A Geometric Interpretation of the Shapley value for TU Games

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Extended abstract. We give the Shapley value of a TU game a new geometric interpretation even if the core of the game is empty. An n-person cooperative game in characteristic function form can be stated as follows: There is a set N of n players, say, player 1, player 2, etc.. We call each possible subset S of the n players N a coalition. To each coalition S we assign a payoff $v(S)$ to be shared by the players in the coalition S. The payoff to the empty coalition is taken to be zero, $v(\emptyset) = 0$. The function v is called the characteristic function of the cooperative game.

The fundamental assumptions of the cooperative TU game theory are the following. (I). In such a cooperative game, we assume that the players are allowed to pre-play the game so that each player knows the payoff $v(S)$ to every coalition S, and that the payoff is transferable among the players. (II). The grand coalition N is formed and all the players in the coalition work cooperatively to get a payoff $v(N)$, then we study how to "fairly" distribute the payoff $v(N)$ among the players.

Among all the solutions to the question in (II), the Shapley value is unquestionably the most central. The Shapley value has significantly influenced recent developments in

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many branches of the social sciences.

If some players are not better off by cooperation, then it is not nature to assume that the grand coalition N is formed. Moreover, if the grand coalition N is not formed then $v(N)$ is not available and it is unnecessary to study how to distribute the payoff $v(N)$ among the players.

In the very original paper entitled "A value for n-person games", Shapley define a game to be any super-additive set function v defined on 2^N such that $v(\emptyset) = 0$. Shapley call the super-additive property, $v(S \cup T) \geq v(S) + v(T)$ whenever $\S \cap T = \emptyset$, the "snowballing" or "bandwagon" effect, i.e. players have incentive to participate in a bigger coalition. Therefore, for Shapley's original definition of game, it is nature to assume that the biggest coalition, grand coalition N, is formed.

Nowadays, some researchers drop the super-additive property from Shapley's original definition of a game , assume that N is formed by law or by chance and define a game to be any set function v defined on 2^N such that $v(\emptyset) = 0$ and call Shapley's original game, with super-additive property, a proper game.

In this article, we adopt the "new" definition of a game. However, in sake of keeping the idea of "snowballing" or "bandwagon" effect, we propose a new class of TU games called coalitional regular in average games, abbreviated as CRIA games.

In the investigation of the solutions of a CRIA game, observing the structure of the core of a game, we introduce the concepts of k^{th} semi-cores and k^{th} quasi-cores of an n-person game, for $k = 1, 2, \cdots, n-1$. When all quasi-cores of an n-person TU game are non-empty, enlightened by the concept of compromise, a middle way between two extremes, we propose the compromise solution of the TU game as the geometric centroid of the $n-1$ mass centers of the quasi-cores of the game. Surprisingly, we find that the compromise solution is exactly the Shapley value. This gives the Shapley value a new geometric interpretation and a new characterization, or say, a new intuitive interpretation, as the compromise solution. In this article, we have a real-world example to explain the intuitive meaning of the compromise solution.

Also, we show that a game is CRIA if and only if none of its k^{th} quasi-cores is empty. Furthermore, a CRIA game might have empty core, therefore, our geometric interpretation is applied to the Shapley value for games with empty core. Finally, our compromise

solution is different from the core-center(Gonzalez-Diaz& Sanchez-Rodriguez, 2007).

Keywords: Shapley value, Core, Centroid, compromise solution.