup \{i\}). \end{array} \tag{5}$$

It is easy to show that the proofs in Section 2 hold true for each coalition restriction $\mathcal{R}$. If $\mathcal{R}$ contains at least one winning coalition then we can show that for $\alpha = 0$ and an individual rational and fair bargaining solution F there is a winning coalitions $S \in \mathcal{R}$ which is both individually and internally stable for any stable power configuration x with respect to F under $\mathcal{R}$.

In this section we do not focus on the adaptation of the respective proofs but we will return to our initial Example 1.1. In the following let the parties be enumerated as follows: (1) CDU/CSU, (2) FDP, (3) SPD, (4) Linke, (5) B90/Grne. The political interests of the five parties in the German parliament make it impossible that FDP and Linke, or CDU/CSU and Linke will ever cooperate. Therefore, let

$$\mathcal{R} = \{S \subseteq N;\ \text{if } 4 \in S \text{ then } 1, 2 \notin S\}.$$

It can be shown that a stable power configuration with respect to E under $\mathcal{R}$ must satisfy the equation system

$$\begin{array}{rcl} x_1(\{1,2\}) & = & \frac{1}{2} + \frac{1-\alpha}{2} x_1(\{1,3,5\}) \\ x_1(\{1,3\}) & = & \frac{1}{2} + \frac{1-\alpha}{2} x_1(\{1,2,5\}) \\ x_1(\{1,3,5\}) & = & \frac{1+\alpha}{3} + \frac{2}{3}(1-\alpha) x_1(\{1,2\}) \\ x_1(\{1,2,5\}) & = & \frac{1+\alpha}{3} + \frac{2}{3}(1-\alpha) x_1(\{1,3\}). \end{array}$$

The unique solution of this system is

$$\begin{array}{r} x_1(\{1,2\}) = x_1(\{1,3\}) = \frac{2-\frac{1}{2}\alpha^2}{2+2\alpha-\alpha^2} \\ x_1(\{1,3,5\}) = x_1(\{1,2,5\}) = \frac{2}{2+2\alpha-\alpha^2}. \end{array}$$

We can further calculate

$$x_5(\{1,3,5\}) = x_5(\{1,2,5\}) = \frac{1-2\alpha}{3} - \frac{1}{3} x_i(\{1,3\}) = -\frac{1}{6}\frac{4\alpha + 9\alpha^2 - 4\alpha^2}{2+2\alpha-\alpha^2} < 0.$$

Hence, $\{1,3,5\}$ and $\{1,2,5\}$ are neither internally stable nor individually rational. Finally we have

$$x_2(\{1,2\}) = x_3(\{1,3\}) = 1 - \frac{2-\frac{1}{2}\alpha^2}{2+2\alpha-\alpha^2}$$

and see that the coalition $\{1,2\}$ and $\{1,3\}$ are the only core stable coalitions. This is in line with reality as the current government is exactly coalition $\{1,2\}$.

The power configuration x is not individual rational since $x_5(\{1,3,5\}) < 0$. Hence, in this case E and $\tilde{E}$ do not coincide. We also consider $\tilde{E}$. Let x be stable with respect to $\tilde{E}$ under $\mathcal{R}$. Then $x_i(S) \leq 1$ for all winning $S \in \mathcal{R}$ and all $i \in S$. Therefore we find

$$x_1(\{1,j\}) = \frac{1}{2} + \frac{1-\alpha}{2} d_1(\{1,j\})$$

for $j = 2,3$. We assume first that $d_1^o(\{1,2\}, x) = x_1(\{1,3\})$ and $d_1^o(\{1,3\}, x) = x_1(\{1,2\})$. In this case x must solve

$$\begin{array}{rcl} x_1(\{1,2\}) & = & \frac{1}{2} + \frac{1-\alpha}{2} x_1(\{1,3\}) \\ x_1(\{1,3\}) & = & \frac{1}{2} + \frac{1-\alpha}{2} x_1(\{1,2\}). \end{array}$$

We find that in this case $x_1(S) = \frac{1}{1+\alpha}$ for all winning coalitions $S \in \mathcal{R}$. Further $x_j(\{1,j\}) = x_j(\{1,5,j\}) = \frac{\alpha}{1+\alpha}$ for $j = 2,3$ and $x_5(S) = 0$ for all winning $S \in \mathcal{R}$. We see that x is stable with respect to $\tilde{E}$. Further, if we change the outside options such that for instance $d_1^o(\{1,2\}, x) = x_1(\{1,3,5\})$, x is still the only solution.

We have mentioned before that we do not have evidence how α should be chosen. However, given the fact that the German cabinet consists of 16 ministers of which 11 are member of CDU/CSU, we can at least get an idea of α. For the constrained egalitarian bargaining solution $\tilde{E}$ we find that α must solve

$$\frac{1}{1+\alpha} = \frac{11}{16}$$

which delivers $\alpha \approx 0.455$. For the egalitarian bargaining solution we find $\alpha \approx 0.487$.

7 Conclusion

Coalition formation in simple games contains two parts: The forming of coalitions and the distribution of power within coalitions. We built a model in which these two parts are interdependent, the distribution of power depends on the forming of coalitions and the coalitions formation depends on the distribution of power. We interpreted the distribution problem as a bargaining (or bankruptcy) problem and showed that under very weak conditions on the bargaining solutions we can find a power configuration which is stable with respect to renegotiations.

We pointed out that essentially two things are crucial for the power of a player within a coalition. First, his marginal contribution, as pivotal player will always be more powerful than a player who is not necessary for the surviving of a coalition. Second, and this is the new approach, his outside option. We can also interpret the outside option as opportunity costs: A player who has a chance to be in a very powerful position in a different winning coalition must somehow be convinced not to leave. In the paper we showed several results for specific convex combinations between these two values. However, we do not have any empirical evidence yet, how they should be weighted.

Besides the very natural motivation of this stable power configuration, we showed that it has further useful properties: First of all, it allows to take into account that there might be coalitions which will never form for any external reasons. Second, under some additional conditions it guarantees the existence of a coalition which is both internally and individually stable for each proper monotonic simple game.

We can think of several challenges which can now be targeted: Empirical evidence for the applicability of the model is the first. In particular, it will be interesting to investigate the implicit values of α. Second, the model might be adapted on general transferable utility game. In this case we would interpret the outside option of a player in a coalition as opportunity costs. Particularly, these costs will depend on the partition rather than on a coalition. A transformation of the game in a hedonic version will therefore lead to a *hedonic game with externalities* Bloch and Dutta (2011).

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