# CONTINUOUS TIME NOISY SIGNALLING

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Abstract. Most real-life signalling à la Spence (1973) is noisy and in many cases signalling takes time. This paper formulates the classic signalling model in continuous time, with Brownian noise obscuring the signal. Adding noise removes the need for belief-based equilibrium refinements, since any signalling level observed by the receiver is on the equilibrium path. The model allows for equilibria where signalling occurs in multiple disjoint intervals of beliefs. There may be no 'most informative' equilibrium with a signalling region containing the signalling regions of all other equilibria. A noisier signal or a less patient sender shrinks the largest range of beliefs for which signalling can be sustained. If the bad type finds signalling sufficiently costly, then the set of equilibria shrinks as the noise increases or patience decreases. Patience and signal precision are interchangeable, as in the previous literature. The bad type prefers the pooling equilibrium, more so when there is little noise or the signaller is patient. The good type may prefer pooling or separating, with separating relatively better under more patience and less noise.

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## 1. Introduction

Consider a long-lived firm advertising its product that each consumer buys only once, e.g. camera, tour, weight loss surgery. The firm knows whether the product is high or low quality, but the consumers do not. Both high and low quality firms care only about the net profit, which is product revenue minus advertising and production costs. The consumers care only about the quality of the product, not the advertising level. Due to the one-time purchase, there is no experience-based learning, but many consumers buy over a long time period, so there is learning from product reviews and media coverage.

The firm may try to bias the information the consumers see about the product by costly advertising, paying for endorsements, posting fake positive reviews and censoring negative ones etc. A firm with low quality finds it more costly to achieve a given level of positive coverage than a high-quality firm, because for a low-quality firm the independent reviews are mostly negative and must be counterbalanced by more marketing by the firm. Creating a favourable media attitude to the product is thus a differentially costly signal, opening an opportunity for signalling along the lines of Spence (1973).

The consumers rationally update their belief about the quality of the product based on the media coverage they see. The coverage depends on many random factors, e.g. how many sources post independent reviews, the reviewers' idiosyncratic preferences for the product, whether the firm's marketing department has a lucky idea. Due to this randomness, a given advertising expenditure by the firm only determines a distribution of favourability levels of reviews. The consumers can thus never be completely sure of the quality level of the product, but as a group, they learn over time. On average, the market is not fooled by advertising.

Other examples of noisy signalling over time include a professional taking regular licensing or certification exams and a manager trying to raise the share price of the managed firm every quarter

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in order to get a bonus. Passing a licensing exam is imperfectly correlated with the professional's competence, which the buyers of the professional's service care about. Conditional on competence, the buyers are not interested in the exam results. Achieving a given probability of passing the exam requires less effort from a more qualified professional. Both qualified and unqualified people are interested in attracting buyers for their services and thus attempt to pass the exam. Since more qualified people pass with a higher probability, the buyers can use the exam results to update their belief about the competence of the professional.

In the case of a manager of a firm, it is easier to generate good results at a firm based on a good business idea, but there is noise in the results. Investors are interested in the future potential of the business idea, not in the current accounting statements. Nonetheless the investors react to current performance, because it signals the manager's effort. The effort in turn depends on the manager's private information about the potential of the firm—it is worth putting in more effort at a good firm.

In such situations of noisy signalling over time, the natural questions are how the amount of signalling depends on the prior over types, the level of noise, the patience of the signaller and the costs of the high and the low type. This paper answers them using a model in continuous time with Brownian noise, where the informed player controls the drift of the noise process. Specific functional forms are necessary to make the model more tractable, since nonzero sum repeated games with incomplete information (such as the repeated signalling game) are notoriously difficult to analyze.

1.1. **Literature.** The model closest to this paper is Dilme (2012), which also considers continuous time signalling with Brownian noise, but without discounting and with the signaller receiving a payoff only when he decides to stop the game. The functional form for the cost of effort is also different in the two papers. Dilme (2012) finds that the more similar the costs of different types, the worse off they are (competition effect) and that the payoff of the signaller does not depend on the volatility of the noise process. These effects are absent in the present paper.

In Daley and Green (2012b) the uninformed players receive information (observations of a diffusion process) exogeneously over time and the informed player decides when to stop the game (execute the trade) and receive a final payoff. Their equilibrium has three regions: if the probability on the good type is high, there is immediate trade, which is efficient. For low probability on the good type, the good type rejects and with positive probability the bad type accepts. For intermediate beliefs, no trade occurs.

The case where the good type of the informed player is a commitment type who does not need incentivizing is examined in Gryglewicz (2009). Both players can stop the game and the payoffs received upon stopping depend on the type of the informed player. The uninformed player optimally stops at a high belief threshold and the low type informed player at a low belief threshold.

Kremer and Skrzypacz (2007) analyze both noisy and noiseless cases when the informed party signals by delaying trade and after a finite amount of time the information is exogeneously revealed. Signalling by delaying trade also occurs in Hörner and Vieille (2009).

Infinitely repeated noiseless signalling in discrete time is considered in Kaya (2009); Roddie (2011). A large set of equilibria is found in Kaya (2009). One of the sources of the multiplicity of separating equilibria is that the signalling cost can be distributed over time in many ways. Roddie (2011) focusses on the least-cost separating equilibrium and finds conditions under which the signalling level is higher than in the one-shot game.

Nöldeke and Van Damme (1990) use a job market signalling model where time is split into periods of length $\Delta$ and then take $\Delta$ to zero. The signal is delay in accepting a wage offer. For fixed $\Delta$, many sequential equilibria are possible. In the limit, the unique equilibrium corresponds to the one in the static game that satisfies independence of never weak best response. Swinkels

(1999) studies a variation of the same model where wage offers to the signalling worker are private. He finds that only the pooling equilibrium is possible.

An early work on one-shot noisy signalling is Matthews and Mirman (1983), where a monopolist tries to deter entry by setting a low price to convince the entrant that the market is unprofitable. The entrant observes the price subject to noise generated by a demand shock. Daley and Green (2012a) consider one-shot signalling where the receivers observe both the signaller's action and a noisy grade with a distribution that depends on the signaller's type and action. Carlsson and Dasgupta (1997) study one-shot signalling games where the receiver has only two actions. They prove the existence of noise-proof equilibrium, which is an equilibrium in the noiseless game that is the limit of equilibria in games perturbed by noise.

Since in a signalling game, belief about the types of the signaller is the central object, signalling games do not fall under the known-own-payoffs assumption of Shalev (1994); Peski (2008), nor do they admit belief-free equilibria along the lines of Hörner and Lovo (2009); Hörner, Lovo, and Tomala (2011) or ex-post public equilibria described in Fudenberg and Yamamoto (2011).

## 2. One-shot noisy signalling

In order to better distinguish which features of repeated noisy signalling are due to noise and which to repetition, this section solves two examples of one-shot noisy signalling. In both cases, the sender has two types, $H$ and $L$, with probability $\mu_0$ of the $H$ type. There is a market of perfectly competitive receivers, each of whom has action set $[0,1]$. The receivers value only the type, not the signal or signaller's action. They obtain value 1 from contracting with the $H$ type and 0 from the $L$ type, and their cost is equal to the offer they make the signaller. These assumptions will also be made in the continuous time model.

The sender moves first, then the signal is generated and observed by the receivers and finally the receivers move. The sender's action determines the signal distribution. Given the signal and a belief about the equilibrium strategy of the types of the signaller, the receivers update via Bayes' rule. Due to competition, their offer to the signaller equals their posterior belief $\mu$ and thus they make zero expected profit. Both types of signaller receive the same benefit $R(\mu)$ from the same offer $\mu$ of the receivers, but the signalling costs differ between types. $R$ is assumed continuous, strictly increasing and weakly concave.

The equilibrium concept is perfect Bayesian equilibrium. The public history is the signal generated by the sender's action. All public histories have positive probability after any action of the sender, so the receivers always use Bayes' rule. There are no off-path beliefs that need to be specified in an ad-hoc way.

### 2.1. Binary action and signal.

The sender has two actions, denoted 1 and 0, interpreted as effort and no effort respectively. The cost of 0 is zero for both $H$ and $L$ type, while the costs of 1 are $A_L > A_H > 0$ for the types. There are two signals, $g$ and $b$ (good and bad). The signal probabilities after effort and no effort are $\Pr(g|1) = \Pr(b|0) = p \in (\frac{1}{2}, 1)$.

If the two types are expected to take the same action, then the receivers' belief does not depend on the signal and remains at $\mu_0$ after both $g$ and $b$. So neither type would take the costly 1 action in this case. If the $L$ type was to take 1 and the $H$ type 0, belief would become less favourable after the $g$ signal. Since the action 1 that increases the probability of $g$ is costly, $L$ would deviate to 0. So the only pure action profiles possible in equilibrium are pooling on 0 and separating with $H$ taking 1, $L$ taking 0.

More generally, it cannot be the case that both types are mixing in equilibrium or that $L$ puts higher probability on 1 than $H$. If $H$ is indifferent between 1 and 0, then $L$ strictly prefers 0 due to the higher cost of 1 to $L$. Recall that the benefit from a higher belief is the same for both types. Similarly, if $L$ is indifferent between 1 and 0, then $H$ strictly prefers 1. Mixed equilibria can therefore only feature $L$ mixing and $H$ taking 1 or $H$ mixing and $L$ taking 0.

If the receivers expect the equilibrium strategies of the types to be such that $H$ puts probability $q_H$ on 1 and $L$ puts probability $q_L$ on 1, then the updated probabilities after $g$ and $b$ are

$$\mu_g = \frac{\mu_0[q_H p + (1-q_H)(1-p)]}{\mu_0[q_H p + (1-q_H)(1-p)] + (1-\mu_0)[q_L p + (1-q_L)(1-p)]} \tag{1}$$

$$\mu_b = \frac{\mu_0[q_H(1-p) + (1-q_H)p]}{\mu_0[q_H(1-p) + (1-q_H)p] + (1-\mu_0)[q_L(1-p) + (1-q_L)p]}. \tag{2}$$

Given the expectations of receivers, a sender of type $\theta$ chooses 1 if $R(\mu_b)+p[R(\mu_g)-R(\mu_b)]-A_\theta \geq R(\mu_b)+(1-p)[R(\mu_g)-R(\mu_b)]$, which is equivalent to $A_\theta \leq (2p-1)[R(\mu_g)-R(\mu_b)]$. For both $H$ and $L$ to follow the strategy the receivers expect, it must be the case that

$$A_H \leq (2p-1)[R(\mu_g)-R(\mu_b)] \leq A_L \tag{3}$$

at the $\mu_g, \mu_b$ resulting from the expected strategy. Call the first inequality in (3) $\text{IC}_H$ and the second $\text{IC}_L$. Note that the IC conditions only depend on the actions the receivers expect the sender to take ($q_H, q_L$ in Eq. (1)), not on the sender's actual choice.

Due to the assumption that $A_L > A_H$, if $\text{IC}_H$ is violated, then $\text{IC}_L$ is slack and if $\text{IC}_L$ is violated, then $\text{IC}_H$ is slack. Since the receivers' expectation of mixing brings both $\mu_g$ and $\mu_b$ closer to $\mu_0$ compared to expectation of separation, it reduces $R(\mu_g)-R(\mu_b)$. It thus makes $\text{IC}_H$ harder and $\text{IC}_L$ easier to satisfy.

The dependence of the equilibrium set on the ICs is characterized in the following proposition.

**Proposition 1.** *(1) Pooling is possible at all parameter values.*
*(2) At parameter values where $IC_H$ is violated under expectation of separation, pooling is the only equilibrium.*
*(3) At parameter values where a separating equilibrium exists, an equilibrium where $H$ mixes also exists.*
*(4) The equilibrium where $L$ mixes exists iff $IC_L$ fails under expectation of separation.*
*(5) At parameter values where $IC_L$ is violated under expectation of separation, there are three equilibria—pooling, $L$ mixing and $H$ playing 1, and $H$ mixing and $L$ playing 0.*

*Proof.* (1) If the receiver expects both types to take action 0, the updated belief after both signals is $\mu_0$. Thus there is no benefit to the sender in choosing 1 and increasing the probability of $g$. So both types will choose 0.
(2) The expectation of separation ($q_H = 1$, $q_L = 0$) creates the maximal benefit to playing 1. If $H$ is not willing to play 1 in this situation, then neither type is willing to play 1 for any other expected $q_H, q_L$.
(3) In a separating equilibrium both ICs must hold. Suppose at $q_H = 1$, $q_L = 0$ the $\text{IC}_H$ condition holds. Then due to the continuity of $\mu_g, \mu_b$ in $q_H, q_L$ and the continuity of $R$ there exists a $q_H > 0$ that together with $q_L = 0$ makes $\text{IC}_H$ hold with equality. In that case $\text{IC}_L$ is slack, so the equilibrium where $H$ mixes exists.
(4) If at $q_H = 1$, $q_L = 0$ the $\text{IC}_L$ condition fails, then due to the continuity of $\mu_g, \mu_b$ in $q_H, q_L$ and the continuity of $R$ there exists a $q_L < 1$ that together with $q_H = 1$ makes $\text{IC}_L$ hold with equality. $\text{IC}_H$ is slack in that case, so the equilibrium where $L$ mixes exists. If $\text{IC}_L$ holds under expectation of separation, there cannot be an equilibrium where $L$ plays 1 with positive probability, because the expectation of separation creates the maximal benefit to playing 1. With $L$ mixing, the benefit is reduced, but the cost stays the same, so $\text{IC}_L$ holds strictly and $L$ chooses 0.

(5) If $IC_L$ is violated under expectation of separation, then $IC_H$ is slack and by continuity of $\mu_g, \mu_b$ in $q_H, q_L$ and the continuity of $R$, there exists $q_H > 0$ that together with $q_L = 0$ makes $IC_H$ hold with equality, and there exists $q_L > 0$ that together with $q_H = 1$ makes $IC_L$ hold with equality. In the first case the result is an equilibrium where $H$ mixes, in the second case an equilibrium where $L$ mixes. Pooling is always possible.

□

The dependence of the ICs on the parameters is clear from (3). As the signal gets more precise, i.e. $p$ increases, $IC_H$ gets easier and $IC_L$ harder to satisfy, since both $(2p-1)$ and $[R(\mu_g) - R(\mu_b)]$ increase in $p$. An increase in $A_H$ makes $IC_H$ harder and an increase in $A_L$ makes $IC_L$ easier to satisfy. With a linear $R$, the payoff difference $R(\mu_g) - R(\mu_b)$ gets smaller as $\mu_0$ moves away from $\frac{1}{2}$, which makes $IC_H$ harder and $IC_L$ easier to satisfy.

From Eq. (3) it can be seen that if $A_H$ is too high, then even under expectation of separation, $H$ does not want to pay the cost of action 1. Then under other expected $q_H, q_L$, neither type wants to take action 1, so no equilibria but pooling are possible. If $p$ is too low or $R$ is too unresponsive to $\mu$, the benefit of taking action 1 is low, so neither type will signal. Again pooling is the only equilibrium. If $A_L$ is too high, then even under expectation of separation, $L$ does not want to pay the cost of action 1. Then under other expected $q_H, q_L$, $L$ strictly prefers action 0, so the equilibrium where $L$ mixes does not exist. If $A_L$ is too low, then under expectation of separation, $L$ would deviate to action 1, so separation cannot be sustained.

With a concave $R$, the $L$ type expected utility is higher in a pooling equilibrium than in the other equilibria. This is because $L$ pays no signalling cost in the pooling equilibrium and expects the receivers' belief to go down if there is any signalling. For the same reason, $L$ prefers the equilibrium where $H$ mixes to the separating equilibrium. The payoff of $L$ from a pooling equilibrium relative to an equilibrium with signalling is the higher the more precise the signal. These results fail with a sufficiently convex $R$, e.g. if $\mu_0 = \frac{1}{2}$, $p = \frac{3}{4}$ and $R(\mu) = \mu^n$, then for $n \geq 4$, the utility of $L$ from pooling is lower than from separating.

The $H$ type utility comparison depends on the parameters. If there is any signalling, then $H$ expects the receivers' belief to go up, but must pay a signalling cost. For a low $A_H$ relative to $p$ and the slope of $R$, the separating equilibrium is the best for $H$. For a high $A_H$, low $p$ and low $R'$, pooling is best for $H$.

To compare the noisy one-shot signalling game to the noiseless case, first the noiseless game corresponding to a given noisy game must be defined. A noisy game is defined by the parameters $A_L$, $A_H$, $\mu_0$, $p$ and $R$. Assume $A_H$, $A_L$ and $\mu_0$ remain the same in the corresponding noiseless case, and of course $p = 1$ there. In that case if $R$ was the same as in the noisy game, then there would be a greater incentive to signal, e.g. if separation was expected in equilibrium, the ICs would be $A_H \leq R(1) - R(0) \leq A_L$. In order to keep the ICs the same, replace $R(\mu)$ by $\tilde{R}(\mu) = \frac{(2p-1)[R(\mu_g)-R(\mu_b)]}{R(1)-R(0)} R(\mu)$. This satisfies $\tilde{R}(1) - \tilde{R}(0) = (2p-1)[R(\mu_g) - R(\mu_b)]$. The values of $A_H$, $A_L$ and $\mu_0$ at which a separating equilibrium exists are the same in the noiseless game with $\tilde{R}$ as in the noisy game with $p$ and $R$.

The set of equilibria in the noiseless game depends on how the belief of the receivers is defined. The pooling equilibrium can be eliminated if beliefs after signal $g$ must rise to 1 and $A_H \leq \tilde{R}(1) - \tilde{R}(0) < A_L$. An equilibrium where both types always signal can be added if belief after $b$ falls to zero and $A_H, A_L \leq \tilde{R}(\mu_0) - \tilde{R}(0)$. It is thus not clear whether noise generally enlarges or reduces the equilibrium set. This may be due to the discreteness of the action set. The next subsection will examine the case where a continuum of actions are available to the signaller. The optimal action can then vary continuously with the parameters of the model.

In the noiseless game with a linear $R$, a lower $\mu_0$ increases the payoff of $H$ from separation relative to pooling, because after a good signal belief jumps to 1. In the noisy game, separation is most attractive for $H$ at intermediate $\mu_0$, because $R(\mu_g) - R(\mu_b)$ is largest there.

2.2. **Continuum of actions, binary signal.** The sender's action $e \in [0, \overline{e}]$ now generates a signal $g$ or $b$ with probabilities $\Pr(g|e) = \lambda e + (1-\lambda)\frac{1}{2}$ and $\Pr(b|e) = \lambda(1-e) + (1-\lambda)\frac{1}{2}$ respectively. Assume $\overline{e} = \frac{1+\lambda}{2\lambda}$ in order to make the probabilities well-defined. Seeing $g$, the receivers update the probability of the $H$ type to $\mu_g = \frac{\mu_0 \Pr(g|e_H^*)}{\mu_0 \Pr(g|e_H^*) + (1-\mu_0)\Pr(g|e_L^*)}$, where $e_\theta^*$ is the action receivers expect type $\theta$ to take in equilibrium. A similar updating rule holds for signal $b$.

Type $\theta$ sender's utility from action $e$ and receivers' belief $\mu$ is $u_\theta(e, \mu) = \mu - \frac{A_\theta}{2}e^2$, with $A_L > A_H > 0$. The sender's benefit from the receivers' belief is thus equal to the belief. Given a best response by the receivers to the realized signal and the sender's equilibrium play, the sender's expected utility from action $e$ is

$$-\frac{A_\theta}{2}e^2 + \mu_b + (\mu_g - \mu_b)\lambda e + (\mu_g - \mu_b)(1-\lambda)\frac{1}{2}.$$

If $e_H^* \leq e_L^*$, then both types of sender will choose $e_\theta = 0$, because there is no benefit to signalling, but there is a cost. Therefore pooling at $e_L^* = e_H^* = 0$ is always an equilibrium.

If the expected actions of the types satisfy $e_H^* > e_L^*$, then $\mu_g > \mu_b$. The marginal cost of signalling is zero at $e = 0$, the marginal benefit is $(\mu_g - \mu_b)\lambda$ everywhere. Due to this, both types will choose $e_\theta > 0$. Given the expected equilibrium actions, the chosen actions satisfy the FOCs

$$A_H e_H = (\mu_g - \mu_b)\lambda, \qquad A_L e_L = (\mu_g - \mu_b)\lambda.$$

The FOCs already allow the comparison of the signalling efforts and expected utilities of the types, formalized in the following proposition.

**Proposition 2.** *The equilibrium signalling efforts of types $H$ and $L$ satisfy $e_H^* = \frac{A_L}{A_H}e_L^*$ and the expected utilities $u_H, u_L$ satisfy $u_H - \mu_b - \frac{1-\lambda}{2}(\mu_g - \mu_b) = \frac{A_L}{A_H}\left[u_L - \mu_b - \frac{1-\lambda}{2}(\mu_g - \mu_b)\right]$.*

Proposition 2 holds for both separating and pooling equilibria. Under pooling, $e_H^* = e_L^* = 0$ and $\mu_g = \mu_b = \mu_0$, so $u_H = u_L = \mu_0$. Under separation, it is clear that $u_H \geq u_L$.

There are no mixed equilibria in this model, because even if mixing is expected in equilibrium, both types will deviate to a pure action. For any expected equilibrium actions, the best response is unique. So only pure actions need be used in updating $\mu_0$.

To find the equilibrium actions, equate the chosen and the expected action, $e_\theta = e_\theta^*$ in the FOCs. The solutions for which $e_H^*, e_L^* \in [0, \frac{1+\lambda}{2\lambda}]$ for at least some $\lambda \in [0,1]$ are

$$e_H^* = \frac{A_H A_L \lambda - \sqrt{A_H A_L [A_H A_L - 4(A_L - A_H)\lambda^2 \mu_0 (1-\mu_0)]}}{2A_H \lambda [A_L \mu_0 + A_H(1-\mu_0)]}$$

$$e_L^* = \frac{A_H A_L \lambda - \sqrt{A_H A_L [A_H A_L - 4(A_L - A_H)\lambda^2 \mu_0 (1-\mu_0)]}}{2A_L \lambda [A_L \mu_0 + A_H(1-\mu_0)]}$$

For any parameter values, at most one separating equilibrium is possible. The conditions on parameters that permit a separating equilibrium to exist are partially characterized in the following proposition.

**Proposition 3.** *If $A_H = 1$, then a separating equilibrium exists for $\lambda \geq \sqrt{\frac{A_L}{A_L + 4\mu_0(1-\mu_0)(A_L - 1)}}$.*

*Proof.* Taking $A_H = 1$, the solutions for $e_H^*, e_L^*$ are in the range $[0, \frac{1+\lambda}{2\lambda}]$ iff $\lambda \geq \underline{\lambda}$, where $\underline{\lambda} = \sqrt{\frac{A_L}{A_L + 4\mu_0(1-\mu_0)(A_L - 1)}}$. For $0 < \mu_0 < 1$ and $A_L > 1 = A_H$, we have $\underline{\lambda} \in (0,1)$. □

It can be seen from Proposition 3 that even with $A_L$ close to $A_H$ and beliefs close to zero or one, there exists a separating equilibrium at $\lambda = 1$.

The comparative statics results yielded by this model are limited to Proposition 4 below. The changes in the efforts as $\mu_0$, $A_L$ or $A_H$ vary do not have clear signs.

**Proposition 4.** *In a separating equilibrium the efforts of both types are increasing in $\lambda$, the precision of the signal.*

*Proof.*

$$\frac{\partial e_H^*}{\partial \lambda} = \frac{A_H A_L^2}{2\lambda^2[A_L\mu_0 + A_H(1-\mu_0)]\sqrt{A_H A_L[A_H A_L - 4(A_L - A_H)\lambda^2\mu_0(1-\mu_0)]}},$$

which is positive. Based on Proposition 2, $\frac{\partial e_L^*}{\partial \lambda} = \frac{A_H}{A_L}\frac{\partial e_H^*}{\partial \lambda}$, which is also positive. □

The expected utility from a pooling equilibrium is $\mu_0$ for both types. The $L$ type prefers pooling to separating, because in a separating equilibrium $L$ pays an effort cost and expects $\mu$ to go down. The expected utilities of the types from a separating equilibrium are complicated functions of the parameters and do not yield clear comparative statics.

With a continuum of sender actions, a noiseless game would have a large set of equilibria and belief refinements would be needed to pick out those of interest. In contrast, the noisy game with a continuum of sender actions only has one or two equilibria, depending on parameter values. Noise thus provides results that are easier to interpret, in addition to making the model more realistic. The drawback is that noise makes the game more difficult to solve.

The preceding one-shot models cannot describe what happens when noisy signalling occurs over a period of time. The next section turns to the main model where the sender has a binary action and controls a Brownian signal process.

## 3. Signalling with a binary action in continuous time

Time is continuous and the horizon is infinite. There is an infinitely lived strategic sender who can be one of two types, $H$ or $L$, with probability $\mu_0$ of $H$. Both types of sender have discount rate $r$ and action set $\{0, 1\}$. The cost of action 0 is zero for both types, but the costs of action 1 to $L$ and $H$ are $A_L > A_H > 0$ respectively.

The sender's action $e_{\theta t}$ controls the drift of a signal process $(X_t)_{t\in\mathbb{R}_+}$ subject to Brownian noise $B_t$. The signal process satisfies $dX_t = e_{\theta t}dt + \sigma dB_t$. The receivers at time $t$ observe $(X_\tau)_{\tau\in[0,t]}$ and, given the equilibrium actions $(e_{H\tau}^*, e_{L\tau}^*)_{\tau\in[0,t]}$ of the types of the sender, form a belief $\mu_t$ that the sender is the $H$ type.

The utility of a sender of type $\theta$ at instant $t$ is $\hat{R}(\mu_t) - A_\theta e_{\theta t}$. Both types of sender have the same benefit $\hat{R}(\mu)$ from belief $\mu$, but the $H$ type has a lower cost of effort. Assume $\hat{R}$ is bounded, strictly increasing and twice continuously differentiable in $\mu$.

The utility function of the sender could be motivated by a continuum of receivers in perfect competition who value the type of the sender, but not the signalling effort or signal realization. The receivers get flow utility 1 from contracting with the $H$ type, but 0 from contracting with the $L$ type. The receivers offer the sender his expected productivity $\mu_t$ at every $t$ and make zero expected profit due to competition.

The focus is on pure-strategy Markov stationary equilibria (the sender's action depends only on the belief of the receivers), where outside an interval of beliefs $(\underline{\mu}, \overline{\mu})$ both types choose 0 and inside that interval at least one type chooses 1. These will be called interval equilibria in what follows. The interval $(\underline{\mu}, \overline{\mu})$ is subsequently referred to as the signalling region and its complement as the pooling region. Existence of an interval equilibrium is shown in the next lemma.

**Lemma 5.** *An interval equilibrium always exists.*

*Proof.* If $e_H^* = e_L^*$ at some $\hat{\mu}$, then the receivers cannot use the signal to distinguish the types of the sender, so belief does not respond to the signal at $\hat{\mu}$. Given this, both types will optimally choose $e_\theta = 0$ at $\hat{\mu}$. Therefore a pooling equilibrium, in which $e_H^* = e_L^* = 0$ for all $\mu$, exists for all parameter values. The signalling region is empty in this case: $\underline{\mu} = \overline{\mu}$. Since the pooling equilibrium always exists and fits the definition of interval equilibrium, an interval equilibrium always exists. □

Solving for an equilibrium has two parts. First, given equilibrium strategies (signalling region) the receivers expect, both types of the sender will solve a control problem to choose their optimal strategy. Second, the chosen strategies are set equal to the expected strategies and the resulting system is solved to find the equilibrium strategies. Before moving to the actual solution, some preliminary observations are in order.

In the pooling region the belief does not change and both types optimally choose $e_\theta = 0$, so if the belief reaches some $\mu$ the pooling region, both types get payoff $\hat{R}(\mu)$ forever. In an interval equilibrium the game essentially ends upon reaching any $\mu$ in the pooling region, with a final payoff $\frac{\hat{R}(\mu)}{r}$ to both types. The pooling region and therefore the final payoff is endogeneous.

To set up the control problems of the types, the process of the state variable must be defined. The usual state variable in the literature is the belief of the uninformed agents (Faingold and Sannikov, 2011; Dilme, 2012). In the signalling region, the receivers update their belief using the standard continuous time Bayes' rule

$$d\mu = \sigma^{-2}\mu(1-\mu)(e_H^* - e_L^*)[dX_t - \mu e_H^* dt - (1-\mu)e_L^* dt]. \tag{4}$$

This is also applicable in the pooling region: if $e_H^* = e_L^*$, then $d\mu = 0$ and belief does not change. In the signalling region it must be that $e_H^* = 1$, $e_L^* = 0$, otherwise belief would fall or remain constant in the costly signal, which would make both types deviate to $e_\theta = 0$. The belief process is well-defined and unique, as shown in the following lemma.

**Lemma 6.** *There is a unique belief process satisfying Eq. (4).*

*Proof.* Fix a control $e_\theta$. If the the initial state $\mu_0$ is constant and the drift and variance in Eq. (4) are bounded and Lipschitz in $\mu$, then by Theorem 3.1 of Touzi (2013), Eq. (4) has a unique strong solution given the controlling process $e_\theta$. Since in the signalling region $e_H^* = 1$ and $e_L^* = 0$, the drift of the belief process is $\sigma^{-2}\mu(1-\mu)(e_\theta - \mu)$ and the variance is $\sigma^{-2}\mu^2(1-\mu)^2$. Both of these are Lipschitz in $\mu$ and bounded. □

For mathematical convenience, the log likelihood ratio $l$ of the types is subsequently used as the state variable, instead of the probability $\mu$ of the $H$ type. Since $l = \ln(\mu) - \ln(1-\mu)$ is infinitely differentiable, Itô's formula can be used to transform the belief process (4) into the log likelihood ratio process

$$\begin{aligned} dl_t &= \sigma^{-2}(e_H^* - e_L^*)[dX_t - \frac{1}{2}e_H^* dt - \frac{1}{2}e_L^* dt] \\ &= \sigma^{-2}(e_H^* - e_L^*)(e_\theta - \frac{1}{2}e_H^* - \frac{1}{2}e_L^*)dt + \frac{e_H^* - e_L^*}{\sigma} dB_t, \end{aligned}$$

where $B_t$ is a standard Brownian motion. Since in the signalling region $e_H^* = 1$ and $e_L^* = 0$, the $l$ process is a simple Brownian motion with drift. The drift is either $\frac{1}{2}$ or $-\frac{1}{2}$, depending on whether $e_\theta = 1$ or 0. The signalling region in log likelihood ratio space is denoted $(\underline{l}, \overline{l})$. The benefit function $\hat{R}(\mu)$ is replaced by $R(l) = \hat{R}\left(\frac{\exp(l)}{1+\exp(l)}\right)$, since $\mu = \frac{\exp(l)}{1+\exp(l)}$.

Both types of the sender maximize their expected discounted payoff by choosing Markov stationary control processes $e_H, e_L$. The solutions to these control problems can be written as value functions. Define $\hat{T}_{t,l_t}$ as the first exit time after $t$ of the $l$ process from $(\underline{l}, \bar{l})$, i.e. $\hat{T}_{t,l_t} = \inf\left\{\tau > t : l_\tau \notin (\underline{l}, \bar{l})\right\} \leq \infty$. In the signalling region, the value function of type $\theta$ is

$$W_\theta(l_t) = \sup_{e_\theta(\cdot)} \mathbb{E}\left[\int_t^{\hat{T}_{t,l_t}} \exp(-rs)\left[R(l_s) - A_\theta e_\theta(l_s)\right] ds + \exp(-r\hat{T}_{t,l_t})\frac{R(l_{\hat{T}_{t,l_t}})}{r}\mathbf{1}\left\{\hat{T}_{t,l_t} < \infty\right\}\right],$$

where $\mathbf{1}\{A\}$ denotes the indicator function for the set $A$. The interpretation of the value function expression is straightforward: the agent gets flow benefit $R(l)$ depending on $l$ and chooses the optimal signalling effort $e_\theta$ at each $l$, which determines the flow cost. When the log likelihood ratio exits the signalling region (if ever), the sender gets $R(l_{\hat{T}_{t,l_t}})$ forever, where $l_{\hat{T}_{t,l_t}}$ is the value of $l$ upon exit, which equals either $\underline{l}$ or $\bar{l}$ due to the continuity of the sample paths of the $l$ process. Some observations about the value functions are formalized in the following lemma.

**Lemma 7.** *$W_\theta$ is finite for $\theta = H, L$. $W_H \geq W_L$, with strict inequality in the signalling region. $W_\theta$ is strictly increasing.*

*Proof.* Due to the boundedness of $R(l)$ and $e_\theta$, discounting ensures that $W_\theta$ is finite—even without the expectation, the integral in the definition of $W_\theta$ is finite for any path of $l$ and any control $e_\theta$.

It is clear that $W_H > W_L$ in the signalling region, because $H$ can follow $L$'s strategy at a strictly lower cost than $L$. Outside the signalling region, $W_H(l) = W_L(l) = \frac{R(l)}{r}$.

To prove $W_\theta$ is strictly increasing, a standard coupling argument is used. Consider two diffusion processes: the $l$ process with optimal effort starting from $l_1$ and the $l$ process under zero effort starting from $l_2 > l_1$. Call the former process $l^{e^*}$ and the latter $l^0$. Define the stopping time $\tau^* = \inf\left\{t > 0 : l_t^0 - l_t^{e^*} = 0\right\}$. The receivers expect the optimal strategy in both cases.

Starting at $l_2$, the strategy $s =$"play 0 until $\tau^*$ and the optimal strategy thereafter" yields a weakly lower payoff than $W_\theta(l_2)$, the payoff to the optimal stationary strategy starting from $l_2$. This holds even though $s$ is not stationary, because if the receivers expect a stationary strategy, then among the optimal strategies for the sender there is a stationary one. The argument is standard—the competitive receivers always play a static best response, which depends on their belief about the type, but not the sender's strategy, so if at some $l$, a sender action $\hat{e}$ is optimal at one point in time, then $\hat{e}$ is optimal at that $l$ at another point in time.

Starting at $l_2$, the strategy $s$ yields a strictly higher payoff than $W_\theta(l_1)$, the payoff to the optimal strategy starting from $l_1$. This is because the revenue $R(l^0)$ is strictly higher than $R(l^{e^*})$ before $\tau^*$ and the same in expectation after $\tau^*$. The cost of $l^0$ is zero while the cost of $l^{e^*}$ is positive before $\tau^*$. The costs of the two strategies are the same in expectation after $\tau^*$. Overall, $W_\theta(l_2) > W_\theta(l_1)$ for $l_1, l_2$ in the signalling region.

If both $l_1, l_2$ are outside the signalling region, then since $R$ was assumed strictly increasing, the payoffs are ordered $W_\theta(l_2) = \frac{R(l_2)}{r} > \frac{R(l_1)}{r} = W_\theta(l_1)$. If $l_2$ is above the signalling region while $l_1$ is in the signalling region, then the expected benefit is strictly higher from $l_2$ onwards and the expected cost is the lowest possible from $l_2$ onwards, so $W_\theta(l_2) > W_\theta(l_1)$. If $l_2$ is in the signalling region while $l_1$ is below the signalling region, then $W_\theta(l_2)$ is higher than the payoff to the strategy of taking zero effort forever starting from $l_2$. The cost of this strategy is the same as the cost of the optimal strategy from $l_1$ onwards, while the benefit is strictly greater, so again $W_\theta(l_2) > W_\theta(l_1)$. ☐

With a concave $R$, the $L$ type strictly prefers less signalling, in the sense that for any log likelihood ratio of the receivers, the payoff of $L$ is higher in a pooling equilibrium than with signalling. Comparing signalling equilibria, $L$'s payoff is higher in an equilibrium with a smaller signalling region.

**Proposition 8.** *Assume $R$ is weakly concave. Then $W_L(l) < \frac{R(l)}{r}$ for all $l$ in the signalling region. For equilibria $1,2$ with $L$ type value functions $W_{L1}, W_{L2}$ and signalling regions $(\underline{l}_1, \bar{l}_1) \subsetneq (\underline{l}_2, \bar{l}_2)$, we have $W_{L1}(l) > W_{L2}(l)$ for all $l \in (\underline{l}_2, \bar{l}_2)$.*

*Proof.* $L$ takes no effort in pooling or signalling equilibria, so the flow cost is the same in both cases. The flow benefit $R$ is increasing in the log likelihood ratio $l$. In a pooling equilibrium $l$ stays constant forever, while in a signalling equilibrium $L$ expects $l$ to strictly decrease. With a weakly concave $R$, there is no benefit from the noise in the $l$ process. This establishes $W_L(l) < \frac{R(l)}{r}$.

For $l \in (\underline{l}_2, \bar{l}_2) \setminus (\underline{l}_1, \bar{l}_1)$, it follows from the above that $W_{L1}(l) = \frac{R(l)}{r} > W_{L2}(l)$. Since $(\underline{l}_1, \bar{l}_1)$ is a proper subset of $(\underline{l}_2, \bar{l}_2)$, at least one of $W_{L1}(\underline{l}_1) > W_{L2}(\underline{l}_1)$, $W_{L1}(\bar{l}_1) > W_{L2}(\bar{l}_1)$ holds (the other may be an equality).

From any point in $(\underline{l}_1, \bar{l}_1)$, the log likelihood ratio process has positive probability of hitting $\underline{l}_1$ and positive probability of hitting $\bar{l}_1$. The flow cost to $L$ is zero in all interval equilibria for all $l$. For the same $l$, the flow benefit to $L$ is the same in all interval equilibria. The distribution over paths of $l$ up to hitting $\underline{l}_1$ or $\bar{l}_1$ starting from $l_0 \in (\underline{l}_1, \bar{l}_1)$ is the same in the two equilibria with signalling regions $(\underline{l}_1, \bar{l}_1)$ and $(\underline{l}_2, \bar{l}_2)$, because in both equilibria in the region $(\underline{l}_1, \bar{l}_1)$, $H$ takes action 1 and $L$ takes 0. Therefore the continuation value comparisons $W_{L1}(\underline{l}_1) \geq W_{L2}(\underline{l}_1)$ and $W_{L1}(\bar{l}_1) \geq W_{L2}(\bar{l}_1)$, at least one of which is strict, determine the payoff comparison $W_{L1}(l) > W_{L2}(l)$ for any $l \in (\underline{l}_1, \bar{l}_1)$. □

To solve the control problems of the types of the sender, the HJB equations are solved and a verification theorem is used to check that the solutions of the HJB equations coincide with the value functions. The HJB equations are

$$rw_H(l) = R(l) + \max\left\{-A_H + \frac{1}{2}w_H'(l)\sigma^{-2}, \quad -\frac{1}{2}w_H'(l)\sigma^{-2}\right\} + \frac{1}{2}w_H''(l)\sigma^{-2}$$
$$rw_L(l) = R(l) + \max\left\{-A_L + \frac{1}{2}w_L'(l)\sigma^{-2}, \quad -\frac{1}{2}w_L'(l)\sigma^{-2}\right\} + \frac{1}{2}w_L''(l)\sigma^{-2}$$

Given the signalling region $(\underline{l}, \bar{l})$ the receivers expect, the optimal strategy of type $\theta$ is to choose

$$e_\theta(l) = \begin{cases} \mathbf{1}\left\{-A_\theta + \frac{1}{2}w_\theta'(l)\sigma^{-2} \geq -\frac{1}{2}w_\theta'(l)\sigma^{-2}\right\} & \text{if } l \in (\underline{l}, \bar{l}) \\ 0 & \text{if } l \notin (\underline{l}, \bar{l}) \end{cases}$$

For $H$ to choose $e_H(l) = 1$ in the signalling region, we need $w_H'(l) \geq A_H\sigma^2$. For $L$ to choose $e_L(l) = 0$, we need $w_L'(l) \leq A_L\sigma^2$. Call these constraints $\text{IC}_H$ and $\text{IC}_L$. After finding the candidate equilibrium strategies, it must be verified that the IC constraints hold at every point in the signalling region.

Proceeding to the second part of the solution, set the chosen actions equal to the equilibrium actions. The HJBs become the pair of linear second-order ODEs

$$rw_H(l) = R(l) - A_H + \frac{1}{2}w_H'(l)\sigma^{-2} + \frac{1}{2}w_H''(l)\sigma^{-2}$$
$$rw_L(l) = R(l) - \frac{1}{2}w_L'(l)\sigma^{-2} + \frac{1}{2}w_L''(l)\sigma^{-2}.$$

This is where using the log likelihood ratio instead of the belief is helpful—in the case of belief, the ODEs would not have constant coefficients. After solving the ODEs for $w_L, w_H$, the IC conditions $w_H'(l) \geq A_H\sigma^2$ and $w_L'(l) \leq A_L\sigma^2$, as well as the smoothness conditions for the verification theorem must be checked at every point in the signalling region.

The solutions to the ODEs are the sum of the general solution of the homogeneous equation and a particular solution of the inhomogeneous equation, $w_\theta = C_{\theta 1}y_{\theta 1} + C_{\theta 2}y_{\theta 2} + y_{\theta p}$. The general solutions for $H$ and $L$ respectively are

$$C_{H1}y_{H1}+C_{H2}y_{H2}=C_{H1}\exp\left(l\frac{-1-\sqrt{1+8r\sigma^2}}{2}\right)+C_{H2}\exp\left(l\frac{-1+\sqrt{1+8r\sigma^2}}{2}\right)$$

$$C_{L1}y_{L1}+C_{L2}y_{L2}=C_{L1}\exp\left(l\frac{1-\sqrt{1+8r\sigma^2}}{2}\right)+C_{L2}\exp\left(l\frac{1+\sqrt{1+8r\sigma^2}}{2}\right).$$

The particular solutions are

$$\begin{aligned}y_{Hp}=&-\frac{A_H}{r}+\frac{2\sigma^2}{\sqrt{1+8r\sigma^2}}\exp\left(l\frac{-1-\sqrt{1+8r\sigma^2}}{2}\right)\int R(l)\exp\left(l\frac{1+\sqrt{1+8r\sigma^2}}{2}\right)dl\\&-\frac{2\sigma^2}{\sqrt{1+8r\sigma^2}}\exp\left(l\frac{-1+\sqrt{1+8r\sigma^2}}{2}\right)\int R(l)\exp\left(l\frac{1-\sqrt{1+8r\sigma^2}}{2}\right)dl\\y_{Lp}=&\frac{2\sigma^2}{\sqrt{1+8r\sigma^2}}\exp\left(l\frac{1-\sqrt{1+8r\sigma^2}}{2}\right)\int R(l)\exp\left(l\frac{-1+\sqrt{1+8r\sigma^2}}{2}\right)dl\\&-\frac{2\sigma^2}{\sqrt{1+8r\sigma^2}}\exp\left(l\frac{1+\sqrt{1+8r\sigma^2}}{2}\right)\int R(l)\exp\left(l\frac{-1-\sqrt{1+8r\sigma^2}}{2}\right)dl,\end{aligned}$$

where the integrals are nonelementary even for simple functional forms of $R$, e.g. for $R(l)=\frac{\exp(l)}{1+\exp(l)}$ or equivalently $\hat{R}(\mu)=\mu$.

Imposing the boundary conditions $w_\theta(\underline{l})=\frac{R(\underline{l})}{r}$ and $w_\theta(\bar{l})=\frac{R(\bar{l})}{r}$, the constants in the general solution for $H$ are

$$C_{H1}=\frac{y_{H2}(\bar{l})[\frac{R(\underline{l})}{r}-y_{Hp}(\underline{l})]-y_{H2}(\underline{l})[\frac{R(\bar{l})}{r}-y_{Hp}(\bar{l})]}{y_{H1}(\underline{l})y_{H2}(\bar{l})-y_{H2}(\underline{l})y_{H1}(\bar{l})}$$

$$C_{H2}=\frac{-y_{H1}(\bar{l})[\frac{R(\underline{l})}{r}-y_{Hp}(\underline{l})]+y_{H1}(\underline{l})[\frac{R(\bar{l})}{r}-y_{Hp}(\bar{l})]}{y_{H1}(\underline{l})y_{H2}(\bar{l})-y_{H2}(\underline{l})y_{H1}(\bar{l})}.$$

The constants for $L$ are determined by a similar expression, replacing the $H$ subscripts with $L$.

Now that all components of the solutions of the HJB equations have been found, it can be verified that they coincide with the value functions.

**Lemma 9.** *The solutions $w_H, w_L$ of the HJB equations equal the value functions $W_H, W_L$ in the signalling region. The Markov controls for the HJB equations maximize the value functions.*

*Proof.* For any signalling region $(\underline{l},\bar{l})$, the solutions of the ODEs are differentiable at least as many times as $R$ on $(\underline{l},\bar{l})$ and continuous on $[\underline{l},\bar{l}]$. Since $R$ was assumed twice continuously differentiable, $w_L$ and $w_H$ are as well. Given the signalling region, $w_H, w_L$ are bounded for any path of $l$ and control $e_\theta$. Therefore $w_H(l)$, $w_L(l)$ are integrable in the probability law of the $l$ process that starts from $l_0$ and is controlled by $e_\theta$, uniformly over Markov controls $e_\theta$. So by Theorem 11.2.2 of Øksendal (2010), $w_L, w_H$ coincide with the value functions $W_L, W_H$.

Under the previous conditions, Theorem 11.2.3 of Øksendal (2010) shows that the optimal Markov control does as well as the optimal nonanticipating control, so if the receivers expect Markov strategies, then both types of the sender have a Markov best response among their best responses.[1] □

[1]This does not imply that the payoffs of all non-Markov equilibria can be attained with Markov equilibria, since in a non-Markov equilibrium the receivers expect non-Markov strategies.

Some comparative statics for payoffs and behaviour can be derived based on the form of $w_\theta$ alone. These are presented in the next proposition.

**Proposition 10.** *In an equilibrium with signalling, starting from $l_0 \in (\underline{l}, \bar{l})$, the dependence of the payoff of type $\theta$ on $r$ and $\sigma^2$ is of the form $W_\theta = \sigma^2 f_\theta(r\sigma^2)$. The set of signalling regions possible in interval equilibria depends on $r$ and $\sigma^2$ only through their product $r\sigma^2$.*

*Proof.* The general solutions $y_{\theta 1}, y_{\theta 2}$ of the ODEs resulting from the HJB equations depend on $r$ and $\sigma^2$ only through the product $r\sigma^2$. The particular solutions depend on $r\sigma^2$, except for a $\sigma^2$ term multiplying the whole expression (the $\frac{A_H}{r}$ term in $y_{Hp}$ can be turned into $\frac{A_H\sigma^2}{r\sigma^2}$). The constants $C_{\theta 1}, C_{\theta 2}$ similarly depend on $r\sigma^2$, except for a $\sigma^2$ term multiplying them, if $\frac{R(l)}{r}$ is rewritten as $\frac{R(l)\sigma^2}{r\sigma^2}$. So the full solutions $w_\theta$, $\theta = H, L$ of the HJB equations depend on $r\sigma^2$, except for a $\sigma^2$ term multiplying them.

Since $W_\theta = \sigma^2 f(r\sigma^2)$, the $\sigma^2$ term is on both sides of the IC conditions $w_H'(l) \geq A_H\sigma^2$ and $w_L'(l) \leq A_L\sigma^2$. Cancelling the $\sigma^2$ terms, the ICs only depend on $r\sigma^2$. □

Proposition 10 echoes the result of Faingold and Sannikov (2011) that the equilibrium is affected by $r$ and $\sigma^2$ only through $r\sigma^2$. Dilme (2012) finds that the value functions are independent of the volatility of the noise, which is a similar result, since $r = 0$ in Dilme's paper.

It is not possible to analytically express the parameters or signalling region for which the IC constraints hold. Only for $A_L$ is there a simple comparative statics result: if $\mathrm{IC}_L$ holds for some $\hat{A}_L$, it holds for all $A_L \geq \hat{A}_L$. The same cannot be said for $A_H$, which enters on both sides of $\mathrm{IC}_H$, unlike in the one-shot binary action model. The comparative statics for other parameters will be based on numerical simulation. Before turning to numerics, simple necessary conditions for the IC constraints are presented in Proposition 11. These provide outer bounds for the region of log likelihood ratios in which signalling can occur. Outside that region, the only interval equilibrium is pooling.

**Proposition 11.** *For $(\underline{l}, \bar{l})$ to constitute a signalling region, $\underline{l}, \bar{l}$ must satisfy*

$$A_H \leq \frac{R(\bar{l}) - R(\underline{l})}{\sigma^2 r(\bar{l} - \underline{l})} \leq A_L.$$

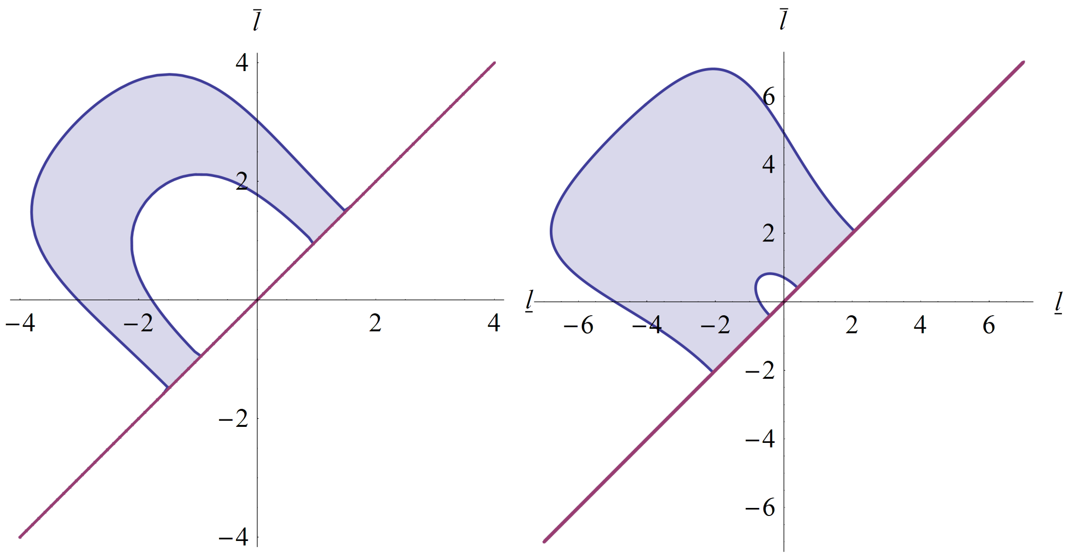
*Signalling regions are bounded away from plus and minus infinity in log likelihood ratio space (away from zero and one in belief space). If $R(l) = \frac{\exp(l)}{1+\exp(l)}$ (benefit to the sender equals the receivers' belief) and $A_L\sigma^2 r < \frac{1}{4}$, then small signalling regions around $l = 0$ are ruled out.*

*Proof.* The necessary conditions are obtained by integrating the ICs over the signalling region:

$$\int_{\underline{l}}^{\bar{l}} w_H'(l)dl \geq A_H\sigma^2(\bar{l} - \underline{l}), \qquad \int_{\underline{l}}^{\bar{l}} w_L'(l)dl \leq A_L\sigma^2(\bar{l} - \underline{l}).$$

Using $\int_{\underline{l}}^{\bar{l}} w_\theta'(l)dl = w_\theta(\bar{l}) - w_\theta(\underline{l})$ and the boundary conditions, the necessary conditions become $A_H\sigma^2(\bar{l} - \underline{l}) \leq \frac{R(\bar{l})-R(\underline{l})}{r} \leq A_L\sigma^2(\bar{l} - \underline{l})$. These conditions bound the average slope of $R(l)$ over the signalling region from below and above. The slope of $R(l)$ must go to zero as $l$ approaches plus or minus infinity, because $R$ is bounded. So for any $A_H\sigma^2 r$ the bounds $\underline{l}, \bar{l}$ of possible signalling regions are bounded above and below by the $H$ type necessary condition.

If $R(l) = \frac{\exp(l)}{1+\exp(l)}$, then the maximum slope of $R(l)$ is $\frac{1}{4}$ and occurs at $l = 0$. In that case for $A_L\sigma^2 r < \frac{1}{4}$, the $L$ type necessary condition rules out small nonempty signalling regions near $l = 0$. □




(A) $A_H = 0.15$, $A_L = 0.2$ and $r = \sigma^2 = 1$ (B) $A_H = 0.1$, $A_L = 0.24$ and $r = \sigma^2 = 1$

FIGURE 1. Necessary conditions for a signalling region (satisfied in the shaded area).

The necessary conditions in Proposition 11 are of a similar form to the IC conditions of the one-shot binary action model given in (3). They are illustrated in Figure 1. For $R(l) = \frac{\exp(l)}{1+\exp(l)}$, $A_H = 0.15$, $A_L = 0.2$ and $r = \sigma^2 = 1$, the left panel depicts in white the values of $\underline{l}$ and $\bar{l} \geq \underline{l}$ for which $(\underline{l}, \bar{l})$ cannot be the signalling region of any interval equilibrium. As $A_L$ rises, the white area inside the horseshoe shape will shrink and when $A_L \sigma^2 r \geq \frac{1}{4}$, it will disappear. As $A_H$ falls, the outer border of the horseshoe will move further from the origin, so wider signalling regions can be sustained in equilibrium. These effects are illustrated on the right panel of Figure 1, where $A_H = 0.1$, $A_L = 0.24$ and $r = \sigma^2 = 1$. Note the different scale of the axes compared to the left panel.

Numerical results on how the set of possible signalling regions depends on the parameters are presented next. Until the end of this section, it is assumed that $R(l) = \frac{\exp(l)}{1+\exp(l)}$, so the sender's benefit from the receivers' belief equals the belief.

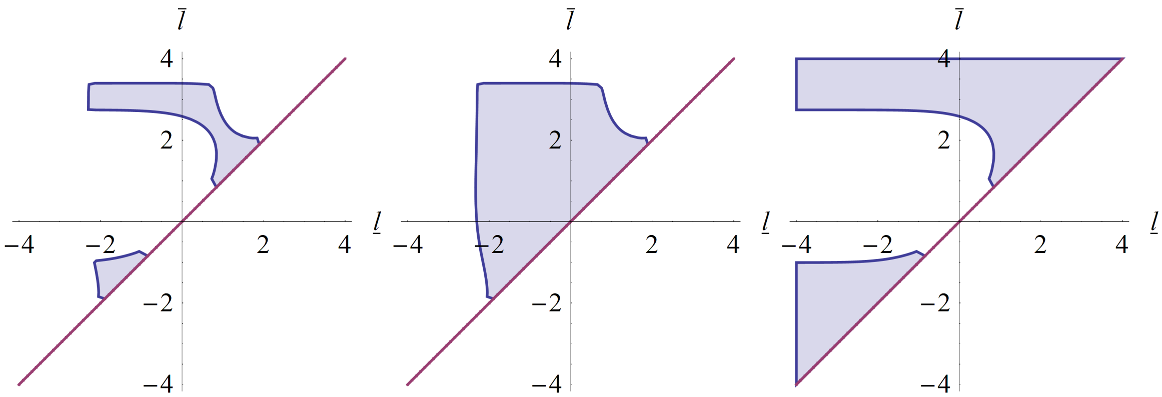
For $A_H = 0.1$, $A_L = 0.24$ and $r = \sigma^2 = 1$, the region where the ICs hold is depicted in the left panel of Figure 2 as the shaded area. For $(\underline{l}, \bar{l})$ to be the signalling region of a separating equilibrium, it is necessary and sufficient that its pair of endpoints belong to the shaded area. The middle panel shows the area where $\text{IC}_H$ holds and the right panel the area where $\text{IC}_L$ holds.

Based on Figure 2, there is in general no signalling region containing all other possible signalling regions. A separating equilibrium can feature two or more disjoint signalling intervals.

If $A_H = 0.15$, $A_L = 0.2$ and $r = \sigma^2 = 1$, then the region where both ICs hold is limited to the diagonal, so the only interval equilibrium is pooling. The region where $\text{IC}_H$ holds for these parameters is depicted in the left panel of Figure 3. The region where $\text{IC}_L$ holds is in the right panel. The intersection of the two regions is the diagonal (empty signalling intervals).

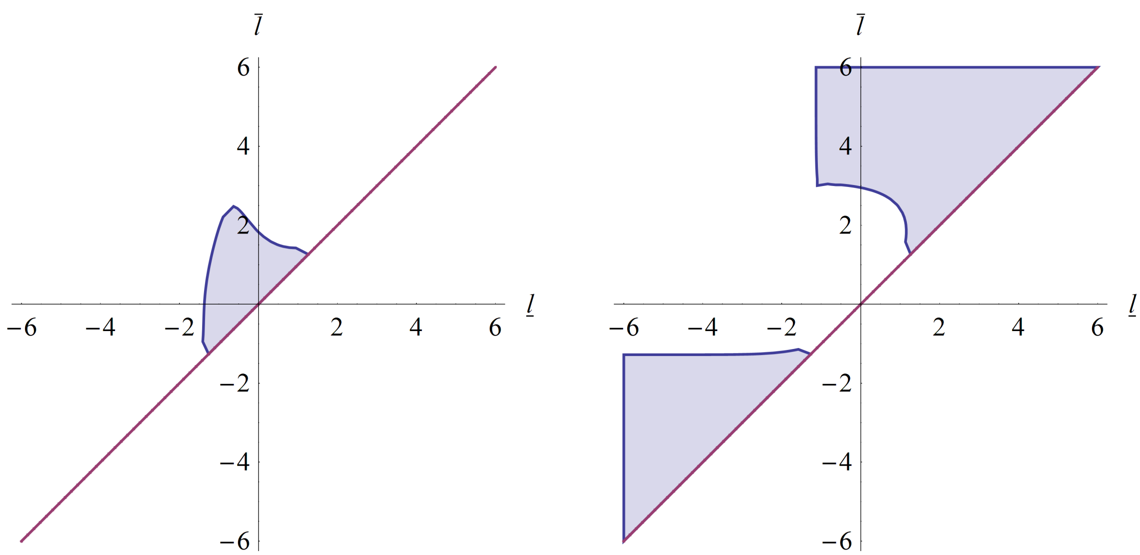
The effect of increased patience or reduced noise on the ICs is shown in Figure 4, where $A_H = 0.1$, $A_L = 0.24$, $r = 1$ and $\sigma^2 = 0.5$. Note the different scale of the axes compared to Figure 2. Since $r$ and $\sigma^2$ affect the ICs only through their product, reducing $\sigma^2$ by half has the same effect as reducing $r$ by half.

Figure 4 clearly shows that the same equilibrium may feature signalling in two disjoint intervals, e.g. for $l \in (-2.5, -2) \cup (2.5, 3)$ in this case. A similar situation is also possible with the parameters




(A) Both ICs hold (B) $\mathrm{IC}_H$ holds (C) $\mathrm{IC}_L$ holds

FIGURE 2. Region where ICs hold (shaded) for $A_H = 0.1$, $A_L = 0.24$ and $r = \sigma^2 = 1$.

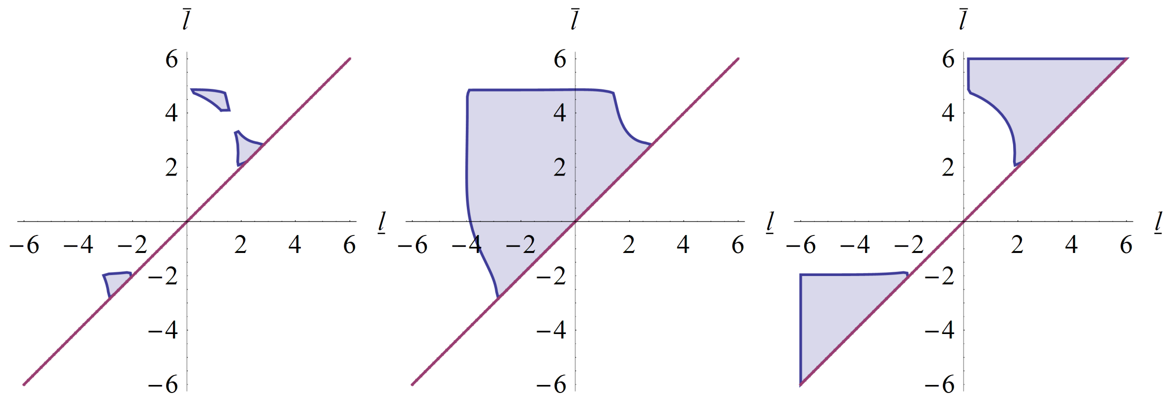



(A) $\mathrm{IC}_H$ holds (B) $\mathrm{IC}_L$ holds

FIGURE 3. Region where ICs hold (shaded) for $A_H = 0.15$, $A_L = 0.2$ and $r = \sigma^2 = 1$.

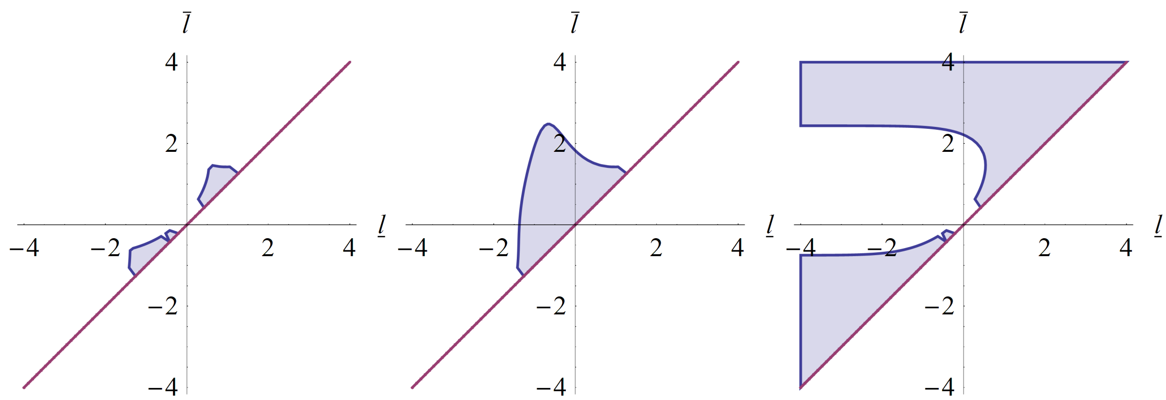
in Figure 2. In fact, whenever a signalling region of positive measure touches the diagonal over an interval of positive length, there exists an equilibrium that has a countably infinite number of disjoint signalling intervals. An example of such an equilibrium could be depicted in the figures as a sequence of points, with the $i$-th at vertical distance $\epsilon 2^{-i}$ from the diagonal and at horizontal distance $\epsilon 2^{-i+1}$ from the $(i-1)$-th.

There need not exist a signalling region containing all others, as in the previous figures. Figure 5 shows that for $A_H = 0.15$, $A_L = 0.28$ and $r = \sigma^2 = 1$, a higher $\underline{l}$ permits a higher $\bar{l}$ for a signalling region. This is due to both ICs depending on both boundaries of the signalling region in such a way that increasing or decreasing either boundary can loosen or tighten either IC, except for increasing $\underline{l}$, which does not tighten $\mathrm{IC}_L$. Figure 6 shows the seven ways in which changing $\underline{l}, \bar{l}$ can lead to violations of the ICs. In the figure, $\mathrm{IC}_H$ is written as $w_H\sigma^{-2} - A_H \geq 0$ and $\mathrm{IC}_L$ as $w_H\sigma^{-2} - A_L \leq 0$. The left column depicts the situation where the IC holds, the right column where it fails after a boundary of the signalling region is changed. In each case only one IC is




(A) Both ICs hold (B) $IC_H$ holds (C) $IC_L$ holds

FIGURE 4. Region where ICs hold (shaded) for $A_H = 0.1$, $A_L = 0.24$, $r = 1$ and $\sigma^2 = 0.5$.




(A) Both ICs hold (B) $IC_H$ holds (C) $IC_L$ holds

FIGURE 5. Region where ICs hold (shaded) for $A_H = 0.15$, $A_L = 0.28$, $r = 1$ and $\sigma^2 = 1$.

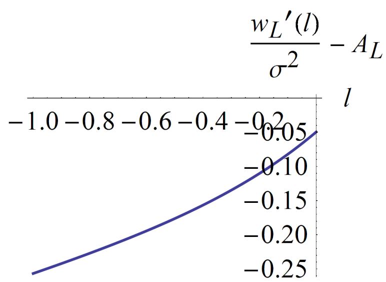
drawn, since the other continues to hold. The figure assumes $R(l) = \frac{\exp(l)}{1+\exp(l)}$. Other forms of $R$ may result in different behaviour.

If $IC_H$ is violated at the boundary that is being changed, the reason is that the slope of $R$ at the changed boundary is smaller and no longer incentivizes $H$ to signal. Similarly if $IC_L$ fails at the changing boundary, it is because the slope of $R$ becomes larger after the change and entices $L$ to signal. If $\bar{l} < 0$, then increasing it increases $R'(\bar{l})$, while if $\bar{l} > 0$, then $R''(\bar{l}) < 0$. Similarly for $\underline{l}$ and $R'(\underline{l})$.

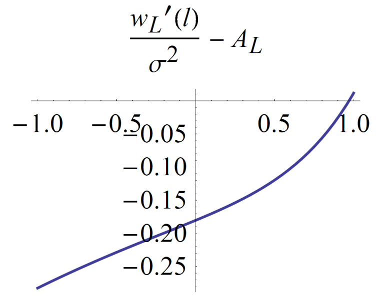
One curious feature of the model is that changing one boundary of the signalling region may lead to a violation of an IC at the other boundary (third row in Figure 6) or in the interior (second row).

As with the necessary conditions for the ICs, $IC_H$ rules out signalling regions that have one boundary too far from $l = 0$ and $IC_L$ rules out too narrow signalling regions close to zero. In addition, $IC_L$ excludes signalling regions with only the upper boundary $\bar{l}$ near zero.

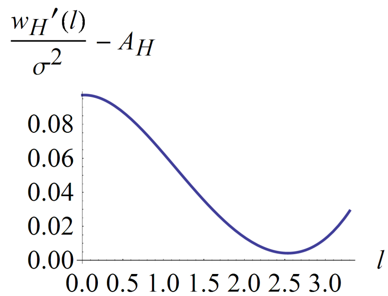
Figure 7 shows that the payoff of the $L$ type in the signalling region of a signalling equilibrium is always strictly below the payoff in a pooling equilibrium at the same log likelihood ratio, as argued in Proposition 8. The topmost curve in Figure 7 is $\frac{\exp(l)}{r(1+\exp(l))}$ and the three bottom curves starting and ending on $\frac{\exp(l)}{r(1+\exp(l))}$ are the solutions of the $L$ type HJB equation if $r = \sigma^2 = 1$, $\bar{l} = 3$ and $\underline{l}$



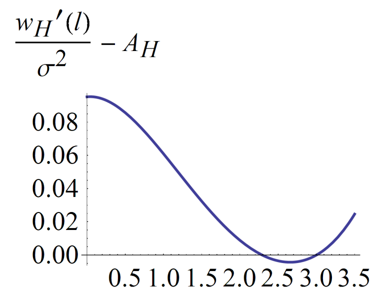

(A) $A_L = 0.4$, $r = \sigma^2 = 1$, $\underline{l} = -1$, $\bar{l} = 0$. $\text{IC}_L$ holds.



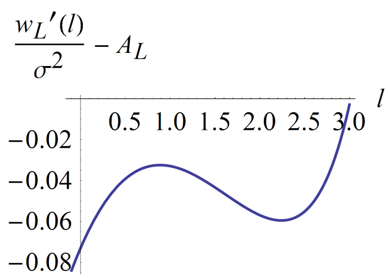

(B) $A_L = 0.4$, $r = \sigma^2 = 1$, $\underline{l} = -1$, $\bar{l} = 1$. $\text{IC}_L$ fails.



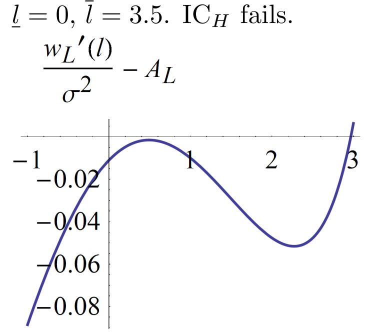

(C) $A_H = 0.1$, $r = \sigma^2 = 1$, $\underline{l} = 0$, $\bar{l} = 3.3$. $\text{IC}_H$ holds.



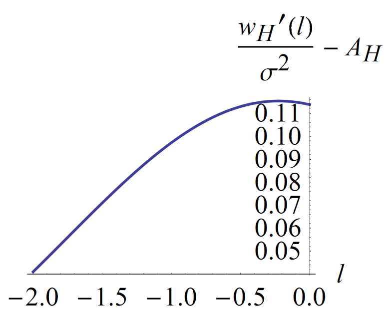

(D) $A_H = 0.1$, $r = \sigma^2 = 1$, $\underline{l} = 0$, $\bar{l} = 3.5$. $\text{IC}_H$ fails.



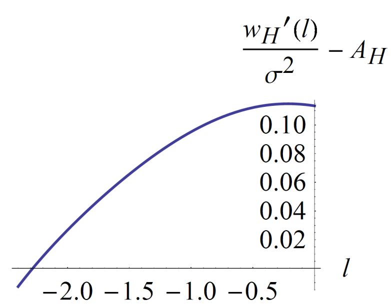

(E) $A_H = 0.2$, $r = \sigma^2 = 1$, $\underline{l} = -0.1$, $\bar{l} = 3$. $\text{IC}_L$ holds.




(F) $A_H = 0.2$, $r = \sigma^2 = 1$, $\underline{l} = -1$, $\bar{l} = 3$. $\text{IC}_L$ fails.



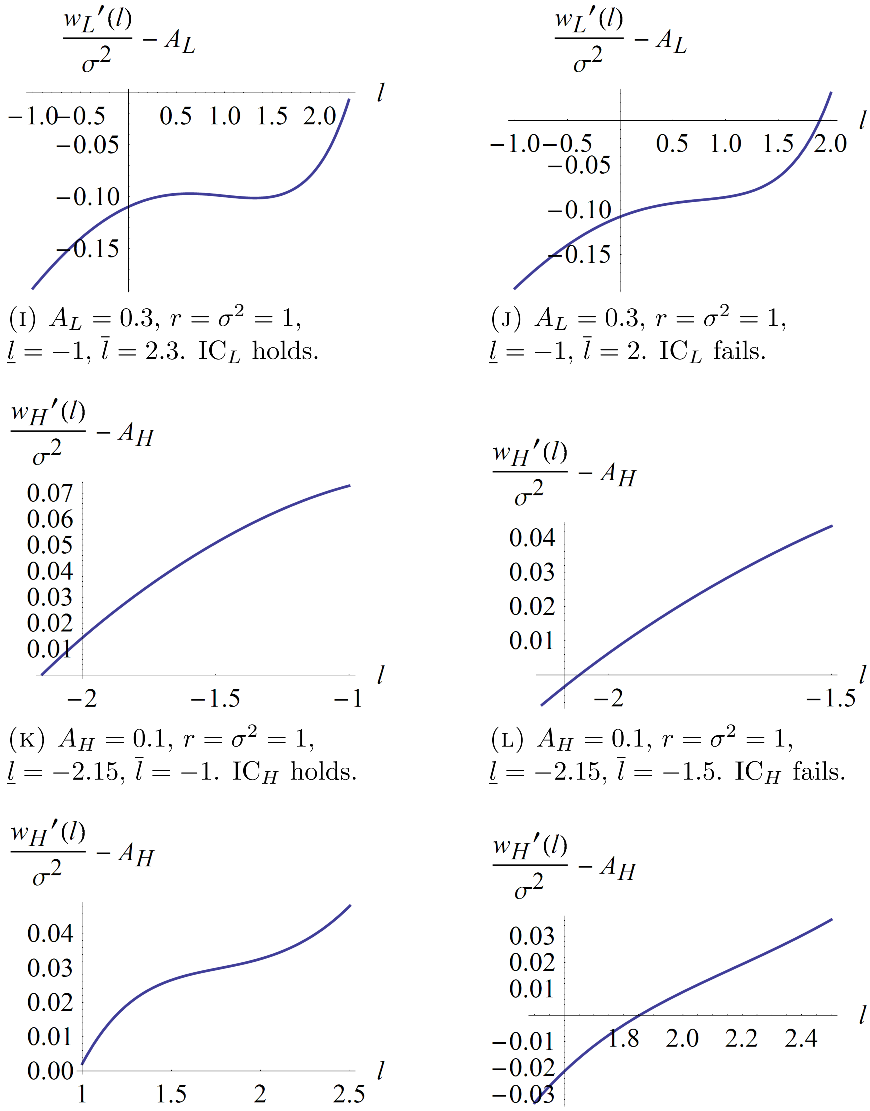

(G) $A_H = 0.1$, $r = \sigma^2 = 1$, $\underline{l} = -2$, $\bar{l} = 0$. $\text{IC}_H$ holds.




(H) $A_H = 0.1$, $r = \sigma^2 = 1$, $\underline{l} = -2.4$, $\bar{l} = 0$. $\text{IC}_H$ fails.

FIGURE 6. Ways in which changing $\underline{l}$ or $\bar{l}$ may violate $\text{IC}_L$ or $\text{IC}_H$.

takes values $-3$, $-1$ and 1. The wider the signalling region, the lower the payoff of $L$. Intuitively, in the signalling region $L$ expects the log likelihood ratio of the receivers to fall, while in a pooling equilibrium the ratio remains constant forever. The payoff is increasing in the log likelihood ratio of the receivers, so in a signalling equilibrium $L$ expects the payoff to fall. The wider the signalling




(I) $A_L = 0.3$, $r = \sigma^2 = 1$, $\underline{l} = -1$, $\bar{l} = 2.3$. IC$_L$ holds.

(J) $A_L = 0.3$, $r = \sigma^2 = 1$, $\underline{l} = -1$, $\bar{l} = 2$. IC$_L$ fails.

(K) $A_H = 0.1$, $r = \sigma^2 = 1$, $\underline{l} = -2.15$, $\bar{l} = -1$. IC$_H$ holds.

(L) $A_H = 0.1$, $r = \sigma^2 = 1$, $\underline{l} = -2.15$, $\bar{l} = -1.5$. IC$_H$ fails.

(M) $A_H = 0.1$, $r = \sigma^2 = 1$, $\underline{l} = 1$, $\bar{l} = 2.5$. IC$_H$ holds.

(N) $A_H = 0.1$, $r = \sigma^2 = 1$, $\underline{l} = 1.5$, $\bar{l} = 2.5$. IC$_H$ fails.

FIGURE 6. Ways in which changing $\underline{l}$ or $\bar{l}$ may violate IC$_L$ or IC$_H$.

region, the greater the fall in the receivers' log likelihood ratio that $L$ expects, so the lower the payoff.

In a given signalling equilibrium the $H$ type payoff can be higher or lower than the pooling payoff $\frac{\exp(l)}{r(1+\exp(l))}$ for different log likelihood ratios. This is shown in Figure 8, where $w_H$ is strictly higher than $\frac{\exp(l)}{r(1+\exp(l))}$ for $l \in (-1.5, -0.2)$ and strictly lower for $l \in (-0.2, 3)$. This comparison of signalling and pooling payoffs accords well with Spence (1973), where for higher fractions of good types in the population, the payoff to the good type from signalling is lower relative to pooling. In Spence's model, the reason is that for a higher prior there is less scope for the belief to rise (the posterior is 1 after the signal). In the present model this mechanism does not work, because the rise in belief after a good signal is highest for intermediate priors.

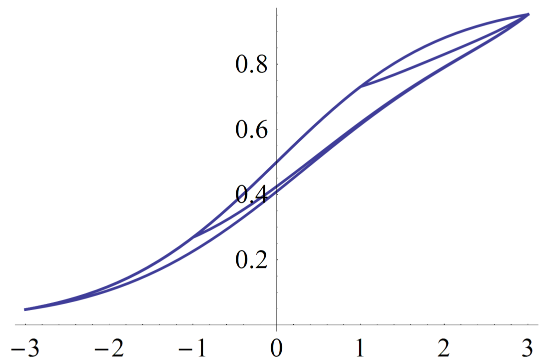



FIGURE 7. From bottom to top: $w_L$ for signalling regions $(-3,3)$, $(-1,3)$, $(1,3)$ and $\emptyset$. The parameters are $r=\sigma^2=1$.

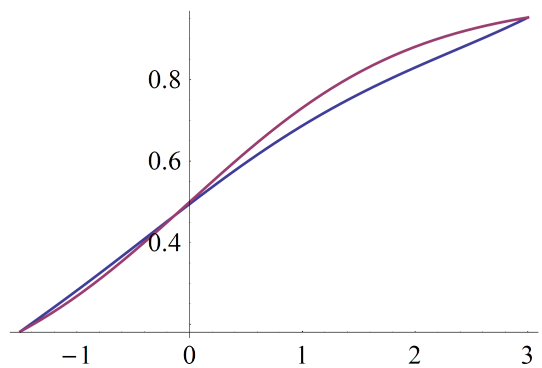



FIGURE 8. $w_H$ for signalling region $(-1.5,3)$ (the curve that is lower on the right), and $\frac{\exp(l)}{r(1+\exp(l))}$. The parameters are $A_H=0.1$ and $r=\sigma^2=1$.

As $r$ or $\sigma^2$ increases, the payoff of $H$ from a signalling equilibrium falls relative to pooling. Intuitively, patience favours signalling and noise favours pooling. Across values of $r$ and $\sigma^2$, the payoff difference between signalling and pooling can be positive or negative.

For the $L$ type, as $r$ increases, the payoff from a signalling equilibrium increases relative to pooling and the level of the payoff falls. Since in the signalling region $L$ expects the receivers' log likelihood ratio (and $L$'s own future payoff) to fall, the more the future payoff matters, the worse off the occurrence of signalling makes $L$. As $\sigma^2$ increases, $L$'s payoff from a signalling equilibrium increases—noise is good for $L$, since the receivers learn about the types more slowly.

The comparative statics are similar in the continuous time and one-shot models with binary action. In both cases, the $L$ type's payoff is higher in a pooling equilibrium than in an equilibrium with signalling. An increase in $A_H$ tightens $\text{IC}_H$, while an increase in $A_L$ loosens $\text{IC}_L$. A more precise signal makes $\text{IC}_H$ easier and $\text{IC}_L$ harder to satisfy, increases the payoff of $H$ and lowers that of $L$ from a signalling equilibrium relative to pooling. $\text{IC}_H$ is harder to satisfy at beliefs further from $\frac{1}{2}$ (log likelihood ratio further from zero), while $\text{IC}_L$ is tighter nearer to belief $\frac{1}{2}$.

The comparative static that differs from the one-shot model is that decreasing $A_H$ does not unambiguously increase the set of signalling regions in the continuous time model.

In the one-shot model with a continuum of actions, the efforts of both types increase in the precision of the signal. In the continuous time binary action model, this corresponds loosely to the expansion of the widest possible signalling region as the noise decreases. A better comparison to the one-shot model with a continuum of actions might be hoped from the continuous time model with a continuum of actions and quadratic cost. As Section 4 shows, this hope is not realized—the quadratic cost case presents rather different behaviour in continuous time than in a one-shot model. One reason might be that the noise structure in the one-shot model is quite different from the Gaussian noise resulting from a Brownian motion in the continuous time model.

## 4. Quadratic cost of signal in continuous time

With quadratic cost of signalling effort, a continuum of different effort profiles of the types of sender constitute equilibria due to unusual mathematical behaviour of the model. The divergence from the usual solution path occurs when the equilibrium condition is imposed, so from the point of view of the types of the sender, the optimization problem is standard. Conceptually, the continuum of equilibria are a continuum of self-fulfilling expectations about the signalling efforts of the types. The setup of the model resembles the binary action case in the previous section.

Time is continuous and the horizon is infinite. There is an infinitely lived strategic sender who is one of two types, $H$ or $L$, with probability $\mu_0$ of $H$. Both types of sender have discount rate $r$ and a bounded action set $[0, \overline{e}]$ at each $t$. The action (signalling effort) of type $\theta$ sender at time $t$ is denoted $e_{\theta t}$.

The sender's effort $e_{\theta t}$ determines the drift a signal process $(X_t)_{t\in\mathbb{R}_+}$, which is subject to Brownian noise $B_t$. The signal process satisfies $dX_t = e_{\theta t}dt + \sigma dB_t$. The receivers at time $t$ observe $(X_\tau)_{\tau\in[0,t]}$ and, given the equilibrium actions $(e^*_{H\tau}, e^*_{L\tau})_{\tau\in[0,t]}$ of the types of the sender, form belief $\mu_t$.

The utility of a sender of type $\theta$ at instant $t$ is $R(\mu_t) - \frac{A_\theta}{2}e^2_{\theta t}$, so the cost of effort is quadratic. Both types of sender have the same benefit $R(\mu)$ from belief $\mu$, but the $H$ type has a lower cost of effort. The benefit function $R$ is assumed bounded, with $R'$ and $R''$ bounded and continuous on $[0, 1]$.

Overall, the game consists of two stochastic control problems, one for each type, related by an equilibrium condition. The solution procedure can be divided in two parts. First, given the equilibrium strategies the receivers expect from the two types of sender, a standard stochastic control problem is solved for each type to find the best response. Second, the chosen strategy is set equal to the expected strategy for each type and the strategies are solved for. The unusual mathematical features arise in the second part, where many strategies satisfy the equilibrium condition that if the receivers expect a certain strategy, then it is optimal for the sender to choose that strategy.

I focus on pure-strategy Markov stationary equilibria (the sender's action depends only on the belief of the receivers), where outside an interval of beliefs $(\underline{\mu}, \overline{\mu})$ both types choose $e_\theta = 0$ and inside that interval at least one type chooses $e_\theta > 0$. Such equilibria are called interval equilibria. The interval $(\underline{\mu}, \overline{\mu})$ is called the signalling region and its complement the pooling region. By the same reasoning as in the binary action case, an interval equilibrium always exists.

**Lemma 12.** *An interval equilibrium always exists.*

*Proof.* Same as for Lemma 5. □

In the pooling region the belief does not change and both types optimally choose $e_\theta = 0$, so if the belief reaches some $\mu$ the pooling region, both types get payoff $R(\mu)$ forever. In an interval equilibrium the game essentially ends upon reaching any $\mu$ in the pooling region, with a final payoff $\frac{R(\mu)}{r}$ to both types.

In the signalling region, the receivers update their belief using the standard continuous time Bayes' rule

$$d\mu = \sigma^{-2}\mu(1-\mu)(e^*_H - e^*_L)[dX_t - \mu e^*_H dt - (1-\mu)e^*_L dt]. \tag{5}$$

This is also applicable in the pooling region: if $e_H^* = e_L^*$, then $d\mu = 0$ and belief does not change. In the signalling region it must be that $e_H^* > e_L^*$, otherwise belief would fall or remain constant in the costly signal, which would make both types deviate to $e_\theta = 0$. As a sufficient condition for the belief process to be well-defined and unique, assume the receivers expect strategies $e_H^*(\mu)$, $e_L^*(\mu)$ that are Lipschitz in $\mu$. It will turn out that the sender has a best response that is Lipschitz, so this assumption can be satisfied.

**Lemma 13.** *If the receivers expect strategies $e_H^*(\mu)$, $e_L^*(\mu)$ that are Lipschitz in $\mu$, then there is a unique belief process satisfying Eq. (5).*

*Proof.* If for any control $e_\theta$, the drift and variance in Eq. (5) are bounded and Lipschitz in belief, then by Theorem 3.1 of Touzi (2013), Eq. (5) has a unique strong solution. Since $e_H^*(\mu)$, $e_L^*(\mu)$ are Lipschitz in $\mu$ by assumption and the action space $[0, \overline{e}]$ is bounded, the drift of the belief process $\sigma^{-2}\mu(1-\mu)(e_H^*(\mu) - e_L^*(\mu))(e_\theta - e_H^*(\mu)\mu - (1-\mu)e_L^*(\mu))$ and the variance $\sigma^{-2}\mu^2(1-\mu)^2(e_H^*(\mu) - e_L^*(\mu))^2$ are both Lipschitz in $\mu$ and bounded. □

Both types of the sender maximize their expected discounted payoff by choosing Markov stationary control processes $e_H, e_L$. The solutions to these control problems can be written as value functions. Define $\hat{T}_{t,\mu_t}$ as the first exit time after $t$ of the $\mu$ process from $(\underline{\mu}, \overline{\mu})$, i.e. $\hat{T}_{t,\mu_t} = \inf\left\{\tau > t : \mu_\tau \notin (\underline{\mu}, \overline{\mu})\right\} \leq \infty$. In the signalling region, the value function of type $\theta$ is

$$V_\theta(\mu_t) = \sup_{e_\theta(\cdot)} \mathbb{E} \int_t^{\hat{T}_{t,\mu_t}} \exp(-rs) \left[R(\mu_s) - \frac{A_\theta}{2} e_\theta^2(\mu_s)\right] ds + \exp(-r\hat{T}_{t,\mu_t}) \frac{R(\mu_{\hat{T}_{t,\mu_t}})}{r} \mathbf{1}\left\{\hat{T}_{t,\mu_t} < \infty\right\},$$

where $\mathbf{1}\{A\}$ denotes the indicator function for the set $A$. The interpretation of the value function expression is straightforward: the agent gets flow benefit $R(\mu)$ depending on $\mu$ and chooses the optimal signalling effort $e_\theta$ at each $\mu$, which determines the flow cost. When the belief exits the signalling region (if ever), the sender gets $R(\mu_{\hat{T}_{t,\mu_t}})$ forever, where $\mu_{\hat{T}_{t,\mu_t}}$ is the value of $\mu$ upon exit. The same observations as in the binary action case can be made about the value functions.

**Lemma 14.** *$V_\theta$ is finite for $\theta = H, L$. $V_H \geq V_L$, with strict inequality in the signalling region. $V_\theta$ is strictly increasing.*

*Proof.* Same as for Lemma 7. □

With a concave $R$, the $L$ type strictly prefers pooling to signalling, in the sense that for any belief of the receivers, the payoff of $L$ is higher in a pooling equilibrium than with signalling.

**Proposition 15.** *If $R$ is weakly concave, then $V_L(\mu) < \frac{R(\mu)}{r}$ for all $\mu$ in the signalling region.*

*Proof.* Same as for Proposition 8. □

To solve the control problems of the types of the sender, the HJB equations are solved and a verification theorem is used to check that the solutions of the HJB equations coincide with the value functions. To use Theorem 11.2.2 of Øksendal (2010) to prove that the solutions $v_H, v_L$ of the HJB equations equal the value functions $V_H, V_L$, it is sufficient that $v_H, v_L$ are twice continuously differentiable on $(\underline{\mu}, \overline{\mu})$, continuous on $[\underline{\mu}, \overline{\mu}]$ and integrable in the probability law of $\mu$ given the starting state $\mu_0$, uniformly over Markov controls $e_H, e_L$. As will be seen, these conditions are satisfied by the solutions of the HJB equations.

Under these conditions, Theorem 11.2.3 of Øksendal (2010) shows that the optimal Markov control does as well as the optimal nonanticipating control, so if the receivers expect Markov strategies, then both types of the sender have a Markov best response. This does not imply that the payoffs of all non-Markov equilibria can be attained with Markov equilibria, since in a non-Markov equilibrium the receivers expect non-Markov strategies.

The HJB equation of type $\theta = H, L$ is

$$rv_\theta(\mu) = \max_{e_\theta}\{R(\mu) - \frac{A_\theta}{2}e_\theta^2 + v_\theta'(\mu)\sigma^{-2}\mu(1-\mu)(e_H^* - e_L^*)\left[e_\theta - \mu e_H^* - (1-\mu)e_L^*\right] + v_\theta''(\mu)\frac{\mu^2(1-\mu)^2(e_H^*-e_L^*)^2}{2\sigma^2}\}.$$

The FOC of the type $\theta$ HJB equation is $-A_\theta e_\theta + v_\theta'\sigma^{-2}\mu(1-\mu)(e_H^* - e_L^*) = 0$, so given the expected equilibrium actions, both types have a unique optimal action $e_\theta = \frac{v_\theta'\mu(1-\mu)(e_H^*-e_L^*)}{A_\theta\sigma^2}$. The SOC is $-A_\theta < 0$ for all $e_\theta$, so the FOCs are necessary and sufficient for a strict global maximum. This is not surprising, because the cost is quadratic and the benefit is linear in the control variable $e_\theta$.

Thus far, the control problems of the two types of the sender were solved for a given pair of equilibrium strategies expected by the receivers. For the second part of the solution of the signalling game, the equilibrium condition $e_\theta = e_\theta^*$ is imposed. The $L$ type FOC then becomes

$$e_L^*(\mu) = \frac{v_L'(\mu)\mu(1-\mu)}{A_L\sigma^2 + v_L'(\mu)\mu(1-\mu)}e_H^*(\mu). \tag{6}$$

Substituting for $e_L^*$ in the $H$ type FOC gives $A_H e_H^* = v_H'\sigma^{-2}\mu(1-\mu)\left[1 - \frac{v_L'\sigma^{-2}\mu(1-\mu)}{A_L + v_L'\sigma^{-2}\mu(1-\mu)}\right]e_H^*$. Therefore for any $\mu$, either $e_H^*(\mu) = 0 = e_L^*(\mu)$ or $A_H A_L + A_H v_L'(\mu)\sigma^{-2}\mu(1-\mu) = A_L v_H'(\mu)\sigma^{-2}\mu(1-\mu)$. The latter is equivalent to

$$v_H'(\mu) = \frac{A_H\sigma^2}{\mu(1-\mu)} + \frac{A_H}{A_L}v_L'(\mu). \tag{7}$$

The only corner solution is $e_H^* = e_L^* = 0$. Other corner solutions would involve $e_H^*(l) > 0$ and $e_L^*(l) = 0$ for some $l$ in the signalling region. The $H$ type control problem has an interior solution if $e_H^*(l) > 0$ and the $H$ type FOC is then satisfied. This implies an equation similar to (6), except derived from the $H$ type FOC, requiring both $e_H^*$ and $e_L^*$ to be positive or both zero.

The relationship between the efforts and payoffs of the two types given by Eqs. (6) and (7) is similar to the one-shot model with quadratic cost, where $e_L^* = \frac{A_H}{A_L}e_H^*$ and the expected utilities of the types are linearly related. This is not surprising, as in both cases the conditions are derived from the FOCs of quadratic problems with a similar structure.

The following Lemma gives conditions on solutions of the HJB equations that are sufficient for these solutions to form an interval equilibrium with a nonempty signalling region (a separating equilibrium). The following Proposition 17 provides restrictions on parameters that are sufficient for the existence of a particular kind of separating equilibrium.

**Lemma 16.** *$e_L^*$, $e_H^*$, $v_L$ and $v_H$ constitute an interval equilibrium with signalling region $(\underline{\mu}, \overline{\mu})$, where $\underline{\mu} < \overline{\mu}$, if all of the following hold*

*(1) $e_L^*$, $e_H^*$, $v_L$ satisfy Eq. (6) for all $\mu \in (\underline{\mu}, \overline{\mu})$,*
*(2) $v_L$ and $v_H$ satisfy Eq. (7) for all $\mu \in (\underline{\mu}, \overline{\mu})$,*
*(3) $v_L, v_H$ are twice continuously differentiable on $(\underline{\mu}, \overline{\mu})$,*
*(4) $v_L, v_H$ are continuous on $[\underline{\mu}, \overline{\mu}]$,*
*(5) $v_L, v_H$ are integrable in the probability law of $\mu$ given the starting state $\mu_0$, uniformly over Markov controls $e_H, e_L$,*
*(6) $e_L^*, e_H^*$ are Lipschitz in $\mu$,*
*(7) $0 < e_L^*, e_H^* \leq \overline{e}$,*
*(8) $v_L \leq v_H$.*

*Proof.* If $\underline{\mu} < \overline{\mu}$, then Eqs. (6) and (7) together are necessary and sufficient for $e_L^*$, $e_H^*$, $v_L$ and $v_H$ to solve the HJB equations and satisfy the equilibrium condition.

If $v_L, v_H$ are twice continuously differentiable on $(\underline{\mu}, \overline{\mu})$, continuous on $[\underline{\mu}, \overline{\mu}]$ and integrable in the probability law of $\mu$ given the starting state $\mu_0$, uniformly over Markov controls $e_H, e_L$, then by Theorem 11.2.2 of Øksendal (2010), $v_L$ and $v_H$ coincide with the value functions $V_L$ and $V_H$. In that case, $e_L^*$, $e_H^*$ are the optimal Markov controls for $V_L$ and $V_H$ and by Theorem 11.2.3 of Øksendal (2010), $e_L^*$ and $e_H^*$ maximize $V_L$ and $V_H$ in the class of all nonanticipating controls.

The Lipschitz condition on $e_L^*, e_H^*$ is sufficient for the belief process to be well-defined (Lemma 13).

The restrictions $0 < e_L^*, e_H^* \leq \overline{e}$ and $V_L \leq V_H$ come from first principles and Lemma 14. □

**Proposition 17.** *$V_H(\mu) = \frac{R(\mu)}{r}$, $V_L(\mu) = \begin{cases} \frac{A_L R(\mu)}{A_H r} - A_L\sigma^2 \left[\ln(\mu) - \ln(1-\mu)\right] & \text{if } \mu \in (\underline{\mu}, \overline{\mu}) \\ \frac{R(\mu)}{r} & \text{if } \mu \notin (\underline{\mu}, \overline{\mu}) \end{cases}$, $e_H^* = \mathbf{1}\left\{(\underline{\mu}, \overline{\mu})\right\}$ and $e_L^*(\mu) = \begin{cases} 1 - \frac{A_H r\sigma^2}{R'(\mu)\mu(1-\mu)} & \text{if } \mu \in (\underline{\mu}, \overline{\mu}) \\ 0 & \text{if } \mu \notin (\underline{\mu}, \overline{\mu}) \end{cases}$ form an interval equilibrium with signalling region $(\underline{\mu}, \overline{\mu})$ if the following hold*

*(1) $\overline{e} > 1$,*

*(2) $\underline{\mu} < \overline{\mu}$ satisfy*

$$R(\overline{\mu}) = \frac{A_L A_H r\sigma^2}{A_L - A_H}\left[\ln(\overline{\mu}) - \ln(1-\overline{\mu})\right], \qquad R(\underline{\mu}) = \frac{A_L A_H r\sigma^2}{A_L - A_H}\left[\ln(\underline{\mu}) - \ln(1-\underline{\mu})\right], \tag{8}$$

*(3) $\frac{R'(\mu)}{r} > \frac{A_H\sigma^2}{\mu(1-\mu)}$ for all $\mu \in (\underline{\mu}, \overline{\mu})$,*

*(4) $\frac{R(\mu)}{r} \leq \frac{A_L A_H\sigma^2}{A_L - A_H}\left[\ln(\mu) - \ln(1-\mu)\right]$ for all $\mu \in (\underline{\mu}, \overline{\mu})$.*

*Proof.* For $V_H$, $V_L$, $e_H^*$ and $e_L^*$ to form an interval equilibrium with signalling region $(\underline{\mu}, \overline{\mu})$, it is sufficient that they satisfy the assumptions of Lemma 16. Here by definition, $e_L^*$, $e_H^*$, $V_L$ satisfy Eq. (6) and $V_L$ and $V_H$ satisfy Eq. (7) for all $\mu \in (\underline{\mu}, \overline{\mu})$.

Since $R$ is assumed bounded and twice continuously differentiable, $V_H$, $V_L$ are twice continuously differentiable on $(\underline{\mu}, \overline{\mu})$, bounded and integrable in the probability law of $\mu$ given the starting state $\mu_0$, uniformly over Markov controls $e_H, e_L$.

The payoff in the pooling region provides the boundary conditions $V_\theta(\underline{\mu}) = \frac{R(\underline{\mu})}{r}$ and $V_\theta(\overline{\mu}) = \frac{R(\overline{\mu})}{r}$. Since $V_\theta$ is twice continuously differentiable on $(\underline{\mu}, \overline{\mu})$, for it to be continuous on $[\underline{\mu}, \overline{\mu}]$ it is sufficient that $\lim_{\mu\to\underline{\mu}^+} V_\theta(\mu) = \frac{R(\underline{\mu})}{r}$ and $\lim_{\mu\to\overline{\mu}^-} V_\theta(\mu) = \frac{R(\overline{\mu})}{r}$. These conditions clearly hold for $V_H = \frac{R(\mu)}{r}$. For $V_L$, they hold iff Eqs. (8) hold. This may be seen by rearranging $\lim_{\mu\to\underline{\mu}^+} V_L(\mu) = \frac{A_L R(\underline{\mu})}{A_H r} - A_L\sigma^2\left[\ln(\underline{\mu}) - \ln(1-\underline{\mu})\right] = \frac{R(\underline{\mu})}{r} = V_L(\underline{\mu})$.

If $\frac{R'(\mu)}{r} > \frac{A_H\sigma^2}{\mu(1-\mu)}$, then $e_L^*(\mu) > 0$. In the signalling region, $e_H^* > 0$ holds by definition. By the assumption $\overline{e} > 1$, we have $e_H^*, e_L^* < \overline{e}$.

$V_H(\mu) \geq V_L(\mu)$ may be written as $\frac{R(\mu)}{r} \geq \frac{A_L R(\mu)}{A_H r} - A_L\sigma^2\left[\ln(\mu) - \ln(1-\mu)\right]$, or equivalently $\frac{R(\mu)}{r} \leq \frac{A_L A_H\sigma^2}{A_L - A_H}\left[\ln(\mu) - \ln(1-\mu)\right]$. □

Slightly perturbing $v_H$ and $e_H^*$ (while ensuring $e_H^* \leq \overline{e}$) in Proposition 17 and again deriving $v_L$ and $e_L^*$ from (7) and (6) results in a different interval equilibrium with the same signalling interval. Proposition 17 essentially says that in a certain parameter region, there is a degree of freedom in specifying $e_L^*$, $e_H^*$ and also in specifying $v_L$, $v_H$. In the signalling region, fixing one of $e_L^*$, $e_H^*$ and one of the other three functions determines the remaining two via Eqs. (6) and (7). The degree of freedom is specific to the model with a quadratic cost of effort and does not have a natural interpretation.

Despite the quadratic cost of effort and the bounded benefit from belief, it is possible to have arbitrarily large signalling efforts in equilibrium. The key is that the difference in efforts also becomes large and the drift and volatility of the belief process are proportional to this difference. With large efforts, beliefs quickly move out of the region where the equilibrium prescribes the large efforts. As the cost of effort gets large, it is only paid for a very short time in expectation. This may be the case for many other cost functions, but only with quadratic cost is the increase in the effort cost exactly offset by the decrease in the duration of the effort.

## 5. Conclusion

This paper presents two models of noisy signalling in continuous time. One has a binary action for both types of the sender, the other has a continuum of actions with quadratic cost. In both cases, equilibrium behaviour (the set of equilibria where signalling occurs in an interval of beliefs) is characterized, as well as the equilibrium payoffs.

Overall, the continuous time signalling model with binary action yields intuitive comparative statics results. More patience, less noise and a lower signalling cost encourage signalling, which is good for incentivizing the good type to signal, but bad because it may encourage the bad type to signal as well. The range of beliefs for which signalling can occur (size of the largest signalling region) expands as the signalling cost of the good type falls.

The payoff of the bad type from a signalling equilibrium is always below that from pooling and decreases as the range of beliefs for which signalling occurs expands. For the good type there is no clear payoff comparison between signalling and pooling equilibria.

The dependence of the payoffs of the types on parameters is intuitive. Patience increases the payoff of the good type from a signalling equilibrium relative to pooling, and noise decreases it. For the bad type the situation is reversed—patience lowers the relative payoff from a signalling equilibrium and noise raises it. All the above results are in line with the one-shot noisy signalling model where the sender has a binary action. The difference from the one-shot model is in the effect of the signalling cost of the good type—decreasing it always expands the set of beliefs where signalling can occur in the one-shot model, but has an ambiguous effect in the continuous time model.

The continuous time noisy signalling model with a continuum of actions and quadratic cost exhibits unusual mathematical behaviour. It has many equilibria even for a fixed region of beliefs where signalling occurs. Generally there is a continuum of possible signalling regions.

The set of separating equilibria expands as the difference of the signalling costs of the types increases, patience increases or noise decreases. In the binary action model, the counterpart of the effect of the difference of the signalling costs is the expansion of the set of separating equilibria as the cost of the good type decreases or the cost of the bad type increases.

The one-shot noisy signalling model with a continuum of actions has fewer equilibria than the corresponding noiseless model. For the one-shot model with binary action, the comparison is not clear. Both noiseless and noisy repeated signalling models have a large set of equilibria, but for different reasons. In the noiseless models, off-path beliefs may be used to provide incentives for a wide range of behaviours and the signalling cost can be distributed in many ways over time. In the noisy models, there are many possible self-fulfilling beliefs about the set of receivers' beliefs at which the sender chooses to signal. With quadratic cost, another source of equilibrium multiplicity is a feature of the mathematics of the problem, without an obvious conceptual foundation.

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