The von Neumann-Morgenstern Stable Sets for 2×2 Games*

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Abstract

We analyze the von Neumann and Morgenstern stable sets for the mixed extension of 2×2 games when only single profitable deviations are allowed. We show that the games without a strict Nash equilibrium have a unique *vN&M* stable set and otherwise they have an infinite number of these sets. Moreover the unique vN&M stable set includes a Pareto optimal strategy profile.

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1 Introduction

In their book von Neumann and Morgenstern [8] first define stable sets for cooperative games and interprete them as "standards of behavior" within society. Then in Sect. 65.1 they propose them to solve abstract systems[1], the setting we make use of in this paper. A salient feature of a stable set is that all its alternatives *protect* one another so that the remaining ones are disregarded with confidence. We believe that the application of this basic notion of stability is well justified when it is unique and guarantees Pareto efficiency. This is the case when it is applied to a class of the mixed extension of the 2×2 games.

Let us justify the application of the von Neumann and Morgenstern (*vN&M*) stable sets to the matching pennies game whose payoff matrix is given by

	heads	*tails*
heads	$+1, -1$	$-1, +1$
tails	$-1, +1$	$+1, -1$

This game has a unique Nash equilibrium (*NE*) in which each player chooses heads and tails with equal probability and the expected payoff is zero. Trembling out of this equilibrium leads players to cycle indefinitely. That is, if players in a non-equilibrium profile alternates their response to the opponent's strategy so that the payoff increases then the *NE* will never be reached. In contrast, consider the set of all strategy profiles in which the two players get zero payoff, say set A. No player has an incentive to move within strategy profiles in this set. From any other strategy profile, however, there is a player with a negative payoff who always improves by deviating to some profile in A. Consequently set A satisfies the so called internal and external stability conditions which define a *vN&M* stable set and that, as we shall see, it is unique for this game. Hence, the *vN&M* stable sets solution stands out as a good candidate to solve this game. Our aim in this paper is to show for which of the mixed extension of 2×2 games the *vN&M* stable sets can be placed as a sound solution.

Greenberg [3] in the Theory of Social Situation (*TOSS*) unifies the analysis of cooperative and noncooperative games giving rise to interesting results in game theory and in economics[2]. In Chapter 7 of *TOSS* it is argued that normal form

[1] An abstract system is a set and a binary relation defined over it.

[2] See, for instance, Luo (2009) [5] and the references therein.

games should capture the notion of negotiation among players when modeling social environments. One of the proposed negotiations is based on the idea that a single player can object to the prevailing strategy profile by threatening the opponents with the possibility of using a different strategy that gives her a higher payoff. For this negotiation and for finite normal form games with pure strategies Greenberg shows that *vN&M* stable sets exist if there are either at most two players or n players with at most two strategies each. (See *Theorems 7.4.5* and *7.4.6* in Greenberg [3]).

In this paper we extend Greenberg's analysis of the *vN&M* stable sets to the mixed extension of the 2×2 games. We find that all 2×2 games (generic and non-generic) without a strict *NE* have a unique *vN&M* stable set and otherwise they have an infinite number of these sets. We also characterize the strategy profiles belonging to the *vN&M* stable sets. It turns out that only strategy profiles which are not dominated[3] by the *NE* and by strategy profiles in which one player's payoff is not contingent on her own strategy belong to *vN&M* stable sets. Unlike the *NE* is not Pareto optimal for the class of games with a unique *vN&M*, we find that this set always contains a Pareto optimal strategy profile in which one player's payoff coincides with some *NE* payoff while her opponent maximizes her own[4]. We conclude by relating our classification of the 2×2 games with some other taxonomies of these games.

The rest of the paper is organized as follows. Section 2 compiles the preliminaries of the paper. Section 3 present the results and some illustrative examples. Section 4 relates our classification of games with other taxanomies and Section 5 contains further research.

2 Notation and definitions

Let $G = \langle N, \{S_i\}_{i=1,2}, \{u_i\}_{i=1,2} \rangle$ be a normal form 2×2 game where $N = \{1, 2\}$ is the set of players, S_i is the set of two actions for player i, and $u_i : S{=}S_1 \times S_2 \longrightarrow \mathbb{R}$, is player i's payoff function. The *mixed extension of game* G is $< N, \{\Delta(S_i)\}_{i=1,2}, \{U_i\}_{i=1,2} >$ where $\Delta(S_i)$ is the simplex of the mixed

[3] A strategy profile dominates another profile if there is a player who can profitable deviate given her opponet's strategy.

[4] For a n-player prisoners' dilemma game Nakanishi [6] has shown the existence and the efficiency of the *vN&M* stable sets.

strategies for player i, and $U_i : \Delta(S) = \Delta(S_1) \times \Delta(S_2) \longrightarrow \mathbb{R}$, assigns to $\sigma \in \Delta(S)$ the expected value under u_i of the lottery over S that is induced by σ, so that $U_i(\sigma) = \sum_{s \in S} (\prod_{j \in N} \sigma_j(s_j) u_i(s))$ where $\sigma_j(s_j)$ is player j's probability of playing action s_j.

A player $i's$ strategy σ_i is a *best response* to player $j's$ strategy σ_j if $U_i(\sigma_i, \sigma_j) \geq U_i(\sigma_i', \sigma_j)$ for all $\sigma_i' \in \Delta(S_i)$. We denote by $BR_i(\sigma_j)$ player i's set of best responses to σ_j.

A strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*)$ is a *NE*, see Nash [7], if σ_i^* is a best response to σ_j^*, *i.e.,* if. $\sigma_i^* \in BR_i(\sigma_j^*)$ for $i = 1, 2$ and $j \neq i$. A *NE* strategy profile σ^* is strict if σ_i^* is the unique best response to σ_j^*, *i.e.*, if $BR_i(\sigma_j^*) = \{\sigma_i^*\}$ for $i = 1, 2$ and $j \neq i$. The set of *NE* strategy profiles is denoted by Σ^*.

Let $\sigma_i, \sigma_i' \in \Delta(S_i)$. Then σ_i strictly dominates σ_i' if $U_i(\sigma_i, \sigma_j) > U_i(\sigma_i', \sigma_j)$ for all $\sigma_j \in \Delta(S_j)$ $(j \neq i)$. A strategy σ_i is strictly dominant if σ_i strictly dominates σ_i' for all $\sigma_i' \in \Delta(S_i)$ $(\sigma_i' \neq \sigma_i)$.

Abusing notation we denote strategies σ_1 and σ_2 as p and q where p and q are, respectively, player 1 and 2's probabilities of playing their "first" action of game G. (By default, $(1-p)$ and $(1-q)$ are player 1 and 2's probabilities of playing their "second" action.)

As indicated in the introduction we follow the negotiation procedure proposed by Greenberg. Suppose that strategy profile σ is proposed to players and that only one player can deviate. Then player i can object σ inducing σ' if she is better off in σ' than in σ. Thus an *abstract system* associated to the mixed extension of a game G is the pair $(\Delta(S), \succ)$ where $\succ$ is the binary relation defined on $\Delta(S)$ so that $\sigma' \succ \sigma$ if there exists player i such that $\sigma_j' = \sigma_j$ for $j \neq i$ and $U_i(\sigma') > U_i(\sigma)$.

Let $A \subseteq \Delta(S)$. Then A is a *vN&M* stable set of $(\Delta(S), \succ)$ if it satisfies the following two conditions:

(i) Internal stability: For all $\sigma \in A$ there not exists $\sigma' \in A$ such that $\sigma' \succ \sigma$.

(ii) External stability: For all $\sigma \notin A$ there exists $\sigma' \in A$ such that $\sigma' \succ \sigma$.

Denoting by $D(A) = \cup_{\sigma \in A} D(\sigma)$ where $D(\sigma) = \{\sigma' \in \Delta(S) : \sigma \succ \sigma'\}$ it is immediate that A is a *vN&M* stable set if and only if $D(A) = \Delta(S) \backslash A$.

3 The vN&M stable set for 2×2 games

In this section we present our results and some illustrative examples.

Theorem 1 *Let* $(\Delta(S), \succ)$ *be the system associated to game* G*.* $(\Delta(S), \succ)$ *has infinite vN&M stable sets if and only if* G *has a strict NE strategy profile and otherwise it has a unique vN&M stable set.*

Proof. Let us consider two cases:

Case 1. *Game* G *has a strict NE strategy profile* σ^*.

Let us assume without loss of generality that $\sigma^* = (1,1)$. Three sub-cases can be distinguished:

a) *Both players have a strictly dominant strategy.* Then $BR_1(q) = \{1\}$ and $BR_2(p) = \{1\}$ for all $p, q \in [0,1]$. Let $\widetilde{\sigma} = (0,0)$ and define $A = \{\sigma \in \Delta(S) : \sigma = \lambda\sigma^* + (1-\lambda)\widetilde{\sigma},\ \lambda \in [0,1]\}$. We first show that A is a *vN&M* stable set and then that there exist infinite *vN&M* stable sets.

Since U_1 is increasing in p for all $q \in [0,1]$ and U_2 is increasing in q for all $p \in [0,1]$, then for each $\sigma \in A$, $D(\sigma) = \{\sigma' \in \Delta(S) : q' = q,\ p' < p\} \cup \{\sigma' \in \Delta(S) : p' = p,\ q' < q\}$. Thus, for all $\sigma, \sigma' \in A$ we have that $\sigma' \notin D(\sigma)$. Moreover for all $\sigma' \notin A$, there exists $\sigma \in A$ such that $\sigma' \in D(\sigma)$. Hence, $D(A) = \Delta(S) \backslash A$ and consequently A is a *vN&M* stable set.

To show that there are infinite *vN&M* stable sets consider a strategy profile $\widehat{\sigma} \notin A$; $1 > \widehat{p} > 0$ and $1 > \widehat{q} > 0$.[5] Reasoning as above we conclude that $\widehat{A} = \{\sigma \in \Delta(S) : \sigma = \lambda\sigma^* + (1-\lambda)\widehat{\sigma},\ \lambda \in [0,1]\} \cup \{\sigma \in \Delta(S) : \sigma = \lambda\widehat{\sigma} + (1-\lambda)\widetilde{\sigma},\ \lambda \in [0,1]\}$ is also a *vN&M* stable set. Therefore, $(\Delta(S), \succ)$ associated to a game in which both players have a strictly dominant strategy has infinite *vN&M* stable sets.

b) *Only one player has a strictly dominant strategy.* Suppose without loss of generality that $BR_1(q) = \{1\}$ for all $q \in [0,1]$. In this case there exists $\overline{p} \in [0,1)$ such that[6]

$$BR_2(p) = \begin{cases} \{0\} & if \quad p < \overline{p} \\ [0,1] & if \quad p = \overline{p} \\ \{1\} & if \quad p > \overline{p} \end{cases}.$$

[5] Note that there are infinite strategy profiles satisfying these conditions.
[6] Given that σ^* is a strict NE strategy profile then $\overline{p} \neq 1$.

Let $\overline{\sigma} = (\overline{p}, 0)$ and let $A = \{\sigma \in \Delta(S) : \sigma = \lambda\sigma^* + (1-\lambda)\overline{\sigma},\ \lambda \in [0,1]\}$. Given that U_1 is increasing in p for all $q \in [0,1]$ and U_2 is increasing in q if $p > \overline{p}$, then arguing as in case **a)** we conclude that A is a *vN&M* stable set.

To show that there are infinite *vN&M* stable sets let us consider a strategy profile $\widehat{\sigma} \notin A$; $1 > \widehat{p} \geq \overline{p}$ and $1 > \widehat{q} > 0$. Reasoning as above we have that $\widehat{A} = \{\sigma \in \Delta(S) : \sigma = \lambda\sigma^* + (1-\lambda)\widehat{\sigma},\ \lambda \in [0,1]\} \cup \{\sigma \in \Delta(S) : \sigma = \lambda\widehat{\sigma} + (1-\lambda)\overline{\sigma},\ \lambda \in [0,1]\}$ is also a *vN&M* stable set. Therefore $(\Delta(S), \succ)$ associated to these games have infinite *vN&M* stable sets.

c) *None of the two players has a strictly dominant strategy.* Then there exist $\overline{p}, \overline{q} \in [0,1)$ such that

$$BR_1(q) = \begin{cases} \{0\} & if \quad q < \overline{q} \\ [0,1] & if \quad q = \overline{q} \\ \{1\} & if \quad q > \overline{q} \end{cases} \text{ and } BR_2(p) = \begin{cases} \{0\} & if \quad p < \overline{p} \\ [0,1] & if \quad p = \overline{p} \\ \{1\} & if \quad p > \overline{p} \end{cases}.$$

Let $\overline{\sigma} = (\overline{p}, \overline{q})$, $\widetilde{\sigma} = (0,0)$ and $A = A_1 \cup A_2$ where $A_1 = \{\sigma \in \Delta(S) : \sigma = \lambda\sigma^* + (1-\lambda)\overline{\sigma},\ \lambda \in [0,1]\}$ and $A_2 = \{\sigma \in \Delta(S) : \sigma = \lambda\overline{\sigma} + (1-\lambda)\widetilde{\sigma},\ \lambda \in [0,1]\}$. In this case U_1 is increasing in p if $q > \overline{q}$ and U_2 is increasing in q if $p > \overline{p}$, while U_1 is decreasing in p if $q < \overline{q}$ and U_2 is decreasing in q if $p < \overline{p}$. It is easy to verify that $D(A) = \Delta(S) \backslash A$. Hence, A is a *vN&M* stable set.

To show that there are infinite *vN&M* stable sets consider a strategy profile $\widehat{\sigma} \notin A$; $1 > \widehat{p} \geq \overline{p}$ and $1 > \widehat{q} \geq \overline{q}$. Reasoning as above we have that

$\widehat{A} = \{\sigma \in \Delta(S) : \sigma = \lambda\sigma^* + (1-\lambda)\widehat{\sigma}, \lambda \in [0,1]\} \cup \{\sigma \in \Delta(S) : \sigma = \lambda\widehat{\sigma} + (1-\lambda)\overline{\sigma}, \lambda \in [0,1]\} \cup \{\sigma \in \Delta(S) : \sigma = \lambda\overline{\sigma} + (1-\lambda)\widetilde{\sigma},\ \lambda \in [0,1]\}$ is also a *vN&M* stable set. Therefore $(\Delta(S), \succ)$ associated to these games have infinite *vN&M* stable sets.

Case 2. *Game G does not have a strict NE strategy profile.*

In trivial games, where all payoffs of at least one player are identical, it is immediate that Σ^* is the unique *vN&M* stable set. Suppose that game G is non-trivial. Let $I_i = \{\sigma \in \Delta(S) : \sigma_j = \sigma_j^*$ and $U_i(\sigma) = U_i(\sigma^*)$ for some $\sigma^* \in \Sigma^*\}$ for $i = 1, 2$ and $j \neq i$. We show that $A = \cup_{i \in N} I_i$ is the unique *vN&M* stable set. Two sub-cases are distinguished:

a) *Only one player has a strictly dominant strategy.* Suppose, as above, without loss of generality that $BR_1(q) = \{1\}$ for all $q \in [0,1]$. Since the game does not have a strict *NE* strategy profile then $BR_2(1) = [0,1]$. Thus $\Sigma^* =$

$\{\sigma \in \Delta(S) : p = 1\}$. As $A = \Sigma^*$ and U_1 is increasing in p for all $q \in [0,1]$ then $D(A) = \Delta(S) \backslash A$ and the result follows.

b) *None of the two players has a strictly dominant strategy.* Without loss of generality two sub-cases are distinguished:

(i) There exist $\overline{p}, \overline{q} \in [0,1]$ such that

$$BR_1(q) = \begin{cases} \{0\} & if \quad q < \overline{q} \\ [0,1] & if \quad q = \overline{q} \\ \{1\} & if \quad q > \overline{q} \end{cases} \text{ and } BR_2(p) = \begin{cases} \{0\} & if \quad p < \overline{p} \\ [0,1] & if \quad p = \overline{p} \\ \{1\} & if \quad p > \overline{p} \end{cases}.$$

Since game G does not have a strict *NE* strategy profile then either $\overline{p} = 1$ and $\overline{q} = 0$ or $\overline{p} = 0$ and $\overline{q} = 1$. Thus $A = \Sigma^*$ and as $D(\Sigma^*) = \Delta(S) \backslash \Sigma^*$ the result follows.

(ii) There exist $\overline{p}, \overline{q} \in [0,1]$ such that

$$BR_1(q) = \begin{cases} \{0\} & if \quad q < \overline{q} \\ [0,1] & if \quad q = \overline{q} \\ \{1\} & if \quad q > \overline{q} \end{cases} \text{ and } BR_2(p) = \begin{cases} \{1\} & if \quad p < \overline{p} \\ [0,1] & if \quad p = \overline{p} \\ \{0\} & if \quad p > \overline{p} \end{cases}.$$

Let $\overline{\sigma} = (\overline{p}, \overline{q})$. Then $\overline{\sigma} \in \Sigma^*$ and we have $I_1 = \{\sigma \in \Delta(S) : q = \overline{q}\}$ and $I_2 = \{\sigma \in \Delta(S) : p = \overline{p}\}$. In this case, U_1 is increasing in p if $q > \overline{q}$ and U_2 is decreasing in q if $p > \overline{p}$. In contrast U_1 is decreasing in p if $q < \overline{q}$ and U_2 is increasing in q if $p < \overline{p}$. It is easy to see that $D(A) = \Delta(S) \backslash A$, *i.e.*, that A is a *vN&M* stable set. To prove that A is unique consider a *vN&M* stable set B. We show by contradiction that $B \subseteq A$. Suppose that there is $\sigma \in B$ such that $\sigma \notin A$. Since A satisfies external stability there exists $\sigma' \in A$ such that $\sigma' \succ \sigma$. Given that $\sigma' \in A$ we have $\sigma' \in I_i$ for some $i \in \{1,2\}$. Then $\sigma'_j = \overline{\sigma}_j$ for $j \neq i$ and since $\sigma' \succ \sigma$ we have $\sigma'_i = \sigma_i$ and $U_j(\sigma') > U_j(\sigma)$. As $\sigma' \notin B$ and B satisfies the external stability condition, there exists $\sigma'' \in B$ such that $\sigma'' \succ \sigma'$. Given that if $\sigma''_j = \sigma'_j = \overline{\sigma}_j$ then $U_i(\sigma'') = U_i(\sigma')$ we have $\sigma''_i = \sigma'_i$ and $U_j(\sigma'') > U_j(\sigma')$. Thus $\sigma'' \succ \sigma$, contradicting the internal stability of B. Consequently $B \subseteq A$ and given that A satisfies external stability it follows that $B = A$. ∎

The following remark establishes which are the strategy profiles belonging to *vN&M* stable sets:

Remark 2 *Only strategy profiles which are not dominated by some NE or by strategy profiles in which the payoff of one player does not depend on his own strategy are in vN&M stable sets.*

Note that in games without a strict *NE* the dominion of A is formed by the strategy profiles in which one player's payoff does not depend on his own strategy is exactly $\Delta(S)\backslash A$, and therefore A is the *vN&M* stable set.

As illustrated in **Example 3**, the *NE* of this game with a unique *vN&M* stable set may not be Pareto optimal. Therefore there is a strategy profile where some agent's payoff can be increased over the *NE* payoff without decreasing the *NE* payoff of her opponent. The question now is whether the *vN&M* stable set contains always a Pareto optimal strategy profile. This is answered in the affirmative in the following proposition.

A strategy profile σ is *Pareto optimal* if there is not another strategy profile σ' such that $U_i(\sigma') \geq U_i(\sigma)$ for $i = 1, 2$ with at least one strict inequality.

Proposition 3 *If* $(\Delta(S), \succ)$ *has a unique vN&M stable set then it includes at least one Pareto optimal strategy profile.*

Proof. If $(\Delta(S), \succ)$ has a unique *vN&M* stable set A then we are in **Case 2** of *Theorem 1.* In trivial games, where all payoffs of at least one player i are identical, it is immediate that a *NE* strategy profile σ^* such that $U_j(\sigma^*) = \max_{\sigma \in \Sigma^*} U_j(\sigma)$ is Pareto optimal. Suppose that game G is non-trivial. We consider the two subcases in **Case 2** of *Theorem 1*:

a) Let $\sigma^* \in \Sigma^*$ such that $U_1(\sigma^*) = \max_{\sigma \in \Sigma^*} U_1(\sigma)$. It is obvious that σ^* is Pareto optimal.

b) This case has two subcases, namely

(i) Reasoning in a similar manner than in **Sub-case a)** the result follows.

(ii) In this case, there exist $\overline{p}, \overline{q} \in [0, 1]$ such that

$$BR_1(q) = \begin{cases} \{0\} & if \quad q < \overline{q} \\ [0,1] & if \quad q = \overline{q} \\ \{1\} & if \quad q > \overline{q} \end{cases} \text{ and } BR_2(p) = \begin{cases} \{1\} & if \quad p < \overline{p} \\ [0,1] & if \quad p = \overline{p} \\ \{0\} & if \quad p > \overline{p} \end{cases}.$$

We have $A = \cup_{i \in N} I_i$ where $I_1 = \{\sigma \in \Delta(S) : q = \overline{q}\}$ and $I_2 = \{\sigma \in \Delta(S) : p = \overline{p}\}$. Let $\widetilde{p}, \widetilde{q} \in \mathbb{R}$ such that $U_1(p, q) = a(p - \widetilde{p})(q - \overline{q}) + c$ and $U_2(p, q) = a'(p - \overline{p})(q - \widetilde{q}) + c'$, ($a > 0$ and $a\acute{} < 0$). Let $\sigma^1, \sigma^2 \in A$ such that $U_1(\sigma^1) = \max_{\sigma \in I_2} U_1(\sigma)$ and $U_2(\sigma^2) = \max_{\sigma \in I_1} U_2(\sigma)$ Supposse without loss of generality that $\sigma^1 = (\overline{p}, 0)$ and $\sigma^2 = (0, \overline{q})$. Then $\widetilde{p} \geq \overline{p}$ and $\overline{q} \geq \widetilde{q}$. If $U_1(\overline{p}, 0) \geq U_1(0, \widetilde{q})$ then σ^1 is a Pareto optimal strategy profile and therefore is not Pareto dominated by any strategy $\sigma = (p, q) \in \Delta(S)$.

This is obvious if $\sigma \in I_2$, that is, $p = \overline{p}$.

Supposse that $p \neq \overline{p}$. If $\widetilde{q} \leq q \leq \overline{q}$ we have $U_1(p,q) < U_1(0,\widetilde{q}) \leq U_1(\overline{p},0)$ for all $q \neq \widetilde{q}$ and $U_1(p,\widetilde{q}) \leq U_1(\overline{p},0)$ and $U_2(p,\widetilde{q}) = U_2(\overline{p},0)$.

Supposse now that $q > \overline{q}$.

If $p > \overline{p}$ then $U_2(p,q) < U_2(p,\overline{q}) \leq U_2(\overline{p},0)$ and if $p < \overline{p}$, $U_1(p,q) < U_1(\overline{p},q) \leq U_1(\overline{p},0)$.

Finally, if $q < \widetilde{q}$ we have $U_2(p,q) < U_2(p,\widetilde{q}) = U_2(\overline{p},0)$ for all $p < \overline{p}$ and $U_1(p,q) < U_1(\overline{p},q) \leq U_1(\overline{p},0)$ for all $p > \overline{p}$.

Hence σ^1 is not Pareto dominated by any strategy in $\Delta(S)$ and therefore σ^1 is a Pareto optimal strategy profile.

If $U_1(\overline{p},0) < U_1(0,\widetilde{q})$ it is very easy to verify that $\overline{pq} > \widetilde{p}\widetilde{q}$ and $U_2(0,\overline{q}) > U_2(\widetilde{p},0)$. Then, reasoning in a similar manner we conclude that σ^2 is a Pareto optimal strategy profile. ■

Note that if a *NE* is not Pareto optimal then there is a Pareto strategy profile in the *vN&M* stable set that maximizes the payoff of the player who does not get the *NE* payoff.

The following examples illustrate the previous results. In the first we present a game with an infinite number of *vN&M* stable sets and we give two of them. In the second the game has a unique *vN&M* stable set with a *NE* that is Pareto optimal. In the third example we have a game whose *NE* is not Pareto optimal and we give the Pareto optimal stategy profile of the *vN&M* stable set.

Example 1 Consider the mixed extension of the following game:

	s_2^1	s_2^2
s_1^1	1,1	0,0
s_1^2	0,0	1,0

Insert Fig. 1 about here.

In this example the players' best responses are:

$$BR_1(q) = \begin{cases} \{0\} & if \quad q < 1/2 \\ [0,1] & if \quad q = 1/2 \\ \{1\} & if \quad q > 1/2 \end{cases} \text{ and } BR_2(p) = \begin{cases} [0,1] & if \quad p = 0 \\ \{1\} & if \quad p > 0 \end{cases}$$

It is easy to check that the set of *NE* strategy profiles is $\Sigma^* = \{(1,1)\} \cup \{(0,q) : 0 \leq q \leq \frac{1}{2}\}$. In *Figure 1* the bold point indicates the unique strict *NE* strategy profile and the thick line segment indicates the non-strict ones. Thus, by the previous theorem, $(\Delta(S), \succ)$ has infinite *vN&M* stable sets. In this game none of the two players has a strictly dominant strategy (see **Case 1 c** in the proof of *Theorem 1*). To give a *vN&M* stable set A, it suffices to consider the set of strategy profiles A_1 such that $(p,q) = \lambda(1,1) + (1-\lambda)(0, \frac{1}{2})$ $(\lambda \in [0,1])$ *i.e.,* $A_1 = \{(p,q) : q = \frac{1}{2} + \frac{1}{2}p,\ 0 \leq p \leq 1\}$, and the set of strategy profiles A_2 such that $(p,q) = \lambda(0, \frac{1}{2}) + (1-\lambda)(0,0)$, $\lambda \in [0,1]$ *i.e.,* $A_2 = \{(0,q) : 0 \leq q \leq \frac{1}{2}\}$. Then $A = A_1 \cup A_2$ (see *Figure 1*). To obtain a second *vN&M* stable set $\widehat{A}$, we consider $\widehat{\sigma} \notin A$; $1 > \widehat{p} \geq 0$ and $1 > \widehat{q} \geq \frac{1}{2}$. For example, taking $\widehat{\sigma} = (\frac{1}{2}, \frac{1}{2})$ we have $\widehat{A} = \{(p,q) : q = p,\ \frac{1}{2} \leq p \leq 1\} \cup \{(p, \frac{1}{2}) : 0 \leq p \leq \frac{1}{2}\} \cup \{(0,q) : 0 \leq q \leq \frac{1}{2}\}$.

Example 2 Consider the mixed extension of the following game:

	s_2^1	s_2^2
s_1^1	1,0	0,1
s_1^2	0,0	1,0

Insert Fig. 2 about here.

In this example player 1's best response coincides with the one in *Example 1* while player 2´s best response is:

$$BR_2(p) = \begin{cases} [0,1] & if \quad p = 0 \\ \{0\} & if \quad p > 0 \end{cases}$$

It is easy to check that the set of *NE* strategy profiles is $\Sigma^* = \{(0,q) : 0 \leq q \leq \frac{1}{2}\}$ which is represented by the thick line segment in *Figure 2*. Since the game does not have a strict *NE* strategy profile, $(\Delta(S), \succ)$ has a unique *vN&M* stable set. None of the two players has a strictly dominant strategy (**Case 2 b** *(ii)* in the proof of *Theorem 1*) and in this case $\overline{p} = 0$ and $\overline{q} = \frac{1}{2}$. To give the only *vN&M* stable set A, we consider the sets $I_1 = \{(p, \frac{1}{2}) : 0 \leq p \leq 1\}$ and $I_2 = \{(0,q) : 0 \leq q \leq 1\}$. Thus, $A = \{(p, \frac{1}{2}) : 0 \leq p \leq 1\} \cup \{(0,q) : 0 \leq q \leq 1\}$ and it is the unique *vN&M* stable set. Note that in this game the *NE* strategy profile $(p^*, q^*) = (0,0)$ is Pareto optimal.

Example 3 Consider the mixed extension of the following game:

	s_2^1	s_2^2
s_1^1	0,-2	-2,0
s_1^2	-3,0	1,-1

where the *NE* strategy profile is $(\overline{p},\overline{q})=(\frac{1}{3},\frac{1}{2})$. In this case the unique *vN&M* stable set is $A=I_1\cup I_2$ where $I_1=\{(p,\frac{1}{2}):0\leq p\leq 1\}$ and $I_2=\{(\frac{1}{3},q):0\leq q\leq 1\}$. We have $U_1(p,q)=6(p-\frac{2}{3})(q-\frac{1}{2})-1$ and $U_2(p,q)=-3(p-\frac{1}{3})(q-\frac{1}{3})-\frac{2}{3}$. Then $\widetilde{p}=\frac{2}{3}$ y $\widetilde{q}=\frac{1}{3}$. Note that $U_1(\frac{1}{3},0)=0$ and $U_1(\frac{1}{3},\frac{1}{2})=-1$, therefore $U_1(\frac{1}{3},0)>U_1(\frac{1}{3},\frac{1}{2})$. The *NE* strategy profile $(\frac{1}{3},\frac{1}{2})$ is not Pareto optimal while $(\frac{1}{3},0)$, which belongs to the *vN&M* stable set, is a Pareto optimal strategy profile.

4 Relationship between 2×2 games classifications

The *vN&M* stable set solution partitions the 2×2 games contingent upon whether it contains a strict *NE* strategy profile. We relate this classification with two others analyzed in the game theory literature.

First, it is well known (see, for instance, Calvo-Armengol [1] and Eichberger *et al.* [2]) that transforming the players' payoff functions as follows:

$$\begin{aligned} u_1'\left(s_1,s_2^1\right) &= u_1\left(s_1,s_2^1\right)-u_1\left(s_1^2,s_2^1\right),\ u_1'\left(s_1,s_2^2\right)=u_1\left(s_1,s_2^2\right)-u_1(s_1^1,s_2^2) \text{ for } s_1\in S_1 \\ u_2'\left(s_1^1,s_2\right) &= u_2\left(s_1^1,s_2\right)-u_2\left(s_1^1,s_2^2\right),\ u_2'\left(s_1^2,s_2\right)=u_2\left(s_1^2,s_2\right)-u_2\left(s_1^2,s_2^1\right) \text{ for } s_2\in S_2 \end{aligned}$$

preserves the best response structure of the game. The transformed 2×2 game becomes:

	s_2^1	s_2^2
s_1^1	a_1,b_1	$0,0$
s_1^2	$0,0$	a_2,b_2

where $a_1=u_1'\left(s_1^1,s_1^1\right),a_2=u_1'\left(s_1^2,s_2^2\right),b_1=u_2'\left(s_1^1,s_2^1\right)$ and $b_2=u_2'\left(s_1^2,s_2^2\right)$. This transformation permits to classify the 2×2 games in terms of their number and nature of *NE* by just examining payoff parameters a_1,a_2,b_1 and b_2 which for generic games are all different from 0 and for non-generic games at least one is zero[7]:

[7] See for instance von Stengel[9]. Roughly speaking, a game is *generic* if it has some neighborhood whose elements have the same number of *NE* as the original game.

Generic games	Conditions	
Dominant solvable	$a_1 a_2<0$ or $b_1 b_2<0$	One pure NE
Coordination	$a_1, a_2, b_1, b_2>0$ or $a_1, a_2, b_1, b_2<0$	Two pure and one mixed NE
Strictly competitive	$a_1, a_2>0$ and $b_1, b_2<0$ or $a_1, a_2<0$ and $b_1, b_2>0$	One mixed NE

Non-generic games	Conditions	
	$a_i=0$ or $b_i=0$ for some i	Two or infinite NE

Thus, *(i)* For generic games only strictly competitive games have a unique *vN&M* stable set which includes the *NE* strategy profile. Dominant solvable and coordination games however, have infinite *vN&M* stable sets. *(ii)* If a non-generic game has a strict *NE* then $(\Delta(S), \succ)$ has infinite *vN&M* stable sets. Otherwise $(\Delta(S), \succ)$ has a unique *vN&M* stable set. Leaving aside trivial games, there are games in which the *vN&M* stable set coincides with the set of *NE,* see **Sub-case 2 a** and **Sub-case 2 b** (i) in the proof of **Theorem 1**.

Second, Germano [4] introduces a procedure for classifying normal form games based on equilibrium correspondences and their discontinuities. In particular, when the procedure is applied to the concept of risk dominant equilibrium within generic 2×2 games, it yields the following equivalence classes: games of the "matching pennies type" on the one hand, and games solvable by iterated strict dominance and coordination games with the exception of those with equal deviation losses for the two strict equilibria, on the other. Although the present work includes generic and non-generic 2×2 games our classification goes along with these two equivalence classes.

5 Further research

Summing up, we have provided a simple classification of the 2×2 (generic and non-generic) games. The nonexistence of a strict *NE* in game G guarantees that $(\Delta(S), \succ)$ has a unique *vN&M* stable set. Otherwise $(\Delta(S), \succ)$ has infinite *vN&M* stable sets.

The natural extension to this work is to analyze the *vN&M* stable sets for finite normal form games, task which does not seem easy to accomplish. We advance a result for an example of two players with more than two strategies.

The "stone, paper and scissor" game is a zero-sum game where the two players simultaneously choose either rock, paper, or scissors and that is represented by the following payoff matrix

	r	p	s
r	0,0	-1,1	1,-1
p	1,-1	0,0	-1,1
s	-1,1	1,-1	0,0

It is well-known that this game has a unique mixed *NE* in which each player plays rock, paper, and scissors with equal probability. Consider the unique *NE* and all the strategies profiles in which the two players have zero payoff, say set A. The remaining strategy profiles gives a positive payoff to one player and a negative one to the other. From any of these profiles the player with negative payoff deviates profitably to a strategy profitable in A which is the unique *vN&M* stable set.

References

[1] Calvo-Armengol, A., (2003) The Set of Correlated Equilibria of 2×2 games, Barcelona Economics Working Paper Series #79.

[2] Eichberger, J. H. Haller and F. Milne. (1993). Naive Bayesian Learning in 2×2 Matrix Games. *Journal of Economic Behavior and Organization* 22, 69-90.

[3] Greenberg, J. (1990) The Theory of Social Situations. *Cambridge University Press.*

[4] Germano, F. (2006) On some Geometry and Equivalence Classes of Normal Form Games *International Journal of Game Theory*, 34, 561-581.

[5] Luo, X. (2009) On the Foundation of Stability *Economic Theory* 40, 185-201. 5.

[6] Nakanishi, N. (2001) On the Existence and Efficiency of the von Neumann-Morgenstern Stable Set in a n-player Prisoners' Dilemma. *International Journal of Game Theory* 2, 291-307

[7] Nash, J. (1951) Non-Cooperative Games. *Annals of Mathematics* 54, 2: 286-295.

[8] von Neumann, J. and O. Morgenstern (1947) Theory of Games and Economic Behavior. *Princeton, NJ: Princeton University Press.*

[9] von Stengel, B. (2002) Computing Equilibria in Two-person Games in *Handbook of Game Theory with Economic Applications*, Vol. 3, 1723–1759.