The Lobbying Game with Asymmetric Information

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Abstract

The lobbying game is a constant-sum Colonel Blotto game with two distinguishing features – the victor of each battlefield is determined probabilistically and payoffs are assigned discretely from the set {win, draw, loss}. At the heart of every lobbying game is the legislative body that the players are lobbying. Its members form a voting system, often one that is weighted. In this scenario, given specifics about the weighted voting system, it is possible to determine the equilibrium strategy that is optimal for both players. However, if one player lacks this information while his adversary possesses it, then a situation of asymmetric information is created. The disadvantaged player must necessarily deviate from the equilibrium strategy. This paper will argue that if the disadvantaged player has knowledge about the ramifications of deviating from the equilibrium strategy in each of the possible distinct weighted voting systems that may arise in the legislature, then he can treat his situation like a game against nature. Three strategy choices will be recommended, and a two-case example using a three-party legislature will be presented. In this way, the player without complete information will be able to exercise some control over his payoff.

1 Introduction

The lobbying game is a variant of the well-studied Colonel Blotto games that models the following scenario. In some town, there is an all-powerful legislative body, with some extremely large number of seats. Each legislator is a member of a political party, and all members of a particular party always vote together as a voting bloc. A measure passes if the proportion of legislators who vote for it is greater than or equal to a predetermined, fixed quota. In other words, the parties in the legislature form a normalized weighted voting system where each party's weight is equal to the proportion of legislators who are part of that party. Two players, perhaps interest groups, with directly opposing interests each have limited, infinitely divisible resources with which to lobby the various political parties in the town. These players will simultaneously allocate all of their resources among the parties. Then, each party will compare the amount of resource committed by each player to them and will vote for each with probability proportional to the amount of resource allocated by that player. Either player achieves victory if he can secure a proportion of votes greater than the quota. If neither player meets this requirement, then the game is a draw.

In general, a Blotto game is where players must divide their resources among several "battlefields" in an attempt to achieve a better result than their opponent or opponents. Several important options about the Blotto game involve the number of players, the number of battlefields, the amount of resource each player possesses, the divisibility of the resource, and most importantly, how payoffs to the players are calculated. The lobbying game is distinguished in this regard by how payoffs are of an all-or-nothing nature – only three outcomes are possible. Additionally, the parties in the lobbying game vote according to a probabilistic method, while most Blotto games have each battlefield deterministically won by the side that allocates a greater amount of resource. Though much theory can be borrowed from Blotto games, these important differences make the lobbying game distinct.

Much work has been done with Blotto games. Coughlin (1992) and similar papers analyze specific forms of the Blotto game and provide general formulas and conclusions. More recent papers, like Weinstein (2006) and Arad and Rubinstein (2012) apply novel ideas to the Blotto game, such as geometry in the former and multi-dimensional iterative reasoning in the latter, in order to shed light on the Blotto game from a new angle. Other papers have sought to explore variants of the Blotto game, such as Coughlin (2012), which analyzes a Blotto game with probabilistic voting that is exactly the lobbying game.

Another class of papers sought to analyze the Blotto game with the help of experiments or studies. These include Montero et al. (2012), Modzelewski, Stein, and Yu (2009), and Chowdhury, Kovenock, and Sheremeta (2009). These papers all ran experiments in the form of Blotto game tournaments, but focused on achieving different goals. Montero et al. (2012) explored apex games in particular, where one voter has a weight so great that the only way to overcome him is to unite all of the smaller voters. Modzelewski, Stein, and Yu (2009) tested for bias against the order in which battlefields were listed. Chowdhury, Kovenock, and Sheremeta (2009) analyzed the Blotto game with asymmetric resources.

Still other papers did not focus on analyzing Blotto games, but rather applied them to various situations in order to analyze those. These include Merolla, Munger, and Tofias (2006), which examines U.S. presidential elections as Blotto games, Mauboussin (2010), which applies the Colonel Blotto game to investing, and Golman and Page (2006), which portrayed Blotto games as games of allocative strategic mismatch and argued for applications in fields ranging from international politics, electoral politics, business, the law, biology, and sports.

The lobbying game has many applications, warranting its study. An equilibrium strategy in the lobbying game is known due to the paper Lake (1979). If one player deviates from the equilibrium strategy while the other continues to play the equilibrium strategy, interesting results can be observed. The motivation for making these observations lies in analyzing situations of asymmetric information. Section 5 of this paper will analyze the situation where one player is unsure of the configuration of the parties to be lobbied in the legislature, while the other player has this knowledge. In this scenario, the player with incomplete information may end up playing a strategy that is not the equilibrium strategy, since the equilibrium strategy is different depending on which weighted voting system the parties in the legislature make up. A general analysis of deviating from the equilibrium strategy will culminate in the analysis of what the player with inferior information should do in a lobbying game with asymmetric information.

2 The Lobbying Game and Weighted Voting Systems

2.1 Formal Statement of the Lobbying Game

In the lobbying game, there will be just two players (known as player A and player B) with equal resources lobbying n parties, which are collectively known as $\mathcal{N} = \{1, 2, \ldots, n\}$. For many of the examples in this paper, the $n = 3$ case will be used, since three is the least number of parties needed to create a situation with enough depth for an example to be meaningful. The parties of the legislature will form a normalized weighted voting system $[q; w_1, w_2, \ldots, w_n]$, where the quota is pre-determined and fixed. If the weights of the parties are known, then each subset of the set of all parties can then be labeled as winning, drawing, or losing for player A. In this constant-sum game, a win will be assigned a payoff of 1, a draw a payoff of $\frac{1}{2}$, and a loss a payoff of 0.

At the start of the lobbying game, player A will divide his resources using his allocation vector, $[a_1, a_2, \ldots, a_n]$, where $\sum_{i=1}^{n} a_i = 1$. Player B has a similar allocation vector $[b_1, b_2, \ldots, b_n]$, where $\sum_{i=1}^{n} b_i = 1$. Then each party in the lobbying game will implement a probablistic decision method: each party will then vote for player A with probability $p_i = \frac{a_i}{a_i + b_i}$. If $a_i = b_i = 0$, then $p_i = \frac{1}{2}$ by default. Though the resulting payoff to player A is random and discrete, an expectation continuous on $[0, 1]$ can be calculated for him with the formula $E = p(\text{win}) + \frac{1}{2} p(\text{draw})$. It is this expectation that this paper is primarily interested in.

2.2 Weighted Voting Systems

Before proceeding to the analysis of the lobbying game, it is important to address weighted voting systems. The weighted voting system formed by the parties in the legislature is the backbone of every lobbying game. Thankfully, there are only a limited number of functionally distinct weighted voting systems for each value of n, the number of parties, though a rule relating n and the number of functionally distinct weighted voting systems is unknown. To follow the conventions in Jiang (2013), under the condition of a pre-fixed quota, if two voting systems are functionally the same, meaning that they are characterized by the same set of winning and losing coalitions, then they are part of the same "region." For small numbers of parties, the weighted voting system can be solved completely, and all possible regions can be identified.

Each region is characterized by its set of minimum winning coalitions, from which the sets of winning and losing coalitions can be derived. For each region, in the context of the lobbying game, each member of the power set of the parties can be labeled as winning, blocking, or losing, defined as follows:

Winning: $\mathcal{C} \in \mathbf{W} \leftrightarrow \sum_{i \in \mathcal{C}} w_i \geq q$
Blocking: $\mathcal{C} \in \mathbf{B} \leftrightarrow 1 - q < \sum_{i \in \mathcal{C}} w_i < q$
Losing: $\mathcal{C} \in \mathbf{L} \leftrightarrow \sum_{i \in \mathcal{C}} w_i \leq 1 - q$

Note that for either player, securing a winning coalition will result in a win while securing a blocking coalition will result in a draw. Additionally, these relations hold, guaranteeing that the game is constant-sum.

(1) $\mathcal{C} \in \mathbf{W} \leftrightarrow \mathcal{N} \backslash \mathcal{C} \in \mathbf{L}$
(2) $\mathcal{C} \in \mathbf{B} \leftrightarrow \mathcal{N} \backslash \mathcal{C} \in \mathbf{B}$

Before analyzing a lobbying game, it is important to note in which region it lies, since that is what determines which coalitions win, draw, or lose for the formula $E = p(\text{win}) + \frac{1}{2} p(\text{draw})$.

2.3 The Geometric Representation of Weighted Voting Systems

The geometric representation of weighted voting systems is an alternative to the traditional algebraic representation that has several advantages. The geometric representation maps all possible n-player weighted voting systems normalized so that the sum of the players' weights is one to a $n - 1$ dimensional simplex. The quota is then attached to the simplex. Most relevant is how regions are simply polytope-like areas within the simplex – hence the name "region." Regions are bounded by hyperplanes representing a coalition's weight being equal to the quota and the sides of the simplex, so their shapes change with the quota. Sometimes, a change in the quota will cause certain regions to disappear completely, and other new regions to suddenly appear. This fact is important since it implies that not all regions exist at all values of the quota, so depending on the value of the quota adopted by the legislature in the lobbying game, not all regions may be possible.

In order to divorce the limitations from needing to append the quota to each simplex, a new complete geometric representation must be introduced. This new representation includes the quota as an additional dimension by stacking the simplexes of the geometric representation. The resulting figure now shows the superregions, all the weighted voting systems that share the same set of minimum winning coalitions, regardless of the quota. The set of superregions represents all possible distinct weighted voting systems for a certain number of players.

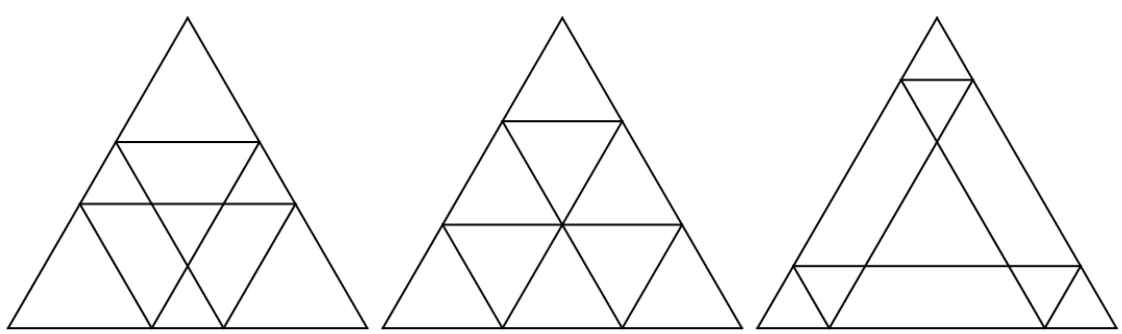
2.4 Three Player Weighted Voting Systems

Since many of the examples in this paper will involve a lobbying game with a three-party legislature, three player weighted voting systems will be analyzed. Three player weighted voting systems have a total of 11 superregions, but at most 10 regions can coexist at any one value of the quota. These 11 superregions can be divided into 5 region types, where two regions or superregions are of the same region type if they can be made to be identical with a permutation of the players. Region types are represented by the superregion that satisfy the canonic rule of players with a lower index being more desirable than players having a higher index. The following table summarizes the five region types of three player weighted voting systems.

Table 1. The Five Unique Region Types for 3-Player Weighted Voting Systems

Index #	Minimum Winning Coalitions	Banzhaf Power Distribution	Quota Range
1	[1]	$[1, 0, 0]$	$(\frac{1}{2} - 1]$
2	[12]	$[\frac{1}{2}, \frac{1}{2}, 0]$	$(\frac{1}{2} - 1]$
3	[12][13]	$[\frac{3}{5}, \frac{1}{5}, \frac{1}{5}]$	$(\frac{1}{2} - 1)$
4	[12][13][23]	$[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$	$(\frac{1}{2} - \frac{2}{3}]$
5	[123]	$[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$	$(\frac{2}{3} - 1]$

The geometric representations of three-player weighted voting systems at three different intervals for the quota are given below. Each polygon within the large simplex is a region.



(a) $q < \frac{2}{3}$ (b) $q = \frac{2}{3}$ (c) $q > \frac{2}{3}$

Figure 1. The Geometric Representation for 3-Player Weighted Voting Systems

Each region type is characterized by its set of minimum winning coalitions. The rest of this paper will refer to these region types by their index number. The Banzhaf power distribution is the power possessed by each player according to the Banzhaf power index, introduced in Banzhaf (1965). It equates power with the proportion of swing votes that a player possesses. An alternate way of measuring voting power is the Shapely-Shubik power index, introduced in Shapley and Shubik (1954), but this paper will not use that power index, for reasons to be made apparent in the next section. The quota range refers to the values of the quota at which the region exists. Most notable in this regard is how the quota ranges of region types 4 and 5 are mutually exclusive, since when one disappears, the other appears, so depending on the value for the quota, only one of these two regions may exist. For this reason, it becomes necessary to split some of the examples into two cases, one with $\frac{1}{2} < q \leq \frac{2}{3}$ and one with $\frac{2}{3} < q \leq 1$. A comparison of region types 4 and 5 in the context of the lobbying game will be conducted later, showing that the two have very different properties despite both being regions where all players have equal power.

3 Theory of the Lobbying Game

3.1 Basic Formulas

This section of the paper will elaborate on the general formulas for the lobbying game. It will begin with a restatement of the lobbying game. There are two players, A and B, who are engaged in a constant-sum game: their payoffs will always add to 1. This paper will be written from player A's perspective by default. The parties to be lobbied are $\mathcal{N} = \{1, 2, \dots, n\}$. The region that the weighted voting system formed by the parties and the quota of the legislature lies in will determine three sets of coalitions: $\mathcal{W}$, the set of coalitions winning for player A; $\mathcal{B}$, the set of "blocking" coalitions that result in a draw; and $\mathcal{L}$, the set of coalitions winning for player B, and hence losing for player A.

Players A and B each need to make an input of a vector of n elements: $[a_1, a_2, \dots, a_n]$ for player A and $[b_1, b_2, \dots, b_n]$ for player B, subject to the normalized budget constraint $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 1$. The possibility of asymmetric budgets will be addressed in section 5. Then, a series of probabilities p_i are defined for each $i \in \mathcal{N}$, where p_i is the probability of party i voting for player A:

$$p_i = \begin{cases} \frac{a_i}{a_i + b_i} & a_i > 0 \text{ or } b_i > 0 \\ \frac{1}{2} & a_i = b_i = 0 \end{cases}$$

The probability of party i voting for player B is the complement of the probability of party i voting for player A, since these two events are mutually exclusive and exhaustive. Define $q_i = 1 - p_i$ to be the probability of party i voting for player B. A coalition can be expressed as a bit vector $\mathcal{C} = \{c_1, c_2, \dots, c_n\}$ where:

$$c_i = \begin{cases} 1 & \text{party } i \text{ is in the coalition} \\ 0 & \text{party } i \text{ is not in the coalition} \end{cases}$$

Then, a series of functions f_i can be defined for each coalition:

$$f_i(\mathrm{C}) = c_i p_i + (1 - c_i) q_i$$

And the expected value formula:

$$E = p(\text{win}) + \frac{1}{2} p(\text{draw})$$

becomes:

$$E = \sum_{\mathrm{C} \in \mathcal{W}} \prod_{i=1}^{n} f_i(\mathrm{C}) + \frac{1}{2} \sum_{\mathrm{C} \in \mathcal{B}} \prod_{i=1}^{n} f_i(\mathrm{C})$$

The above formula is the one that was used to perform all of the calculations in the next two sections. Each superregion has its own formula for the expectation, but the same expectation function can be used for every weighted voting game in one superregion.

3.2 The Equilibrium Strategy

A theorem determining the equilibrium strategy in the lobbying game was first proven in Lake (1979). It roughly states that the equilibrium strategy in the lobbing game is to assign resources in a way that is proportional to each party's Banzhaf power.

The calculations done in Leech (1990), a paper exploring the intricacies of and differences between the Banzhaf and Shapley-Shubik power indices, lead to this result in an extremely elegant manner – a major assumption of that paper was that a vector of p_i's could be calculated for the parties, where p_i is the probability that a party votes a certain way. One conclusion of that paper was that if the probabilites were drawn from a random distribution that has a mean of $\frac{1}{2}$, the expected gain each player brings to the coalition is equal to his Banzhaf power. Since the probabilities must have a mean of $\frac{1}{2}$ due to symmetry in the lobbying game where both players have equal resources, the expected gain that a player will get from obtaining a party is equal to that party's Banzhaf power, so each player should distribute his resources according to the Banzhaf power of the parties, which is exactly the theorem from Lake (1979).

Now one can understand why the use of the Shapley-Shubik power index is not appropriate in the context of the lobbying game. The equbilirum strategy depends only on the Banzhaf power distribution of the parties. In a more traditional Blotto-style lobbying game where each player's payoff is directly proportional to the number of votes he receives, the equilibrium strategy is to allocate resources in a way that is proportional to each party's weight. Why a different strategy takes over when the goal changes to meeting a target quota can be easily explained. Consider a party with very little but still positive weight who is a dummy in the weighted voting system. In the modified lobbying game, both players would want to assign a nonzero quantity of resource to this party since winning that party would increase a player's payoff. In the lobbying game, however, winning a dummy cannot produce an increase in one's payoff – there is no coalition in which the dummy is a swing voter, so the capture of its vote will help neither player. A party's worth to the player then becomes not its weight but its capacity to push the player over the quota – which is exactly its Banzhaf power.

By the definition of an equilibrium strategy, if one player plays the equilibrium strategy, then the other player's best response is the same strategy. The next section explores the consequences of deviating from the optimal strategy, first in general and then for each of the five region types that the weighted voting system formed by the quota and parties could be part of, in order to prepare for the analysis of the lobbying game with asymmetric information.

4 Deviation from the Equilibrium Strategy

4.1 General Discussion and Methodology

The analysis of this section will be mostly graphical. The images will graph Player A's expectation for each of his allocations when player B's strategy is pre-determined. Graphs will be drawn in the form of a prism, much like the complete geometric representation, except the simplex will represent not the distribution of weights, but player A's allocation. The additional dimension will represent the expectation.

In each graph, the optimal point is located, using the standard method of critical points. For graphs representing scenarios where the second player plays the equilibrium strategy, it is expected that the optimal strategy for player A is to mirror the equilibrium strategy, resulting in an expectation of $\frac{1}{2}$. All other strategies are expected to be inferior, but an important fact to note is that all expectation functions must be continuous.

One should see how deviating from the equilibrium strategy punishes a player for doing so, because in the analysis of what a player should do if he is unsure of which region the weighted voting game formed by the parties and quota lies in, a method will be described that requires an understanding of the effects of deviating from the equilibrium strategy in each one of the superregions.

4.2 Example of Three-Party Lobbying Game

In the three-party lobbying game, player A and player B only have three parties to lobby, so their allocation vectors are $[a_1, a_2, a_3]$ and $[b_1, b_2, b_3]$. This paper is primarily interested in how each of player A's allocations fares against player B playing the equilibrium strategy. By the definition of the equilibrium strategy in a symmetric game, player A's expectation can be no greater than $\frac{1}{2}$, but it is interesting to explore the shape of the surface described by the function. The expectation will be graphed on a triangular prism, where the bases are 2-dimensional simplexes.

The three-party lobbying game has 11 superregions divided into 5 region types. Since superregions of the same region type behave in the same way except permuted, translating to a single rotation in the graphs, it is only necessary to explore the 5 region types. The expectations in this section were calculated using the formulas presented in the previous section.

First is region type 1, the region where party 1 is party whose sole vote determines the outcome. No ties are possible in this region. Since the equilibrium strategy is $[1, 0, 0]$, the expectation function for this region is:

$$E_{\text{Region Type 1}} = \frac{a_1}{1 + a_1}$$

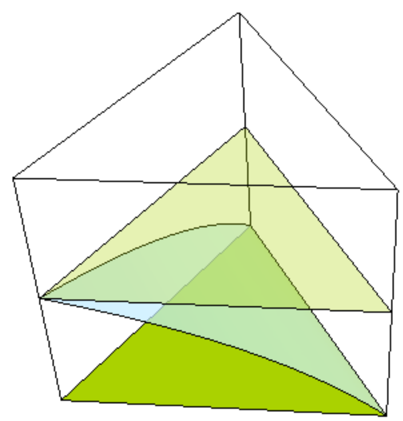


Figure 2. Region Type 1

The graph shows the outline of a triangular prism – this is the domain and range of the graph. The base of the prism, the simplex, has a side length of $\frac{2}{\sqrt{3}}$ and a height of 1. The darker green simplex is where the expectation is 0, the lighter green simplex is where the expectation is $\frac{1}{2}$, and the top of the prism is where the expectation is 1. The light blue surface is the graph of the expectation over the simplex. Each side of the simplex is labeled with a party: the right-side slanted side with party 1, the horizontal side with party 2, and the left-side slanted side with party 3. At points inside the simplex, the amount of resource given to each party is equal to the distance from that point to the side labeled with that party. Therefore, vertices represent points at which one player is given the entirety of player A's resources.

As expected, the equilibrium strategy in region type 1 is for player A to allocate all of his resources to party 1. The expectation declines gradually and then more sharply as resources are diverted away from party 1. If an unknown legislature has a high probability of being in a region of type 1, then an extremely skewed allocation will be favored.

Second is region type 2, the region where both parties 1 and 2 are needed to pass a measure. A tie is possible if party 1 votes for one player and party 2 votes for the other. The expectation function for this region is:

$$E_{\text{Region Type 2}} = \frac{a_1 + a_2 + 4a_1a_2}{1 + 2a_1 + 2a_2 + 4a_1a_2}$$

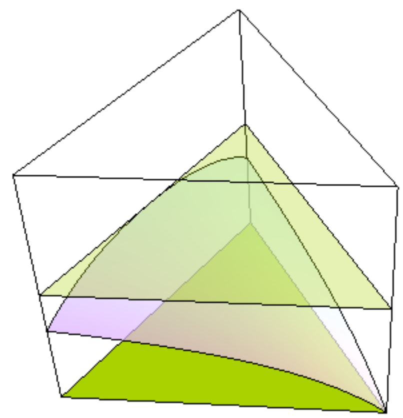


Figure 3. Region Type 2

The equilibrium strategy is where this surface is tangent to the plane representing the expectation equaling $\frac{1}{2}$ – this occurs at the allocation $[\frac{1}{2}, \frac{1}{2}, 0]$. This result makes sense since parties 1 and 2 are equally important in this region type while party 3 is irrelevant. The expectation declines moderately if player A favors one of party 1 and 2 over the other – even if he allocates all of his resources to one of these two parties, he will enjoy a modest expectation of $\frac{1}{3}$. By allocating any part of his resources to party 3, however, player A will experience a severe decline in his expectation. If this region is predicted to be likely when the superregion the lobbying game is in is unknown, it will discourge allocating very much to party 3.

Third is region type 3, which is perhaps the most interesting region. Here, a player needs party 1 and either party 2 or party 3 to win. A draw results if one player acquires party 1 and the other acquires parties 2 and 3. The expectation function for this region is:

$$E_{\text{Region Type 3}} = \frac{5(15a_2a_3 + a_1(1 + 10a_3 + 10a + 2(1 + 5a_3)))}{2(3 + 5a_1)(1 + 5a_2)(1 + 5a_3)}$$

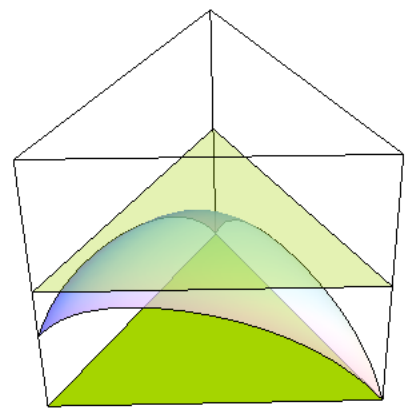


Figure 4. Region Type 3

The maximum of the expectation function occurs at the allocation $[\frac{3}{5}, \frac{1}{5}, \frac{1}{5}]$. In this region, erring towards party 1 is less punishing than erring towards parties 2 or 3; even if player A allocates all of his resources to party 1, he still enjoys a modest expectation of $\frac{5}{16}$. This region type is interesting because it is the only one where the Banzhaf power index and Shapley-Shubik power index disagree. If player B allocates his resources according to the Shapley-Shubik power index instead, $[\frac{2}{3}, \frac{1}{6}, \frac{1}{6}]$, the following results are obtained:

$$E_{\text{Region Type 3 Shapley}} = \frac{3(a_1 + 12a_1a_2 + 12(a_1 + 2a_2 + 6a_1a_2)a_3)}{2(2 + 3a_1)(1 + 6a_2)(1 + 6a_3)}$$

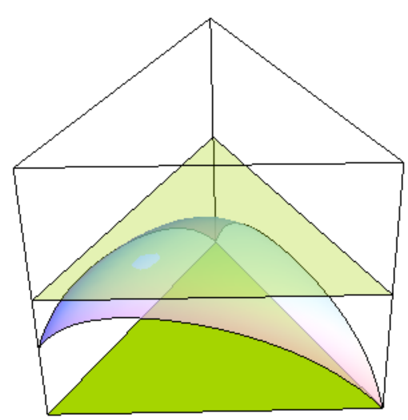


Figure 5. Region Type 3, Shapley-Shubik strategy

The graph is strikingly like the one for the Banzhaf power index, but close examination reveals that this strategy is flawed. If player A allocates his resources according to the Banzhaf power index while player B allocates his according to the Shapley-Shubik power index, then player A will enjoy an expectation of approximately 0.503045, which is greater than $\frac{1}{2}$. If player A instead employs the strategy that best exploits the Shapley-Shubik strategy – this is accomplished by taking the Banzhaf strategy and increasing the amount allocated to parties 2 and 3 by approximately 0.000246, and reducing the amount allocated to party 1 by double that amount – he will only enjoy an expectation of 1.5×10^{-7} more than what he could have secured by using the Banzhaf strategy. Thus, the equilibrium strategy does a remarkably good job of exploiting strategies that deviate from it by a small amount, like the strategy of allocating according to Shapley-Shubik power.

Fourth is region type 4, the region of majority rule. Ties are impossible in this region. The expectation for this region is:

$$E_{\text{Region Type 4}} = \frac{9a_1a_2 + 9a_1a_3 + 9a_2a_3 + 27a_1a_2a_3}{(1+3a_1)(1+3a_2)(1+3a_3)}$$

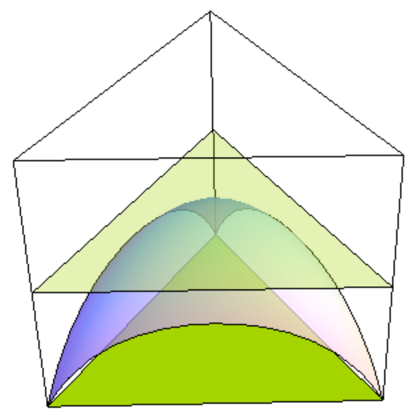


Figure 6. Region Type 4

Since all parties play symmetric roles in this superregion, the equilibrium strategy is to allocate evenly between all three parties. Any deviation from player A will be met with a sharp decline in expectation. This region only exists when $q \le \frac{2}{3}$, because when the quota exceeds that value, region type 5 takes its place.

Last is region type 5, the region requring unanimity. The expectation for this region is:

$$E_{\text{Region Type 5}} = \frac{3a_1 + 3a_2 + 3a_3 + 9a_1a_2 + 9a_1a_3 + 9a_2a_3 + 54a_1a_2a_3}{2(1+3a_1)(1+3a_2)(1+3a_3)}$$

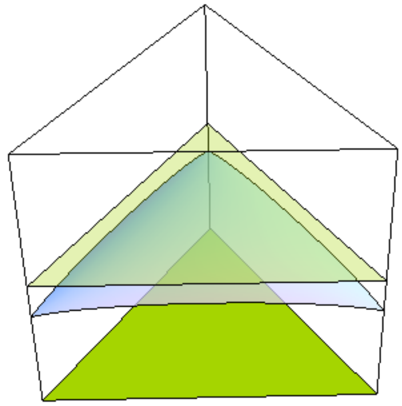


Figure 7. Region Type 5

This region shares the same equilibrium strategy as region type 4, but one will immediately notice that deviating from the equilibrium strategy does not have a very devastating impact on the expectation. Indeed, the expectation from player A allocating all of his resources to one party still results in a respectable expectation of $\frac{3}{8}$. This property can be explained by noticing how a tie is extremely likely in this region. Since six out of eight of the possible coalitions are in $\mathcal{B}$, the expectation is driven towards $\frac{1}{2}$.

The graph below illustrates the difference between the expectation in region type 5 and the expectation in region type 4:

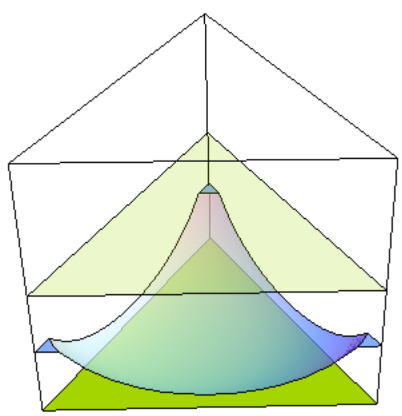


Figure 8. Difference between Region Type 5 and Region Type 4

One can see that the difference is positive except at $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ and that the difference grows larger towards the corners and then levels off. Region type 4 represents a strong weighted voting game, so the probability of a draw is zero, while region type 5 requires unanimity, so there are only two coalitions that do not result in a draw. If a player is put in a situation where he does not know which region the weighted voting system is in but his opponent does, he would prefer a quota greater than $\frac{2}{3}$, so that he will suffer less from deviating from the equilibrium strategy.

4.3 Overall Conclusions about Deviation from the Equilibrium Strategy

This section explored the consequences of deviating from the equilibrium strategy in the lobbying game, with the case of three parties as an example. The graphs of the expectation functions for Player A in the case where Player B plays the Banzhaf strategy showed that the maximum expectation indeed occurs at the Banzhaf power distribution, confirming that the Banzhaf strategy is an equilibrium.

In regions of all types in three-player weighted voting systems, the Banzhaf power distribution is the same as the Shapley-Shubik power distribution, except for region type 3. Checking the graph of the expectation function for Player A against a Player B who plays the Shapley-Shubik strategy reveals that it fails the conditions for an equilibrium player A has a strategy different from the Shapley-Shubik strategy that results in an expectation greater than 0.5; hence, this strategy is not an equilibrium.

From the comparison between region type 4 and region type 5, it was concluded that in regions where it is more difficult to pass legislation (e.g. higher quota, fewer winning coalitions, more players with veto power), the repercussions of deviating from the equilibrium strategy are less, while in strong voting games, deviating from the equilibrium strategy results in large decreases in expectation.

5 The Lobbying Game with Asymmetric Information

5.1 Introduction and Practical Analogue

The lobbying game with asymmetric information models a situation where two players with opposing interests need to lobby various parties, but one player is unsure of the exact configuration of the legislative body that the parties are part of. Image this scenario. Player A and player B have opposing interests and equal, infinitely divisible resources to allocate among the n parties that form a legislature. However, it is pre-election season. Though the parties are not due to vote on the issue of interest until after the election, the players must allocate their resources before the election. Player B has conducted an extensive survey that will allow him to accurately predict the results of the election. Thus, he will always play the equilibrium strategy for the region he knows that the legislature will end up in. Knowing that player B will play the aforementioned strategy, how should player A allocate his resources?

In general, player A may not want to play an equilibrium strategy for a particular superregion, since this strategy will likely perform poorly in another superregion – the section exploring deviation from the equilibrium strategy revealed that large deviations from the equilibrium result in a much lower expectation, so that is not optimal. Rather, the goal is to find some kind of middle ground between all of the superregions deemed likely.

The three-party lobbying game will be used as an example to illustrate the proposed methodologies.

5.2 The Scenario of Perfect Estimation

If player A can estimate the probability that the legislature will lie in each particular superregion, such as by using previous election data, then a solution becomes obvious. Player A will still be disadvantaged compared to his adversary, since player B knows exactly which superregion the legislature will lie in, while player A can only determine a probability distribution, but now that player A has information about the probability that the legislature will fall in each superregion, he can take a weighted average of the expectations of all the superregions to create a new expectation function. This expectation function can be calculated because player B's strategy is known for each of the superregions, and since its value depends solely on player A's allocation of resources, he can maximize his expectation.

As an example, consider a case where the quota is fixed at $\frac{7}{10}$. Player A estimates that a party 1 dictatorship (region type 1), party 1 and 2 joint rule (region type 2), and unanimous rule (region type 5) are equally likely. Hence, the expection can be calculated with the following equation:

$$E = \frac{1}{3}E_{\text{Region Type 1}} + \frac{1}{3}E_{\text{Region Type 2}} + \frac{1}{3}E_{\text{Region Type 5}}$$

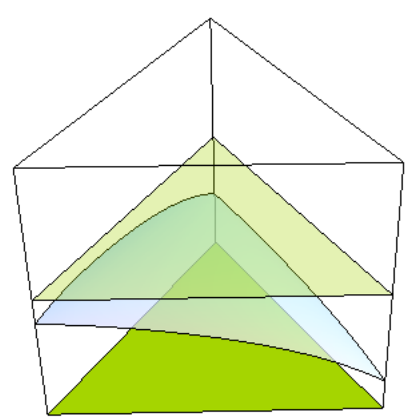


Figure 9. Example for Perfect Estimation

It is easy to find that the maximum of the expectation subject to the constraints of a legal allocation occurs at the point $[0.708282, 0.260279, 0.031439]$, which produces an expectation of 0.436542 against Player B's strategy.

When Player A can estimate the probability of each superregion occurring, then his problem can be solved by taking a weighted average of the expectation functions for all the regions. This method, however, has the drawback of relying on the accuracy of player A's probability estimations. Additionally, the resulting expectation function being maxed is an average of multiple other expectation functions, so if the legislature ends up in some superregions, player A may face an expectation lower than the one he estimated. Conversely, in other superregions, player A may enjoy a higher expectation than estimated.

5.3 Equating the Lobbying Game with Asymmetric Information with a Game Against Nature

If player A cannot estimate the probabilities of each of the superregions occuring, such as if he has no previous election data to work with, or a change in the political landscape has rendered previous data unreliable in making future predictions, then the method presented in the previous subsection cannot be used. Fortunately for player A, his predicament can be equated with a game against nature. In such a game, there is only one sentient player who is trying to maximize his payoff. The second player is replaced by an entity called nature, who makes her decision, called the state of nature, completely randomly in a way unknown to the first player. Since which region the new legislature will be in is unknown, and player B's strategy depends solely on which region the new legislature will fall in, a piece of information known to him, the region of the new legislature and player B's strategy can be lumped together as nature.

There are already some well-known recommendations for strategies to employ in a game against nature; a few were discussed in Straffin's *Game Theory and Strategy*. One of these suggestions was made by Pierre-Simon Laplace in 1812. His recommendation was to assume that all of the states of nature were equally likely. When one has no information about the probabilities of the states of nature, then this approach is fundamentally appealing. It does, however suffer from the problem of column duplication – if the same state of nature is listed twice, it does not make it any more likely, but the Laplace strategy will interpret it as so. Since the Laplace strategy assigns probabilities to the superregions when none existed before, the method adopted in the previous subsection can be used again.

Due to the symmetrical nature of the expectation functions in the lobbying game, when the expectation functions for all 10 regions are summed and averaged, the maximum will be at $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$, featuring an expectation of 0.347524, or barely more than $\frac{1}{3}$.

Player A will want to somehow even his chances. An interesting idea to explore is that if it were possible for player A to secretly increase his resources, how much of an increase in his resources would he need to neutralize player B's advantage from knowing the superregion that the legislature will lie in. In this way, one can measure the value of player B's survey. It is necessary to secretly perform this increase, since playing the equilibrium strategy for the case of equal resources is no longer optimal in the situation of asymmetric resources. As it turns out, player A will need to increase his resources to 1.91301 times player B's resources and then allocate them equally between the three parties to increase his expectation to $\frac{1}{2}$. Thus, player B's survey is worth an amount equal to more than 91% of the budget for lobbying of the two players.

5.4 The Canonic Three-Party Lobbying Game as an Example

5.4.1 Outline of the Example

It was too intuitive for player A to adopt the allocation $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ in the previous subsection. A more complex and more interesting variant is if player A can be sure that only the five canonical regions are capable of occuring. One can imagine that the parties in the legislature have a strict hierarchy of strength, and a stronger party cannot play a lesser role in the legislature than a weaker party. Here, symmetry is destroyed so the analysis becomes more complicated. One thing to notice, however, is that since region type 4 and region type 5 cannot exist at the same values of the quota, two cases must be explored, one with $q \leq \frac{2}{3}$ and one with $q > \frac{2}{3}$.

In each of these cases, three subcases will be explored, detailing three different strategies that player A may choose to adopt. The first of these will be the Laplace strategy, as defined in the previous subsection. In short, it recommends to treat all regions as equally likely. The second of these will be the strategy recommended by Abraham Wald in 1950. He recommended a maximin strategy – in other words, he suggested to always assume the worst-case scenario. This strategy has the advantage of estimating a worst-case expectation – by following this strategy, player A is guaranteed an expectation of no lower than his estimate. Finally, this paper introduces a variant on the Laplace strategy that will be called the geometric strategy. Rather than assuming that all regions are equally likely, this strategy will assume that all canonic weight vectors are equally likely. This assumption will have the effect of assuming that the probability of a region occuring is proportional to its area in the geometric representation. In higher dimensions, the hypervolume can be calculated and used in the same way as the area in two-dimensions for this strategy. One advantage that the geometric representation has over the algebraic representation of weighted voting systems is the ability to estimate the probability of a region occuring in this way.

In the next subsubsections, all numbers will be reported to six decimal places.

5.4.2 The Case of $q \leq \frac{2}{3}$

Since $q \leq \frac{2}{3}$, the only regions that can occur are the regions representing region types 1, 2, 3, and 4. As an example, this paper will use the case of $q = 0.6$.

Laplace's suggestion results in the following expectation function:

$$E = \frac{1}{4}E_{\text{Region Type 1}} + \frac{1}{4}E_{\text{Region Type 2}} + \frac{1}{4}E_{\text{Region Type 3}} + \frac{1}{4}E_{\text{Region Type 4}}$$

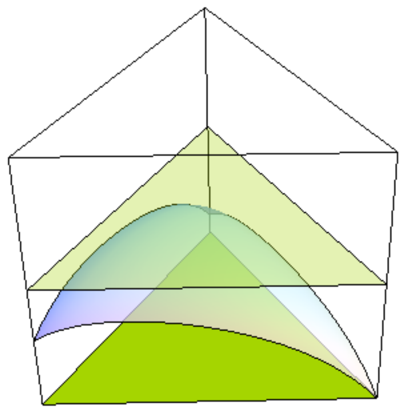


Figure 10. $q = 0.6$, Laplace strategy

It is easy to calculate that the optimal point lies at a distribution of $[0.576379, 0.273216, 0.150405]$, with an expectation of 0.437271. The expectation in this case reflects the expectation given that Laplaces assumptions are true; the actual expectation may be much lower. If player A wishes to enjoy an expectation of $\frac{1}{2}$, then he must increase his resources to 1.244055 times that of player B's.

Wald's suggestion leads to an expectation function calculated by taking the minimum of the expectations of all the possible regions. In this case:

$$E = \text{Min}(E_{\text{Region Type 1}}, E_{\text{Region Type 2}}, E_{\text{Region Type 3}}, E_{\text{Region Type 4}})$$

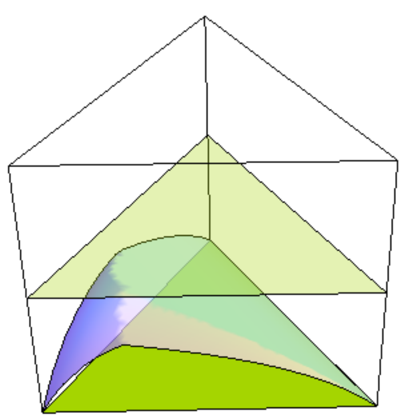


Figure 11. $q = 0.6$, Wald strategy

A quick check gives the maximum point to be at $[0.674382, 0.162826, 0.162792]$, with an expectation of 0.400877. The expectation in this case refers to the expectation if Walds assumptions are true. In other words, it is a lower bound on the expectation of player A. By following Wald's suggestion, player A can guarantee that he will enjoy an expectation of at least $\frac{4}{10}$. If player A wishes to guarantee an expectation of at least $\frac{1}{2}$, he must secretly increase his resources to 1.416665.

To implement the modified Laplace strategy based on the geometric representation of weighted voting systems, it is necessary to calculate the areas of the portions of the regions in the canonic region of the simplex. The figure below shows the geometric representation of three-player weighted voting systems when the quota is 0.6. The canonic portion of the simplex is outlined in red. This area interesects with the four aforementioned regions, and their areas that lie within the canonic portion are listed in the table below.

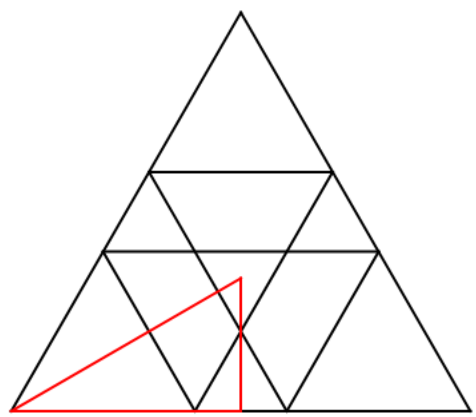


Figure 12. $q = 0.6$, Canonic region of the geometric representation

Table 2. Areas of the canonic regions when $q = 0.6$

Region Type #	Area Occupied	Percent Occupied in Canonic Region
1	$\frac{2}{25\sqrt{3}}$	48%
2	$\frac{1}{50\sqrt{3}}$	12%
3	$\frac{3}{50\sqrt{3}}$	36%
4	$\frac{1}{150\sqrt{3}}$	4%
Total	$\frac{1}{6\sqrt{3}}$	100%

The expectation can now be calculated because values have been assigned as the probability of each region occuring.

$$E = 0.48E_{\text{Region Type 1}} + 0.12E_{\text{Region Type 2}} + 0.36E_{\text{Region Type 3}} + 0.04E_{\text{Region Type 4}}$$

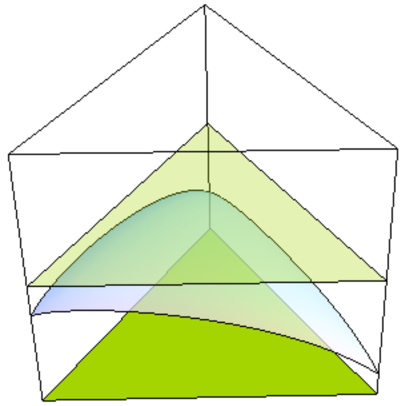


Figure 13. $q = 0.6$, Geometric strategy

It is easy to calculate that the optimal distribution is $[0.763666, 0.147550, 0.088784]$ resulting in an expectation of 0.442926. To enjoy an expectation of $\frac{1}{2}$, player A's resources must total 1.239245.

5.4.3 The Case of $q > \frac{2}{3}$

This will be the case representative of scenarios with $q > \frac{2}{3}$, so the only regions that can occur are the regions representing region types 1, 2, 3, and 5. Let $q = 0.7$.

In this case, if one follows Laplaces suggestion, then the expectation function is:

$$E = \frac{1}{4}E_{\text{Region Type 1}} + \frac{1}{4}E_{\text{Region Type 2}} + \frac{1}{4}E_{\text{Region Type 3}} + \frac{1}{4}E_{\text{Region Type 5}}$$

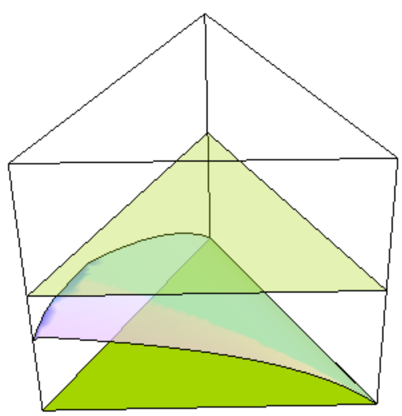


Figure 14. $q = 0.7$, Laplace strategy

It is possible to calculate that the optimal point lies at a distribution of [0.654354, 0.238478, 0.107168], with an expectation of 0.447297. To make his expectation equal with player B's, player A must increase his resources to 1.250765.

Again, some players may prefer the more risk-adverse strategy prescribed by Wald. The expectation function will be:

$$E = \mathrm{Min}(E_{\text{Region Type 1}}, E_{\text{Region Type 2}}, E_{\text{Region Type 3}}, E_{\text{Region Type 5}})$$

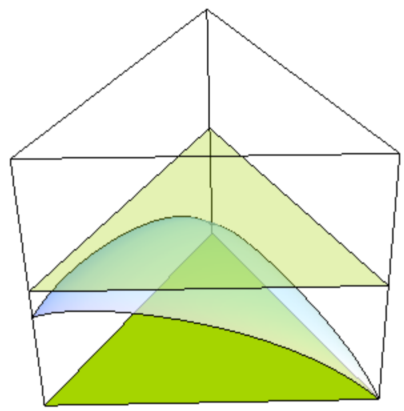


Figure 15. $q = 0.7$, Wald strategy

In this case, the prescribed strategy is [0.764871, 0.177571, 0.057558], with an expectation of 0.433386. To guarantee an expectation of at least $\frac{1}{2}$, player A must increase his resources to 1.398150.

Finally, the geometric strategy will take advantage of the properties of the geometric representation. The figure below shows the geometric representation of three-player weighted voting systems when the quota is 0.7, with the canonic portion of the simplex is outlined in red. This area interesects with the regions representing region types 1, 2, 3, and 4, and their areas that lie within the canonic portion are listed in the table below.

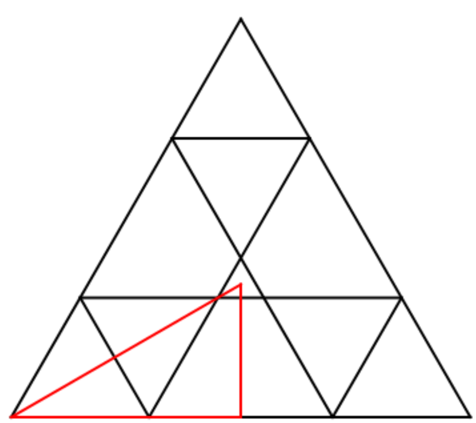


Figure 16. $q = 0.7$, Canonic region of the geometric representation

Table 3. Areas of the canonic regions when $q = 0.7$

Region Type #	Area Occupied	Percent Occupied in Canonic Region
1	$\frac{9}{200\sqrt{3}}$	27%
2	$\frac{3}{40\sqrt{3}}$	45%
3	$\frac{9}{200\sqrt{3}}$	27%
4	$\frac{1}{600\sqrt{3}}$	1%
Total	$\frac{1}{6\sqrt{3}}$	100%

Since values have been assigned as the probability of each region occuring, the expectation can now be calculated.

$$E = 0.27E_{\text{Region Type 1}} + 0.45E_{\text{Region Type 2}} + 0.27E_{\text{Region Type 3}} + 0.01E_{\text{Region Type 4}}$$

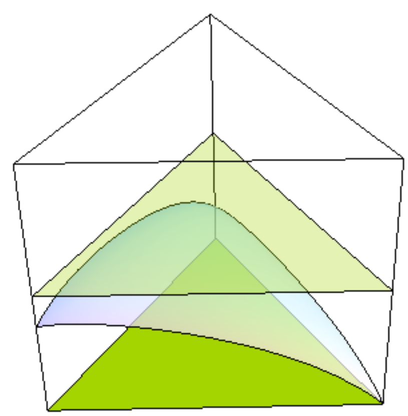


Figure 17. $q = 0.7$, Geometric strategy

The optimal point is $[0.685582, 0.278690, 0.035738]$, with an expectation of 0.448044. Resources totaling 1.225365 are needed to produce an expectation of $\frac{1}{2}$.

5.5 Summary of Findings

A situation of asymmetric information was created to study the effects of Player A not knowing which region the partys voting system is in while Player B knows this information and plays the equilibrium strategy for each region. The three-party case was explored as an example again.

In subsection 5.2, a method was described that could easily determine Player As optimal strategy and his expectation if he can assign probabilities to the regions.

In subsection 5.3, it was shown that if player A cannot assign probabilities to the regions, then he can treat the game like a game against nature. The simplest suggestion in this case is Laplaces equal probability assumption, which results assigns probabilities to the regions and allows the method in case 1 to be applied. It was shown that player A will need to distribute his resources evenly between the three parties and suffer an expectation of slightly more than $\frac{1}{3}$. If he wishes to enjoy an expectation of $\frac{1}{2}$, then he must increase his resources to 1.91301 times player Bs.

In subsection 5.4, symmetry was destroyed by exploring only the canonic part of the simplex as possible weighted voting systems, and two subcases were considered based on the quota. Laplaces assumption was tested again for each subcase, but it was then argued that assigning probabilities using region area may be more appropriate, thus introducing the new geometric strategy. Region area analysis revealed that Player A ought to invest more resources in party 1, and if he does, he will enjoy a slightly higher expectation than in the Laplace case. When the quota is higher, replacing region type 4 with region type 5, it was found that Player A should increase his investment in party 1 to maximize his expectation, which will be increased from the previous scenario of a lesser quota. Thus, a higher quota is more favorable to the player with less information.

Another strategy was explored in subsection 5.4, that prescribed by Wald. This strategy is unique in that it achieves the highest guaranteed level of expectation. Walds strategy lied in between the Laplace and geometric strategies in terms of favoring the most influential party. When the quota was greater, Walds strategy benefited with a 0.03 increase in the guaranteed level of expectation, showing once again that a higher quota is more favorable to the player with less information.

The table below lists the cases by quota and strategy and shows the recommended allocation, calculated expectation, and amount of resource needed to increase player A's expectation to $\frac{1}{2}$ for each case, summarizing the results of subsection 5.4.

Table 4. Summary results from subsection 5.4

Quota	Strategy	Recommended Allocation	Expectation	Equalization
0.6	Laplace	$[0.576379, 0.273216, 0.150405]$	0.437271	1.244055
0.6	Wald	$[0.674382, 0.162826, 0.162792]$	0.400877	1.416665
0.6	Geometric	$[0.763666, 0.147550, 0.088784]$	0.442926	1.239245
0.7	Laplace	$[0.654354, 0.238478, 0.107168]$	0.447297	1.250765
0.7	Wald	$[0.764871, 0.177571, 0.057558]$	0.433386	1.398150
0.7	Geometric	$[0.685582, 0.278690, 0.035738]$	0.448044	1.225365

6 Conclusion

The lobbying game, a variant of the Blotto games, was introduced and explored. The equilibrium strategy in the lobbying game, allocating proportionally to Banzhaf power, was stated in Lake (1979) and reproduced here. Deviating from the equilibrium strategy was explored, and it was found that depending on the characteristics of the region that the weighted voting system of the lobbying game is in, allocating more towards a particular party has differing effects. Particularly, in regions with a less strong voting system, meaning that it is more difficult to pass a measure due to winning coalitions requiring more members, deviating from the equilibrium strategy holds less noticeable consequences. An example supporting this conjecture was seen with the comparison of region types 4 and 5 in the three-party lobbying game. Lake's theorem held true in each of the five examples that were explored, and the strategy of allocating proportionally to Shapley-Shubik power was determined to be suboptimal.

All of the preceding analysis served to set up the analysis of the lobbying game with asymmetric information, the game where one player lacked information about which region the weighted voting system of the lobbying game was in. Without this critical piece of information, that player was forced to deviate from the equilibrium strategy, since it could not be identified. A solution to the lobbying game with asymmetric information in the case of perfect estimation was introduced, and this solution was combined with the idea that the lobbying game with asymmetric information could be treated like a game against nature to provide a method for recommending a strategy in the lobbying game with asymmetric information. Three methods of recommending a strategy were introduced, including the Laplace strategy, the Wald strategy, and the geometric strategy. The first and third strategies were more optimistic than the Wald strategy but suffered from having to make an assumption about the probabilities of each region occuring when this information could not be confirmed. The Wald strategy was more pessimistic than the other two and resulted in a lower expectation, but this expectation had the great advantage of being a worst-case estimate, meaning that the player's actual expectation could only be higher. The geometric strategy was novel in its taking advantage of the geometric representation of weighted voting systems, showing another of its advantages over the algebraic representation.

A scenario for the lobbying game and the lobbying game with asymmetric information was contrived, but they have many real applications, ranging from politics to sports. Most obviously, the lobbying game could be applied to a contest between opposing candidates in an election where the candidates must allocate their campaign funds and time to various districts. If the weights of the districts are known, then the candidates are able to play the equilibrium strategy; if not, then they may wish to follow the suggestions outlined in section 5.

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