Cost Sharing with Production Constraints
Extended Abstract

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Abstract

Two common assumptions in many works on cost sharing are 1. the lack of reciprocal production constraints; namely, it is assumed that if a bundle is to be produced then any "smaller" bundle can be produced, and 2. the cost function is differentiable. This is obviously not the case in many cost problems of interest. Haimanko ([6]-[7]) addressed the second matter, but not the first, and his methods may not be extended to treat the first matter. We consider two classes of cost problem whose sets of producible bundles are centrally symmetric convex bodies, and whose cost functions have (generically) major non-differentiabilities. The cost functions in the first class are convex exhibiting non-decreasing marginal costs to scale, and those in the second class are piece-wise linear. We show existence and uniqueness of a cost allocation mechanism, satisfying standard axioms, on these classes.

1 Introduction

Allocating the joint cost of producing a bundle of infinitely divisible consumption goods is a common practical problem with no obvious solution. Billera, Heath, and Raanan [2] were the first to apply Aumann and Shapley [1] theory of nonatomic games to set equitable telephone billing rates that share services cost among users. Billera and Heath [3] and Mirman and Tauman [10] offered an axiomatic justification of Aumann-Shapley (A-S) prices using economic terms only. Tauman [14] proved that A-S prices naturally extend the average cost prices from a single product to an arbitrary finite number of products with nonseparable production cost functions.

The A-S prices are characterized (see [3] and [10]) by five simple axioms- cost sharing, additivity, rescaling invariance, monotonicity, and consistency, the latter tying the pricing of multiple commodities with that of a single one. Other characterizations of A-S prices were also proposed by Young [15], Hart and Mas-Colell [8], and Monderer and Neyman [11].

The A-S price mechanism, however, can only be applied when a cost function is differentiable, while typical cost functions possess major nondifferentiabilities. For instance, if a cost of producing a bundle of outputs is given by the least expensive configuration of inputs (minimal factor cost), then it generically non-differentiable. Haimanko [6]-[7] characterized a price mechanism on classes of cost problems possessing major non-differentiabilities, by the axioms of cost sharing, additivity, rescaling invariance, monotonicity, and consistency. Haimanko called this mechanism the Mertens mechanism due to its affinity with the Mertens [9] value.

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However, all the aforementioned works share a crucial assumption– if a bundle of commodities is to be produced, then any “smaller” bundle is assumed feasible. Namely, it is assumed that the set of production outputs is a cube. However, in many economic applications this assumption might not hold, and the set of production outputs may not be assumed to be a cube.

In this paper we exhibit a price mechanism on the classes of cost problems whose cost functions are convex with non-decreasing marginal costs to scale or piecewise linear, and whose set of production outputs are convex, closed under lattice operations, and obey an additional technical assumption. This price mechanism satisfies the five axioms– cost sharing, additivity, rescaling invariance, monotone price, and consistency. As in Haimanko [6], our mechanism has an affinity with the Mertens [9] value and we refer to it as the *Mertens mechanism.* Our main result is Theorem 2.3, which states that this mechanism is uniquely characterized by the five axioms on these classes of cost problems.

Theorem 2.3 follows mainly from recent developments in the representation theory of positive projection (see [4]) and the theory of values of nonatomic market games (see [5]).

2 Definitions and The Results

2.1 Cost Problems

We denote by $\mathbb{R}_+$ the nonnegative real numbers and by $\mathbb{R}_{++}$ the strictly positive real numbers. A compact and convex set $K \subseteq \mathbb{R}^k_+$ is a *production set* iff $0_k \in K$, it is centrally symmetric, and it is closed under lattice operations. Denote by $\mathcal{K}^k_p$ the set of all production sets $K \subseteq \mathbb{R}^k_+$ and let $\mathcal{K}_p = \bigcup_{k=1}^{\infty} \mathcal{K}^k_p$. A *cost problem* is a pair (f, K) with $K \in \mathcal{K}^k_p$ for some $k \in \mathbb{N}$, and $f : K \to \mathbb{R}$ with $f(0_k) = 0$. As $K \subseteq \mathbb{R}^k_+$ is a compact lattice then there is a unique $a(K) \in K$ s.t. $a(K) \geq x$ for every $x \in K$. We interpret $a(K)$ as the quantities of commodities $1, ..., k$ that are produced, K is the set of bundles whose production is feasible, and $f(x)$ is the cost of producing the bundle $x \in K$. For any class F of cost problems, denote by F^k its subset of problems (f, K) with $K \in \mathcal{K}^k_p$.

A price mechanism on a class of cost problems F is a function $\phi : F \to \bigcup_{k=1}^{\infty} \mathbb{R}^k$ s.t. the range of $\phi\big|_{F^k}$ is contained in $\mathbb{R}^k$. If $(f, K) \in F^k$ and $1 \leq j \leq k$ we denote by $\phi_j((f, K))$ the jth coordinate of $\phi((f, K))$. If $x, y \in \mathbb{R}^k_+$ denote $x * y = (x_1 y_1, ..., x_k y_k)$, and $(y * f)(x) = f(y * x)$ for a function f on $\mathbb{R}^k_+$. If $x \in \mathbb{R}^k_{++}$ denote $x^{-1} = (x_1^{-1}, ..., x_k^{-1})$. For $m \geq k \geq 1$ and a partition $(S_1, ..., S_k)$ of $\{1, ..., m\}$ let $\pi^* : \mathbb{R}^m \to \mathbb{R}^k$ be given by $\pi^*_i(x) = \sum_{j \in S_i} x_j$.

Definition 2.1. A price mechanism ϕ on F is:

1. *cost sharing* iff for every $(f, K) \in F$

$$f(a(K)) = a(K) \cdot \phi((f, K)); \tag{2.1}$$

2. *additive* iff for every $(f_1, K), (f_2, K) \in F$ s.t. $(f_1 + f_2, K) \in F$ we have

$$\phi((f_1 + f_2, K)) = \phi((f_1, K)) + \phi((f_2, K)); \tag{2.2}$$

3. *consistent* iff for every $1 \leq k \leq m$, every partition $\pi = \{S_1, ..., S_k\}$ of $\{1, ..., m\}$ s.t. $(f \circ \pi^*, K), (f, \pi^*(K)) \in F$,

every $1 \le i \le k$, and every $j \in S_i$ we have

$$\phi_j((f,K)) = \phi_i((f',K')); \tag{2.3}$$

4. *rescaling invariant* iff for every $(f,K) \in F^k$ and $\alpha \in \mathbb{R}^k_{++}$ with $(\alpha * f, \alpha^{-1} * K) \in F^k$

$$\phi((\alpha * f, \alpha^{-1} * K)) = \alpha * \phi((f,K)); \tag{2.4}$$

5. *monotone cost* iff for every $(f,K),(g,K) \in F^k$ s.t. $f-g$ is monotonically non-decreasing on K we have for every $x \in K$

$$\phi((f,K)) \cdot x \ge \phi((g,K)) \cdot x. \tag{2.5}$$

2.2 The Classes F_c and F_l

Given a cost problem (f,K) with $K \subseteq \mathbb{R}^k_+$, a point x in the relative interior of K, and $y \in AF(K)$, the affine space generated by K, the *directional derivative* of f at x in the direction y is

$$df(x,y) = \lim_{\varepsilon \searrow 0} \frac{f(x+\varepsilon y) - f(x)}{\varepsilon}. \tag{2.6}$$

The limit exists for every convex function f, and hence for every linear combination of convex functions. For $k \ge 1$ let F^k_c be the set of cost problems (f,K) with f being a non-decreasing Lipschitz continuous convex function with $f(0_k) = 0$, exhibiting *non-decreasing marginal cost to scale*, namely, for x in the relative interior of K, $y \in AF(K)$, and $t \ge 1$ s.t. tx is in the relative interior of K it holds that

$$df(x,y) \le df(tx,y). \tag{2.7}$$

Let F^k_l be the set of cost problems (f,K) with f being non-decreasing, continuous, piecewise linear function with $f(0_k) = 0$. For $* \in \{c,l\}$ let $F_* = \bigcup\limits_{k=1}^{\infty} F^k_*$.

2.3 The Mertens Mechanism

Given a cost problem (f,K) with convex f, and x in the relative interior of K, the function $df(x,\cdot)$ is convex and finite on $AF(K)$ by [13, Theorem 23.1]. Hence its directional derivative exist at any point $y \in AF(K)$ in any direction $z \in AF(K)$. We shall denote it by

$$df(x,y,z) = \lim_{\varepsilon \searrow 0} \frac{df(x,y+\varepsilon z) - df(x,y)}{\varepsilon}. \tag{2.8}$$

For any cost problem (f,K) with convex f consider the following cost mechanism. For $y \in AF(K)$ let $\|y\|_K = \max\{\langle x,y\rangle : x \in 2K - a(K)\}$. Then $\|\cdot\|_K$ is a norm on $AF(K)$ and there is a probability distribution P_K on $AF(K)$ whose Fourier transform is $\mathcal{F}[P_K](y) = \exp(-\|y\|_K)$ ([12, Lemma 1]). For any $1 \le j \le k$ define

$$(\phi_M)_j((f,K)) = \int_{AF(K)} \left(\int_0^1 df(ta(K), y, e_j)dt \right) dP_K(y), \tag{2.9}$$

with e_j the jth unit vector. We call ϕ_M the *Mertens mechanism*, due to its affinity with the Mertens [9] value.

Lemma 2.2. *The Mertens mechanism satisfies the axioms (1)-(5)*

Our main result in this paper is:

Theorem 2.3. *If ϕ is a cost sharing, additive, rescaling invariant, monotone price, and consistent price mechanism on F_c (F_l) then $\phi = \phi_M$.*

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