Stable Coalitions with Power Accumulation (Incomplete version, do not cite)

Karl Jandoc and Ruben Juarez *

Department of Economics and UHERO, University of Hawaii

2424 Maile Way, Honolulu, HI 96822 (Corresponding Author: rubenj@hawaii.edu)

April 2013

Abstract

Agents endowed with power compete for a divisible resource by forming coalitions with other agents. The coalitions with the largest power wins the resource and divide it between its members.

We study four models of coalition formation where agents might or might not accumulate power and agents might or might not participate in further coalition formation processes. An axiomatic approach is provided by focusing in variations of two main axioms: *self-enforcement*, which requires that no further deviation happens after a coalition has formed, and *rationality* which requires that players pick the coalition that gives them their highest payoff.

For four different cases, we determine existence of stable coalitions that are *self-enforcing* and *rational* for different sharing rules. The stable coalitions found can be implemented as a coalition-proof subgame perfect Nash Equilibrium.

Keywords: Coaliton Formation, Power Accumulation, Self Enforcement.
JEL Classification C70 · D71 · D85

*Financial support from the AFOSR Young Investigator Program is greatly appreciated. Any errors or omissions are the authors' own.

1 Introduction

In many social situations, decisions are made within the context of a group. Indeed, the character and composition of these groups shape economic, political and societal outcomes. Cartels, lobby groups, customs unions, armed groups, political parties are obvious examples of groups, also called *coalitions*, that influence outcomes towards their favor (Ray [7]).

Despite the importance of coalition formation in many economic situations, the literature remains disunified (Ray and Vohra [8]). For instance, recent literature on coalition formation has focused mainly on the purely hedonic aspect, in which the payoff to a coalition member depends only on the composition of members of the coalition to which he belongs (Dreze and Greenberg [4], Bogomolnaia and Jackson [3], Sonmez, Banerjee and Konishi [9]). Bloch and Dutta [2] point out that an important aspect of coalition (and network) formation is the ability of the different groups to change a particular "social state". This is captured in their idea of a "effectivity relation" that measures a coalition's ability to change from a status quo into a different social state. This can be interpreted as a coalition's "power", which measures its ability to impose its will in a particular coalition formation game. Piccione and Razin [6] examine how power relations determine the ranking of agents in society. The identity of the coalitions (as characterized by the power of players within that coalition) determine the social order and thus the structure of society.

Furthermore, in a typical non-democracy where there is a heterogeneity of powers among agents and agreements are not binding, coalitions form to maintain power in order to impose its will in a given society. Over time, however, coalitions may disintegrate and new factions may form to overthrow the existing ruling coalition. The task of this paper is to find coalitions that are stable from the moment of inception.

Acemoglu et al. [1] [hereafter AES] studies a model of coalition formation where agents are endowed with power and form coalitions with the goal of becoming the most powerful coalition. The winning coalition will split a given resource in proportion to their power. Agents outside the winning coalition are "killed" and these players would not participate in future coalition formation process. AES's main stability concept, *self-enforcement*, requires that no subcoalition is powerful enough to ensure further deviations. Self-enforcement is a very powerful property that ensures coalition form at stage one.

However, the environment of AES is severely limited in many fronts. First, in many economic and political interactions, there are many ways how an "outsider" can influence the balance of power inside the coalition. In a static setting, Juarez [5] discusses an equilibrium notion where players outside the equilibrium coalition must not have sufficient power to be a threat to this coalition. The "no-threat equilibrium" does not presuppose that outside players will not have an influence on the stability of a coalition that has formed.

Second, there may be instances where a winning coalition would accumulate power over time. In many nondemocratic societies, a ruling coalition can perpetuate himself in authority by the fact that it can use the state's resources to consolidate and accumulate power to further dominate the rest of society.

Third, AES limit their discussion on a narrow class of sharing rules for the resource in play. The sharing rule used by AES is one of proportional-like sharing, where players with higher power gets a substantially higher share of the prize. Other rules, like equal division of the resources (henceforth called "equal-sharing"), are also appealing.

We therefore envision four scenarios that would be pertinent in modeling coalition formation in the context of non-democracies:

1. Power of the ruling coalition does not accumulate and non-winning players

are killed.

2. Power of the ruling coalition does not accumulate and non-winning players are not killed.

3. Power of the ruling coalition accumulates and non-winning players are killed.

4. Power of the ruling coalition accumulates and non-winning players are not killed.

AES falls in category 1 in this list. In this paper we study how stable coalitions that are self-enforcing form in the four scenarios above for different sharing rules.

2 The Model

Consider the set $N = \{1, \dots, n\}$ of initial agents who are endowed with powers $\pi = [\pi_1, \dots, \pi_n]$ respectively. A coalition C is a subset of N, $C \subset N$. The set of coalitions are all possible subsets of N, denoted by 2^N. A coalition formation game is a pair (S, π) where $S \subseteq N$ and $\pi \in \mathbb{R}_+^S$. The set of coalition formation games is denoted by $\mathbf{G}$. We assume that power is additive, that is, the power of coalition S is the sum of all powers of the players inside the coalition, $\pi(S) = \sum_{i \in S} \pi_i$. We denote as π_S the restriction of the vector $\pi \in \mathbb{R}_+^N$ over coalition S.

Definition 1 *Given a game (T, π), the set of winning coalitions is:*

$$W_{(T,\pi)} = \{S \subset T | \pi(S) > \pi(T \setminus S)\}$$

Definition 2 *A sharing rule is a function $\xi : 2^N \to \mathbb{R}_+^N$ such that:*

- $\sum_{i \in S} \xi_i(S) = 1$*; and*
- *(Cross-Monotonicity) If* $T \subset S$ *and* $i \in T$ *then* $\xi_i(T) > \xi_i(S)\ \forall\ S, T \in 2^N$

Cross-monotonicity of a sharing rule requires that the share of the prize of a player i in coalition S would be higher if he is part of any subcoalition of S that deviates compared to the share he will get if he stayed on coalition S.[1]

Throughout the paper we devote special attention to simple (and commonly used) sharing rules such as equal sharing and proportional sharing (see Juarez [5]), or a convex combination of the two (hereafter called "combination sharing"). That is,

$$\xi_i(S) = \begin{cases} \dfrac{I}{|S|} & \text{if equal sharing} \\ \dfrac{\pi_i}{\pi(S)} & \text{if proportional sharing} \\ \lambda \cdot \dfrac{I}{|S|} + (1-\lambda) \cdot \dfrac{\pi_i}{\pi(S)},\ \lambda \in (0,1) & \text{if combination sharing} \end{cases}$$

Note that these three basic sharing rules are cross-monotonic. Each player inside a winning coalition S that forms splits a prize I according to a predetermined sharing rule $\xi(S)$.

In the sections that follow, we use a property of sharing rules that the basic sharing rule may or may not satisfy. The property that we examine is consistent ranking of coalitions by players. Suppose agents i and j are players who belong to the intersection of coalitions S and T. A rule satisfies consistent ranking if whenever agent i prefers S over T, then agent j also prefers S over T. In other words, between competing coalitions, a coalition S is picked if all players in the intersection unanimously pick S over a competing coalition.

Definition 3 *(Consistent Ranking) The sharing rule* ξ *satisfies consistent ranking if for any two players* i *and* j*, and coalitions* S *and* T *such that* $i, j \in S \cap T$*,*

[1] This is contrary to the case where there are externalities, where agents might gain by associating with other agents of similar characteristics, see Juarez [5].

if $\xi_i(S) \geq \xi_i(T)$, *then* $\xi_j(S) \geq \xi_j(T)$

Under consistent ranking, we are able to assign a number denoted by $R_i(\cdot)$ by player i. For any player i, a rank $R_i : 2^N \to \mathbb{R}_+$ is a number such that for any coalitions $S, T \in 2^N$ we have $R_i(S) > R_i(T) \Leftrightarrow \xi_i(S) > \xi_i(T)$.

Note that the definition of consistent ranking implies the existence of a global ranking R for the society that coincides with individual rankings. That is, $R_i(S) \geq R_i(T) \Leftrightarrow R(S) > R(T)$ where $R(\cdot)$ is the rank of the society over coalitions.

Also note that equal sharing and proportional sharing satisfy consistent ranking. Indeed, under equal-sharing, agents' share increase as they move to coalitions of smaller sizes. Similarly, under proportional sharing, agents' share increase as they move to coalitions of smaller power.

The major task of this paper is to examine the equilibrium under the different sharing rules and, if it exists, characterize the stable coalitions that emerge.

2.1 Dynamic Coalition Formation

Let $t = 0, 1, \ldots$ denote the t discrete rounds of the game. We define a *transition correspondence* than maps from the set of coalition formation games to a particular set of coalitions.

Definition 4 *A* **transition correspondence**[2] *is a continuous*[3] *and scale-invariant*[4] *correspondence* $\phi : \mathbf{G} \to 2^N$ *such that* $\forall\ (X, \pi_X) \in \mathbf{G}$*:* $\phi(X, \pi_X) \subset W_{(X,\pi_X)}$.

The transition correspondence describes the movement from one coalition to another coalition throughout the rounds. In particular, the game at round t

[2]This definition includes two compelling axioms from AES, *Inclusion* and *Power.*

[3]A correspondence is continuous if for any generic vector of power there is always a neighborhoor around the power vector of every agent where the correspondence does not change.

[4]In the vector of power.

will be denoted by $\left(S^t, \pi^t\right)$.

In the next sections we will define the evolution of S^t given the transition correspondence ϕ. We look at the cases where $S^t = N$ or $S^t \in \phi(S^{t-1}, \pi^{t-1})$. The first case, $S^t = N$, depicts a scenario where agents outside of the formed coalitions are not killed, that is, they may participate in future coalition formation games in subsequent rounds. On the other hand, the case where $S^t \in \phi(S^{t-1}, \pi^{t-1})$ is where agents are killed and cannot participate in subsequent rounds.

The manner by which the power of the players accumulate will be affected by both the sharing rule ξ and the transition correspondence ϕ. Through subsequent stages, the power accumulation function for player i at stage t will be defined as

$$\pi_i^t = \begin{cases} \pi_i^{t-1} + \xi_i(S^{t-1})I & \text{if } S^{t-1} \in \phi(S^{t-1}, \pi^{t-1}) \\ \pi_i^{t-1} & \text{otherwise} \end{cases}$$

Note that in the case when agents are killed, the power of agents that are not part of the winning coalition will be irrelevant.

3 Case 1: $I = 0$ and $S^t \in \phi(S^{t-1}, \pi^{t-1})$

3.1 Desirable Properties

This section resembles the AES main features where if a coalition S forms, then players outside S are killed in the sense that they could not further participate in any future coalition formation process (i.e., this is the case where $S^t \in \phi(S^{t-1}, \pi^{t-1})$). The main task is to find rules that are "self-enforcing," that is we are interested in finding coalitions that do not have the incentive or the power to deviate in future rounds of the game.

Axiom 1 *(Self-enforcement) The transition correspondence ϕ satisfies self-enforcement if for any game $(X;\pi_X) \in \mathbf{G}$ and $Y \in \phi(X;\pi_X)$, then $Y \in \phi(Y;\pi_Y)$.*

When there is no confusion, given a rule ϕ and a game $(S,\pi_S) \in \mathbf{G}$, we say that the coalition S is self-enforcing if $S \in \phi(S,\pi_S)$.

Self Enforcement requires that given any starting coalition X, a coalition Y is part of the mapping from X only if there would be no further deviations into subcoalitions of Y once Y forms.

Since the sharing-rule is cross-monotonic, we expect that in the presence of self-enforcing coalitions that are strict subsets of the grand coalition, the grand coalition will not be chosen, since all the agents gain by choosing its subset. This is reflected in the definition of a minimalistic transition correspondence.

Definition 5 (Minimalistic)

The transition correspondence ϕ is Minimalistic if for the game $(S,\pi) \in \mathbb{G}$ there exists $T \subsetneq S$, such that $T \in \phi(T,\pi_T)$ and $T \in W(S,\pi)$, then $S \notin \phi(S;\pi)$.

Definition 6 *Consider two transition correspondence ϕ and $\tilde{\phi}$. We say that ϕ is superior to $\tilde{\phi}$ if for any game (N,π), $T \in \tilde{\phi}(N,\pi)$ and $S \in \phi(N,\pi)$ such that $\xi_i(T) \geq \xi_i(S)$ for some $i \in T \cap S$ if and only if $T \in \phi(N,\pi)$.*

If a transition correspondence is superior to another then it always picks outcomes that are preferred by common agents being chosen.

Definition 7 *A transition correspondence ϕ is coalitionally stable if for any problem (N,π) such that $S \in \phi(N,\pi)$ then there is not a coalition T such that $T \notin \phi(N,\pi)$ $T \in W_{(N,\pi)}$, $T \in \phi(T;\pi_T)$ and $\xi_i(T) \geq \xi_i(S)$ for all $i \in T$.*

A transition correspondence is coalitionally stable if there is no other coalition that is winning and self-enforcing that is not chosen which gives an improvement to the agents.

Within the class of consistent ranking solutions, coalitional stability is equivalent to the rationality introduced axiom by AES. Their analysis considered cases where an agent prefers to be in a coalition where he has a higher relative power[5]. This is described by sharing rules such as proportional sharing. We extend the analysis of AES by considering sharing rules that satisfy or do not satisfy the consistent ranking property and by modifying the Rationality Axiom accordingly.

Axiom 2 *(Rationality) For any $X \in 2^N$, for any $Y \in \phi(X;\pi_X)$ and for any $Z \subset X$ such that $Z \in W_X$ and $Z \in \phi(Z;\pi_Z)$, we have that $Z \notin \phi(X;\pi_X) \Leftrightarrow R_i(Y) > R_i(Z)\ \forall i \in Y \cap Z$*

Rationality requires that for any two coalitions Z and Y that are both winning and self-enforcing from a coalition X, Z will not be chosen if and only if Y is ranked higher than Z by players in their intersection.

3.2 Result with Consistent Ranking

This sections sets out to identify a mapping ϕ^* that satisfies Axioms 1 and 2. We assume that the transition correspondence ϕ^* takes on the specific form:

$$\phi^*\left(S^t, \pi^t\right) = \underset{M \in Q(S^t) \cup \{S^t\}}{\arg\max} \; R(M)$$

where

$$Q(S^t) = \{T \subset S^t \mid |T| < |S^t| \textit{ and } T \in W_{(S^t, \pi_{S^t})}, T \in \phi(T;\pi_T)\}$$

This mapping defines for the coalition S^t a set $Q(S^t)$ of proper subcoalitions, which are both winning in S^t and self-enforcing. We then pick the highest ranked coalition in all the coalitions contained in $Q(S^t)$. If $Q(S^t)$ is empty then we pick coalition S^t itself. Note that ϕ^* satisfies the definition of a transition

[5]this is implicit in the Rationality Axiom (Axiom 4) in AES.

correspondence since it maps into winning coalitions within S^t and by picking a coalition M in Q we are picking subsets of S^t such that $\phi(S^t; \pi_{S^t}) \neq \emptyset$ and it has a maximum rank over a finite set.

The following proposition will show that the imposed properties on the sharing rules together with the axioms on ϕ will generate a unique mapping that will constitute a stable equilibrium.

Proposition 1 *Consider a sharing rule that satisfies consistent ranking. Then, the following conditions are equivalent for the transition correspondence ϕ that is self-enforcing:*

i. *ϕ is superior to any other transition correspondence that is self-enforcing and minimalistic,*

ii. *ϕ is coalitionally stable,*

iii. *ϕ is rational,*

iv. *$\phi = \phi^*$. is the unique mapping that satisfies Self-enforcement and Rationality.*

Proof. Let $S \in 2^N$. Consider the mapping ϕ below which is obtained inductively:

- For $|S| = 1$, $\phi(S; \pi_S) = S$
- Suppose that this has been defined for $|S| = k-1$. We then define $\phi(S; \pi_S)$ for $|S| = k$ as:

  $$\operatorname*{arg\,max}_{M \in Q(S) \cup \{S\}} \quad R(M)$$

  where

  $$Q(S) = \{T \subset S \mid |T| < |S| \text{ and } \mathrm{T} \in \mathrm{W}_{(\mathrm{S}, \pi_{\mathrm{S}})}, \mathrm{T} \in \phi(\mathrm{T}; \pi_{\mathrm{T}})\}$$

To show that $\phi(S;\pi_S)$ maps into self-enforcing coalitions, take any $X \in \phi(S;\pi_S)$. There are two cases, either $X = S$ or $X \in Q$. If $X = S$, then $X \in \phi(S;\pi_S) = \phi(X;\pi_X)$. If $X \in Q$, then $X \in \phi(X;\pi_X)$ by definition of Q. Rationality is satisfied by the following: take $Y \in \phi(S;\pi_S)$, $Z \subset S$ such that $Z \in W_S$ and $Z \in \phi(Z;\pi_Z)$.

($\Rightarrow$) since $Y \in \phi(S;\pi_S)$ we have that:

$$Y \in \underset{M \in Q(S) \cup \{S\}}{\arg\max} \; R(M)$$

Notice that since Z is winning and self-enforcing within S, then $Z \in Q(S) \cup \{S\}$. Then if $Z \notin \underset{M \in Q(S) \cup \{S\}}{\arg\max} \; R(M)$, it follows that $R(Z) < R(Y)$.

($\Leftarrow$) If $R(Y) > R(Z)$ then:

$$Y \in \underset{M \in Q(S) \cup \{S\}}{\arg\max} \; R(M)$$

and

$Z \notin \underset{M \in Q(S) \cup \{S\}}{\arg\max} \; R(M)$. It then follows that $Z \notin \phi(S;\pi_S)$

We prove uniqueness by induction. The mapping is unique for $|S| = 1$ since this maps into a singleton. Our induction hypothesis is that up to $|S| = k-1$ there exists a unique mapping. We then try to prove that for $|S| = k$ there can be no two mappings that satisfy the two axioms.

Suppose not. Suppose there exists a $\phi(S;\pi_S)$ and $\tilde{\phi}(S;\pi_S)$, $\phi(S;\pi_S) \neq \tilde{\phi}(S;\pi_S)$ satisfying the two axioms. Let $T \in \tilde{\phi}(S;\pi_S)$ and $T \notin \phi(S;\pi_S)$. Then we know that:

- $T \in \tilde{\phi}(S;\pi_S) \Rightarrow T \in \tilde{\phi}(T;\pi_T)$ by Axiom 1.
- $T \in \tilde{\phi}(S;\pi_S) \Rightarrow T \in W_{(S,\pi_S)}$ by the definition of the transition correspondence.

We have the following cases:

CASE A. $T \neq S$ and $|T| < |S| = k$

- Take any coalition X. If $|X| = 1$ then $\phi(X;\pi_X) = \tilde{\phi}(X;\pi_X)$ because $\phi(X;\pi_X) \neq \emptyset$ and $\phi(X;\pi_X)$ maps into a subset of coalition X. The same is true for $\tilde{\phi}(X;\pi_X)$. Therefore for coalition sizes $s < k$ we have $\phi(X;\pi_X) = \tilde{\phi}(X;\pi_X)$ and for coalition of size k we have $\phi(X;\pi_X) \neq \tilde{\phi}(X;\pi_X)$.
- Thus, since $|T| < k$, $T \in \tilde{\phi}(T;\pi_T) = \phi(T;\pi_T) \Rightarrow T \in \phi(T;\pi_T)$.
- Since $T \notin \phi(S;\pi_S)$, by Axiom 2, there exists a $Y \in \phi(S;\pi_S)$ such that $R_i(Y) > R_i(T) \;\; \forall\, i \in Y \cap T$.
- By crossmonotonicity and by Axiom 1, $Y \neq S \Rightarrow Y \in \phi(Y;\pi_Y) = \tilde{\phi}(Y;\pi_Y)$.
- Also, $Y \in \phi(S;\pi_S) \Rightarrow Y \in W_{(S,\pi_S}$.

CASE A.1. $Y \in \tilde{\phi}(S;\pi_S)$. Since $R_i(Y) > R_i(T)\; \forall i \in T \cap Y$, and $T \in \tilde{\phi}(T;\pi_T)$, $T \in W_{(S,\pi_S)}$ we have that $T \notin \tilde{\phi}(T;\pi_T)$, a contradiction.

CASE A.2. $Y \notin \tilde{\phi}(S;\pi_S)$. Since $Y \in \tilde{\phi}(Y;\pi_Y)$ and $Y \in W_{(S,\pi_S)}$, by Axiom 2 we have that $R_i(Y) < R_i(T)\; \forall i \in T \cap Y$, a contradiction.

CASE B. $T = S$

- Suppose that S is the only coalition in which they differ and suppose further that $S \in \phi(S;\pi_S)$ but $S \notin \tilde{\phi}(S;\pi_S)$.
- Since S is the only coalition in which they differ, we have $\tilde{\phi}(S;\pi_S) \subseteq \phi(S;\pi_S)$.
- By cross-monotonicity, there exists a $\tilde{T}$, where $\tilde{T} \in \tilde{\phi}(S;\pi_S)$ and $|\tilde{T}| < k$ in which $\tilde{T} \in \phi(S;\pi_S)$.
- Since $\tilde{T} \subseteq S$, by cross-monotonicity $R_i(\tilde{T}) > R_i(S)\; \forall i \in \tilde{T} \cap S$. This contradicts the assumption that $S \in \phi(S;\pi_S)$.

■

3.3 Results without Consistent Ranking

An example of a sharing rule that does not satisfy consistent ranking is combination sharing. This is illustrated in the next example:

Example 1 *Suppose we have two coalitions* $S = \{1, 2, 3, 4\}$ *with power vector* $\pi_S = [.006, .02, .48, .004]$ *and* $T = \{2, 3, 5\}$ *with power vector* $\pi_T = [.02, .48, .02]$. *Let the sharing rule be combination sharing with* $\lambda = .15$ *and prize* $I = 1$. *Since* $S \cap T = \{2, 3\}$, *we should look at the share of these players inside both coalitions. The shares of the prize of players 2 and 3 in coalitions S and T are the following:*

$\xi_2(S) = .071; \xi_2(T) = .083$

$\xi_3(S) = .84; \xi_3(T) = .83$

Thus, player 2 prefers coalition T to S while player 3 prefers coalition S to T.

The basic tension in this type of sharing rule is that, depending on the parameter λ, some players prefer to be in larger coalitions if they have higher relative power within that coalition while others prefer to be in a smaller coalition. In the example above, player 3 wants to be in a coalition with a larger size while player 2 prefers to be in a coalition of smaller size.

There are a class of games, however, where combination sharing will yield consistent ranking. In particular, when we restrict the class of games characterized by a power profile that is *size monotonic in the extremes* (SME), then combination sharing is consistent for any value of λ.

Definition 8 *Consider a power profile* $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$ *arranged in descending powers with* π_{median} *as the median player's power where:*

$$\pi_{median} = \begin{cases} \dfrac{\pi_{\frac{|S|}{2}} + \pi_{\frac{|S|}{2}+1}}{2} & \text{if |S| is even} \\ \pi_{\frac{|S|+1}{2}} & \text{if |S| is odd} \end{cases}$$

Define the upper extreme set of coalition S as:

$UE(S) = \{i \in S | \pi_i > \pi_{median}\}$

We also define the lower extreme set of coalition S as:

$LE(S) = \{i \in S | \pi_i < \pi_{median}\}$

Definition 9 *(Size Monotonicity in the Extremes) Consider sets $A \subset UE(S)$ and $B \subset LE(S)$. A coalition S is size monotonic in the extremes (SME) if $|B| > |A|$ implies that $\pi(B) > \pi(A)$*

Note that SME implies that $\pi_1 + \pi_2 + \cdots + \pi_k < \pi_{|S|-k+1} + \pi_{|S|-k+2} + \cdots + \pi_{|S|}$ $\forall\ k < \frac{|S|}{2}$.

We define a game $\bar{G} = (N, \pi)$ to be an SME game if the coalition N is SME for the power profile π. A feature of SME games is that subcoalitions with larger size will also have larger power. Thus, the tension between coalition size and power for the convex combination sharing rule will disappear.

Lemma 1 *Suppose $\bar{G} = (N, \pi)$ is an SME game, then for any coalitions $A, B \subset N$ where $|A| > |B|$ we must have $\pi(A) > \pi(B)$.*

Proof. Arrange players in descending power such that $\pi_1 > \pi_2 > \cdots > \pi_n$. Suppose that we have two coalitions A and B such that $|A| > |B|$ but $\pi(B) > \pi(A)$. Notice that the power of coalition A is greater or equal than the $|A|^{th}$ lowest powered coalition $C = \{i_n, i_{n-1}, \ldots, i_{n-|A|+1}\}$, that is, $\pi(A) \geq \pi(C)$. Also notice that the power of coalition B is less or equal than the $|B|^{th}$ highest powered coalition $D = \{i_1, i_2, \ldots, i_{|B|}\}$. All of these together imply $\pi(D) \geq \pi(B) > \pi(A) \geq \pi(C)$ or $\pi(D) > \pi(C)$. Since $|D| < |C|$, this contradicts SME. ■

Proposition 2 *Under combination sharing, if we restrict the class of games to SME games then ϕ^* is the unique mapping that satisfies Self-enforcement and Rationality for any λ.*

Proof. By Lemma 1 we know that combination sharing is consistent (for any value of λ) if the class of games is SME. We can then apply Proposition 1 to show that ϕ^* is the unique mapping that satisfies Self-enforcement and Rationality. ∎

3.4 Correspondences without Rationality

Rationality implies that agents who are selected by a transition correspondence have common preferences among self-enforcing coalitions. This is not particularly appealing in many scenarios, for instance when the sharing rule does not satisfy consistent ranking.

In this section, we study transition correspondences that do not satisfy rationality.

Definition 10 *The transition correspondence satisfies the weak axiom of revealed preferences (WARP) if $T \in \phi(S;\pi)$ and Q is winning and self-enforcing in (S,π), and if $Q \in \phi(\tilde{S};\pi)$ and T is winning and self-enforcing in $(\tilde{S},\pi)$, then $T \in \phi(\tilde{S};\pi)$.*

We now show that, given a fixed sharing rule, self-enforcing transition correspondence that are minimalistic coincide on the coalitions that are self-enforcing.

Lemma 2 *Consider the cross-monotonic sharing rule and transition correspondences ϕ and $\tilde{\phi}$ that are self-enforcing and minimalist, then the sets of coalitions that are self-enforcing coincide. That is,*

$$\{S | S \in \phi(S)\} = \{T | T \in \tilde{\phi}(T)\}.$$

Proof.

Consider the sets $A^u = \{S | S \in \phi(S), |S| \leq u\}$ and $B^u = \{T | T \in \tilde{\phi}(T) \leq u\}$.

We will prove by induction on the size of u that $A^u = B^u$.

This is clearly true if $u = 1$, because any singleton coalition is self-enforcing.

For the induction hypothesis, assume that $A^{u-1} = B^{u-1}$.

Consider $S \in A^u$. Then $S \in \phi(S)$. Therefore, there is no $Q \subsetneq S$ such that $Q \in W(S, \pi)$ and $Q \in \phi(Q)$. Therefore, since $A^{u-1} = B^{u-1}$, there is no $Q \subsetneq S$ such that $Q \in W(S, \pi)$ and $Q \in \tilde{\phi}(Q)$. Hence, $S \in \phi(S)$ and $S \in B^u$. Thus $A^u \subset B^u$. We can similarly prove that $B^u \subset A^u$. ■

This proposition is two-fold. On one hand, it implies that the cross-monotonic sharing rule will uniquely determine the set of coalitions that are self-enforcing. On the other hand, it implies that in order to construct an arbitrary self-enforcing transition correspondence, we just need to choose any of its winning coalitions that are self-enforcing and are different than the grand coalition.

For instance, one rule to select the value of the transition correspondence at coalitions that are not self-enforcing, is the sequential dictator, where agents pick their most prefered coalition among the self-enforcing coalitions, and we break ties by continuing in the order.

Proposition 3 *Suppose that the transition correspondence ϕ satisfies the WARP and is minimalistic, then there exists a ranking on the set of self-enforcing coalitions R such that if $S \notin \phi(S)$, then $\phi(S) = \arg\max_{\{T | T \in \phi(T) \text{ and } T \in W(S,\pi)\}} R(T)$.*

4 Case 2: $I = 0$ and $S^t = N$

4.1 Desirable Properties

In this section we consider the case where a players outside the forming coalition are not killed, that is, they can participate in the further coalition formation games. With this threat, the players inside a forming coalition should consider their coalition's relative strength against the powers of people outside of it. We modify the Rationality Axiom as

Axiom 2.2 *(Rationality 2) If* $S \in W_{(N,\pi)}$ *but* $S \notin \phi(N;\pi) \iff \forall T \in \phi(N,\pi)$*,* $\xi_i(T) > \xi_i(S)\ \forall\ i \in T \cap S$

4.2 Results

The following proposition shows for sharing rules with consistent ranking, a unique mapping exists that satisfies Rationality 2. In fact this mapping produces a coalition that is included in the *no threat equilibrium* (NTE) set (see Juarez [5]).

Definition 11 *Under the No-Threat Equilibrium (NTE), if a group of agents find it profitable to deviate from a coalition, then there is another group of agents who can react to that deviation in a way that harms the agents who originally deviated.*

Example 2 *Consider the vector of power* $\pi = [.41, .34, .12, .10, .08]$*. The coalition* $S = \{2,3,5\}$ *is an NTE equilibrium since if, say,* $T = \{2,3\}$ *deviates from* S*, then player 5 can form with 4 and 1 and defeat* T*.*

We define the mapping to be

$$\tilde{\phi}^*\left(N, \pi^t\right) = \underset{M \in W_{(N,\pi)}}{\arg\max}\ R(M)$$

Proposition 4 *If a sharing rule satisfies consistent ranking then* ϕ^* *is the unique mapping that satisfies Rationality 2. Moreover the coalition obtained from this mapping is a NTE.*

Proof. Suppose we have a coalition $S \in \tilde{\phi}^*(N,\pi)$ that does not satisfy Rationality 2, that is, S is not unique. Then there exists another $T \in \tilde{\phi}^*(N,\pi)$ such that $\xi_i(T) > \xi_i(S)$ for some $i \in S \cap T$ and $\xi_j(S) > \xi_i(T)$ for some $j \in S \cap T$. This contradicts consistent ranking. Thus, $S \in \tilde{\phi}^*(N, \pi)$ is unique.

Cross-monotonicity implies that $S \in \tilde{\phi}^*(N, \pi)$ is minimally winning. Suppose S is not minimally winning, then there exists $T \subset S$ such that $T \in W_{(N,\pi)}$ which is not preferred by people in T, that is, $\xi_i(T) < \xi_i(S)$. This contradicts cross-monotonicity.

Since $S \in \tilde{\phi}^*(N, \pi)$ is the unique minimally winning coalition, by Juarez [5] we have that this is a NTE.

■

Hence, sharing rules such as equal sharing and proportional sharing always generate a coalition characterized by the mapping that satisfies Rationality 2. There are several differences between the coalition obtained by this mapping compared to the one obtained in Case 1. First, a coalition of size two can be stable under the NTE.

5 Case 3: $I > 0$ and $S^t \in \phi(S^{t-1}, \pi^{t-1})$

5.1 Desirable Properties

Case 1 defined a unique mapping under the condition that power was not accumulating with each stage a coalition wins. The following extends the analysis to the case where the power of a coalition accumulates by players splitting a prize I that then adds to their power as they continue to the next round. Thus, the equilibrium concept of self-enforcement needs to be modified for this situation. A coalition S is self-enforcing if it is self-enforcing in the initial round and no further deviation is possible even after adding powers to the players inside S. Thus, no subcoalition has enough power to deviate and win.

Axiom 1.1 *Let ϕ be a transition correspondence. A coalition is* **internally self-enforcing** *(ISE) if for any coalition $S \in \phi(N; \pi_N)$ then $S \in \phi(S; \pi_S + I * \xi(S))$ $\forall\ I \geq 0$ where $I * \xi(S)$ is the accumulated power from the prize shared by the players inside the coalition.*

When there is no confusion, we say that a coalition S is an *ISE coalition* if it is generated by the transition correspondence satisfying Axiom 1.1. Notice that this Axiom is similar to self-enforcement in Case 1 when $I = 0$. Ideally, we would want a mapping ϕ that satisfies Internal Self-Enforcement and Rationality. We ask whether sharing rules with consistent ranking will always guarantee existence of this mapping. Proportional sharing will always induce an internally self-enforcing coalition. This is because once a (self-enforcing) coalition S^* forms at the initial stage, the relative power of each player $i \in S^*$ is unaffected by adding the share of the prize $\frac{\pi_i}{\pi_S} \cdot I$.

Proposition 5 *Under proportional sharing, there exists a unique mapping that satisfies ISE (Axiom 1.1) and Rationality (Axiom 2). This mapping coincides with ϕ^* in Case 1.*

However, we show that not all sharing rules that have consistent ranking will generate a coalition that is internally self-enforcing for any vector of powers. In particular, equal sharing does not satisfy ISE, as shown in the example below.

Example 3 *Consider the power profile $\pi = [.20, .15, .14, .13, .12, .11, .10, .05]$ with $I = .1$ with equal sharing. Under this sharing rule, if a coalition S forms and continues to form forever, then the relative power of each player at the limit approaches $\frac{1}{S}$ and hence all internally self-enforcing coalition must be of size 2^{m-1} (this is proven later below). In this example, a coalition of size 3 cannot form since does not have enough power to do so (the three highest powered players only have .49). A 7-person coalition will not be internally* *self-enforcing since if we add the share of the prize to the players then a 3-person coalition can deviate. For instance, if $S = \{1, 2, 3, 4, 5, 6, 7\}$ forms, then after adding $\frac{1}{7}$ to each players' power then $T = \{1, 2, 3\}$ can deviate since $\pi(T) = .49 + .04 > \pi(S \setminus T) = .46 + .057$. The grand coalition is not stable since at the limit when relative powers are equalized, a 7-person coalition can deviate and be internally self-enforcing.*

A way out of this dilemma is to impose some restrictions on admissible power profiles. Indeed, restricting powers of a subcoalition to be *size monotonic in the extremes* (as defined in Section 2) will be a necessary and sufficient condition for a mapping to exist under equal sharing.

5.2 Results

The results in this section discuss the existence of a mapping $\tilde{\phi}$ for a feasible domain of games $\tilde{G}$ under equal sharing. We say that $\tilde{G}$ is a feasible domain of games if $(S, \pi) \in \tilde{G}$ implies that $(S, \pi + \xi(S) \cdot I) \in \tilde{G}\ \forall I > 0$

Proposition 6 *Consider a feasible domain of games $\tilde{G}$ and a mapping $\tilde{\phi}$: $\tilde{G} \to 2^N$. Under equal sharing, if $\tilde{\phi}$ satisfies ISE and Rationality then:*

1. *$\tilde{G}$ should only contain games such that they have a winning subcoalition of size $2^m - 1$ that is SME*
2. *$\tilde{\phi}(\cdot)$ is the coalition with the fewest number of agents of size $2^m - 1$ that is SME*

Proof. ($\Rightarrow$) The proof proceeds in several steps:

Step 1: If a coalition S is picked by any mapping ϕ and continues to form, then over time the relative power of $i \in S$ approaches $\frac{1}{|S|}$.

Proof of Step 1: With equal sharing, the relative power of player i in a coalition S that continues to form through the k^{th} stage is $\frac{\pi_i^0+\frac{kI}{|S|}}{\sum_{i\in S}\pi_i^0+kI}$. Evaluating this expression as $k\to\infty$ by using l'Hospital's rule yields:

$\lim_{k\to\infty}\frac{\pi_i^0+\frac{kI}{|S|}}{\sum_{i\in S}\pi_i^0+kI}=\lim_{k\to\infty}\frac{\frac{I}{|S|}}{I}=\lim_{k\to\infty}\frac{1}{|S|}=\frac{1}{|S|}$

Step 2: Any coalition that is chosen by a mapping ϕ that satisfies ISE should be of size 2^m-1.

Proof of Step 2. We shall prove this by induction on the size of coalition S, $|S|=2^m-1+r$ where $r\in[0,2^m-1]$. Let the base of induction be $m=1$. In this case,

$$|S|=\begin{cases}1 \text{ if } r=0\\ 2 \text{ if } r>0\end{cases}$$

We know that $|S|=1$ is an ISE coalition since a singleton maps into itself. On the other hand, if $|S|=2$, then the player i such that $\pi_i>\pi_j$ can always deviate from S and be self-enforcing (since he is a singleton coalition). Thus S where $|S|=2$ is not an ISE coalition.

Let our induction hypothesis be that this is true for $m=h$. That is,

$$|S|=2^h-1+r\begin{cases}S \text{ an ISE coalition if } r=0\\ S \text{ not an ISE coalition if } r>0\end{cases}$$

We now show that this relationship remains true for $m=h+1$.

If $r=0$, then:

- By Step 1 the relative power of $i\in S$ is $\frac{1}{2^{h+1}-1}$ as the rounds approach infinity. That is, $\lim_{k\to\infty}\frac{\pi^k}{\pi^k(S)}=\left[\frac{1}{2^{h+1}-1},\frac{1}{2^{h+1}-1},\ldots,\frac{1}{2^{h+1}-1}\right]$
- A coalition T that wishes to deviate from S must be as at least 2^h-1+r,

where $2^h - 1 \leq 2^h - 1 + r \leq 2^{h+1} - 1$. Note that a $|T| = 2^h - 1$ will not be winning since $\pi(N \setminus T) > \pi(T)$

- In this case, by Step 1 we know that if T continues to form then the relative power of $i \in T$ will approach $\frac{1}{2^h-1+r}$ in the limit.
- By the same reasoning, a coalition V, where $|V| = 2^h - 1$ can deviate from coalition T. This will be an ISE coalition by our induction hypothesis. Thus T is not an ISE coalition. Therefore, S where $|S| = 2^{h+1} - 1$ is an ISE coalition.

If $r > 0$, then:

- By Step 1 the relative power of $i \in S$ is $\frac{1}{2^{h+1}-1+r}$ as the rounds approach infinity. That is, $\lim_{k\to\infty} \frac{\pi^k}{\pi^k(S)} = \left[\frac{1}{2^{h+1}-1+r}, \frac{1}{2^{h+1}-1+r}, \ldots, \frac{1}{2^{h+1}-1+r}\right]$
- A coalition T where $|T| = 2^h - 1$ can deviate from S. From our induction hypothesis T will be an ISE coalition. Therefore S where $|S| = 2^h - 1 + r$ cannot be an ISE coalition if $r > 0$.
- Finally, by Rationality we know that under equal sharing the smallest ISE coalition will yield the highest share of the prize for all players. Thus, the coalition generated by the mapping ϕ will be the smallest coalition of size $2^m - 1$.

Step 3: The power profile within the ISE coalition S^* is SME.

Proof of Step 3. Suppose not. Suppose that for $A \subset UE(S^*)$, $B \subset LE(S^*)$ we have that if $|B| > |A|$ then $\pi(A) > \pi(B)$. We know from Step 2 that $|S^*| = 2^m - 1$. Take sets A and $Q = \{UE(S^*) \setminus A\}$ from the upper extreme set and B and $V = \{LE(S^*) \setminus B\}$ from the lower extreme set of S^*. Since $\pi(Q) > \pi(V)$ (because $|Q| > |V|$ and $Q \subset UE(S^*)$) and $\pi(A) > \pi(B)$ then we have that $\pi(A \cup Q) > \pi(B \cup V) + \pi_{median}$. Thus, coalition $(A \cup Q)$ is winning in S^* and has size $2^{m-1} - 1$, thus it is an ISE coalition and can deviate.

To prove part 1, suppose that a game $g = (N, \pi)$ produces a set W of winning subcoalitions of size $2^m - 1$ (note that this set is non-empty) but whose elements are not SME. Then by Step 3 we know that if we pick any element $S \in W$ there will be a coalition $T \subset S$ that is winning within S and is ISE. Thus there will be no coalition within the game $g = (N, \pi)$ that is ISE. Therefore, Part 1 of the proposition must hold for a mapping $\tilde{\phi}$ to satisfy ISE.

The proof of Part 2 of the proposition follows from Steps 1-3.

■

For the case of combination sharing, if $\tilde{\phi}$ should satisfy ISE and Rationality then it must be the case that $\tilde{G}$ should only contain games that are SME games (as defined in Section 2). This is to guarantee that there is consensus among players of the coalition to be picked. In this case it is also the smallest coalition of size $2^m - 1$.

Proposition 7 *Consider a feasible domain of games $\tilde{G}$ and a mapping $\tilde{\phi}$: $\tilde{G} \to 2^N$. Under combination sharing, if $\tilde{\phi}$ satisfies ISE and Rationality then:*

1. *$\tilde{G}$ should only contain SME games that have a winning subcoalition of size $2^m - 1$*
2. *$\tilde{\phi}(\cdot)$ is the smallest coalition of size $2^m - 1$.*

Proof. Step 1: show convergence of relative power to $\frac{1}{|S|}$ Step 2: show that if grand coalition is SME then any subcoalition SME - to show that the steps in last proposition can be used Step 3: show that mapping satisfies rationality - use lemma 1 Step 4: show that only coalitions of size $2^m - 1$ is ISE -use proof in last proposition ■

6 Case 4: $I > 0$ and $S^t = N$

6.1 Desirable Properties

New problems crop up in the final case where agents are not killed and power accumulates. In particular, there are two issues. First, as players accumulate power, one player may have high enough relative power to be a dictator. Second, there may be potential "jumping" between coalitions. The following examples illustrates these issues for proportional sharing.

Example 4 *Assume proportional sharing. Let the prize be $I = 1$ and the initial powers be $\pi^0 = (.26, .25, .20, .19, .10)$. The set of winning coalitions at $t = 0$ are:*

$$W_0 = \{[1,2],[1,2,3],[1,2,4],[1,2,5],[1,3,4],[1,3,5],[2,3,4],[2,3,5],[3,4,5]$$
$$,[1,2,3,4],[1,2,4,5],[1,2,3,5],[1,3,4,5],[2,3,4,5],[1,2,3,4,5]\}$$

Unlike in the equal sharing case, $S = [1,2]$ cannot be self-enforcing. To see this, suppose S continues to form. Both will accumulate power according to the proportion $\frac{.26}{.51}$ for player 1 and $\frac{.25}{.51}$ for player 2. However, at the 25^{th} round, player 1's accumulated power is now 13.005 which is higher than the combined powers of $[2,3,4,5]$ which is 12.99. In this instance, player 1 can unilaterally deviate and be a dictator.

Example 5 *Assume proportional sharing. Let the prize be $I = 1$ and the initial powers be $\pi^0 = (.23, .21, .20, .19, .17)$. The set of winning coalitions at $t = 0$ are:*

$$W_0 = \{[1,2,3],[1,2,4],[1,2,5],[1,3,4],[1,3,5],[2,3,4],[2,3,5],[3,4,5]$$
$$,[1,2,3,4],[1,2,4,5],[1,2,3,5],[1,3,4,5],[2,3,4,5],[1,2,3,4,5]\}$$

Suppose, for instance, the coalition [1,2,3,4] forms at $t = 0$. Then at $t = 1$ after they have accumulated power, there is an incentive for players 2,3,4 to dump player 1 in favor of player 5. This will happen since players 2,3,4 will get a higher share of the prize by aligning with 5 and they have sufficient power to do so. At later rounds, however, the same incentive to dump the highest-powered player for the outsider (provided that they have sufficient power to win) will still be there, and thus there will be a phenomenon of "jumping" to different coalitions.

This case is also true for combination sharing as long as $\lambda < 1$.

We first introduduce the concept of refinement. Given the vector of power π_X, a refinement of π_X is a vector of powers $\bar{\pi}_Y$ such that the power of its agents is formed by breaking down the power of the agents in X. That is, there exists a partition of the agents Y, $(P_1, \ldots, P_{|X|})$, such that $\sum_{j \in S_i} \bar{\pi}_j = \pi_i$ for every $i \in X$.

We introduce the concept of external self-enforcement to incorporate the features that agents are not killed and that power accumulates throughout rounds.

Axiom 3.1 (Externally self enforcement) *Let ϕ be a transition correspondence. A mapping is* **externally self-enforcing** *(ESE) if it is anonymous and $S \in \phi(N; \pi)$ only if $S \in \phi(N; (\pi_S, \bar{\pi}_{-S}) + \xi(S) \cdot I)$ $\forall$ $I > 0$* for every refinement $\bar{\pi}_{-S}$ of π_{-S}.

When there is no confusion, we say that a coalition S is an *ESE coalition* if it is generated by the transition correspondence satisfying ESE. This modification of the self-enforcement axiom shows that coalitions that are externally self enforcing should map into the same coalition even though players from $N \setminus S$ (players outside S) can still form coalitions and threaten S, or can refine into smaller agents, and even if the powers within the forming coalition accumulates through rounds.

We also modify our definition of Rationality to accommodate the same features.

Axiom 3.2 (Rationality 3) *Suppose that* $T \notin \phi(N;\pi)$ *but* $T \in \phi(N; (\pi_T, \bar{\pi}_{-T}) + \xi(T)I)$ *for all* $I > 0$ *and all refinement* $\bar{\pi}_{-T}$ *, then if* $S \in \phi(N;\pi)$ *then we have that* $\xi_i(S) > \xi_i(T)$ $\forall i \in S \cap T$.

Note that the trivial correspondence $\phi(N;\pi) = N$ satisfies ESE and rationality 3.

6.2 Results

This subsection formalizes our results for this case. We find that for equal sharing a mapping that satisfies external self-enforcement always exists while this is not true for combination sharing. For the equal sharing case, we define the mapping ϕ^{**} as the smallest winning coalition of size 2^m:

$$\phi^{**} = \arg\min_{Q} \ m$$

where $Q \in \{S \in 2^N$ such that $S \in W_{(N,\pi)}$ and $|S| = 2^m\}$

Proposition 8 *Under equal sharing, the correspondence* ϕ^{**} *is the only non-trivial mapping that satisfy ESE and Rationality 3.*

Proof. ($\Rightarrow$) The proof proceeds in several steps:

Step 1: If a coalition S is picked by any mapping ϕ and continues to form, then over time the relative power of $i \in S$ approaches $\frac{1}{|S|}$.

Proof of Step 1: Similar to the proof of Step 1 in Proposition 4.

Step 2: Any coalition that is chosen by the mapping ϕ^{**} should be of size 2^m.

Proof of Step 2. We know that $|S| = 1$ is an ESE coalition since a singleton maps into itself provided that $\pi(S) > \pi(N \backslash S)$, that is, S is a dictator. Knowing this, we shall prove Step 2 by induction on m where m is such that $|S| = 2^m + r$ where $r \in [0, 2^m - 1]$. Let the base of induction be $m = 1$. In this case,

$$|S| = \begin{cases} 2 \text{ if } r = 0 \\ \\ 3 \text{ if } r > 0 \end{cases}$$

We know that $|S| = 2$ is an ESE coalition since a deviation into a singleton coalition by a non-dictator is not winning, that is, $\pi_i + \frac{kI}{2} < \pi_j + \frac{kI}{2} + \pi(N \setminus S)\ \forall\ i, j \in S$. On the other hand, if $|S| = 3$, then at the limit we can have 2 players deviating from S which will be an ESE coalition. This is because at the limit the relative power $\frac{\pi_i}{\pi(S)} \to \frac{1}{3}\ \forall i \in S$ and $\frac{\pi_j}{\pi(S)} \to 0\ \forall j \in N \setminus S$ and therefore 2 players from S can deviate. Hence, S such that $|S| = 3$ is not an ESE coalition.

Let our induction hypothesis be that this is true for $m = h$. That is,

$$|S| = 2^h + r \begin{cases} \text{ESE if } r = 0 \\ \\ \text{not ESE if } r > 0 \end{cases}$$

We now show that this relationship remains true for $m = h + 1$.

If $r = 0$, then:

- By Step 1 the relative power of $i \in S$ is $\frac{1}{2^{h+1}}$ as the rounds approach infinity.
- A coalition T that wishes to deviate from S must satisfy two conditions: (1) it should be winning at the game $(N, \pi + \xi(T))I\ \forall I > 0$ and; (2) it must be of size smaller than S (otherwise players in the intersection $S \cap T$ will be better off staying with S)

- Moreover, at the limit any coalition deviating from S should include at least 2^h players from S. Otherwise

  $V \subset S$ where $|V| = 2^h$. Otherwise, any coalition that does not contain V will not be winning.

- Suppose that a coalition $T = (V \cup W)$ where $|V| = 2^h$, $W \subseteq (N \setminus V)$, $|W| = r$. By the induction hypothesis this will not be an ESE coalition.

- If V deviates, then this will not be winning since $\pi(V) < \pi(N \setminus V)$ where

  $\pi(V) = \sum_{i \in V} \pi_i + \frac{kI}{|S|} \cdot 2^h$
  and
  $\pi(N \setminus V) = \sum_{j \in (N \setminus V)} \pi_j + \frac{kI}{|S|} \cdot 2^h$

  If $\pi(V) > \pi(N \setminus V)$, this implies that $\sum_{i \in V} \pi_i > \sum_{j \in (N \setminus V)} \pi_j$ which means that V was winning in the initial round. This is a contradiction since V could have just formed instead of S. Thus, the coalition $T = V$ where $|T| = 2^h$ that deviates from S where $|S| = 2^{h+1}$ will not be winning. Therefore, S is an ESE coalition.

If $r > 0$, then:

- By Step 1 the relative power of $i \in S$ is $\frac{1}{2^{h+1}+r}$ as the rounds approach infinity.

- A coalition T where $|T| = 2^h$ can deviate from S and will be winning. From our induction hypothesis T will be an ESE coalition. Therefore S where $|S| = 2^h + r$ cannot be an ISE coaltion if $r > 0$.

- Finally, by Axiom 2.3 (Rationality) we know that under equal sharing the smallest self-enforcing coalition will yield the highest share of the prize for all players. Thus, the coalition generated by the mapping ϕ will be the smallest coalition of size 2^m.

■

We make several remarks on the transition correspondence ϕ^{**}. Although the mapping produces a coalition that is externally self-enforcing under equal sharing, it may not necessarily be efficient. That is, there is potentially a deviation (not necessarily to an externally self-enforcing coalition) that could potentially make all players in the coalition who deviated better off. To see this, suppose that ϕ^{**} maps into coalition S where S is an 8-person coalition from the initial population of 11 players. In the limit, suppose coalition T deviates from S where T is composed of four players from S and the remaining 3 players from $N \setminus S$. After forming this 7-person coalition, the original four players that were part of S that deviated will have enough power at the limit to deviate from this 7-person coalition. Notice this path is an improvement from S for the four players that deviated but also an improvement for the three players in $N \setminus S$. Thus, this off-equilibrium behavior can potentially make a subset of players in S better off.

The next propositions show why ESE is not satisfied by combination sharing, except by the trivial correspondence.

Proposition 9 *Under combination sharing, the trivial correspondence $\phi(N, \pi) = N$ is the only mapping that satisfies ESE and Rationality 3.*

Proof. Suppose $S \in \phi(N; \pi)$. Thus, for players in $i \in S$, at every round power will accumulate by $\pi_i^t = \pi_i^{t-1} + \lambda \cdot \frac{I}{|S|} + (1 - \lambda)\frac{\pi_i^{t-1}}{\pi^{t-1}(S)}$.

If λ is close enough to zero then there will exist a round k and a coalition $T = S - \{i\}$, $i \in S$ such that for a player $j \in N \setminus S$, $T \cup \{j\}$ is winning at that round, that is, $T \cup \{j\} \in W_{(N,\pi+\xi(S))}$. This coalition will be preferred $\forall\ i \in T$ because $|S| = |T \cup \{j\}|$ but $\frac{\pi_i(T\cup\{j\})}{\pi(T\cup\{j\})} > \frac{\pi_i(S)}{\pi(S)}$.

Therefore, there will be no mapping such that ESE is satisfied. ■

7 Non-cooperative Game Treatment

We now provide a non-cooperative treatment of the game described by the axiomatic analysis of the earlier sections. We devide this section into two ortoghonal cases, when agents are killed and when they are not.

7.1 Agents are killed

Let the game be $g = (N, \pi)$. At every stage t, there will be a coalition that forms. Let $N^0 = N$ be the initial coalition with power profile π^0.

The non-cooperative game is as follows:

1. Nature picks agenda setter $a^{t,q} \in N^t,\ q = 1$
2. $a^{t,q}$ makes proposal $P^{t,q} \in 2^{N^t}$ where $a^{t,q} \in P^{t,q}$
3. Nature randomly selects a voter within $P^{t,q}$ denoted by $V^{t,q,1}$ who chooses yes or no: $\bar{V}(V^{t,q,1}) \in \{yes, no\}$
   - we proceed to the second voter $V^{t,q,2}$ and obtain his vote
   - proceed similarly for all $|P^{t,q}|$ players
   - denote by $Q^{t,q} = \{i \in P^{t,q} | \ \bar{V}(i) = yes\}$
   - if $Q^{t,q} \in W_{(N_t, \pi_{N_t})}$, proceed to step 4
   - otherwise proceed to step 5
4. If $P^{t,q} = N^t$, the game proceeds to Step 6 with agents' power as $\pi_i^{t+1} = \pi_i^t + \xi_i(N^t)I \ \forall \ i \in N^t$. Otherwise, $N^t \setminus P^{t,q}$ are killed and game returns to Step 1 with $N^{t+1} = P^{t,q}$ with agents' power as $\pi_i^{t+1} = \pi_i^t + \xi_i(P_{t,q})I \ \forall \ i \in P^{t,q}$.
5. If $q < |N^t|$, Nature randomly picks next agenda setter $a^{t,q+1} \in N^t$ among members in N^t who have not proposed, then proceeds to step 2 with q increased by 1. If $q = |N^t|$, proceed to step 6.

6. N^t is the coalition that forms.

7. After repeating this infinitely, the sequence of winning coalitions and their respective powers are $\{(N^1, \pi^1), (N^2, \pi^2), \ldots\}$. Consider this sequence with the relative powers $\{(N^1, \tilde{\pi}^1), (N^2, \tilde{\pi}^2), \ldots\}$ where $\tilde{\pi}_i^t = \frac{\pi_i^t}{\pi^t(N)}$.

   - If $\lim_{t\to\infty}(N^t, \tilde{\pi}^t)$ exists, let $(N^*, \tilde{\pi}^*) = \lim_{t\to\infty}(N^t, \tilde{\pi}^t)$.
   - If $\lim_{t\to\infty}(N^t, \tilde{\pi}^t)$ does not exists, let $N^* = \emptyset$
   - The payoff of agents in this game is:

$$U_i = \begin{cases} \xi_i(N^*, \tilde{\pi}^*) - \epsilon \sum \mathbf{I}_{N^{t+1}(i) \neq N^t(i)} & \text{if } i \in N^* \\ 0 \quad \text{if } i \notin N^* \end{cases}$$

The aggregate payoff is simply the share of the agent if he is part of a the limit coalition N^* minus a cost ϵ that is incurred if there is a different coalition that transitions from one period to another. The aggregate payoff can be justified in several ways. First, expropriation by the winning coalition of any player that is not part of the winning coalition at a particular stage will justify why non-members will get zero aggregate payoff. Alternatively, if player i were extremely risk averse in the sense that he would always prefer to be a part of the winning coalition at every stage, then not being a part of a winning coalition would drive his aggregate payoff to zero.

Proposition 10

- *Consider the transition correspondece ϕ^* from Case 1 that is rational and self enforcing. Consider the initial game (N, π). Then $S^* \in \phi^*(N, \pi)$ if and only if there exists a set of strategies that implements S^* as a Coalition-Proof Subgame Perfect Nash Equilibrium of the game (*) above.*

- *Consider the transition correspondece ϕ^* from Case 3 that is internally self enforcing and satisfies Rationality 2. Consider the initial game (N, π)*

and $S^* \in \phi^*(N,\pi)$. Then there exists a set of strategies that implements S^* as a Coalition-Proof Subgame Perfect Nash Equilibrium of the game above.

7.2 Agents are not killed

If agents are not killed, consider the set of agents N with the initial power vector π^0. At every stage t, there will be a coalition, S^t, that forms, and a power vector for each stage denoted by π^t. The stage game is as follows:

1. Nature picks agenda setter $a^{t,q} \in N,\ q = 1$
2. $a^{t,q}$ makes proposal $P_{t,q} \in 2^N$ where $a^{t,q} \in P^{t,q}$
3. Nature randomly selects a voter within $P^{t,q}$ denoted by $V^{t,q,1}$ who chooses yes or no: $\bar{V}(V^{t,q,1}) \in \{yes, no\}$
   - we proceed to the second voter $V^{t,q,2}$ and obtain his vote
   - proceed similarly for all $|P^{t,q}|$ players
   - denote by $Q^{t,q} = \{i \in P^{t,q} |\ \bar{V}(i) = \{yes\}\}$
   - if $Q^{t,q} \in W_{(N,\pi_t)}$, proceed to step 4
   - otherwise proceed to step 5
4. If $P^{t,q} = N$, then proceed to step 6. Otherwise, $S^t = P^{t,q}$ and next period begins and the game returns to Step 1 with agents' power as $\pi_i^{t+1} = \pi_i^t + \xi_i(S^t)I\ \forall\ i \in S^t$ and $\pi_i^{t+1} = \pi_i^t\ \forall\ i \notin S^t$.
5. If $q < |N|$, nature randomly picks next agenda setter $a^{t,q+1} \in N$ who have not proposed, then proceeds to step 2 with q increased by 1. If $q = |N|$, proceed to step 6.
6. $S^t = N$, then next period begins with $\pi_i^{t+1} = \pi_i^t + \xi_i(N)I$ for all $i \in N$ as the new vector of powers.

7. After repeating this infinitely, the sequence of winning coalitions and their respective powers are $\{(S^1, \pi^1), (S^2, \pi^2), \dots\}$. Consider this sequence with the relative powers $\{(S^1, \tilde{\pi}^1), (S^2, \tilde{\pi}^2), \dots\}$ where $\tilde{\pi}_i^t = \frac{\pi_i^t}{\pi^t(N)}$.

   - If $\lim_{t\to\infty}(S^t, \tilde{\pi}^t)$ exists, let $(S^*, \tilde{\pi}^*) = \lim_{t\to\infty}(S^t, \tilde{\pi}^t)$.
   - If $\lim_{t\to\infty}(S^t, \tilde{\pi}^t)$ does not exists, let $S^* = \emptyset$
   - The payoff of agents in this game is:

$$U_i = \begin{cases} \xi_i(S^*, \tilde{\pi}^*) - \epsilon \sum \mathbf{I}_{S^{t+1}(i) \neq S^t(i)} & \text{if } i \in S^* \\ 0 \quad \text{if } i \notin S^* \end{cases}$$

Proposition 11 • *Consider the transition correspondece ϕ^* from Case 2 that is rational. Consider the initial game (N, π) and $S^* \in \phi^*(N, \pi)$. Then there exists a set of strategies that implements S^* as a Coalition-Proof Subgame Perfect Nash Equilibrium of the game above.*

- *Consider the transition correspondece ϕ^* from Case 4 that is externally self enforcing and satisfies Rationality 3. Consider the initial game (N, π) and $S^* \in \phi^*(N, \pi)$. Then there exists a set of strategies that implements S^* as a Coalition-Proof Subgame Perfect Nash Equilibrium of the game above.*

8 Proofs

8.1 Propositions 9 and 10

8.1.1 Proof of Proposition 9

We now define the set of strategies σ that would implement the Nash Equilibrium. The set of strategies are the same for cases 1 and 3, and very similar to the strategies used to prove proposition 10.

Let $S^* \in \phi(N,\pi)$. Consider the period t and $i \in S^*$. Then the strategy σ_i of agent i equals to:

$$\sigma_i = \begin{cases} \text{if } i \text{ is a proposer : always propose } S^* \text{regardless of history} \\ \text{if } i \text{ is a responder : vote yes} \Leftrightarrow \text{ proposed coalition is } S^* \end{cases}$$

On the other hand, there are two cases if $i \notin S^*$. For Case 1, where power does not accumulate:

$$\sigma_i = \begin{cases} \text{if } i \text{ is a proposer : propose} \underset{\{C: C\in\phi(C,\pi_C),\ C\in W_{(N^t,\pi^t)}\}\cup\{N^t\}}{\arg\max} \xi_i(C) \\ \text{if } i \text{ is a responder : vote yes } \Leftrightarrow \text{ proposed coalition is } C \subseteq N^t, \\ \qquad C \in W_{(N^t,\pi^t)},\ C \in \phi(C,\pi_C) \text{ and } i \in C \end{cases}$$

For Case 3, where power accumulates:

$$\sigma_i = \begin{cases} \text{if } i \text{ is a proposer : propose} \underset{\{C: C\in\phi(C,\pi_C+I\xi(C))\ \forall\ I\geq 0,\ C\in W_{(N^t,\pi^t)}\}\cup\{N^t\}}{\arg\max} \xi_i(C) \\ \text{if } i \text{ is a responder : vote yes } \Leftrightarrow \text{ proposed coalition is } C \subseteq N^t, \\ \qquad C \in W_{(N^t,\pi^t)}, C \in \phi(C,\pi_C + I\xi(C))\ \forall I \geq 0 \text{ and } i \in C \end{cases}$$

First, note that the set of strategies above implement S^*. This is clear because S^* is a winning coalitions and the agents in S^* will refuse any proposal other than S^*.

Second, note that no agent has the incentive to deviate from σ. To see this, if an agent $i \notin S^*$ changes his strategy the agents in S^* will continue rejecting proposals different than S^*. On the other hand, suppose agent $j \in S^*$ changes his strategy and it implements coalition T. Then, by definition of the other strategies T is self-enforcing at every period, because agents only accept offers that are self-enforcing. By definition, $R_j(S^*) \geq R_j(T)$. Therefore, agent j did not improve his utility.

8.1.2 Proof of Proposition 10

The strategies used in Proposition 10 is similar to the previous strategies in Proposition 9 if the agent is part of the limit coalition S^*. However, if he is not part of S^*, then he proposes an externally self-enforcing coalition which maximizes his payoffs if he is a proposer, or he votes yes to any externally self-enforcing coalition that includes him if he is a voter.

As with Proposition 9, there are two cases if $i \notin S^*$. For Case 2, his strategy equals:

$$\sigma_i = \begin{cases} \text{if } i \text{ is a proposer : propose } \underset{C:\ C \in W_{(N^t,\pi^t)}}{\arg\max} \ \xi_i(C) \\ \text{if } i \text{ is a responder : vote yes } \Leftrightarrow \text{ proposed coalition is } C \subset N^t, \\ \qquad C \in W_{(N^t,\pi^t)}, \text{ and } i \in C \end{cases}$$

For Case 4:

$$\sigma_i = \begin{cases} \text{if } i \text{ is a proposer : propose } \underset{C: C \in \phi(N, \pi_N + I\xi(C))\ \forall I \geq 0,\ C \in W_{(N^t,\pi^t)}}{\arg\max} \ \xi_i(C) \\ \text{if } i \text{ is a responder : vote yes } \Leftrightarrow \text{ proposed coalition is } C \subset N^t, \\ \qquad C \in W_{(N^t,\pi^t)},\ C \in \phi(N, \pi_N + I\xi(C))\ \forall\ I \geq 0 \text{ and } i \in C \end{cases}$$

These set of strategies above implement S^* since S^* is a winning coalition and the agents in S^* will refuse any proposal other than S^*.

Second, note that no agent has the incentive to deviate from σ. To see this, if an agent $i \notin S^*$ changes his strategy the agents in S^* will continue rejecting proposals different than S^*. On the other hand, suppose agent $j \in S^*$ changes his strategy and it implements coalition T. Then, by definition of the other strategies T is externally self-enforcing at every period. If we are in Case 2, then T is not a minimally winning coalition of minimal size or weight, therefore j doesn't improve his utility. If we are in Case 4, then coalition T is not the smallest externally self-enforcing coalition of size 2^m, and thus j will not improve his utility.

References

[1] D. Acemoglu, G. Egorov, and K. Sonin. Coalition Formation in Non-Democracies. *Review of Economic Studies*, 75(4):987–1009, 2008.

[2] F. Bloch and B. Dutta. Formation of Networks and Coalitions. *Mimeo University of Warwick*, 2010.

[3] A. Bogomolnaia and M. Jackson. The Stability of Hedonic Coalition Structures. *Games and Economic Behavior*, 38(2):201–230, 2002.

[4] J. Dreze and J. Greenberg. Hedonic Coalitions: Optimality and Stability. *Econometrica*, 48(4):987–1003, 1980.

[5] R. Juarez. Coalition Formation in Tournament Games with Externalities: The No-Threat Equilibrium. *Mimeo University of Hawaii at Manoa*, 2012b.

[6] M. Piccione and R. Razin. Coalition Formation under Power Relations. *Theoretical Economics*, 4(1):1–15, 2009.

[7] D. Ray. *A Game-Theoretic Perspective on Coalition Formation.* Oxford University Press, New York, 2007.

[8] D. Ray and R. Vohra. *Coalition Formation.* North-Holland, 2013.

[9] T. Sonmez, S. Banerjee, and H. Konishi. Core in a Simple Coalition Formation Game. *Social Choice and Welfare*, 18(1):135–153, 2001.