COMPRESSED EQUILIBRIUM IN LARGE REPEATED GAMES OF INCOMPLETE INFORMATION

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Abstract. Due to their many applications, large Bayesian games have been a subject of growing interest in game theory and related fields. But to a large extent, models (1) have been restricted to one-shot interaction, (2) are based on an assumption that player types are independent and (3) assume that the number of players is known.

The current paper develops a general theory of Bayesian repeated large games that avoids some of these difficulties. To make the analysis more robust, it develops a concept of compressed equilibrium which is applicable to a general class of Bayesian repeated large anonymous games.

1. Introduction

Studies of large (many players) anonymous games provide general models for the analysis of interaction in economics, political science, computer science, biology and more. Early studies of large games focused on cooperative (non strategic) games, as surveyed in the book of Aumann and Shapley [4]. Our interest here is in the more recent literature that deals with large strategic games.

Indeed, the last three decades produced substantial literature that deals with large strategic games in a variety of applications that include: markets [26], bargaining [22], auctions [28], voting [13], electronic commerce [14], market design [8, 6], and more. One generally-observed phenomenon is that model predictions are more robust when the number of players is large.

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However, the analysis of such games is often subject to three restrictive assumptions. First, models are of one-shot games that exclude studies of repeated interaction. Second, preferences and information are either completely known or are statistically independent across players, again excluding many important applications. Finally, it is generally assumed that the exact number of players is known to the players and other analysts. We postpone to later in this introduction the significant exception of Green [17] and follow up literature, who do study *repeated* complete-information games.

Our current research focusses on a robust theoretical model of large repeated games of incomplete information that overcomes such limitations. The current paper reports on the properties of an equilibrium that emerges out of this model. A companion paper [20] reports on learning and stability properties of this new equilibrium.

1.1. **Approaches to the study of large strategic games.** Our interest is in strategic games that share the following features:

- There is a large but finite number of players, n, which may be only partially known to the players and other game analysts.
- Players are of unknown types that describe their preferences and information. Their types may be statistically correlated through unknown states of nature.
- The players are anonymous: each knows his own parameters, can only observe aggregate data regarding the opponents, and his payoff depends on this observed data and his own individual parameters and choices.
- The game is repeated m times; m may be only partially known to the players and other game analysts.

Models of large games have adopted two type of approaches.

The *continuum approach* was initiated in Schmeidler [30], who studied the set of equilibria of a game with a continuum of players, $\mathcal{E}^{\infty}$. Schmeidler's main result is that one-shot continuum games with complete information have a pure strategy equilibrium.[1] The more

[1]See Khan and Sun [2] for a (somewhat outdated) survey of the substantial follow-up literature

recent *asymptotic approach,* on the other hand, studies properties of the sets of equilibria of the n-person games, $\mathcal{E}^n$, as n becomes infinitely large. Kalai [19] studies large one-shot games with incomplete information and and shows, under an assumption of type independence, that the equilibria of such games satisfy strong robustness properties. Additional recent contributions to this growing literature include Cartwright and Wooders [11], Gradwohl and Reingold [16, 15], Azrieli and Shmaya [7], Carmona and Podczeck [10], and Azevedo and Budish [6].

In order to connect the two approaches above, recent papers study the relationship of the $lim_{n\to\infty}\mathcal{E}^n$ to $\mathcal{E}^\infty$, see Carmona and Podczecky's paper [10] and the reference there.

While the approaches above are useful for the study of large games, they fall short of describing the family of large games described above due to the lack of dynamics and the restricted informational assumptions. They are also deficient in the following respects: (1) The set of equilibria, $\mathcal{E}^n$, may fail to provide a proper description of the outcomes of the n peron game when the number of players, n, is unknown. (2) If $\mathcal{E}^n$ does not provide a proper description of the outcomes of the n-person game, then the comparisons of $lim_{n\to\infty}\mathcal{E}^n$ with $\mathcal{E}^\infty$ may be meaningless. (3) Without this comparison it is not clear that $\mathcal{E}^\infty$ provides a proper approximation for the outcome of the finite large game.

Indeed, lack of knowledge of the number of players is a difficult issue in game theory. This was pointed out by Myerson [24, 25] who viewed it as an issue of incomplete information, and offered a clever solution for the special case in which the number of players is randomly generated by a Poisson distribution.

The continuum approach suggests an alternative way of dealing with lack of knowledge of the number of players n, by simply disallowing equilibria that depend on n. For example, in a driving game where players choose between driving on the $right$ or on the $left$, all-drive-on-the-$right$ and all-drive-on-the-$left$ are equilibria of the continuum game and of all the finite n player games. But equilibrium selections that depend on the number of players n, for example all-drive-on-the-$right$ when n is even and all-drive-on-the-$left$ when it is *odd*, cannot be recognized in the continuum game.

The idea of restricting players to equilibria that do not depend on n is appealing, but the continuum approach suffers from the conceptual difficulties already mentioned, and additional technical difficulties. For one, unlike a standard assumption of game theory that players behavior is decentralized, in the a continuum game players *joint* behavior must be mathematically measurable. Also, it is difficult to perform statistical analysis of a continuum of random variables as needed when studying mixed strategies and even pure stratagies in Bayesian games. In fact, there are conceptual difficulties in even defining the notion of mixed strategies in the continuum setup.

The equilibria studied in this paper also disallow behavior that depends on the number of players. But keeping the number players finite, it avoids the conceptual and techincal difficulties above. Before describing the equilibrium notion, we introduce a model of a game which does not include the number of players and periods as fixed primitives.

1.2. **Anonymous games of proportions.** We study anonymous symmetric games with "abstract" number of players and "abstract" number of repetitions. The basic model starts with a fixed "skeleton" game that specifies the characteristics of an individual player (actions, types, etc.) and his payoff dependency on the *proportions* of types and actions of his opponents. The skeleton game is then augmented, to yield a family of standard symmetric repeated games, by specifying pairs of positive integers n and m that represent the number of players in the game and the number of times the game is repeated.

One advantages of this model is in the ease of performing asymptotic analysis, as one can vary the number of players and repetitions while holding "everything else" fixed. It also permits a "skeleton" definition of repeated-game individual strategies that do not depend on the number of opponents and repetitions. Such strategies enable the construction of compressed equilibrium, described below, that does not depend on the number of players and repetitions.

The concluding section of the paper shows how the assumption of player symmetry may be relaxed.

1.3. **Compressed optimization.** The *compressed* view combines ideas from both the asymptotic and the continuum approaches. Briefly stated, a payoff maximizing player facing a large number of conditionally-independent opponents may use *compressed probability assesments* for events in the game by making the following simplifying assumptions: (1) Similar to the price taking assumption in economics, his actions do not affect the general outcomes of the game. In addition (2), the player replaces probability distributions about the opponents' random empirical distribution of types and actions by their expected values. Under *compressed optimization* the player chooses strategies that maximize his expected payoffs realtive to his compressed probability assesments.

While the above view ecompasses modeling benefits of the continuum model, it bypasses the technical and conceptual difficulties discussed above. Some of the gains are illustrated by the following example.

Example 1.1. Optimal Charity Contribution. Each of $n+1$ individuals, $i = 0, 1, ..., n$, is about to make a contribution to a certain charity at possible dollar levels $x^i = 0, 1, 2, ..., d$. Wishing to be at the 0.9 quantile of all contributors but not higher, Pl.0's VM utility function is $u(x^0, \mathbf{e}) = 1 - (\widehat{\mathbf{e}} \langle x^0 \rangle - 0.9)^2$, where $\mathbf{e}$ describes the empirical distribution of opponent's contributions and $\widehat{\mathbf{e}}$ is its cummulative distribution, i.e., $\widehat{\mathbf{e}} \langle x^0 \rangle$ is the proportion of opponents who contribute x^0 or less.

If Pl.0 knew $\mathbf{e}$, he would choose x^0 to be a 0.9 quantile of it. But not knowing $\mathbf{e}$, he assumes that there is a vector of independent probability distributions $\overline{f} = (f^1, ..., f^n)$ that describes the probabilities of individual contributions, i.e., $f^i(x^i)$ is the probability that contributor i contributes \$$x^i$. To choose his contribution optimally, Pl.0 should select an x^0 that maximizes:

$$\sum_{\mathbf{e_n}} u(x^0, \mathbf{e_n}) \mathcal{P}_{\overline{f},n}(\mathbf{e_n}) = \sum_{\mathbf{e_n}} [1 - (\widehat{\mathbf{e}}_n \langle x^0 \rangle - 0.9)^2] \mathcal{P}_{\overline{f},n}(\mathbf{e_n}).$$

The sum is taken over all possible empirical distributions of contributions $\mathbf{e_n}$ that can be generated by n contributors, and $\mathcal{P}_{\overline{f},n}(\mathbf{e_n})$ is the computed probabilities of obtaining the empirical distributions $\mathbf{e_n}$ when the individual contributions are drawn according to $\overline{f}$.

Under compressed optimization, on the other hand, Pl.0 compresses the probabilities of contributions in $\overline{f}$ to their average, $\kappa_{\overline{f}}(x) \equiv \frac{1}{n}\sum_i f^i(x)$, to obtain the expected proportion of opponents who contribute at every level x. For the cumulative distribution $\widehat{\kappa}_{\overline{f}}$ he selects an x^0 that maximize:

$$u(x^0, \kappa_{\overline{f}}) = 1 - (\widehat{\kappa}_{\overline{f}}\left\langle x^0 \right\rangle - 0.9)^2$$

In other words, he simply chooses x^0 to be a 0.9 quantile of $\kappa_{\overline{f}}$.

As discussed in this paper, under sufficient conditions the simplification above is valid, i.e., it is assymptotically optimal for large n's. But there may be other uncertainties that connot be eliminated by compression to expected values. For example, suppose Pl.0 believes that there are two possible profiles of contribution probabilities, $\overline{f}_1$ and $\overline{f}_2$; and that based on some unknon state of nature his opponents make their choices accoding to the profile $\overline{f}_1$, with probability θ_1, or according to the profile $\overline{f}_2$, with probability $\theta_2 = 1 - \theta_1$.

Now, the optimal contribution of Pl.0 is any x^0 that maximizes:

$$\theta_1 \sum_{\mathbf{e_n}} u(x^0, \mathbf{e_n})\mathcal{P}_{\overline{f}_1,n}(\mathbf{e_n}) + \theta_2 \sum_{\mathbf{e_n}} u(x^0, \mathbf{e_n})\mathcal{P}_{\overline{f}_2,n}(\mathbf{e_n}),$$

and his *compressed* optimal choice is any x_0 that maximizes:

$$\theta_1 u(x^0, \kappa_{\overline{f}_1}) + \theta_2 u(x^0, \kappa_{\overline{f}_2}).$$

In other words, he still replaces the random empirical distributions by their (conditional) expected values, but he does not replace the unkown θ's by their expected value.

1.4. **Compressed equilibrium.** An assumption that all the players perform compressed optimization leads to a well defined and easier to compute notion of *compressed equilibrium.* The gained simplification from the substitutions of expected values is especially important for *repeated* games. Here, in addition to simplifying the assessments of payoffs, the players and analysts simplify the process of updating beliefs, as they transit from one stage of the game to the next.

These transitions are one of the issues that make the study of repeated large games more challenging than studies of one-shot large games. In one-shot models it is sufficient to assume that the payoff functions are continuous as the number of players increases. But in repeated interaction, in which the outcome of a stage is used by the players to determine their next-stage actions, it is important that the outcome observed at the end of each stage is continuous as the number of players increases. For this reason, the model presented in this paper assumes that there is smoothing of publicly-observed period outcomes in a form of noise or other random variables. This idea is common in game theory, tracing back at least to Green's paper [17].

1.4.1. *Myopicity.* Another property that carries over from Green's model to the current paper is that compressed equilibria are *myopic*. Specifically, in checking whether his strategy is optimal, it suffices for a player to check it only on a period-by-period basis: Are the actions he chooses for any given period optimal for that period (based on the information he has going into it and ignoring his payoffs in future periods)? Clearly, myopicity offers a signifcant simplification in the analysis of equilibrium.

1.4.2. *Uniform properties of compressed equilibria.* Under the assumptions that the number of players or repetitions in the game is unknown, it is important to have strategic concepts that do not depend on these parameters. Indeed, in the family of games studied in this paper, compressed equilibria posses uniformity properties in the number of players and in the number of repetitions.

With regards to the number of players, a compressed equilibrium in a game with n players is a compressed equilibrium of the same game with any number of players (even though it is a good approximation to real equilibrium only when n is large).

With regards to the number of repetitions, a strategy profile is a compressed equilibrium in a game repeated m times, if and only if it is a compressed equilibrium of the game repeated $m' \leq m$ times.

Combining and extending the above, we show the existence of compressed equilibria that are uniform in both, the number of players and the number of repetitions of the game.

1.4.3. *Validation of the compressed view.* Are compressed concepts useful for the analysis of repeated large anonymous games? The answer is positive in the following sense. First, a main theorem in this paper shows that assessed compressed probabilities of play paths of the game become uniformly accurate as the number of players increases. This means that compressed best-response analysis is essentially valid for all sufficietly large n's: If strategy g is better than strategy h in the compressed sense, then it is essentially better in the real sense, and uniformly so for all sufficiently large values of n. Among several implications, every compressed equilibrium of the repeated game is an approximate equilibium for all the versions of the game with sufficiently many players.

1.4.4. *Universality of compressed equilibria.* The validation argument above provides a strong rationale for playing a compressed equilibrium in a fixed repeated game when a player knows that the number of opponents is sufficiently large. However, for such strong rationale to hold, the needed "sufficiently large" number of players must increase as the number of repetions of the game increases.

The reason for this monotonicty stems from competing implications of laws of large numbers. On the positive side, in every stage of the game laws of large numbers permit the player to replace the aggregate play of the opponents by their expected values. But no matter how many opponents the player has, there may still be a small probability that this substitution fail. This means that for any fixed number of opponents, as the game is made longer the probability that the approximation fail in at least one of the periods may increase to one.

To overcome the above difficulty, one may adopt different approaches. One possiblity is to try to weaken the notion of correct approximation. For example, it may be sufficient in applications to have correct approximations in a high proportion of periods, rather than in every period.

In this paper, however, we keep the criterion of sure correct approximation for all periods of play, and make up for it by requiring more precise period approximation. In particual, this is obtained if for larger number of repetitions we restrict our attention to games with larger number of players, as described below.

Consider a family of games G, with various values for the number of players n and for the number of repetitions m. A profile of strataegies κ is said to be (asymptotically) *universal equilibrium for the family* G, if for every ε there is a critical number of players N_ε such that in any game in G with more than N_ε players, all the players are epsilon best responding (in the standard sense) by using their κ strategies. In other words, when playing a universal equilibrium, a player is assured that he is epsilon optimizing uniformly for all sufficiently large n's, no matter which game in the family is played.

We consider families of games in which $n > m^{2+\delta}$ for arbitrarily small positive δ's. For any such family we show that all the compressed equilibria are universal in the sense above.

1.5. **Relationship to Green** [17]**.** Green studies pure strategies in anonymous, repeated, complete-information games. He formalizes the idea of price taking in market games, and introduced a general framework of repeated games with random outcome. Green and Sabourian [29] derive conditions under which the Nash Correspondence is continuous, i.e., that equilibrium profile in the non-atomic game is a limit of equilibrium profiles in the games with increasing number of players. Advances of the current paper over Green's include the following features: incomplete information, mixed strategies, lack of knowledge of the number of opponents and repetitions; and avoidence of technical dificulties of the continuum.

2. The Model, Notations, and Conventions

2.1. **Games and strategies.** Throughout the paper we restrict ourselves to a fixed game-skeleton Γ, used to construct repeated games $\Gamma^{m,n}$ that vary in the number of players n and the number of repetitions m. The game skeleton and the constructed games are all (ex-ante) symmetric in the players.

Definition 2.1. [game] A *game skeleton* is an eight tuple of components $\Gamma = (S, \theta, T, \tau, A, X, \chi, u)$ with the following interpretation:

- S is a finite set that contains the possible *states of nature*; $\theta \in \Delta(S)$ is a *prior* probability distribution over S.[2]
- T is a finite set of possible player's *types*; $\tau : S \to \Delta(T)$ is a stochastic *type-generating function*.
- A is a finite set of possible player's *actions*.
- X is a Borel space of *outcomes*, and $\chi : S \times \Delta(T \times A) \to \Delta(X)$ is a Borel function that generates probability distributions over outcomes.
- $u : T \times A \times X \to [0, 1]$ is a Borel function that describes player's *payoffs*.

The game $\Gamma^{m,n}$ has a set of n players, $N = \{0, 1, ..., n-1\}$, and is played repeatedly in m stages as follows: First a state of nature $s \in S$ is randomly chosen according to θ. Then every player is informed of his type, which is drawn randomly and independently of the other players according to the distribution τ_s. Continuing inductively, at every stage $k = 0, 1, \ldots, m-1$ every player chooses an action from A, and a public outcome x_k is randomly chosen according to χ_{s,e_k} , where $e_k \in \Delta(T \times A)$ is the realized empirical distribution of players type and period actions:

$$e_k[t, a] = (\text{the number of players of type } t \text{ who chose the action } a \text{ at day } k)/n.$$

All the players are informed of the realized outcome x_k before proceeding to the next stage.

We use the notations $\Gamma^{m,\cdot}$,$\Gamma^{\cdot,n}$,$\Gamma^{\cdot,\cdot}$, to denote respectively games with arbitrary number of players, with arbitrary number of repetitions and with arbitrary number of both.

[2]As usual, $\Delta(B)$ is the set of all probability distributions over the set B.

This framework extends the model in the papers of Green and Sabourian in several ways, among which, it allows for incomplete information and for the use of mixed strategies as described next.

A repeated-game strategy for $\Gamma^{\cdot,\cdot}$ (or a *strategy* for short) is a function $f : T \times (A \times X)^{<\mathbb{N}} \to \Delta(A)$.[3] It describes the probability that a player chooses his action based on his type and every history of his own individual actions and observed outcomes. More specifically, $f(t, a_0, x_0, \ldots, a_{k-1}, x_{k-1})$ is the player's k-th period mixed action when he is of type t, and in the previous periods he took the actions $a_0, ...a_{k-1}$ and observed the outcomes $x_0, \ldots, x_{k-1}$.

Notice that as defined, a strategy f may be used by any player in any game with arbitrary number of opponents and arbitrary number of repetitions.

A *reactive strategy* is one in which the player does not condition on his own past actions, i.e., $f : T \times X^{<\mathbb{N}} \to \Delta(A)$. [4] Reactive strategies were introduced in Kalai and Stanford [18] for repeated games with complete information.

2.2. **Nash Equilibrium.** To define such an equilibrium profile $\bar{f} = (f^0, \ldots, f^{n-1})$, it suffices to define the payoffs of a player under unilateral or no deviations from it as we do below.

However, to simplify notations below and later in the paper, we write the definitions only for one representative player, player 0, with the understanding that the same definitions apply to all the players. Players $1, \ldots, n-1$ are referred to as player 0's 'opponents'.

Also below and later in the paper, we use bold face letters to denote random variables that assume values from corresponding sets. For example, $\mathbf{S}$ is the random variable that describes a randomly-selected state from the set of possible states S. We use superscripts for players' names and subscripts for period numbers.

Let $\bar{f} = (f^0, \ldots, f^{n-1})$ be a reactive strategy profile. Assume that player 0 considers playing a (not necessarily reactive) strategy g and the opponents follow the profile $\bar{f}$. A

[3] For any set B, $B^{<\mathbb{N}} = \bigcup_{m=0}^{\infty} B^m$.

[4] In much of what follows, the restriction to reactive strategies can be eliminated at the cost of using more complex notations.

random $(g, \bar{f})$-*play* in the n-player game is a collection $\left(\mathbf{S}, \mathbf{T}^i, \mathbf{A}^i_k, \mathbf{X}_k\right)_{i \in N, k=0,1,\ldots}$ of random variables, representing the state of nature, types, actions and outcomes, such that:

- The state of nature $\mathbf{S}$ is distributed according to θ.
- Conditional on $\mathbf{S}$, the players types $\mathbf{T}^i$ are i.i.d with the distribution $\tau_{\mathbf{S}}$.
- Conditional on the history of stages $0, \ldots, k-1$, players choose period k actions $\mathbf{A}^i_k$ independently. Player 0 uses the distribution $g(\mathbf{T}^0, \mathbf{A}^0_0, \mathbf{X}_0, \ldots, \mathbf{A}^0_{k-1}, \mathbf{X}_{k-1})$ and each of his opponents $i \in N \setminus \{0\}$ uses the distribution $f^i(\mathbf{T}^i, \mathbf{X}_0, \ldots, \mathbf{X}_{k-1})$.
- The outcome $\mathbf{X}_k$ of day k is drawn randomly according to $\chi_{\mathbf{S}, \mathbf{d}_k}$, where $\mathbf{d}_k$ is the (random) empirical type-action distribution at stage k given by

$$\mathbf{d}_k[t, a] = \#\{i \in N | \mathbf{T}^i = t, \mathbf{A}^i_k = a\}/n. \tag{1}$$

In equations,

$$\begin{aligned}
&\mathbb{P}\left(\mathbf{S} = s, \mathbf{T}^i = t^i \ i \in N\right) = \theta[s] \cdot \prod_{i \in N} \tau_s[t^i]. \\
&\mathbb{P}\left(\mathbf{A}^i_k = a^i \ i \in N \,\middle|\, \mathbf{S}, \mathbf{T}^i, \mathbf{A}^i_l, \mathbf{X}_l \ l < k, i \in N\right) = \\
&\qquad g(\mathbf{T}^0, \mathbf{A}^0_0, \mathbf{X}_0, \ldots, \mathbf{A}^0_{k-1}, \mathbf{X}_{k-1})[a^0] \cdot \prod_{i \in N \setminus \{0\}} f^i\left(\mathbf{T}^i, \mathbf{X}^0, \ldots, \mathbf{X}_{k-1}\right)[a^i] \\
&\mathbb{P}\left(\mathbf{X}_k \in \cdot \,\middle|\, \mathbf{S}, \mathbf{T}^i, \mathbf{A}^i_l \ l \le k, i \in N, \mathbf{X}_0, \ldots, \mathbf{X}_{k-1}\right) = \chi_{\mathbf{S}, \mathbf{d}_k}.
\end{aligned} \tag{2}$$

Define

$$U^{g, \bar{f}}_{m,n} = \mathbb{E} \sum_{k=0}^{m-1} u\left(\mathbf{T}^0, \mathbf{A}^0_k, \mathbf{X}_k\right) \tag{3}$$

to be player 0's payoff in $\Gamma^{m,n}$.

A reactive profile $\bar{f}$ is an ϵ-equilibrium ($\epsilon \ge 0$) in $\Gamma^{m,n}$, if $U^{f^0, \bar{f}}_{m,n} + \epsilon \ge U^{g, \bar{f}}_{m,n}$ for every strategy g, and with the same restriction applied to all the other players.

3. The Compressed View of a Game

The compressed view of a game $\Gamma^{m,n}$ is a simplified way by which players and other analysts may assess the probabilities of plays and the associated payoffs. In addition to their simplicity, concepts used under this approach are uniform in the number of players and the number of repetitons of the game. For example, as discussed later in this section, there is a repeated game strataegy κ that is a compressed equilibrium of all the games $\Gamma^{m,n}$ with any values of m and n.

In the sections that follow, we argue that the compressed view is effective for the purpose of analysing large games: As the number of players increases, compressed probabilities and payoffs become accurate in approximating their real counter parts. This means that for large games compressed best-response and compressed equilibrium are valid concepts in the analysis of how to play a game. Moreover, combined with their uniform applicability described in the previous paragraph, they prsent a robust tool for the analysis of large repeated games, especially when the number of players or the number of repetitions is not fully known.

As discussed later in the paper, the compressed view exhibits natural connections to models of non-atomic games with a continuum of players of the type studied by Schemeidler [30] for one-shot games with complete information. But a major difference is that the compressed view applies directly to n-person games, without being sidetracked to imaginary games with a continuum of players. In doing so, it avoids mathematical difficulties associated with the continuum, e.g., measurability conditions and continuum of independent random variables.

Described by the formal definitions below, the compressed view adopts the following simplifying assumptions about each player in the game: (1) In spirit similar to price-taking behavior in economic models, the player's actions have negligable effect on the probabilities of events related to current and future outcomes of the game. (2) In computing probabilities of events in the game, the player replaces random empirical distributions of types and actions with their expectations, while leaving the rest of the analysis unchanged.

3.1. **compressed strategies.**

Definition 3.1. For a profile of reactive strategies $\overline{f} = (f_1, ..., f_n)$ in the n-person m-times repeated game. The compressed strategy of $\overline{f}$ is defined by: $\kappa_{\overline{f}}(t, x_0, \ldots, x_{k-1}) = \frac{1}{n} \sum_{i=0}^{n-1} f^i(t, x_0, \ldots, x_{k-1})$.

The following direct observations are useful in the interpretation and forthcoming discussion of compressed strategies and equilibrium:

- Being the average of the probabilities of taking an action a, $\kappa_{\overline{f}}(t, x_0, \ldots, x_{k-1})[a]$ is the *expected proportion of* a choosers among the players of type t after they observe the outcomes $x_0, \ldots, x_{k-1}$. Alternatively, it is the probability that a randomly chosen player from this group would choose a.
- $\kappa_{\overline{f}}$ itself is a reactive strategy.

Definition 3.2. For a reactive strategy g define the n-person version of g by $\overline{g^n} = (g, ..., g)$.

- Every reactive strategy g is a compressed strategy. In particular, for any number of players n, $g = \kappa_{\overline{g^n}}$.
- Since strategies in our model are defined unconditionally on the number of players and repetitions, any reactive strategy g is a compressed strategy in all the games $\Gamma^{\cdot,\cdot}$.
- For any reactive/compressed strategy κ we may think of the class of strategy profiles $[\kappa]$ which are equivalent to each other in the sense that they all compress to κ. Statements in the sequel that refer to κ may refer to all the profiles in $[\kappa]$.
- Despite the fact that a reactive strategy and a compressed strategy are the same mathematical object, in the discussion that follows we selectively use one term over the other to emphasize the relevant interpretations in different contexts.

3.2. **Compressed probabilities and payoffs.** We consider a game in which the players use reactive strategies that compress to the strategy κ, and a representative player, player 0, who considers the use of any (reactive or not) strategy g. As before, we denote the state of nature by the random variable $\mathbf{S}$, player 0's type and period actions by $\mathbf{T}^0$ and by $\mathbf{A}_0^0, \mathbf{A}_1^0, \ldots$, and the period outcomes by the variables $\mathbf{X}_0, \mathbf{X}_1, \ldots$.

Definition 3.3. For κ and g as above, we call the sequence of random variables $\mathbf{S}, \mathbf{T}^0, \mathbf{A}_0, \mathbf{X}_0, \mathbf{A}_1, \mathbf{X}_1, \ldots$, with the *compressed probability distribution* defined inductively below, the random *compressed* (g, κ)*-play.*

$$\mathbb{P}\left(\mathbf{S} = s, \mathbf{T}^0 = t\right) = \theta[s] \cdot \tau_s[t] \ ,$$

$$\mathbb{P}\left(\mathbf{A}_k^0 = a \,\middle|\, \mathbf{S}, \mathbf{T}^0, \mathbf{A}_0^0, \mathbf{X}_0, \ldots, \mathbf{A}_{k-1}^0, \mathbf{X}_{k-1}\right) = g\left(\mathbf{T}^0, \mathbf{A}_0^0, \mathbf{X}_0, \ldots, \mathbf{A}_{k-1}^0, \mathbf{X}_{k-1}\right)[a] \ , \tag{4}$$

$$\mathbb{P}\left(\mathbf{X}_k \in \cdot \,\middle|\, \mathbf{S}, \mathbf{T}^0, \mathbf{A}_0^0, \mathbf{X}_0, \ldots, \mathbf{A}_{k-1}^0, \mathbf{X}_{k-1}, \mathbf{A}_k^0\right) = \chi_{\mathbf{S}, \mathbf{e}_k}(\cdot) \ ,$$

where $\mathbf{e}_k \in \Delta(T \times A)$ is the random empirical distribution of type and actions in period k:

$$\mathbf{e}_k[t, a] = \tau_{\mathbf{S}}[t] \kappa(t, \mathbf{X}_0, \ldots, \mathbf{X}_{k-1})[a].\text{[5]} \tag{5}$$

Remark 3.4. The notion of compressed play expressed in (5) captures the two notions of simplification mentioned above.

First, since g is not included in the expressions $\mathbf{e}_k[t, a]$, Player 0 is a *stochastic outcome taker*: In addition to not influencing the state of nature $\mathbf{S}$, his actions do not affect the probabilities of $\mathbf{X}_0, \mathbf{X}_1, \ldots$. This is a generalization of the economic price-taking property in Green's (1978) paper to a *stochastic* setup. Moreover, the application of the concept is not restricted to market games.

Second, the expression above significantly simplifies the stochastic process by replacing random empirical distributions with their expected values. In particular, the expressions for the $\mathbf{e}_k[t, a]$s are based on the assumption that the random proportion of players with a type action pair $[t, a]$ must equal its theoretical probabilities given by τ and κ.

[5]Notice that the random variables that represent the state of nature, player 0 type, and actions and the outcomes are distributed as in an n-player random (g, κ)-play. Only the random frequencies of realized opponents' types and the random frequencies of the actions chosen by the opponents in each period k are replaced by their determinstic expected values, i.e., the theoretical distributions $\tau_{\mathbf{S}}$ and $\kappa(t, \mathbf{X}_0, \ldots, \mathbf{X}_{k-1})$.

Definition 3.5. For κ and g as above, define player 0's *compressed payoff* in the m-stage game $\Gamma^{m,\cdot}$ by

(6)
$$U_{m,\mathcal{COMP}}^{g,\kappa} = \mathbb{E} \sum_{k=0}^{m-1} u\left(\mathbf{T}^0, \mathbf{A}_k^0, \mathbf{X}_k\right),$$

where $\mathbf{S}, \mathbf{T}^0, \mathbf{A}_0, \mathbf{X}_0, \mathbf{A}_1, \mathbf{X}_1, \ldots$ is a compressed (g, κ)-play.

The definition above is naturally extended to profiles of strategies $\bar{f}$ by substituting $\kappa_{\bar{f}}$ for κ.

3.3. **Compressed best response and equilibrium.**

Definition 3.6. Let κ be a compressed strategy. We say that a strategy g is a compressed best response to κ if for all strategies h, $U_{m,\mathcal{COMP}}^{g,\kappa} \geq U_{m,\mathcal{COMP}}^{h,\kappa}$.

Since players are outcome takers and since in our model the payoff function is separable, it follows that a compressed best rensponse is always myopic, so player 0 doesn't have to include the length of the game m and the history of the play in his period considerations. Thus, we get the following equivalent characterization of compressed best-response strategy:

Proposition 3.7. *Let κ be a compressed strategy. A strategy g is compressed best response to κ if and only if for every type t and for every sequences of outcomes $x_0, \ldots, x_{k-1}$ and actions $a_0, \ldots, a_{k-1}$ (with $k < m$) one has*

(7)
$$[g\left(t, a_0, x_0, \ldots, a_{k-1}, x_{k-1}\right)] \subseteq argmax_{a \in A} \mathbb{E}\left(u(t, a, \mathbf{X}_k \,\middle|\, \mathbf{T}^0 = t, \mathbf{X}_0 = x_0, \ldots, \mathbf{X}_{k-1} = x_{k-1})\right),$$

where $\mathbf{S}, \mathbf{T}^0, \mathbf{X}_0, \ldots, \mathbf{X}_{k-1}$ is the compressed κ-play and $[g\left(t, a_0, x_0, \ldots, a_{k-1}, x_{k-1}\right)]$ is the support of $g\left(t, a_0, x_0, \ldots, a_{k-1}, x_{k-1}\right)$.

In particular, an outcome-taking player always have reactive best-response strategies and there is no significant loss in restricting the definition of compressed equilibrium to reacctive strategies.

Definition 3.8. A reactive/compressed strategy κ is a *compressed equilibrium* of the m-stage repeated game $\Gamma^{m,\cdot}$, if κ is compressed best response to itself, i.e., $U^{\kappa}_{m,\mathcal{COMP}} \equiv U^{\kappa,\kappa}_{m,\mathcal{COMP}} \geq U^{g,\kappa}_{m,\mathcal{COMP}}$ for every strategy g.

Notice that the notions of compressed payoffs, compressed best response, and compressed equilibrium do not depend on the number of players. In particular, in order to play compressed-optimally a player does not have to know the number of opponents and their individual strategies. All he has to know is the aggregate data expressed by the compressed strategy.

The following equivalent characterization of compressed equilibrium follows from Proposition 3.7.

Corollary 3.9. *A reactive strategy κ is compressed equilibrium of $\Gamma^{m,\cdot}$ if and only if for every type t and every sequence of outcomes $x_0, \ldots, x_{k-1}$ (with $k < m$) one has*

$$(8) \qquad [\kappa\,(t, x_0, \ldots, x_{k-1})] \subseteq \mathit{argmax}_{a\in A}\mathbb{E}\left(u(t, a, \mathbf{X}_k \,\middle|\, \mathbf{T}^0 = t, \mathbf{X}_0 = x_0, \ldots, \mathbf{X}_{k-1} = x_{k-1})\right),$$

where $\mathbf{S}, \mathbf{T}^0, \mathbf{X}_0, \ldots, \mathbf{X}_{k-1}$ is the compressed κ-play and $[\kappa\,(t, x_0, \ldots, x_{k-1})]$ is the support of $\kappa\,(t, x_0, \ldots, x_{k-1})$

Corollary 3.9 is analogous of Theorem 5 in Green's paper [17], but it is extended for the case of incomplete information: at every stage the player's plays an equilibrium in the corresponding one-shot game with incomplete information. Green [17], Sabourian [29] and Al-Najjar and Smorodinsky [1] prove, under various condition, that this property (which Al-Najjar and smorodinsky call *myopic play)* holds approximately for every equilibrium of the repeated game. For our purpose, it is enough to prove the myopic property in the compressed view of the game, which is significantly simpler.

Again, the definitions above, applicable to a single compressed strategy κ, are applicable to any profile $\bar{f}$ that compresses to κ. In particular, we have the following explicit definition.

Definition 3.10. A profile of strategies $\overline{f} = (f_1, ..., f_n)$ is a compressed equilibrium if its compression $\kappa_{\overline{f}}$ is a compressed equilibrium.

Remark 3.11. Note that for every reactive profile $\overline{f} = (f_1, \ldots, f_n)$ one has $[\kappa_{\overline{f}}(t, x_0, \ldots, x_{k-1})] = \bigcup_i [f_i(t, x_0, \ldots, x_{k-1})]$. It follows from Proposition 3.7 that $\overline{f} = (f_1, \ldots, f_n)$ is compressed equilibrium if and only if each f_i is a best response to the compressed strategy $\kappa_{\overline{f}}$. And even more explicitly, to play optimally a player needs only to check that any action he intends to take in every period is myopically optimal relative to $\kappa_{\overline{f}}$.

Definition 3.12. A compressed equilibrium κ is *uniform in the number of periods*, if it is a compressed equilibrium of $\Gamma^{m,\cdot}$ for every $m \geq 0$.

This type of uniformity is reminiscent of the notion of uniform equilibrium, studied by Aumann and Maschler [5] for games with a high but unknown discount parameter. However, the uniformity presented here is stronger since there is no restriction to large m's to parallel their restriction to high discount paramenters.

Proposition 3.13. *There exists a compressed equilibrium* uniform *in both, the number of players and the number of periods .*

Before proceeding with important properties of the compressed model, we compare it to an older model of large games.

4. The compressed view vs. continuum model

To highlight the relationship between the compressed view and the continuum model, it is sufficient to consider, as the continuum model literature generally does, a one-shot game with complete information.

Schmeidler [30] starts with a measureable space I of players equipped with a non-atomic measure λ. He defines the notion of a strategy profile as a measurable function $f : I \to \Delta(A)$ and assume that the payoff for every player depends on his own action and the aggregation $\int f(i)\ \lambda(\mathrm{d}i)$. Following Schmeidler, Rashid [27] and Carmona [9] study an n-player analogue of Schmeidler's non-atomic game, where every player i chooses a distribution $f(i) \in \Delta(A)$ and then the payoff of every player depends on his own action and the aggregation $\frac{1}{n}\sum_i f(i)$. Dubey, Mas-Colell and Shubik [12] build on Schmeidler's model of game with continuum of players to show that the noncooperative equilibrium outcome of a market game is a competitive equilibrium.

One fundamental difference between the papers above (abbreviated by SRC) and our approach is the following: In the SRC papers players' payoffs depends on the aggregation of the players' *mixtures*, no randomization takes place. In contrast, in our framework the payoffs depend on the aggregation of the random *realized actions*. As in standard game theory, our players choose their actions independently (conditional on past observations) according to their individual strategies. The compressed view, according to which the players assume that the aggregation of the mixtures is the realized aggregation of pure actions, only represents the way the players perceive the game.

The mathematical results in the following sections present strong justifications for this perception. On the other hand the SRC, and the Mas-Collel paper described below, take these type of results foregranted, without explicit statements and justifications.

Mas-colell [23] formulates Schmeidler's Theorem in terms joint empirical distribution of types and actions. He does not explicitly define a strategy, but every such distribution is equivalent to a strategy in the sense of this paper: the strategy is the conditional distribution of actions given types.[6] Under this equivalence, a compressed equilibrium is an equilibrium as defined by Mas-Colell.

In addition to the above, there are the differences already mentioned: (1) We work entirely within the finite-number-of-players framework. (2) We allow for mixed strategies and incomplete information with interdependent types, and (3) we study repeated, as opposed to one-shot games.

[6]This is similar to the equivalence between distributional strategies of Milgrom and Weber and behavioral strategies.

5. Validation of the compressed view

Returning to the earlier setting, let $\bar{f} = (f^0, \dots, f^{n-1})$ be a reactive profile of strategies for the game $\Gamma^{m,n}$ with compression κ $(= \kappa_{\overline{f}}$). We assume that one of the players, say player 0, considers playing any (reactive or not) strategy g and that the opponents follow the profile $\bar{f}$. For every m denote by $\mathcal{P}^{g,\kappa}_{m,\mathcal{COMP}}$ the joint distribution over the sequence of random outcomes of the compressed κ-play given by (4), and denote by $\mathcal{P}^{g,\bar{f}}_{m,n}$ the joint distribution over the sequence of random outcomes of the actual n-player $(g, \bar{f})$-play given in (2). [7]

We say that the outcome generating function χ of Γ is Lipschitz, if for some constant L, for every s, $\|\chi(s,e) - \chi(s,e')\| \leq L \cdot \|e - e'\|_1$ where $\|\chi(s,e) - \chi(s,e')\|$ is the total variation distance between $\chi(s,e)$ and $\chi(s,e')$.[8]

Theorem 5.1. *Let $\overline{f}$, κ and g be as above, and assume that the outcome generating function χ is Lipschitz. Then*

$$\|\mathcal{P}^{g,\kappa}_{m,\mathcal{COMP}} - \mathcal{P}^{g,\bar{f}}_{m,n}\| < C \cdot m\sqrt{\frac{\log n}{n}},$$

where C is a constant that depends on the parameters of the game skeleton (the number of actions, the number of types and the Lipschitz constant), and not on m and n.[9]

Here $\|\mathcal{P}^{g,\kappa}_{m,\mathcal{COMP}} - \mathcal{P}^{g,\kappa}_{m,n}\|$ is the total variation distance between $\mathcal{P}^{g,\kappa}_{m,\mathcal{COMP}}$ and $\mathcal{P}^{g,\kappa}_{m,n}$. When the distance is small, compressed forecasts of outcomes are close to the correct forecasts, which means that no statistical test can distinguish between the correct forecasts and the compressed forecasts. Thus outcome sequences that are generated by the n-player model validate the compressed perception of the game.

The following corollary states that player 0's compressed assesments of payoffs become accurate as n increases.

[7] Note however that $\mathcal{P}^{g,F}_{m,\mathcal{COMP}}$ is independent of g.

[8] The total variation distance between a pair μ, μ' of distribution over X is given by $\|\mu - \mu'\| = \sup_B |\mu(B) - \mu(B')|$ where the suprimum ranges over all Borel subsets B of X.

[9] The proof establishes the bound $C \leq 2|T| + 3|T| \cdot |A| \cdot L$. See also remark 10.6 for a somewhat sharper asymptotic

Corollary 5.2. *Let $\overline{f}$, κ and g be as above, then $|U^{g,\kappa}_{m,\mathcal{COMP}} - U^{g,\bar{f}}_{m,n}| < C \cdot m\sqrt{\frac{\log n}{n}}$ for every m, n, where C is a constant that depends only on the parameters of the game skeleton.*

6. Relating Compressed Equilibria to Standard Concepts

Recall that for every strategy profile $\overline{f}$ and any $\epsilon > 0$, a strategy g is an ϵ best response to $\overline{f}$, if $U^{g,\overline{f}}_{m,n} \leq U^{h,\overline{f}}_{m,n} + \epsilon$ for all strategies h. In a parallel manner, g is an ϵ compressed best response to $\kappa_{\overline{f}}$, if $U^{g,\kappa_{\overline{f}}}_{m,\mathcal{COMP}} \geq U^{h,\kappa_{\overline{f}}}_{m,\mathcal{COMP}} + \epsilon$.

The next corollary illustrates that a compressed best response in the m-times repeated game is essentially a real best response, and uniformly so for all the games with sufficiently many players.

Corollary 6.1. *Consider the m-times repeated game. For any $\epsilon > 0$ there is a positive integer N with the following properties:*

1. if a strategy g is a compressed best response to a compressed strategy κ, then in all the games with more than N players g is a (real) ϵ best response to any profile $\overline{f}$ with $\kappa_{\overline{f}} = \kappa$.

Conversely, for such large n's every strategy g which is (real) best response to a profile of strategies $\overline{f}$ is an ϵ compressed best response to $\kappa_{\overline{f}}$.

Thus, for the purpose of computing essentialy-optimal play, the player must only know that the number of players n is large, without having to know its value. Moreover, he only needs to know the compressed strategy, without having to know the individual strategies.

Corollary 6.2. *Let κ be a compressed equilibrium of a game $\Gamma^{m,\cdot}$, then for every game $\Gamma^{m,n}$, κ is a $\left(C \cdot m\sqrt{\frac{\log n}{n}}\right)$-equilibrium, where C is a constant that depends only on the parameters of the game skeleton.*

Corollary 6.2 is a strong version of theorems that relate equilibria in games with continuum of players to approximate equilibria games with finite number of players. There are several such theorems in the literature, mostly for one-shot games of complete information and pure strategies. See Carmona and Podczeck [10] for a recent result. The main concern of these papers is the justification of the equilibria in the game with continuum of players as limits of equilibria of the finite player games. In contrast, for us the compressed view is conceptually simpler since it stays entirely with games of finitely many players. Moreover, due to the uniform property of the best response strategies, the justification is stronger in the senses dicussed above.

Closer in spirit to our motivation is a recent paper of Bodoh-Creed [8]. He analyzes equilibria in large interdependent values uniform price auction model where bidders have arbitrary preferences for multiple units using a nonatomic limit game.

Remark 6.3. The fact that κ is an ϵ-equilibrium implies that every player has a chance of at most $\sqrt{\epsilon}$ to reach a node in the game in which he doesn't play $\sqrt{\epsilon}$-best response.

The notion of universal equilibrium, discussed below, is useful for games that are long and include many players. When one strategy is equilibrium for all such games, it's use does not require that the players have precise knowledge of the number of repetitions or the number of players. The theorem below is applicable to games in which the number of players is substantially larger than the number of repetitions.

Definition 6.4. Let κ be a repeated game strategy, and let $\widehat{\Gamma}$ be any subset of the games $\Gamma^{\cdot,\cdot}$. We say that κ is a *universal* (asymptotic) *equilibrium of the family* $\widehat{\Gamma}$, if for every $\epsilon > 0$ there exists a natural number N such that for all the games $\Gamma^{m,n} \in \widehat{\Gamma}$ with $n > N$ and for every strategy g,

$$U^{g,\kappa}_{m,n} \leq U^{\kappa,\kappa}_{m,n} + \epsilon \tag{9}$$

Theorem 6.5. *Let $\widehat{\Gamma}$ be a set of games $\Gamma^{m,n}$ such that $m < \delta(n)$, where $\delta : \mathbb{N} \to \mathbb{N}$ is some function such that $\delta(n) \cdot \sqrt{\frac{\log n}{n}} \xrightarrow[n\to\infty]{} 0$. Let κ be any uniform compressed equilibrium as in Definition 3.8, Then κ is universal equilibrium of $\widehat{\Gamma}$.*

Since Theorem 3.13 establishes the existence of uniform compressed equilibrium, the theorem above is an existence theorem for universal equilibrium on the class of games it describes.

7. Illustrative Example: repeated selection of hat color

Let $C = \{red(r), blue(b)\}$ and $I = \{follower(f), defector(d)\}$ denote respectively hat colors and player's social inclinations, and consider the following game skeleton Γ.

First, an unkown state of nature $s \in S \equiv C$ is drawn with equal probabilities, $\theta(r) = \theta(b) = 1/2$.

Then, conditional on the realized state s, players' types $t = (c, i) \in T \equiv C \times I$ are drawn i.i.d. according to the following probabilities. Regardless of the realized s, the inclination i is drawn with probabilities $\tau(f) = .90$ and $\tau(d) = .10$. On the other hand, the prefered color c is drawn with probabilities that do depend on s: $\tau(r) = .60$ and $\tau(b) = .40$, if $s = r$; but with the reverse probabilities $\tau(r) = .40$ and $\tau(b) = .60$, if $s = b$.

Thus, as assumed in the general model, players types are interdependent, but they are independent conditional on the unknown state of nature s.

Next, the set of period actions $A \equiv C$. At the end of each period the realized empirical proportions of color choices, $e = (e(r), e(b))$, is randomly rounded to one decimal to result in *reported empirical proportion* $x = (x(r), x(b))$ as follows.

First, there is a restriction that $x(r) \in X = \{0.0, 0.1, 0.2, \ldots, 1.0\}$. The rounding off probabilities, $\chi(e)$ in the terminology of the formal model, are linear. For example, if the real realized proportion of red choosers $e(r) = 0.29$, then the reported proportion $x(r)$ is chosen to be 0.30 with probability .9, and and 0.20 with probability .1; then $x(b) \equiv 1 - x(r)$.

The period payoff of a player of type $t = (c, i)$ who chooses the action a is $u(t, a, x) = \delta_{a=c}0.05 + \delta_{i=f}x(a) + \delta_{i=d}x(a')$ where a' is the opposite color to a. In other words he is paid .05, if he chooses his preferred color (zero otherwise); plus the reported proportion of players that he matches, if he is a follower; or the reported proportion of players that he mismatches, if he is a defector.

The game is played as follows. First a state of nature and player types are drawn as described above and each player is informed of his type. At the begining of each period every player chooses a hat color and at the end of the period rounded off proportions are drawn and announced. Players then collect their period payoffs.

Coordinated learning in a compressed equilibrium. While there are several types of compressed equilibrium in this example, we are interested in the one that involves *coordinated learning* as described below.

In period 0, each player chooses his preferred color. Let x_0 be the rounded reported proportion of choices at the end of period 0. Then in all periods $k \geq 1$, if $x_0(r) \geq 0.5$, the followers choose red and defectors choose blue; and if $x_0(r) < 0.5$, the followers choose blue and defectors choose red.

Using the myopic characterization in Corollary 3.9, it is easy to see that the strategy profile above is a compressed equilibrium. For period 0, there are compressed probabilities of .50 each that the empirical distribution of hat colors be either .60 to .40 or .40 to .60. Moreover, with probability one these will remain the reported proportion after rounding. Since his own actions do not alter these reported proportions, a player's inclination has no effect on his expected payoff and his best response is to choose his preferred color.

In every subsequent period, Bayesian updating will lead to an empirical color distribution of .90 to .10, with the .90 placed on the most popular choice in period 0. So color preferences have no effect relative to inclinations, and the compressed equilibrium strategy is optimal relative to itself.

As follows from the uniform approximation by compressed best response, the strategy above is an ϵ best response to the strategies of the opponents for sufficiently large n's.

For small n's, on the other hand, even if the proportion of red choosers reported at the end of period zero reflects correctly the true color perferences, there is a non-negligible probability that the proportion of followers is smaller than the proportion of defectors, in which case the equilibrium continuation strategies are not compatible with the players incentives.

Moreover, with small n's even the incentive to play the equilibrium strategies in period 0 may not hold. A defector who prefers red may have the incentive to choose blue, if by doing so he sways the continuation game to have the majority play red and the minority play blue, his preferred color.

8. Counter-examples

The examples below illustrate the importance of two issues that arise in our approximation results: (1)The need to bound the number of game repetitions m relative to the number of players n. (or the need to have larger number of players for longer games). (2) The need to smooth the outcome functions using noise, such as the random rounding in the illustrative example above.

Both issues can be discussed even for games of complete information. But to keep the presentation short here, we stay with our illustrative example above. The first example shows the role of the bound on the length of the game, m, in Theorem 5.1

Example 8.1. In the illustrative example, consider the strategy κ by which at period $k = 0$ the players randomize $(1/2, 1/2)$ (on red and blue respectively), and in every round $k \geq 1$ they randomizes $(1/2, 1/2)$, if $x_{k-1}(r) = 0.5$; and they play red otherwise. The event that the players randomize $(1/2, 1/2)$ forever has compressed probability one.

But in the real game with any finite number of players n, in every period there is a small probability that the reported rounded off proportion or red choosers will be different from 1/2, trigerring a continuation play of all red. And no matter how small this trigger probability is, it will happen with probability one if the game is played repeatedly with no limit on the number of periods m. Thus, large m's will lead to severe discrepancies between the compressed and the real probabilities.

The next example shows that Theorem 5.1 fails without the assumption of norm continuity of the outcome generating function.

Example 8.2. Modify the illustrative example to have the set of possible outcomes $X = [0, 1]$, and assume that the proportion of players who choose red is reported precisely (without any noise) at the end of each period. This corresponds to the outcome generating function

$\chi(e) = \delta_e$ that puts probability one on the true empirical distribution, a function which is not continuous in the total variation distance on $\Delta(X)$.

Consider again the strategy κ by which in period 0 players choose randomly $(1/2, 1/2)$. In every period $k \geq 1$ they randomize $(1/2, 1/2)$ again, if the (reported) proportion of red choosers in the previos period $x_{k-1}(r) = 1/2$ exactly; but they all choose red otherwise.

The event that the players continue to randomize $(1/2, 1/2)$ forever has compressed probability one. But the real probability that the actual (and reported) proportions in period 0 are $(1/2, 1/2)$ is close to zero (especially for large n's). Thus with probability close to one they will continue to play red forever from period 1 onwards, contradicting the theorem in the paper.

9. Extensions and variations

9.1. **Extensions.** Some restrictions in this paper were done to keep the simplicity of presentation and notation. Three of these restrictions may be relaxed as follows.

9.1.1. *Asymmetric games, allowing for player roles* . Symmetry of players may be relaxed by letting players be of different roles: male and females, sellers and buyers, members of different biological species, etc.

This can be done by expanding the notion of game skeleton (Definition 2.1) to include a finite set of player *roles* R and a distribution $\rho \in \Delta(R)$, which represents the proportion of players in each role. The distribution must consist of rational numbers and be combined with restrictions on the possible number of players n that guarantee integer number of players in each role. For example, we may consider games with even number of players $n = 2h$ with $\rho[m] = \rho[f] = 1/2$, so that we have h *males* and h *females.*

In addition to type-independence conditional on the state of nature, assumed in the current model, player types would be independent, conditional on the state of nature *and* their roles. The rest of the definitions and results of this paper carry through.

9.1.2. *Discounting vs finite-long games.* Much of the theory presented here may be extended from finitely repeated games to infinitely repeated games with discounting, in which statements related to the length of the game are replaced by statement about the future discount factor.

For example, Corollary 6.2 has a natural analogue for discounted games: A compressed equilibrium is an approximate equilibrium when the number of players is large and the discount factor is bounded away from 1.

9.1.3. *Weighted players.* The symmetry of players' impact on the period outcomes in the games $\Gamma^{n,\cdot}$ can be relaxed, as long as the impact of the players diminished as n increases. This may be relevant in biological games, congestion games, market games and other applications where players's size needs not be the same.

Fix a skeleton game Γ. For every m, n and every distribution of *player weights* $w \in \Delta(\{0, 1, \ldots, n-1\})$, let $\Gamma^{m,w}$ be similar to the game $\Gamma^{m,n}$ defined in Section 2.1, but subject to the following modifications: (1) the average used in computing the empirical distribution e_k of players types and actions should be weighted by w, and (2) the average used to compute the compressed strategy should also be weighted by w. Under these modification similar results to ours holds if the $\|\|_\infty$-norm of w is sufficiently small.

9.2. **Additional directions of research.** The mathematics involved in the main results reported in this paper require specific assumptions, such as noise in the period reported outcomes and limitations on the length of the game. From a technical point of view, further investigation of these and alternative models may be important.

9.2.1. *Alternatives to the noise model.* As Example 8.2 shows, Theorem 5.1 relies on the introduction of noise. The implication of noise is that even if the players' behavior is not continuous in the actual history, it is continuous in the "noisy history."

An analogue to theorem 5.1 without noise can be proved if we assume that the players strategies are continuous in the history of actual empirical choices. However, additional restrictions on the game should be satisfied so that compressed equilibrium in continuous strategies exist.

9.2.2. *Removing the limitations on m.* As Example 8.1 shows, the number of players required in Theorem 5.1 grows as the number of periods grow. This is because every additional period may add probability of failure of the law of large number in the empirical distribution of actions. When players behavior is assumed to depend on the entire past history, a failure of the compressed view in one period may produce failure in all subsequent periods.

But the coordinated learning equilibrium in our illustrative example, was not subject to this difficulty. It provided a good approximation of the probabilities of play regardless of the number of repetitions. This uniform good approximation is due to two features of the equilibrium: (1) future play depends only on the outcome of the first period, and (2) no new learning takes place beyond the first period.

One may try to construct a general class of equilibria that exhibit such features. More specifically, restrict attentions to strategies that depend only on a fixed number of initial periods, say periods $0, \dots, K$, then construct an analogue of Theorem 5.1 in which for large enough n's (that depend on K and ϵ, and not on the total number of game repetitions m) the future play depends only on the outcomes of the first K periods.

It may be required, however, that all the relevant learning that takes place during the play of the game should happen with high probability in the first K periods.

10. Proofs

10.1. **Preliminaries.**

10.1.1. *Notations.* For every Borel spcae Z we denote by $\Delta(Z)$ the space of probability distributions over Z. We let $\|\mu - \mu'\|$ denote the total variation distance between the distributions $\mu, \mu' \in \Delta(Z)$. If Z is finite then we view $\Delta(Z)$ as a subset of $\mathbb{R}^Z$, in which case $\|\mu - \mu'\| = \|\mu - \mu'\|_1/2$ where $\|\|_1$ denotes the L^1 norm. In the case of finite Z we also denote by $\|\mu - \mu'\|_\infty$ the L^∞ distance given by $\|\mu - \mu'\|_\infty = \max_{z \in Z} |\mu[z] - \mu'[z]|$ and we denote by $[\mu]$ the support of μ.

10.1.2. *Concentration of Empirical Distribution.* We use the following propositions.

Proposition 10.1. *Let Z be a finite set. Then for every Z-valued independent random variables $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$*

$$\mathbb{P}\left(\|\mathbf{m} - \mathbb{E}\mathbf{m}\|_\infty > \eta\right) < 2|Z| \cdot e^{-2\eta^2 n},$$

where $\mathbf{m} \in \Delta(Z)$ is the random empirical distribution of $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ given by $\mathbf{m}[z] = \frac{1}{n}\#\{i | Z_i = z\}$.

Proof. It follows from Chernoff Bound [3, Corollary A.1.7] that

$$\mathbb{P}\left(\|\mathbf{m}[z] - \mathbb{E}\mathbf{m}[z]\| > \eta\right) < 2e^{-2\eta^2 n}$$

for every $z \in Z$. Therefore

$$\mathbb{P}\left(\|\mathbf{m} - \mathbb{E}\mathbf{m}\|_\infty > \eta\right) \leq \sum_{z \in Z} \mathbb{P}\left(\|\mathbf{m}[z] - \mathbb{E}\mathbf{m}[z]\| > \eta\right) \leq 2|Z|e^{-2\eta^2 n}.$$

∎

Proposition 10.2. *Let Z be a finite set. Then for every Z-valued independent random variables $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$*

$$\mathbb{E}\big\|\mathbf{m} - \mathbb{E}\mathbf{m}\big\|_1 < |Z|/\sqrt{n},$$

where $\mathbf{m} \in \Delta(Z)$ is the random empirical distribution of $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ given by $\mathbf{m}[z] = \frac{1}{n}\#\{i | Z_i = z\}$.

Proof. For every $z \in Z$ the random variable $\mathbf{m}[z]$ is the average of n independent variables with values in $\{0, 1\}$. Therefore

$$\mathbb{E}\big|\mathbf{m}[z] - \mathbb{E}\mathbf{m}[z]\big| \leq \sqrt{\mathrm{Var}(\mathbf{m}[z])} < 1/\sqrt{n}.$$

It follows that

$$\mathbb{E}\big\|\mathbf{m}-\mathbb{E}\mathbf{m}\big\|_1 < \sum_{z\in Z}\mathbb{E}\big|\mathbf{m}[z]-\mathbb{E}\mathbf{m}[z]\big| < |Z|/\sqrt{n}.$$

∎

10.1.3. *Coupling.* Let X be a Borel space and μ and ν two probability distributions over X. A *coupling* of μ and ν is a pair $(\mathbf{X},\mathbf{Y})$ such that $\mathbf{X}$ is distributed μ and $\mathbf{Y}$ is distributed ν. The following lemma shows how coupling of random variables is useful in order to study distance between distributions.

Proposition 10.3. [21, Theorem 5.2]*Let X be a Borel space and $\mu,\nu\in\Delta(X)$. Then*

(1) *If $(\mathbf{X},\mathbf{Y})$ is a coupling of (μ,ν) then $\mathbb{P}(\mathbf{X}\neq\mathbf{Y})\geq\|\mu-\nu\|$.*
(2) *There exists an* optimal *coupling $(\mathbf{X},\mathbf{Y})$ of (μ,ν) such that $\mathbb{P}(\mathbf{X}\neq\mathbf{Y})=\|\mu-\nu\|$.*

10.2. **Proof of Proposition 3.13.** A *one-shot strategy* in Γ is given by a function $f:T\to\Delta(A)$. In the one-shot compressed setup, if players play according to f then the state of nature $\mathbf{S}$, a player 0's type $\mathbf{T}$, and the outcume $\mathbf{Y}$ are random variables with joint distribution

$$\mathbb{P}\left(\mathbf{S}=s,\mathbf{T}=t,\mathbf{Y}\in B\right)=\theta(s)\cdot\tau_s[t]\cdot\chi_{d_f(s)}(B), \tag{10}$$

where $d_f(s)\in\Delta(T\times A)$, the *empirical distribution of players type-action under f when the state of nature is s*, is given by

$$d_f(s)[t,a]=\tau_s[t]\cdot f(t)[a]. \tag{11}$$

We call a triple $(\mathbf{S},\mathbf{T},\mathbf{X})$ with distribution (10) a *random one-shot f-play.*

A one-shot strategy f is a *compressed one-shot equilibrium* if

$$[f(t)]\subseteq\mathrm{argmax}_{a\in A}\mathbb{E}(u(t,a,\mathbf{X}|\mathbf{T}=t)$$

for every type t, where $\mathbf{S},\mathbf{T},\mathbf{X}$ is a random one-shot f-play.

Lemma 10.4. *Every one-shot game admits a compressed one-shot equilibrium*

Proof. Since T is finite, the set of one-shot strategies is compact and convex and the best response correspondence has closed graph and convex values. Therefore it has a fixed point by Kakutani's fixed point theorem. ∎

Proof of Proposition 3.13. We construct by induction the compressed strategy κ and the compressed κ-play process $\mathbf{S}, \mathbf{X}_0, \mathbf{X}_1, \dots$ so that the condition of Corollary 3.9 is satisfied, as follows: Let $\mathbf{S}$ be S-valued random variable with distribution θ. For every k let $\kappa(t, x_0, \dots, x_{k-1})$ be such that $f(t) = \kappa(t, x_0, \dots, x_{k-1})$ is a one-shot equilibrium in the game with prior $\mathbb{P}\left(\mathbf{S} = \cdot | \mathbf{X}_0 = x_0, \dots, \mathbf{X}_{k-1} = x_{k-1}\right)$ (The function κ can be selected to be measurable by von-Neumann's Selection Theorem) and let $\mathbf{X}_k$ be a random variable such that

$$\mathbb{P}\left(\mathbf{X}_k \in \cdot | \mathbf{S} = s, \mathbf{X}_0 = x_0, \dots, \mathbf{X}_{k-1} = x_{k-1}\right) = \chi_{d_f(s)},$$

where $d_f(s) \in \Delta(T \times A)$ is given by (11). ∎

10.3. **Proof of Theorem 5.1.** We prove the theorem using a lemma that couples a play in the actual game with a compressed play. In order to distinguish between corresponding entities in the actual play and in the compressed play, we denote the random variables that represent player 0 actions in the compressed play by $\mathbf{B}^0_0, \mathbf{B}^0_1, \dots$ and the random variables that represent the outcomes in the compressed play by $\mathbf{Y}_0, \mathbf{Y}_1, \dots$.

Lemma 10.5. *Let Γ be a game skeleton. For every strategy g and a reactive strategy profile $\bar{f} = (f_1, \dots, f_n)$ of Γ with compression $\kappa = \kappa_{\bar{f}}$ there exist random variables $\mathbf{S}, \mathbf{T}^i, \mathbf{A}^i_k, \mathbf{X}_k, \mathbf{Y}_k$ for $i \in N$ and $k = 0, 1, \dots$ such that*

- $(\mathbf{S}, \mathbf{T}^0, \mathbf{B}^0_0, \mathbf{Y}_0, \mathbf{B}^0_1, \mathbf{Y}_1, \dots)$ *is a compressed random (g, κ)-play.*
- $(\mathbf{S}, \mathbf{T}^i, \mathbf{A}^i_k, \mathbf{X}_0, \mathbf{X}_1, \dots)$ *is a random $(g, \bar{f})$-play of $\Gamma^{\cdot,n}$.*
- *For every m one has*

$$\text{(12)} \quad \mathbb{P}\left(\mathbf{A}^0_0 = \mathbf{B}^0_0, \mathbf{X}_0 = \mathbf{Y}_0, \dots, \mathbf{A}^0_m = \mathbf{B}^0_m, \mathbf{X}_m = \mathbf{Y}_m\right) > 1 - (2|T| + 3|T| \cdot L \cdot |A|) \cdot m\sqrt{\frac{\log n}{n}}.$$

Proof. We couple a compressed (g, κ)-play $(\mathbf{S}, \mathbf{T}^0, \mathbf{B}_k^0, \mathbf{Y}_k)$ and a $(g, \bar{f})$-play $(\mathbf{S}, \mathbf{T}^i, \mathbf{A}_k^i, \mathbf{X}_k)$ as follows:

- The state of nature $\mathbf{S}$ is distributed according to θ.
- Conditioned on $\mathbf{S}$, the players' types $\mathbf{T}^i$ for $i \in N$ are i.i.d $\tau_{\mathbf{S}}$.
- Conditioned on the history of stages $0, \ldots, k-1$, the players in the actual play choose their day k actions independently, where player 0 mixes according to g and opponent $i \in N \setminus \{0\}$ mixes according f_i. Player 0's action B_k^0 in the compressed play is optimally coupled with his action A_k^0 in the actual play according to Proposition 10.3.
- The outcomes $\mathbf{X}_k$ and $\mathbf{Y}_k$ of day k in the actual and compressed plays are randomized according to the corresponding empirical distribution of day k types and actions. Moreover, these outcomes are optimally coupled according to Proposition 10.3.

In equations,

$$\mathbb{P}\left(\mathbf{S} = s, \mathbf{T}^i = t^i \ i \in N\right) = \theta[s] \cdot \prod_i \tau_s[t^i]$$

$$\mathbb{P}\left(\mathbf{A}_k^i = a^i \ i \in N \,|\mathcal{F}_k\right) = g\left(\mathbf{T}^0, \mathbf{A}_0^0, \mathbf{X}_0, \ldots, \mathbf{A}_{k-1}^0, \mathbf{X}_{k-1}\right)[a^0] \cdot \prod_{i \in N\setminus\{0\}} f^i\left(\mathbf{T}^i, \mathbf{X}_0, \ldots, \mathbf{X}_{k-1}\right)[a^i]$$

$$\mathbb{P}\left(\mathbf{B}_k^0 = a \,|\mathcal{F}_k\right) = g\left(\mathbf{T}^i, \mathbf{B}_0^0, \mathbf{Y}_0, \ldots, \mathbf{B}_{k-1}^0, \mathbf{Y}_{k-1}\right)[a]$$

$$\mathbb{P}\left(\mathbf{B}_k^0 \neq \mathbf{A}_k^0 \,|\mathcal{F}_k\right) = \|g\left(\mathbf{T}^i, \mathbf{B}_0^0, \mathbf{Y}_0, \ldots, \mathbf{B}_{k-1}^0, \mathbf{Y}_{k-1}\right) - g\left(\mathbf{T}^0, \mathbf{A}_0^0, \mathbf{X}_0, \ldots, \mathbf{A}_{k-1}^0, \mathbf{X}_{k-1}\right)\|$$

$$\mathbb{P}\left(\mathbf{X}_k \in \cdot \,\middle|\, \tilde{\mathcal{F}}_k\right) = \chi_{\mathbf{S},\mathbf{d}_k}$$

$$\mathbb{P}\left(\mathbf{Y}_k \in \cdot \,\middle|\, \tilde{\mathcal{F}}_k\right) = \chi_{\mathbf{S},\mathbf{e}_k}$$

$$\mathbb{P}\left(\mathbf{X}_k \neq \mathbf{Y}_k \,\middle|\, \tilde{\mathcal{F}}_k\right) = \|\chi_{\mathbf{S},\mathbf{d}_k} - \chi_{\mathbf{S},\mathbf{e}_k}\|$$

where $\mathcal{F}_k$ is the *sigma*-algebra generated by $\mathbf{S}, \mathbf{T}^i, \mathbf{A}_l^i, \mathbf{B}_l^0, \mathbf{Y}_l, \mathbf{X}_l$ for $i \in N$ and $l < k$, which represents the $k-th$ day history, $\tilde{\mathcal{F}}_k$ is the σ-algebra generated by $\mathcal{F}_k$ and the k-th day actions $\mathbf{A}_k^i$ and $\mathbf{B}_k^0$, and $\mathbf{d}_k$ and $\mathbf{e}_k$ are the random empirical distributions of day k type and actions in the actual and finite plays, given by (1) andn (5) respectively.

The construction renders $(\mathbf{S}, \mathbf{T}^0, \mathbf{B}_k^0, \mathbf{Y}_k)$ a compressed (g, κ)- play and $(\mathbf{S}, \mathbf{T}^i, \mathbf{A}_k^i, \mathbf{X}_k)$ a $(g, \bar{f})$- play in $\Gamma^{\cdot,n}$.

Let G_0 be the event that the empirical distribution of types is η-concentrated around its expectation, i.e.,

$$G_0 = \{\|\mathbf{m} - \tau_{\mathbf{S}}\|_\infty < \eta\},$$

where $\mathbf{m} \in \Delta(T)$ is the random empirical distribution of $\mathbf{T}_1, \ldots, \mathbf{T}_n$ given by

$$\mathbf{m}[t] = \frac{1}{n}\#\{i|T^i = t\}. \tag{13}$$

and η is TBD. Since $\mathbf{T}_1, \ldots, \mathbf{T}_n$ are independent conditioned on $\mathbf{S}$ it follows from Proposition 10.1 that

$$\mathbb{P}(G_0) > 1 - 2|T|e^{-2\eta^2 n}. \tag{14}$$

For every $k = 0, 1, 2, \ldots$ consider the event

$$G_k = G_0 \cap \{\mathbf{Y}_0 = \mathbf{X}_0, \ldots, \mathbf{Y}_{k-1} = \mathbf{X}_{k-1}\}.$$

Note that from the coupling condition of $\mathbf{B}_k^0$ and $\mathbf{A}_k^0$ it follow by induction that $\mathbf{B}_k^0 = \mathbf{A}_k^0$ on G_k and therefore

$$G_m \subseteq \left\{\mathbf{A}_0^0 = \mathbf{B}_0^0, \mathbf{X}_0 = \mathbf{Y}_0, \ldots, \mathbf{A}_m^0 = \mathbf{B}_m^0, \mathbf{X}_m = \mathbf{Y}_m\right\}.$$

Thus, to prove the lemma it is sufficient to establish the bound from below on $\mathbb{P}(G_m)$. To do that we prove that $\mathbb{P}\left(G_{k+1}\,|G_k\right)$ is high for every k.

Conditioned on $\mathcal{F}_k$ the type of opponent $i \in N \setminus \{0\}$ is $\mathbf{T}^i$ and his day k action is randomized according to $f^i\left(\mathbf{T}^i, \mathbf{Y}_0, \ldots, \mathbf{Y}_{k-1}\right)$. The empirical distribution $\mathbf{d}_k$ is induced by the type-action of all the opponents and the contribution of player 0. Therefore

$$\Big|\mathbb{E}\left(\mathbf{d}_k[t, a]\,|\mathcal{F}_k\right) - \mathbf{m}[t] \cdot \kappa\left(t, \mathbf{Y}_0, \ldots, \mathbf{Y}_{k-1}\right)[a]\Big| \leq 1/n \tag{15}$$

for every $(t,a) \in T \times A$ where $\mathbf{m} \in \Delta(T)$ is the random empirical distribution of types given by (13). It follows that on G_k

$$\begin{aligned}
&\left\|\mathbb{E}\left(\mathbf{d}_k \middle| \mathcal{F}_k\right) - \mathbf{e}_k\right\|_1 = \sum_{t,a} \left|\mathbb{E}\left(\mathbf{d}_k[t,a] \middle| \mathcal{F}_k\right) - \mathbf{e}_k[t,a]\right| \\
&\le \sum_{t,a}\Big(\left|\mathbf{m}[t]\cdot\kappa\left(t,\mathbf{Y}_0,\ldots,\mathbf{Y}_{k-1}\right)[a] - \tau_{\mathbf{S}}[t]\cdot\kappa\left(t,\mathbf{Y}_0,\ldots,\mathbf{Y}_{k-1}\right)[a]\right| + 1/n\Big) \qquad (16) \\
&\le \left\|\mathbf{m}-\tau_{\mathbf{S}}\right\|_1 + |T|\cdot|A|\cdot 1/n < |T|\cdot\eta + |T|\cdot|A|/n,
\end{aligned}$$

where the first inequality follows from (15) and (5) and the last inequality from the fact that $\|\mathbf{m}-\tau_{\mathbf{S}}\|_1 \le |T|\cdot\|\mathbf{m}-\tau_{\mathbf{S}}\|_\infty < \eta$ on G_k since $G_k \subseteq G_0$.

Since the players type-actions are independent conditioned on $\mathcal{F}_k$ it follows from Proposition 10.2 that

$$\mathbb{E}\Big(\|\mathbf{d}_k - \mathbb{E}\left(\mathbf{d}_k \,|\mathcal{F}_k\right)\|_1 \,\Big|\mathcal{F}_k\Big) < |A|\cdot|T|/\sqrt{n}.$$

From the last inequality and (16) it follows that

$$\mathbb{E}\Big(\|\mathbf{d}_k - \mathbf{e}_k\|_1 \,\Big|\mathcal{F}_k\Big) < |A|\cdot|T|/\sqrt{n} + |T|\cdot\eta + |T|\cdot|A|/n$$

On G_k. From the coupling condition of $\mathbf{X}_k$ and $\mathbf{Y}_k$, the Lipschitz condition on χ and the last inequality it follows that on G_k

$$\begin{aligned}
&\mathbb{P}\left(\mathbf{Y}_k \neq \mathbf{X}_k \middle| \mathcal{F}_k\right) = \mathbb{E}\left(\mathbb{P}\left(\mathbf{Y}_k \neq \mathbf{X}_k \middle| \tilde{\mathcal{F}}_k\right) \middle| \mathcal{F}_k\right) = \\
&\mathbb{E}\left(\|\chi_{\mathbf{S},\mathbf{e}_k} - \chi_{\mathbf{S},\mathbf{d}_k}\| \middle| \mathcal{F}_k\right) \le L\cdot\mathbb{E}\Big(\|\mathbf{d}_k - \mathbf{e}_k\|_1 \,\Big|\mathcal{F}_k\Big) \le L\cdot|A|\cdot|T|/\sqrt{n} + L|T|\eta + L|T|\cdot|A|/n.
\end{aligned}$$

Since G_k is $\mathcal{F}_k$-measurable it follows from the last inequality that

$$\mathbb{P}\left(G_{k+1}\middle|G_k\right) = 1 - \mathbb{P}\left(\mathbf{Y}_k \neq \mathbf{X}_k \middle| G_k\right) \ \ge 1 - L\cdot|A|\cdot|T|/\sqrt{n} + L|T|\eta + L|T|\cdot|A|/n.$$

From the last inequality and (14) it follows that

$$\mathbb{P}(G_m) \ge 1 - 2|T|e^{-2\eta^2 n} - m\cdot L\cdot|T|\cdot(|A|/\sqrt{n} + \eta + |A|/n). \qquad (17)$$

The last inequality holds for every η. Choosing $\eta = \sqrt{\frac{\log n}{n}}$ we get that

$$\mathbb{P}(G_m) \geq 1 - 2|T|/n^2 - m \cdot L \cdot |T| \cdot \left(|A|/\sqrt{n} + \sqrt{\frac{\log n}{n}} + 1/n \right)$$
$$> 1 - (2|T| + 3|T| \cdot L \cdot |A|) m \sqrt{\frac{\log n}{n}},$$

as desired ∎

Remark 10.6. By choosing $\eta = C \cdot \sqrt{n}$ in (17) we get that we can replace the conclusion of Lemma 10.5 with the assertion that for every $\epsilon > 0$

$$\mathbb{P}\left(\mathbf{A}_0^0 = \mathbf{B}_0^0, \mathbf{X}_0 = \mathbf{Y}_0, \ldots, \mathbf{A}_m^0 = \mathbf{B}_m^0, \mathbf{X}_m = \mathbf{Y}_m\right) > \epsilon + Cm/\sqrt{n},$$

where C is a constant that depends on ϵ and the parameters of the game.

Proof of Theorem 5.1. Let $C = 2|T| + 3|T| \cdot L \cdot |A|$. Lemma 10.5 establishes a coupling of the compressed random outcomes $(\mathbf{Y}_0, \ldots, \mathbf{Y}_{m-1})$ and the random outcomes $(\mathbf{X}_0, \ldots, \mathbf{X}_{m-1})$ in $\Gamma^{\cdot,n}$ such that

$$\mathbb{P}(\mathbf{X}_0 = \mathbf{Y}_0, \ldots, \mathbf{X}_{k-1} = \mathbf{Y}_{k-1}) > 1 - C \cdot m \sqrt{\frac{\log n}{n}}.$$

Therefore it follows from Proposition 10.3 that $\|\mathcal{P}^{g,F}_{m,\mathcal{COMP}} - \mathcal{P}^{g,F}_{m,n}\| < C \cdot m \sqrt{\frac{\log n}{n}}$ as desired. ∎

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