Sequential Assignment Problems and Two Applications

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Abstract

We study matching markets from practice, where a set of objects are assigned to a set of agents in two-steps. The placement of students to the exam and mainstream public schools in the U.S. and the appointment of teachers to the public schools in Turkey until recently are two examples of such markets. We analyze the mechanisms currently in use in both markets and show that they fail to satisfy desirable fairness and welfare criteria. Moreover, they give participants perverse incentives: misreporting preferences can be beneficial and improved performance on admission test may worsen the assignment of a participant. We characterize the subgame perfect Nash equilibria of the induced preference revelation games in both markets which motivate us to propose an alternative method, applicable to both markets, through which assignments take place in a single step. This may also help explain why Turkish ministry of education abandoned their two-step assignment system.

1 Introduction

In this paper we study indivisible good allocation problems, where a set of agents are to be assigned a set of (indivisible) objects in a sequential fashion in two-steps, and each agent is entitled to receive at most one object. More specifically, there is a first step of assignments in which all agents actively participate by reporting their rank-ordered preferences when only a subset of objects is available. The first step is then followed by a second step of assignments in which the set of objects that were unavailable in the first step are assigned to those agents who were unassigned in the first step. We study two applications of this problem from practice: student placement to exam and regular public schools in the U.S. and the appointment of teachers to state schools in Turkey.

In Boston and New York City, there are two types of public schools: exam and regular (mainstream) schools.[1] Students who wish to apply to exam schools take a centralized test and are then ranked based on their scores. Meanwhile, regular schools rank students based on certain predetermined criteria, i.e. proximity and sibling status. The admissions for both type of schools are processed separately. In general, the admission decisions to the test schools are completed well before any students are assigned to the regular schools. In particular, students are assigned

[1] There are different types of regular schools.

to exam schools via a serial dictatorship mechanism induced by the test scores in a first step, and the unassigned students are then assigned to regular schools via a student-proposing deferred acceptance mechanism (Gale and Shapley, 1962) in a second step.

In Boston there are three exam schools.[2] Around 25% of the seventh grade students enroll to the exam schools.[3] Sixth grade students take the centralized exam before December and apply to be accepted one of these schools in the following year. Students are ranked based on their exam score and their GPA in the previous year. The assignment of the students are done according to this ranking and the ordered list submitted by the students. Successful students receive their acceptance letters from the exam schools by mid-March. Sixth graders can also apply to be transferred to another regular school after mid-March. The assignment for the regular schools are done via DA mechanism.

In New York City there are nine exam schools.[4] The assignments to the exam and regular schools are done sequentially. However, students submit their preferences over exam and regular school at the same time. Every year between 25,000 and 30,000 student take the Specialized high school admission test (SHSAT) which is used for the assignment to the exam schools. Students who take this test submit two different list to the central authority. In the first one they only rank exam schools and in the second one they rank regular schools which do not require a test score. The assignment for the specialized high school is done by a serial dictatorship in which students are ordered based on their score on SHSAT. The assignment for the regular schools is done by DA mechanism. The two mechanisms are run at the same time. The central authority aims to place the students who are given an offer at a specialized high school firstly. Only students who have an offer from both exam and regular schools are informed of their placement from regular schools together with their exam school assignment. They are asked to pick one of these two offers. Around 5,000 students are enrolled to the exam school every year. Then students who are not assigned in this step is considered and DA mechanism used for the assignment to the regular schools.

There are 4000-5000 students each year in this situation. Only 50% of these students have an offer from their most preferred regular school. Around 70% of these students take the exam schools.

In Turkey, the assignment of teachers to teaching positions in state schools takes place via a centralized process overseen by the Turkish Ministry of Education (TMoE). Every year the TMoE offers a standardized test to those university graduates who wish to serve in state sponsored jobs. Although this test is mostly taken by new university graduates, many who have graduated in the past are also eligible to take the test if they wish so.[5] In a given year, the appointment of

[2] These schools are Boston Latin Academy, Boston Latin School and the John D. O'Bryant School of Mathematics and Science.

[3] In 2012-2013 school year 836 of 3,795 seventh grade students have enrolled to exam schools.

[4] These schools are Bronx High School of Science, Brooklyn Latin School, Brooklyn Technical High School, High School for Math, Science and Engineering at City College, High School of American Studies at Lehman College, Queens High School for the Sciences at York College, Staten Island Technical High School and Stuyvesant High School.

[5] Some of them can be currently employed as a teacher.

teachers to state schools are based solely on the candidates' performance on that year's test. There are two types of teaching positions in each specialization. These are the *tenured* positions which offer a life time employment guarantee and *contractual* positions which can be taken only for a fixed number of years (typically for only a few years) and the conditions of employments are based on a specific contract mutually agreed upon by the school and the teacher. Although an otherwise identical tenured position is generally preferable to a contractual position, it is also common to observe strong preference for contractual positions in major metropolitan cities such as Istanbul over tenured positions in smaller cities or rural areas.

In a given year the TMoE first announces the list of all available tenured positions in each school and each specialization throughout Turkey. Then each applicant, be it a new graduate or an existing contractual teacher, submits rank-ordered preference lists over the available tenured positions before a certain deadline announced by the ministry.[6] In this first step, existing contractual teachers who are seeking a new position are also restricted to rank-list only tenured positions.[7] Applicants are then assigned to the available positions by a serial dictatorship mechanism induced by the test scores. If a contractual teacher is unassigned in the first step, then she retains her current job assignment. Otherwise, she is appointed to a tenured position and a contractual position at her old school opens up for reappointment. Typically within a few weeks after the first step, the TMoE announces the list of available contractual positions. And in this second step, only the unassigned new graduates are permitted to apply to these contractual positions. Applicants are again assigned via a serial dictatorship mechanism induced by the test scores. The number of teachers assigned to tenured and contractual positions in 2009 and 2010 is presented in Table 1. For instance in December 2009, 8,850 tenured positions were filled by applicants in the first step. 6,323 applicants who were assigned to tenured positions were existing teachers working in contractual positions. These contractual positions which became available as a consequence of assignments of existing teachers to the tenured positions were filled in the same month.

Time of the Assignment	Type of the Positions	Number of Positions Filled
February 2009	Tenured	8,285
March 2009	Contractual	6,323
December 2009	Tenured	8,850
December 2009	Contractual	6,323
June 2010	Tenured	10,000
July 2010	Contractual	9,000
December 2010	Tenured	30,000
December 2010	Contractual	6,843

Table 1. Number of Teachers Assigned to Tenured and Contractual Positions (2009-2010)

[6] Existing teachers employed in tenured positions are not allowed to participate this assignment procedure.

[7] In other words, any contractual position currently filled by an applicant cannot be rank-listed by any applicants including its current occupant.

We show that the student assignment system in the U.S. and the teacher appointment system in Turkey share a number of serious deficiencies. Among other shortcomings, both systems fail to generate Pareto-efficient or fair assignments, and both systems induce strategic action on the part of applicants while deciding what rank-order lists to submit in each step. Our goal in this paper is to investigate general sequential assignment systems when the mechanisms used in each step satisfy certain properties that are by and large deemed desirable in the matching literature.

We argue that the deficiencies of the systems in the U.S. and Turkey are not specific to these particular markets. Our analysis indicates that there may indeed be a fundamental problem with achieving distributional and strategic goals via sequential assignment systems that employ mechanisms that satisfy even very mild requirements. More specifically, we show that if $\Psi = (\varphi^1, \varphi^2)$ is a sequential assignment system which employs mechanisms φ^1 and φ^2 in steps 1 and 2 such that φ^1 is individually rational and non-wasteful and φ^2 is non-wasteful, then such a system is prone to manipulation and cannot generate efficient outcomes; and still worse, it may even be wasteful (Theorem 1). Similarly, if $\Psi = (\varphi^1, \varphi^2)$ is a sequential assignment system which employs mechanisms φ^1 and φ^2 such that φ^1 is fair (with respect to a given priority structure) and non-wasteful and φ^2 is non-wasteful, then such a system neither generates fair outcomes nor respects improvements in the priority structure (Theorem 2).

We also characterize the subgame perfect Nash equilibria (SPNE) induced by a sequential preference revelation game of a sequential assignment system. We find that when both φ^1 and φ^2 are individually rational, non-wasteful and either population monotonic and non-bossy or fair, then every SPNE outcome of the preference revelation game associated with system Ψ leads to a non-wasteful and individually rational matching (Theorem 3). On the other hand, when both φ^1 and φ^2 are individually rational, non-wasteful, population monotonic, and minimally fair, then every SPNE outcome of the preference revelation game associated with system Ψ leads to a matching that has no priority violations (Theorem 4). As corollaries of these results, we provide a detailed account on the characteristics of the set of SPNE for each of the two applications that motivate our study.

It is natural to ask whether these sequential assignment systems suffer from the serious problem if all the available positions in the first step are preferred to the all available positions in the second step by all agents. The answer is no. In particular, under this preference domain we can consider the steps separately. However, in the real life we observe that some of the available objects in the second step are preferred to the some of the available objects in the first step. In particular, 15% of teachers assigned to a contractual position in December 2010 could have been assigned to a tenured position in the previous step if they had listed those schools in their preference list. Similarly, around 20% of the students who get both exam and regular school offer in NYC choose to go to the regular one.

Our analysis points to clear disadvantages of sequential assignment systems and provides justification for the alternative use of single step assignment systems when possible. This conclusion is also supported by the recent transition of the TMoE from the system analyzed here to a simpler single step assignment system.[8] More broadly, these observations motivate us to advocate the use of a suitable adaptation of Gale and Shapley's celebrated Deferred Acceptance (DA) mechanism to the specific context as a single step assignment system. In particular, in the context of teacher assignment the dominant strategy outcome of DA Pareto dominates any SPNE of the old assignment system.

1.1 Related Literature

The main characteristic of the type of problems we study here that distinguishes them from the vast majority of the problems considered in the literature is that they involve sequential assignment of indivisible resources. Whereas the set of agents and resources are predetermined and fixed in a standard assignment problem, in a sequential assignment problem agents and resources considered within a step may as well depend on the decisions made in a previous step. Yet, the two type of problems still share a number of common features.

There is a sense in which a *school choice problem with exam and regular schools* (SCPwERS) can be thought as a combination of two separate school choice problems (Abdulkadiroğlu and Sönmez, 2003) when the assignment in each step is considered independently. On the other hand, the *teacher assignment problem* (TAP) in Turkey, described above, inherits properties of the Student Placement Problem (SPP) due to Balinksi and Sönmez (1999) and the House Allocation Problem with Existing Tenants (HAPwET) due to Abdulkadiroğlu and Sönmez (1999). As in the context of SPP, in TAP too applicants are ranked based on their test scores, and *fairness* (i.e., favoring applicants with better test scores) is a central goal. And, as in HAPwET, some of the applicants- the contractual teachers- have private endowments-the contractual positions they currently occupy- that may later become available for reassignment to other applicants.

A paper that is closely related to ours is Ergin and Sönmez (2006), where the authors characterize the set of NE of the widely-used Boston mechanism and show that this set coincides with the set of stable matchings. We find that while this conclusion need not hold generally for any sequential assignment problem, in the context of TAP (but not in that of SCPwERS) the set of SPNE of the sequential preference revelation game is also equal to the stable set.[9]

The only paper, that we are aware of, to consider sequential assignment is Westkamp (2012), where the author studies the German college admissions system which operates through a combination of Boston and college-proposing deferred acceptance mechanism. Similar to Ergin and Sönmez (2006), Westkamp (2012) too characterizes the set of SPNE of this game as being the stable set. While we also provide characterizations of SPNE for both of our applications, we show that equivalence of SPNE to the stable set may not always be guaranteed. Most notably, differently than Westkamp (2012), our focus here is on properties of a general sequential assignment system and on showing that the sources of the deficiencies may be inherently related to

[8] As far as we are aware, this transition took place without the involvement of any economists in the decision process.

[9] In the context of TAP, stability is characterized by the combination of individual rationality, fairness, and nonwastefulness.

the sequential nature of the assignment system. As such, we show that these deficiencies may be impossible to avoid regardless of what specific mechanism is used in any step.

The rest of the paper is organized as follows. Section 2 introduces the formal model. Section 3 provides a detailed description of the sequential systems in the U.S. and Turkey. Section 3 presents impossibility results concerning general sequential systems. Section 4 characterizes the SPNE of general sequential systems as well as those of the two motivating applications. Section 5 presents a simple alternative to sequential systems. Section 6 concludes.

2 Model

Let $I^* = \{i_1, i_2, ..i_n\}$ be the set of all agents, $S^* = \{s_1, s_2, ..., s_m\}$ be the set of all objects, $q^* = (q_{s_1}, q_{s_2} ..., q_{s_m})$ be the capacity vector for all objects. Let $\succ^* = (\succ_{s_1}, \succ_{s_2}, ..., \succ_{s_m})$ denote a priority profile, where $\succ_s$ is the strict priority order for object s such that $\emptyset \succ_s i$ means that agent i is not acceptable for object s. We allow for an object to be socially or privately owned. Let $h^* = (h(i))_{i \in I^*}$ be an ownership profile, where $h(i)$ is the object for which agent i has the property right (i.e., her endowment) such that $h(i) = \emptyset$ means that agent i has no property right over any object. Each agent i can own at most one object, i.e. $|h(i)| \leq 1$. On the other hand, an object can be owned by more than one agent.

Each agent i has a strict (i.e., complete, transitive, and antisymmetric) preference relation P_i over $S \cup \{\emptyset\}$ where $\emptyset$ represents being unassigned. Let R_i denote the associated at least as good as relation of agent i. We thus have

$$s \; R_i \; s' \Leftrightarrow s \; P_i \; s' \text{ whenever } s \neq s'.$$

A **sequential assignment problem**, or a *problem* for short, is a 6-tuple $(I, S, P, q, \succ, h)$ where $I \subseteq I^*$, $S \subseteq S^*$, $P = (P_i)_{i \in I}$ is a preference profile, $q = (q_s)_{s \in S}$, $\succ = (\succ_s)_{s \in S}$ and $h = (h(i))_{i \in I}$.

Fix a problem $(I, S, P, q, \succ, h)$. A matching is a function $\mu : I \rightarrow S \cup \emptyset$ such that the number of agents assigned to a object s does not exceed the total number of the copies of s and each agent can be assigned to at most one object, i.e., $|\mu^{-1}(s)| \leq q_s$ and $|\mu(i)| \leq 1$ for all $s \in S$ and $i \in I$. Let $\mathcal{M}$ be the set of all matchings. A matching μ is **non-wasteful** if there exists no object agent pair (i, s) such that $|\mu^{-1}(s)| < q_s$, $s \; P_i \; \mu(i)$, and $i \succ_s \emptyset$.[10] A matching μ is **individually rational** if no agent is assigned to an object either she finds worse than being unassigned option or she is unacceptable for. Formally, a matching μ is individually rational if $\mu(i) \; R_i \; \emptyset$ and $i \succ_{\mu(i)} \emptyset$ for all $i \in I$. A matching μ **Pareto dominates** another matching μ' if each agent weakly prefers her assignment in matching μ to her assignment in μ' and at least one agent i strictly prefers her assignment in μ to her assignment in μ'. A matching μ is **Pareto efficient** if it is not Pareto dominated. A matching μ is **fair** if whenever an agent prefers some other agent's assignment to her own, then the other agent has a higher priority for that object than herself.

[10] This is different from the standard non-wastefulness notion (see Balinski and Sonmez (1999)).

Formally, if μ is fair then for every $i, j \in I$, $\mu(j)\ P_i\ \mu(i)$ implies $j \succ_{\mu(j)} i$. A matching μ is **stable** if it is non-wasteful, individually rational and fair. A matching μ is **mutually fair** if there does not exist an agent-object pair (i, s) such that (1) i ranks s at the top of his preference list (2) $\mu(i) \neq s$ (3) there exists another student i' with lower priority for s than i and $\mu(i') = s$.[11] A matching μ **serially fair** if it satisfies fairness whenever $\succ_s = \succ_{s'}$ for all $s, s' \in S$.[12]

A **mechanism** φ is a mapping that associates a matching to a given problem. Denote the outcome selected by mechanism φ for problem $(I, S, P, q, \succ, h)$ by $\varphi(I, S, P, q, \succ, h)$ and the match of agent $i \in I$ by $\varphi_i(I, S, P, q, \succ h)$.

A **sequential assignment system,** or a *system* for short, is a pair of mechanisms $\Psi = (\varphi^1, \varphi^2)$ such that

1. φ^1 operates on the restricted problem $(I,\ S^1, P^1, q|S^1, \succ |S^1, h)$ whose primitives are the set of all agents, a subset of all objects available for assignment in the first step (defined by the application), and the preferences and priorities over available objects, and[13]

2. φ^2 operates on the restricted problem $(I^2,\ S^2, P^2, q^2, \succ |S^2, h)$ whose primitives are the set of all agents (without property rights) who are unassigned in the first step, the set of all objects available for assignment in the second step (defined by the application), the preferences, and priorities over available objects. More precisely,

$$\begin{aligned} I^2 &= \{i : \varphi_i^1(I, S^1, P^1, q|S^1, \succ |S^1, h) = \emptyset \text{ and } h(i) = \emptyset\},\ S^2 = S \backslash S^1, \\ q_s^2 &= q_s - |\{i : h(i) = s \text{ and } \varphi_i^1(I, S^1, P^1, q|S^1, \succ |S^1, h) = \emptyset\}| \ \ \forall s \in S^2. \end{aligned}$$

It is important to note that since the objects assigned in the first step are no longer available for allocation in the second step, the problem in the second step (including the participating agents as well as available objects) is "endogenously" determined through the assignments made in the first step. Then, the assignment of agent i for a problem under system Ψ is formally defined as:

$$\Psi_i(I, S^1, S^2, P^1, P^2, q, \succ, h) = \begin{cases} \varphi_i^1(I, S^1, P^1, q|S^1, \succ |S^1, h) \text{ if } \varphi_i^1(I, S^1, P^1, q|S^1, \succ |S^1, h) \neq \emptyset, \\ h(i) \text{ if } \varphi_i^1(I, S^1, P^1, q|S^1, \succ |S^1, h) = \emptyset \text{ and } h(i) \neq \emptyset, \\ \varphi_i^2(I^2, S^2, P^2, q^2, \succ |S^2, h) \text{ otherwise.} \end{cases}$$

A mechanism or a system is said to be **non-wasteful {fair} [individually rational]** if it selects a non-wasteful {fair} [individually rational] matching for a given problem.

[11] Mutual fairness is also used by Morrill (2012).

[12] Note that both mutual fairness and serial fairness are very weak form of fairness. Boston mechanism, Top Trading and Cycles mechanism, Serial Dictatorship mechanism which fail to be fair satisfy mutual and serial fairness.

[13] The notations $q|S^1$ and $\succ |S^1$ respectively denote the restrictions of q and $\succ$ to the set of objects in S^1. Here, $P^1 = (P_i^1)_{i \in I}$ is the preference profile over the available objects in step 1. Similarly, $P^2 = (P_i^2)_{i \in I^2}$ is the preference profile over the available objects in step 2.

A *mechanism is strategy-proof* if it is always a dominant strategy for each agent to report his preferences truthfully. Formally, for every $i \in I$ and every P_i', and every P, we have

$$\varphi_i(I, S, P, q, \succ, h) \ R_i \ \varphi_i(I, S, P_i', P_{-i}, q, \succ, h).$$

A *system is strategy-proof* if no agent ever gains by ranking available objects non-truthfully in each step she participates. Formally, for every $i \in I$, every pair (P_i', P_i'') and every P, we have

$$\Psi_i(I, S^1, S^2, P|S^1, P|(S^2, I^2), q, \succ, h) \ R_i \ \Psi_i(I, S^1, S^{2'}, (P_i', P_{-i})|S^1, (P_i'', P_{-i})|(S^{2'}, I^{2'}), q, \succ, h).$$

We say that $\tilde{\succ}$ is an improvement in the priorities for agent $i \in I$ if (1) $i \succ_s j \Longrightarrow i \tilde{\succ}_s j$ for all $s \in S$, (2) there exists at least one agent j' and school s' such that $j \succ_s i \tilde{\succ}_s j$, and (3) $k \succ_s j' \Longleftrightarrow k \tilde{\succ}_s j'$ for all $s \in S$ and $j, k \in I \backslash \{i\}$. A mechanism φ **respects improvements in the priorities** if $\tilde{\succ}$ is an improvement in the priorities for agent $i \in I$, then $\varphi_i(I, S, P, q, \tilde{\succ}, h)$ R_i $\varphi_i(I, S, P, q, \succ, h)$ for any $i \in I$.

In the rest of the paper for a given problem we fix the set of agents, objects, quotas, priority order and the ownership profile and represent the outcome of a system for a given problem by $\Psi(P^1, P^2)$.

3 Two Applications

3.1 School Choice Problem with Exam and Regular Schools (SCPwEXRS)

A **school choice problem with exam and regular schools**, or a problem for short, consists of[14]

1. A set of schools $S = \{s_1, s_2, ..., s_m\}$. S is composed of two disjoint sets, i.e. exam and regular schools. Let S^e be the set of exam schools and S^r be the set of regular schools and $S = S^e \cup S^r$.

2. A capacity vector $q = (q_s)_{s \in S}$ where q_s is the number of available seats in $s \in S$.

3. A set of students $I = \{i_1, i_2, ..., i_n\}$.

4. A preference profile $P = (P_i)_{i \in I}$ where P_i is the strict preference of i over $S \cup \emptyset$.

5. A priority order $\succ = (\succ_s)_{s \in S}$ where $\succ_s$is the strict priority order of applicants in I for school s.

6. An ownership profile $h = (h(i))_{i \in I}$ where $h(i)$ is the object for which agent i has the property right.

[14]We are using a similar notation with Balinski and Sonmez (1999).

All the available seats in both type of schools are social endowments. Therefore, $h(i) = \emptyset$ for all $i \in I$. Let $c(i)$ be the test score of applicant $i \in I$ and c be the test score profile of all applicants, $c = (c(i))_{i\in I}$. In the school choice problem with exam and regular schools only the exam schools rank the students based on their exam score. That is, for each $s \in S^e$ $i \succ_s j$ if and only if $c(i) > c(j)$. On the other hand, the regular schools use some predetermined exogenous rules (proximity, sibling) while ranking students. All exam schools rank students in the same order. However, the priorities of students for any two different regular school do not have to be the same.

The current system used in Boston is a **serial dictatorship followed by deferred acceptance mechanism** (SD-DA) and works as follows:

Step 1:

- Only exam schools, S^e, are available for assignment in this step and all students can participate in. Students submit their preferences over the set S^e and $\emptyset$. Let $P^1 = (P^1_i)_{i\in I}$ be the list of submitted preference in step 1. Therefore, the problem considered in Step 1 is $(I, S^e, P^1, q|S^e, \succ |S^e, h)$.[15]

- Serial dictatorship mechanism is applied to the problem $(I, S^e, P^1, q|S^e, \succ |S^e, h)$: The agent with the highest score is assigned to his top choice in the list he submitted, the next agent is assigned to his top choice among the remaining schools, and so on.

- Let μ_1 denote the assignment in step 1.

Step 2:

- The problem considered in Step 2 is $(I^2, S^2, P^2, q^2, \succ |S^2, h)$. I^2, S^2 and q^2 are calculated as described in Section 2. Note that $S^2 = S^r$, $q^2 = (q_s)_{s\in S^r}$ and all the unassigned students in the first step participate in.

- Student proposing deferred acceptance mechanism is used in the placement process:

  - Each agent $i \in I^2$ applies to the top ranked school in P^2_i. Each school $s \in S^2$ tentatively accepts all best offers up to its quota q^2_s according to its priority order. The rest are rejected.

  - In general; Each agent $i \in I^2$ applies to the highest ranked school in P^2_i which has not rejected him yet. Each school, which holds tentatively accepted offers or receives new offers in this round, tentatively accepts all best acceptable offers, among the new and previously held ones, up to its quota according to its priority order. The rest are rejected.

- Let μ_2 be the final assignment in Step 2.

[15] Since only the exam schools are considered in this step priority order for all available schools will be the same and it is equivalent to the order of test scores.

The placement of agent $i \in I$ induced by the SD-DA is:

$$\mu(i) = \begin{cases} \mu_1(i) \text{ if } \mu_1(i) \neq \emptyset, \\ \mu_2(i) \text{ otherwise.} \end{cases}$$

3.2 Teacher Assignment Problem (TAP)

A **teacher assignment problem**, or a problem for short, consists of[16]

1. A set of schools $S = \{s_1, s_2, ..., s_m\}$. S is composed of two disjoint sets, i.e. contractual and tenured schools. Let S^c be the set of contractual schools and S^t be the set of tenured schools and $S = S^c \cup S^t$.

2. A capacity vector $q = (q_s)_{s \in S}$ where q_s is the number of available seats in $s \in S$.

3. A set of students $I = \{i_1, i_2, ..., i_n\}$. I is composed of two disjoint sets, i.e. existing teachers and new graduates. Let I^e be the set of existing teachers and I^n be the set of new graduates and $I = I^e \cup I^n$.

4. A preference profile $P = (P_i)_{i \in I}$ where P_i is the strict preference of i over $S \cup \emptyset$.[17]

5. A priority order $\succ = (\succ_s)_{s \in S}$ where $\succ_s$is the strict priority order of applicants in I for school s.

6. An ownership profile $h = (h(i))_{i \in I}$ where $h(i)$ is the object for which agent i has the property right.

Each $i \in I^e$ has property right over a school in S^c, $\sum_{i \in I^e} 1(\ h(i) = s) = q_s$ for all $s \in S^c$. For each new graduate $i \in I^n$ $h(i) = \emptyset$. The number of available seats in $s \in S^c$ is $q_s = |h^{-1}(s)|$. All the available seats in tenured schools are social endowments.

The strict priority order of applicants in I for each school s, $\succ_s$, is determined according to the centralized test score of each agent and the property rights. Each tenured school $s \in S^r$ ranks applicants based on their test score: $i \succ_s j$ if and only if $c(i) > c(j)$. Each existing teacher currently working in a contractual school $s \in S^c$ is given right to keep his position unless he is assigned to a better school. That is, each contractual school $s \in S^c$ ranks its current teachers at the top of its priority order. All the other applicants are ranked based on their test score. Each contractual school $s \in S^c$ considers each existing teacher working in another contractual school as unacceptable. The priority order of each contractual school $s \in S^c$ as:

- For all $i, j \in I$ such that $h(i) = s$, $h(j) \neq s$ then $i \succ_s j$

- For all i, $j \in I$ such that $h(i) = h(j)$, $i \succ_s j$ if and only if $c(i) > c(j)$

[16] We are using a similar notation with Balinski and Sonmez (1999).

[17] All existing teachers are assumed to prefer their current position to $\emptyset$.

- For each $s \in S^c$ and all $i \in I^e$ such that $h(i) \neq s$ then $\emptyset \succ_s i$
- For each $s \in S^c$ and all $i \in I^n$ and $j \in I^e$ such that $h(j) \neq s$, $i \succ_s j$.

As one can notice, in the teacher assignment problem the priority order can be constructed by using the test scores and the ownership profile. Alternatively, we can define the TAP as (I, S, P, q, c, h). To be consistent with the general framework we define problem as $(I, S, P, q, \succ, h)$.

The system that was in use in Turkey until very recently is the **two-step serial dictatorship mechanism** (TSSD) and works as follows:

Step 1:

- Only tenured schools, S^t, are available for assignment in this step and all teachers can participate in. Teachers submit their preferences over the set S^t and $\emptyset$. Let $P^1 = (P_i^1)_{i\in I}$ be the list of submitted preference in step 1. Therefore, the problem considered in Step 1 is $(I, S^t, P^1, q|S^t, \succ |S^t, h)$.[18]
- Serial dictatorship mechanism is applied to the problem $(I, S^t, P^1, q|S^t, \succ |S^t, h)$: The agent with the highest score is assigned to his top choice in the list he submitted, the next agent is assigned to his top choice among the remaining schools, and so on.
- Let μ_1 denote the assignment in step 1.

Step 2:

- The problem considered in Step 2 is $(I^2, S^2, P^2, q^2, \succ |S^2, h)$. I^2, S^2 and q^2 are calculated as described in Section 2. Note that $S^2 = S^c$ and all the unassigned new graduates participate in. [19]
- Serial dictatorship mechanism is used in the assignment process: The agent with the highest test score is assigned to his top choice in the list he submitted. The number of available seats in that school is updated and if it falls to zero that school is removed. The agent with the second highest test score is assigned to his top choice among the remaining schools, and so on.
- Let μ_2 be the final assignment in Step 2.

The placement of agent $i \in I$ induced by the TSSD is:

$$\mu(i) = \begin{cases} \mu_1(i) \text{ if } \mu_1(i) \neq \emptyset \\ h(i) \text{ if } \mu_1(i) = \emptyset \text{ and } h(i) \neq \emptyset \\ \mu_2(i) \text{ otherwise.} \end{cases}$$

[18] Since only the tenured schools are considered in this step priority order for all available schools will be the same and it is equivalent to the order of test scores.

[19] Since only the new graduates can participate in this step each school $s \in S^2$ ranks the agents in I^2 based on test scores.

4 Deficiencies of General Sequential Systems

We next show that sequential systems, regardless of the specific mechanisms used in each step, may be inherently flawed. To this end, we offer some impossibility results. In Theorem 1, we show that if a non-wasteful mechanism is used in both steps and the mechanism used in the first step is individually rational, then that system is vulnerable to manipulation and fails to be Pareto efficient, or even non-wasteful.

Theorem 1 *Let $\Psi = (\varphi^1, \varphi^2)$ be a system. If φ^1 is non-wasteful and individually rational and φ^2 is non-wasteful then Ψ fails to be strategy-proof, Pareto efficient and non-wasteful.*

Proof. We consider two different cases. In the first case, all schools are social endowment. In the second case, we allow some schools to be owned by agents.

Case 1: There are three schools $S = \{s_1, s_2, s_3\}$ with one available seat and two agent $I = \{i_1, i_2\}$. Let $S^1 = \{s_2, s_3\}$, $h(i_1) = \emptyset$ and $h(i_2) = \emptyset$. The priority structure of each school is the same and given as $i_1 \succ_s i_2 \succ_s \emptyset$ for all $s \in S$. Let the true preferences be as follows:

$$s_2 P_{i_1} s_3 P_{i_1} s_1 P_{i_1} \emptyset$$
$$s_1 P_{i_2} s_2 P_{i_2} s_3 P_{i_2} \emptyset$$

In the first step, if both agents act truthfully and submit their ranking lists by keeping the relative order of available schools and $\emptyset$ in P then there are two non-wasteful and individually rational matchings: $\mu_1^1 = \begin{pmatrix} s_2 & s_3 \\ i_1 & i_2 \end{pmatrix}$ and $\mu_1^2 = \begin{pmatrix} s_2 & s_3 \\ i_2 & i_1 \end{pmatrix}$. No matter which one of these two matchings is selected in the first step none of the agents can participate the second step and s_1 is available in the second step. Therefore, the unique non-wasteful matching selected in the second period is $\mu_2^1 = \mu_2^2 = \begin{pmatrix} s_1 \\ \emptyset \end{pmatrix}$. That is, any system satisfying conditions mentioned in the statement of Theorem 1 assigns i_2 to either s_2 or s_3. Let $\widetilde{\Psi}$ be a system selecting (μ_1^1, μ_2^1) and $\overline{\Psi}$ be a system selecting (μ_1^2, μ_2^2). That is, the outcome of $\widetilde{\Psi}$ is $\mu^1 = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & i_1 & i_2 \end{pmatrix}$ and the outcome selected by $\overline{\Psi}$ is $\mu^2 = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & i_2 & i_1 \end{pmatrix}$.

Pareto Efficiency: The outcomes of both $\widetilde{\Psi}$ and $\overline{\Psi}$ are Pareto dominated by the following matching: $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_2 & i_1 & \emptyset \end{pmatrix}$. Therefore both mechanisms fail to be Pareto efficient. It is worth to mention that μ' is a fair matching. Hence, the outcomes of $\widetilde{\Psi}$ and $\overline{\Psi}$ mechanisms are Pareto dominated by a fair matching.

Non-wastefulness: In the outcomes of both $\widetilde{\Psi}$ and $\overline{\Psi}$ the available seat of s_1 is wasted since i_2 prefers s_1 to both his assignment under $\widetilde{\Psi}$ and $\overline{\Psi}$ and i_2 is acceptable for s_1.

Strategy-proofness: Suppose i_2 submits the following preference lists in the first step: $\emptyset P'_{i_2} s_2 P'_{i_2} s_3$. There is a unique individually rational and non-wasteful matching in step 1: $\mu'_1 = \begin{pmatrix} s_2 & s_3 \\ i_1 & \emptyset \end{pmatrix}$ and both $\widetilde{\Psi}$ and $\overline{\Psi}$ selects μ'_1. Based on the matching selected in the first step i_2 can participate second step and s_1 is available in the second step. When i_2 submits $P''_{i_2} : s_1 P''_{i_2} \emptyset$ then there is a unique non-wasteful matching: $\mu'_2 = \begin{pmatrix} s_1 \\ i_2 \end{pmatrix}$. That is, (P'_{i_1}, P''_{i_1}) pair is a profitable deviation for i_2 under $\widetilde{\Psi}$ and $\overline{\Psi}$.

Case 2: Consider the same example. We only change the example by giving the property rights of school s_1 to i_1. That is, $h(i_1) = s_1$ and $h(i_2) = \emptyset$. Given the set of available schools in step 1 is not changed then in the first step in any non-wasteful matching i_1 will be assigned to another school and he will give up his property rights for s_1. Therefore, s_1 will be socially endowed in the second period as in the Case 1. One can follow the steps in Case 1 and show that the impossibility result is robust to the ownership structure, i.e. whether all schools are socially endowed or not. ■

In Theorem 1, we do not need the individual rationality of φ^1 to show that impossibility of having a non-wasteful and Pareto efficient system. The next result says that a sequential system may perform poorly as far as fairness and respecting improvements when mild requirements are imposed on the mechanisms used in each step.

Theorem 2 *Let $\Psi = (\varphi^1, \varphi^2)$ be a system. If φ^1 is non-wasteful and serially fair and φ^2 is non-wasteful, then Ψ is not fair and does not respect improvements in the priority order.*

Proof. As in the proof of Theorem 1, we consider two different cases. In the first case, all schools are social endowment. In the second case, we allow some schools to be owned by agents.

Case 1: There are three schools $S = \{s_1, s_2, s_3\}$ with one available seat and three agents $I = \{i_1, i_2, i_3\}$. Let $S^1 = \{s_2, s_3\}$, and $h(i_1) = h(i_2) = h(i_3) = \emptyset$. The priority structure of each school in S is the same and given as $i_1 \succ_s i_2 \succ_s i_3 \succ_s \emptyset$ for all $s \in S$. Let the true preferences be as follows:

$$s_2 P_{i_1} s_3 P_{i_1} s_1 P_{i_1} \emptyset$$
$$s_1 P_{i_2} s_2 P_{i_2} s_3 P_{i_2} \emptyset$$
$$s_1 P_{i_2} s_2 P_{i_2} s_3 P_{i_2} \emptyset$$

When all agents act truthfully all systems satisfying conditions mentioned in the statement of Theorem 2 select the following matching: $\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_3 & i_1 & i_2 \end{pmatrix}$.

Fairness: Agent i_2 prefers s_1 to $\mu(i_2)$ and has higher priority for s_1 then $\mu^{-1}(s_1) = i_3$. Therefore, any system selecting μ cannot be fair.

Respecting improvement: Now consider the following priority order for all $s \in S^1 : i_1 \succ'_s i_3 \succ'_s i_2 \succ'_s \emptyset$. When all agents act truthfully any systems satisfying conditions in Theorem 2 selects the following outcome: $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_2 & i_1 & i_3 \end{pmatrix}$. That is, when agent i_2 has lower priority then he becomes better off. Therefore, any system satisfying conditions mentioned in the statement of Theorem 2 does not respect improvement in the priority order. If all schools in S^1 rank the students based on the test scores then we can also say that any system satisfying conditions mentioned in the statement of Theorem 2 does not respect improvement in the priority order.

Case 2: Consider the same example. We only change the example by giving the property rights of school s_1 to i_1 and keep everything else the same. That is, $h(i_1) = s_1$. Given the set of available schools in step 1 is not changed then in the first step in any serially fair mechanism will assign i_1 to another school and he will give up his property rights for s_1. Therefore, s_1 will be available in the second period as in the Case 1. One can follow the steps in Case 1 and show that the impossibility result is robust to the ownership structure, i.e. whether all schools are socially endowed or not. ■

In TAP and SCPwEXRS different combinations of DA and SD mechanisms are used. DA and SD mechanisms satisfy the conditions given in the statements of Theorem 1 and Theorem 2. Hence, Theorems 1 and 2 have the following immediate corollaries for the two applications we have considered.

Corollary 1 *SD-DA used in SCPwEXRS is manipulable, wasteful, not fair, leads to avoidable welfare loss and does not respect improvements in the priority order (test scores).*

Corollary 2 *TSSD used in TAP is manipulable, wasteful, not fair, leads to avoidable welfare loss and does not respect improvements in the priority order (test scores).*

In the teacher assignment system, the sequence in which assignment is done cannot be changed since in order to have an available contractual position first a contractual teacher should be assigned to another school. On the other hand, in the school choice system the order can be changed by first assigning students to the regular schools and then assign the remaining students to the exam schools. Then, one can wonder whether the deficiencies of the two step mechanism used in school choice system are a consequence of considering exam schools before the regular school. It is easy to show that even if we first consider the regular school then the exam schools then we face the same deficiencies. Under this case the mechanism used to place students will be deferred acceptance followed by serial dictatorship mechanism (DA-SD).

Corollary 3 *DA-SD in SCPwEXRS is manipulable, wasteful, not fair, leads to avoidable welfare loss and does not respect improvements in the priority order.*

In the following two examples, we illustrate how SD-DA and TSSD mechanisms fail to satisfy the desired properties.

Example 1 *Let $S = \{s_1, s_2, s_3, s_4\}$, $S^e = \{s_3, s_4\}$, $S^r = \{s_1, s_2\}$, $I = \{i_1, i_2, i_3, i_4\}$ and $h(i_1) = h(i_2) = h(i_3) = h(i_4) = \emptyset$. All schools have one available seat, $q_s = 1$ for all $s \in S$. Let true preferences and test scores be as follows:*

$$\begin{array}{lrcl} s_2 P_{i_1} s_3 P_{i_1} s_1 P_{i_1} s_4 P_{i_1} \emptyset & c(i_1) & = & 90 \\ s_1 P_{i_2} s_4 P_{i_2} s_2 P_{i_2} s_3 P_{i_2} \emptyset & c(i_2) & = & 88 \\ s_3 P_{i_3} s_1 P_{i_3} s_2 P_{i_3} s_4 P_{i_3} \emptyset & c(i_3) & = & 85 \\ s_4 P_{i_4} s_2 P_{i_4} s_1 P_{i_4} s_3 P_{i_4} \emptyset & c(i_4) & = & 70 \end{array}$$

The set of available schools in step 1 is $S^1 = \{s_3, s_4\}$. The outcome selected in Step 1 when all the agents act truthfully is $\mu_1(i_1) = s_3$, $\mu_1(i_2) = s_4$, $\mu_1(i_3) = \emptyset$ and $\mu_1(i_4) = \emptyset$. In step 2 the set of the available schools and set of the applicants allowed to participate are: $S^2 = \{s_1, s_2\}$ and $I^2 = \{i_3, i_4\}$. The outcome selected in Step 2 when all the agents act truthfully is $\mu_2(i_3) = s_1$ and $\mu_2(i_4) = s_2$. The final outcome of the SD-DA mechanism is $\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_1 & i_2 \end{pmatrix}$.

SD-DA is not Pareto efficient: *There exists another matching $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_2 & i_1 & i_3 & i_4 \end{pmatrix}$ that Pareto dominates the outcome of the SD-DA mechanism, μ. It is worth to mention that μ' is a fair matching. That is, the outcome of the current mechanism is Pareto dominated by a fair matching.*

SD-DA is not Strategy-proof: *If i_2 ranks s_4 below $\emptyset$ in his list in Step 1 then the final outcome will be $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_2 & i_4 & i_1 & i_3 \end{pmatrix}$ and i_2 will be strictly better-off.*

SD-DA does not respect improvements: *If we take $c'(i_2) = 75$ then the outcome will be $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_2 & i_4 & i_1 & i_3 \end{pmatrix}$ and $\mu'(i_2) P_{i_2}\ \mu(i_2)$. That is, when i_2 gets higher score he is assigned to a less preferred school.*

SD-DA is not fair: *$\mu(i_4) P_{i_1} \mu(i_1)$ and i_1 has higher priority for $\mu(i_4) = s_2$.*

SD-DA is wasteful: *Consider the same example with only one agent, $I = \{i_1\}$. SD-DA mechanism assigns i_1 to s_3. But i_1 prefers s_2 to its match s_3 and s_2 has an empty under the outcome of SD-DA.*

Consider the same example with the following modification, $S^r = \{s_3, s_4\}$ and $S^e = \{s_1, s_2\}$. All the other things are kept the same. Then it is easy to see that DA-SD mechanism suffers from the same deficiencies as the SD-DA mechanism. ■

Example 2 *Consider Example 1 with the following modifications: $h(i_1) = s_1$, $h(i_2) = s_2$, $h(i_3) = h(i_4) = \emptyset$. Take the same test scores for students i_1, i_3 and i_4. Only change the test score of i_2 to $c(i_2) = 80$. Then TSSD mechanism selects the following matching: $\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_4 & i_2 & i_1 & i_3 \end{pmatrix}$.*

TSSD is not Pareto efficient: *There exists another matching* $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_1 & i_2 \end{pmatrix}$ *that Pareto dominates the outcome of the TSSD mechanism* μ. *It is worth to mention that* μ' *is a fair matching. That is, the current mechanism is Pareto dominated by a fair matching.*

TSSD is not Strategy-proof: *If* i_3 *ranks* s_4 *below* $\emptyset$ *in the submitted preferences in Step 1 then the final outcome will be* $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_1 & i_2 \end{pmatrix}$ *and* i_3 *will be strictly better-off. Moreover none of the agents will be hurt.*

TSSD does not respect improvements: *If we take* $c'(i_3) = 75$ *then the outcome will be* $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_3 & i_4 & i_1 & i_2 \end{pmatrix}$ *and* $\mu'(i_3)P_{i_3}\ \mu(i_3)$. *That is, when* i_3 *gets higher score he is assigned to a less preferred school.*

TSSD is not fair: $\mu(i_4)P_{i_3}\mu(i_3)$ *and* i_3 *has higher priority than* i_4 *for* $\mu(i_4) = s_1$.

TSSD is wasteful: *Consider the same example with only two agents,* $I = \{i_1, i_3\}$ *and* $S = \{s_1, s_3, s_4\}$ *where* $h(i_1) = s_1$ *and* $h(i_2) = \emptyset$. *The preference of agents are*

$$\begin{array}{lrcl} s_3 P_{i_1} s_1 P_{i_1} s_4 & c(i_1) & = & 90 \\ s_3 P_{i_3} s_1 P_{i_3} s_4 & c(i_3) & = & 85 \end{array}$$

The matching selected by the TSSD is $\mu'' = \begin{pmatrix} s_1 & s_3 & s_4 \\ \emptyset & i_1 & i_3 \end{pmatrix}$. *But* i_3 *prefers* s_1 *to its match* s_4 *and* s_1 *has an empty seat under* μ''. ■

5 Equilibrium Analysis of the Preference Revelation Games

In Section 4, we have shown that current systems in Turkey and US are not strategy-proof. Moreover, it may not be difficult for the agents to manipulate these systems. In this section we analyze the properties of the preference revelation game associated with the current systems. Since both systems are composed of two steps we look at the subgame perfect Nash equilibrium (SPNE). Here, the set of actions in each step is all the possible orders of available objects in that step and $\emptyset$. The outcome of the game is determined by the system.

We analyze the game under complete information of payoffs, available strategies and priorities. Agents are assumed to play simultaneously and the outcome of the first step is publicly announced. We first introduce two axioms that we will use in our analysis.

Fix the set of schools, quota vector, priorities and ownership profile. Let I be the initial set of students and $A(I)$ be a subset of I and represent the set of students listing a school from S at the top of their preferences, $A(I) = \{i \in I : sP_i\emptyset \text{ for some } s \in S\}$. Then, $I\backslash A(I)$ is the set of students from I who list $\emptyset$ at the top of their preferences. Then, a problem is a pair of $(I, ((P_i)_{i\in A(I)}, (\widetilde{P}_j)_{j\in I\backslash A(I)}))$. A mechanism φ is **population monotonic** if $A(I') \subseteq A(I)$ and $I' \subseteq I$ then for all $i' \in A(I')$

$$\varphi_{i'}(I', ((P_i)_{i\in A(I')}, (\widetilde{P}_j)_{j\in I'\setminus A(I')}))R_{i'}\varphi_{i'}(I, ((P_i)_{i\in A(I)}, (\widetilde{P}_j)_{j\in I\setminus A(I)})).$$

That is, if the number of applicants ranking a real school at the top of the preference list decreases then applicants ranking a real school at the top of the preference list cannot be worse off. The number of applicants ranking a real school at the top of the preference list can decrease as a consequence of either departure of some agents or increase in the number of agents ranking $\emptyset$ at the top.

In Section 3 two applications of the sequential assignment problem are explained. In both applications the set of priority profile is restricted. For instance, in TAP the schools rank all applicants except the existing teachers in the same order. Moreover, only one applicant can be ranked at the top of priority order of more than two schools. Then it is natural to analyze the mechanisms under the possible set of priority profiles. Let $\succ$ be the class of all possible priority profiles. A mechanism φ is **weakly fair** if it selects a mutually fair outcome for any problem $(I, S, q, P, \succ |(I, S), h)$ such that $\bigcup_{i\in I} h(i) \cap S = \emptyset$ and $\succ \in \succ$.

Before analyzing the games played in both problems we first show that in the general setting if an agent can gain from misreporting then this implies that she will also be weakly better off by ranking all available schools below $\emptyset$ in the first step and reporting true relative preferences over the available schools in the second step.

Let $\widetilde{P}_i$ be the agent i's true preference over the objects and $\emptyset$. Denote agent i's true relative preference over the available schools in step $t \in \{1, 2\}$ including $\emptyset$ with $\widetilde{P}_i^t$.

Proposition 1 *Let $\Psi = (\varphi^1, \varphi^2)$ be a system such that both φ^1 and φ^2 are strategy-proof, individually rational and φ^1 is non-wasteful and population monotonic. If there exists a pair (Q_i^1, Q_i^2) such that $\Psi_i((Q_i^1, \widetilde{P}_{-i}^1), (Q_i^2, \widetilde{P}_{-i}^2))P_i\Psi_i(\widetilde{P}^1, \widetilde{P}^2)$ then $\Psi_i((\widetilde{Q}_i^1, \widetilde{P}_{-i}^1), (\widetilde{Q}_i^2, \widetilde{P}_{-i}^2))R_i\ \Psi_i((Q_i^1, \widetilde{P}_{-i}^1), (Q_i^2, \widetilde{P}_{-i}^2))$ where $\widetilde{Q}_i^1 = \emptyset\widetilde{Q}_i^1 x$ for all $x \in S^1$ and $\widetilde{Q}_i^2 = \widetilde{P}_i^2$.*

Proof. Suppose not. First note that $\Psi_i(\widetilde{P}^1, \widetilde{P}^2)$ cannot be less preferred to $\emptyset$ by i due to individual rationality. Therefore, $\Psi_i((Q_i^1, \widetilde{P}_{-i}^1), (Q_i^2, \widetilde{P}_{-i}^2))P_i\emptyset$. Suppose $\Psi_i((Q_i^1, \widetilde{P}_{-i}^1), (Q_i^2, \widetilde{P}_{-i}^2))$ is an object available in step 1. That is, $\varphi^1(Q_i^1, \widetilde{P}_{-i}^1) = \Psi_i((Q_i^1, \widetilde{P}_{-i}^1), (Q_i^2, \widetilde{P}_{-i}^2))$. Therefore, i is assigned in step 1 and does not participate in the second step. Given that φ^1 is strategy-proof, i cannot get better school than $\varphi^1(\widetilde{P}^1)$ in the first step. Then, $\varphi^1(\widetilde{P}^1)P_i\emptyset$ since $\varphi^1(Q_i^1, \widetilde{P}_{-i}^1)P_i\emptyset$ and $\varphi^1(\widetilde{P}^1)$ cannot be less preferred to $\varphi^1(Q_i^1, \widetilde{P}_{-i}^1)$. Hence, $\Psi_i(\widetilde{P}) = \varphi^1(\widetilde{P}^1)$. This is a contradiction. Therefore, $\Psi_i((Q_i^1, \widetilde{P}_{-i}^1), (Q_i^2, \widetilde{P}_{-i}^2))$ should be a school available in step 2 and $\Psi_i((Q_i^1, \widetilde{P}_{-i}^1), (Q_i^2, \widetilde{P}_{-i}^2)) = \varphi_i^2(\widetilde{Q}_i^2, \widetilde{P}_{-i}^2)$. When i submits $\widetilde{Q}_i^1 = \emptyset\widetilde{Q}_i^1 x$ for all $x \in S^1$ he will not be assigned in the first step. Otherwise, individual rationally would be violated. Moreover, the set of agents assigned to an object under both $\varphi^1(Q_i^1, \widetilde{P}_{-i}^1)$ and $\varphi^1(\widetilde{Q}_i^1, \widetilde{P}_{-i}^1)$ are the same since φ^1 is population monotonic and non-wasteful. Then the same set of agents will participate the second step and the quotas of each object in S^2 will be the same when i submits Q_i^1 and $\widetilde{Q}_i^1$. Due to the strategy-proofness i cannot get a better school than $\varphi^2(\widetilde{P}^2)$ in both $\varphi^2(Q_i^2, \widetilde{P}_{-i}^2)$ and $\varphi^2(\widetilde{Q}_i^2, \widetilde{P}_{-i}^2)$. Recall that $\widetilde{Q}_i^2 = \widetilde{P}_i^2$. Therefore, $\varphi_i^2(\widetilde{P}^2)R_i\varphi_i^2(\widetilde{Q}_i^2, \widetilde{P}_{-i}^2)$ and $\Psi_i((\widetilde{Q}_i^1, \widetilde{P}_{-i}^1), (\widetilde{Q}_i^2, \widetilde{P}_{-i}^2))R_i\Psi_i((Q_i^1, \widetilde{P}_{-i}^1), (Q_i^2, \widetilde{P}_{-i}^2))$. ■

In Section 1 we have shown that TSSD is not strategy proof. Although TSSD is not strategy-proof, Proposition 1 shows that not all of the applicants can benefit from misreporting their preferences. In particular, existing teachers cannot benefit from misreporting.

Corollary 4 *Under TSSD, existing teachers cannot benefit from misreporting.*

In the following proposition we show that if a system satisfies the conditions in Proposition 2 then the only way to manipulate the mechanism is truncating the reported preferences in step 1.

Proposition 2 *Let $\Psi = (\varphi^1, \varphi^2)$ be a system such that both φ^1 and φ^2 are strategy-proof and φ^1 is individually rational, non-wasteful and resource monotonic. If whenever there exists a preference pair (Q_i^1, Q_i^2) such that $\Psi_i((Q_i^1, \widetilde{P}_{-i}^1), (Q_i^2, \widetilde{P}_{-i}^2))P_i\Psi_i(\widetilde{P}^1, \widetilde{P}^2)$ then there exists a school $s \in S^1$ where $\emptyset Q_i^1 s$ and $s\widetilde{P}_i^1\emptyset$.*

Proof. In order to prove the proposition we show that there does not exists a preference pair $(\widetilde{Q}_i^1, \widetilde{Q}_i^2)$ such that $s\widetilde{Q}_i^1\emptyset$ if and only if $s\widetilde{P}_i^1\emptyset$ and $\Psi_i((\widetilde{Q}_i^1, \widetilde{P}_{-i}^1), (\widetilde{Q}_i^2, \widetilde{P}_{-i}^2))P_i\Psi_i(\widetilde{P}^1, \widetilde{P}^2)$. We first consider the case where $\varphi_i^1(\widetilde{P}^1) = \emptyset$. If $\varphi_i^1(\widetilde{P}^1) = \emptyset$ then $\varphi_i^1(\widetilde{Q}_i^1, \widetilde{P}_{-i}^1) = \emptyset$. This is a consequence of strategy-proofness. Then consider the following preference profile $\widehat{Q}_i^1 : \emptyset\widehat{Q}_i^1 s$ for all $s \in S^1$. Due to non-wastefulness and population monotonicity, the same set of students are assigned to the same set of objects under $\varphi^1(\widehat{Q}_i^1, \widetilde{P}_{-i}^1)$, $\varphi^1(\widetilde{P}^1)$ and $\varphi^1(\widetilde{Q}_i^1, \widetilde{P}_{-i}^1)$. Then the same set of agents will participate the second step and the quotas of each object in S^2 will be the same when i submits $\widetilde{P}_i^1$ and $\widetilde{Q}_i^1$. Since the mechanism used in second step is strategy-proof i cannot benefit from misreporting in the second step. Therefore we should consider the case $\varphi_i^1(\widetilde{P}^1) \neq \emptyset$. Due to the strategy-proofness if $\varphi_i^1(\widetilde{Q}_i^1, \widetilde{P}_{-i}^1)(i) \neq \emptyset$ then $\varphi_i^1(\widetilde{P}^1)P_i\varphi_i^1(\widetilde{Q}_i^1, \widetilde{P}_{-i}^1)$. On the other hand $\varphi_i^1(\widetilde{Q}_i^1, \widetilde{P}_{-i}^1) = \emptyset$ cannot be true due to the strategy-proofness of φ^1. ■

Both DA and SD mechanisms are strategy-proof, non-wasteful, population monotonic and individually rational. Therefore, all the three systems, TSSD, DA-SD and SD-DA, satisfy the conditions mentioned in Proposition 1 and Proposition 2.

In the following theorems we provide general results on the equilibrium analysis of systems.

Theorem 3 *Let $\Psi = (\varphi^1, \varphi^2)$ be a system such that both φ^1 and φ^2 are individually rational, non-wasteful and either population monotonic and non-bossy or fair. Every SPNE outcome of the preference revelation game associated with Ψ leads to a non-wasteful and individually rational matching.*

Proof. Let $Q = (Q_i^1, Q_i^2)_{i \in I}$ be an SPNE profile and μ be the associated equilibrium outcome.

If μ is individually irrational then there exists $i \in I$ such that $\emptyset P_i \mu(i)$. Then submitting $P_i' = \emptyset P_i' x$ for all $x \in S^t$ in step $t \in \{1, 2\}$ is a profitable deviation for agent i. Therefore, Q cannot be SPNE profile which is a contradiction.

Suppose μ is wasteful. Then, there exists $i \in I$ such that $sP_i\mu(i)$ and $i \succ_s \emptyset$ and $|\mu^{-1}(s)| < q_s$. We consider two possible cases and we show that if μ is wasteful then there exists a profitable deviation for at least one agent and therefore Q cannot be SPNE profile.

Case 1: Suppose s is available in the first step. Since φ^1 is non-wasteful s should be ranked after $\emptyset$ in Q_i^1. Consider following preference profile $P_i' = sP_i'\emptyset P_i'x$ for all $x \in S^1\backslash\{s\}$. Due to the individual rationality i can be assigned to s or $\emptyset$ when he deviates and submits P_i'. If the first case is true we are done. If the second case is true then all the seats of s are filled by other students who rank s before $\emptyset$. Denote the new matching by μ_1. Due to the non-wastefulness of φ^1 all students in $\mu_1(s)$ should have been assigned to a school in μ and they weakly prefer μ to μ_1. Moreover, all the seats of their assignment in μ should be filled filled by other students under μ_1. We first show that this cannot be true if φ^1 is fair. Not to violate fairness all agents assigned to the assignments of agents in $\mu_1(s)$ under μ should have higher priority than agents in $\mu_1(s)$ and they should prefer μ to μ_1. Not to violate individual rationality all of these agents should be assigned to a school in μ. If we continue we will finally show that either μ_1 or μ fails to be fair. Now we show that i cannot be assigned to $\emptyset$ when he deviates given φ^1 is non-bossy and resource monotonic. Suppose not. Due to the non-bossiness and individual rationality μ_1 will be selected when i ranks $\emptyset$ at the top of his submitted preferences. We know that some of the students in $\mu_1(s)$ prefer μ to μ_1. Therefore resource monotonicity of φ^1 is violated.

Case 2: Suppose $s \in S^2$. If $\mu(i) \in S^2 \cup \emptyset$ then we refer to the Case 1. By using the same steps in Case 1 one can show that (Q_i^1, P_i') is a profitable deviation for i. If $\mu(i) \in S^1$ then we show that submitting $\widetilde{P}_i^1 = \emptyset\widetilde{P}_i^1x$ for all $x \in S^1$ in the first step and $\widetilde{P}_i^2 = s\widetilde{P}_i^2x$ for all $x \in S^2 \cup \emptyset$ in the second step is a profitable deviation for student i. First consider the case where φ^1 is fair. In this case, all students assigned to a school when i submits Q_i^1 will also be assigned to a school in step 1 when i submits $\widetilde{P}_i^1$. This is also true when φ^1 is resource monotonic. Therefore, in the second step the set of agents is a subset of $I^2 \cup \{i\}$ and set of available seats weakly increases compared to the case in which i plays Q_i. Suppose i is not assigned to s. Then due to the individual rationality he will be assigned to $\emptyset$. Denote the selected matching in the second step by μ_2. If φ^2 is fair the students who participate in step 2 except i cannot fill all the available seats of s. Otherwise fairness would be violated. If φ^2 is resource monotonic and non-bossy then μ_2 will be selected when i ranks $\emptyset$ at the top of his preferences. We know that some of the students in $\mu_2(s)$ prefer μ to μ_2. Therefore resource monotonicity of φ^2 is violated. ■

Theorem 4 *Let $\Psi = (\varphi^1, \varphi^2)$ be a system such that φ^1 is individually rational, non-wasteful, population monotonic, non-bossy and either weakly fair and only the socially owned objects are available in step 1 are social endowment or mutually fair. Every SPNE outcome of the preference revelation game associated with Ψ leads to a matching μ in which there does not exist (i, j, s) triple such that*

- $s \in S^1$ *and* $j \in \mu^{-1}(s)$,
- $sP_i\mu(i)$ *and* $i \succ_s j$.

Proof. Let $Q = (Q_i^1, Q_i^2)_{i\in I}$ be an SPNE profile and μ be the associated equilibrium outcome. Suppose there exist two agents $i, j \in I$ such that $s \in S^1$, $sP_i\mu(i)$, $j \in \mu(s)$ and $i \succ_s j$. Note that Q_i^1 cannot be equivalent to $Q' : sQ'\emptyset Q'x$ for all $x \in S^1\backslash\{s\}$. Otherwise, mutual fairness (or weak fairness given only the socially owned objects are available in step 1) of φ^1 would be violated. We claim that submitting Q' is a profitable deviation for some agents. We first start checking whether it is a profitable deviation for i.

Denote the matching selected by φ^1 in step 1 for preference profile (Q', Q_{-i}^1) by $\widetilde{\mu}_1$. Since φ^1 is individually rational then i will be either assigned to s or $\emptyset$. If i is assigned to s then Q' is a profitable deviation for i and Q cannot be SPNE. If i is assigned to $\emptyset$ then all agents assigned to s should have higher priority than i and all the seats of s are filled. Otherwise mutual fairness (or weak fairness given only the socially owned objects are available in step 1) and/or non-wastefulness of φ^1 would be violated. Suppose $\widetilde{\mu}_1 = \emptyset$ then consider the profile (Q'', Q_{-i}^1) where i submits $Q'' : \emptyset Q''x$ for all $x \in S^1$. Due to non-bossiness and individual rationality the allocations of all agents are unchanged. Given the number of agents ranking a real object decreases due to the population monotonicity all the agents in $\widetilde{\mu}_1^{-1}(s)$ should be weakly better off compared to μ_1. Moreover, the ones who were not assigned to s in μ_1 should be strictly better off under $\widetilde{\mu}_1$. Moreover, there exists a student i_1 such that $\widetilde{\mu}_1(i_1) = s \neq \mu_1(i_1)$, $sQ_{i_1}^1\mu_1(i_1)Q_{i_1}^1\emptyset$. Otherwise, non-wastefulness and/or individual rationality would be violated.

Given $i \succ_s j$ and $i_1 \succ_s i$, i_1 has higher priority than at least one student in $\mu_1^{-1}(s)$ for school s. Therefore, $Q_{i_1}^1 \neq Q'$ otherwise mutual fairness (or weak fairness given only the socially owned objects are available in step 1) would be violated. Now consider the following preference profile $(Q', Q_{-i_1}^1)$ where i_1 submits Q' and all other players keep their strategies in Q. Denote the matching selected by φ^1 for preference profile $(Q', Q_{-i_1}^1)$ by $\widetilde{\mu}_2$. Since φ^1 is individually rational then either $\widetilde{\mu}_2(i_1) = s$ or $\widetilde{\mu}_2(i_1) = \emptyset$. If $\widetilde{\mu}_2(i_1) = s$ then Q' is a profitable strategy for i_1. Otherwise, all students in $\widetilde{\mu}_2^{-1}(s)$ have higher priority than i_1 and $|\widetilde{\mu}_2^{-1}(s)| = q_s$. Now consider the profile $(Q'', Q_{-i_1}^1)$ where i_1 submits Q''. Due to the non-bossiness the allocations of all agents are unchanged. Due to population monotonicity all the students in $\widetilde{\mu}_2^{-1}(s)$ should be weakly better off compared to μ_1. Moreover, agents in $\widetilde{\mu}_2^{-1}(s)$ who were not assigned to s in μ_1 strictly prefer $\widetilde{\mu}_2$ to μ_1. Moreover, there exists a student i_2 such that $\widetilde{\mu}_2(i_2) = s \neq \mu_1(i_2)$, $sQ_{i_2}^1\emptyset$ and i_2 prefers s to $\mu_1(i_1)$. Otherwise, non-wastefulness and/or individual rationality would be violated.

In each repetition of this procedure it is easy to see there does not exist a student in $\widetilde{\mu}_n^{-1}(s)$ with lower priority than all the students in $\widetilde{\mu}_{n-1}^{-1}(s)$ and $|\widetilde{\mu}_{n-1}^{-1}(s)| = |\widetilde{\mu}_n^{-1}(s)| = q_s$. Due to the finiteness of students at some point we will never observe the case where a student is assigned to $\emptyset$ when he submits Q' and all the others keep their strategies in Q. Therefore, Q cannot be a SPNE strategy profile. ■

Theorem 5 *Let $\Psi = (\varphi^1, \varphi^2)$ be a system such that φ^1 is individually rational, population monotonic and φ^2 is individually rational, population monotonic, non-wasteful and weakly fair. Every SPNE outcome of the preference revelation game associated with Ψ leads to a matching μ*

in which there does not exist (i,j,s) triple such that

- $s \in S^2$ *and* $j \in \mu^{-1}(s)$,
- $sP_i\mu(i)$ *and* $i \succ_s j$.

First note that according to our definition in Section 2 in any system agents with endowment are not allowed to participate in step 2. Let $Q = (Q_i^1, Q_i^2)_{i\in I}$ be an SPNE profile and μ be the associated equilibrium outcome. Suppose there exist two agents $i,j \in I$ and $s \in S^2$ such that $s \in S^2$, $sP_i\mu(i)$, $j \in \mu(s)$ and $i \succ_s j$. There are two possible cases: (1) $\mu(i) \in S^1$ and i does not participate in the second step or (2) i participate the second step but his strategy is not $Q' : sQ'\emptyset Q'x$ for all $x \in S^2\backslash\{s\}$. Note that, if $i \in I^2$ and $Q_i^2 = Q'$ then weak fairness of φ^2 would be violated. We show that Q cannot be a SPNE.

Proof. First suppose that agent i participates the second step. We claim that there exists an agent among the ones assigned in the second step who can be better of by only deviating in the second step and submitting Q'. Note that when nobody deviates from his strategy in the first step the set of schools and agents in the second step will not change. Therefore, we can prove that some agents participating the second step can benefit from deviating to Q' by following the same steps in the proof of Theorem 4.

Now suppose that there does not exist an agent i' such that (1) $\mu(i') \in S^2$, (2) $sP_{i'}\mu(i')$ and (3) $j \succ_s i'$ for all $j \in \mu(s)$. Then agent i should be assigned in step 1. We claim that ranking $\emptyset$ at the top of the submitted preference list in the first step and submitting Q' in the second step is a profitable deviation for i. To see this first note that i will be assigned to $\emptyset$ in the first step. Due to the population monotonicity and individual rationality the agents assigned to a school in S^1 under Q^1 will be assigned to weakly better schools in S^1 when i deviates. No new agents other than i can participate the second step. Moreover, the number of available seats in each object will not decrease. Then when i submits Q' he will be assigned to either s or $\emptyset$. Denote the matching by $\overline{\mu}_2$. If $\overline{\mu}_2(i) = s$ then we are done. If $\overline{\mu}_2(i) = \emptyset$ then all agents in $\overline{\mu}_2^{-1}(s)$ should have higher priority than i for s and all the seats of s are filled. Otherwise weak fairness and/or non-wastefulness would be violated. We continue with $\overline{\mu}_2(i) = \emptyset$. Now consider the case where i ranks $\emptyset$ at the top of his list in the second step. Due to non-bossiness the allocations of all agents are unchanged. Moreover there exists at least one agent i_1 such that $\overline{\mu}_2(i) = s \neq \mu(i_1)$ since agent j is not assigned to s anymore and all the seats of s is filled. Given the number of agents ranking a real object weakly decreases due to the population monotonicity all the agents in $\widetilde{\mu}_2^{-1}(s)$ should be weakly better off compared to μ. Moreover, the ones who were not assigned to s in μ should be strictly better off due to the strict preferences. However, this violates the fact that i_1 prefers his assignment in μ to s. Therefore, i cannot be assigned to $\emptyset$ when he submits Q' in the second step. ■

5.1 Subgame Perfect Nash Equilibria of SD-DA in SCPwEXRS

In this subsection, we analyze the SPNE of the current sequential assignment system used in SCPwEXRS. Recall that in SCPwEXRS serial dictatorship mechanism is applied in the first step and deferred acceptance mechanism is applied in the second step. Serial dictatorship mechanism is individually rational, non-wasteful, population monotonic, non-bossy and strategy-proof. Moreover, it selects a fair (mutually fair) outcome when only the exam schools are available. Deferred acceptance mechanism is individually rational, non-wasteful, strategy-proof and fair (mutually fair). By following Theorem 3 one can see that every SPNE outcome of SD-DA leads to a non-wasteful and individually rational matching under the true preferences.

Corollary 5 Every SPNE outcome of the preference revelation game associated with SD-DA leads to a non-wasteful and individually rational matching under agents' true preferences.

Proof. Follows from Theorem 3. ■

In the school choice problem, not all SPNE of the preference revelation game associated with SD-DA leads to a fair matching under agents' true preferences. However, SPNE of the preference revelation game associated with SD-DA leads to a matching where the priorities of the exam schools are respected under agents' true preferences.

Corollary 6 *Every SPNE outcome of the preference revelation game associated with SD-DA leads to a matching μ in which there does not exist (i, j, s) triple such that*

- $s \in S^e$ *and* $j \in \mu^{-1}(s)$,
- $sP_i\mu(i)$ *and* $i \succ_s j$.

Proof. Follows from Theorem 4. ■

Recall that only the students who have not been assigned to an exam school participate in the second step. Since it is the last step and we are using a strategy-proof mechanism to assign the participants to the available schools agents cannot benefit from misreporting. That is, it is weakly dominant strategy for all students to submit true preference over the available schools in step 2. Without loss of generality in the rest of this subsection we assume that students act truthfully in the second step of SD-DA.

Let (S, I, P, P_S, q) be the associated college admission problem of the school choice problem with exam and regular schools, $(S, I, P, \succ, q, h)$ where for each school s iP_sj if and only if $i \succ_s j$. In particular, the unique difference between college admission problem and school choice problem is that in college admission problem schools are active and have preferences over students, $P_S = (P_s)_{s\in S}$, whereas in school choice problem schools are passive and considered as objects to be consumed. A matching μ in school choice problem is individually rational, non-wasteful and fair if and only if it is stable for its associated college admission problem (Balinski and Sönmez, 1999). Moreover, for each college admission problem there exists a unique stable matching which is preferred to any other stable matching by all students. This matching is called student optimal stable matching. Similarly, student optimal stable matching is individually rational, non-wasteful, fair and preferred to any other individually rational, non-wasteful and fair matching by all students. In the following proposition, we show that in any school choice problem with exam and regular schools there exists at least one SPNE outcome of preference revelation game associated with SD-DA mechanism which is (weakly) preferred to any individually rational, non-wasteful and fair matching by all students.

Proposition 3 *In any SCPwEXRS, there always exists at least one SPNE outcome of preference revelation game associated with SD-DA mechanism which (weakly) Pareto dominates any individually rational, non-wasteful and fair matchings*

Proof. We show the existence of a SPNE outcome which (weakly) Pareto dominates the student optimal stable matching[20]. Denote the student optimal stable matching with μ. Then consider the following strategy profile:

- Student i submits $\widetilde{P}_i^1 : \mu(i)\widetilde{P}_i^1 x$ for all $x \in S^e \backslash \{\mu(i)\}$ in the first step if $\mu(i) \in S^e$
- Student i submits $\widetilde{P}_i^1 : \emptyset \widetilde{P}_i^1 x$ for all $x \in S^e$ in the first step if $\mu(i) \in S^r$,
- Student i submits his true preferences over the regular schools in the second step whenever he is active, i.e. $\widetilde{P}_i^2 = P_i|S^r$.

Denote the outcome of SD-DA mechanism under this preference profile by ν. It is easy to see that the students assigned to the exam schools in μ are assigned to the same school under this preference profile.

We need to first show that this preference profile is SPNE. First look at the subgames in the second step. Each subgame can be considered as an independent school choice problem. Truthtelling is weakly dominant strategy under DA mechanism. Therefore, submitting true preferences in the second step is a Nash equilibrium in each subgame.

Now we analyze the strategies in the first step. First consider a student i who is assigned to a regular school. Given that μ is fair and non-wasteful all the seats of the exam schools that i prefers to $\mu(i)$ are filled by students with better exam score under this preference profile. Therefore, i cannot be assigned to a better exam school by deviating in the first step. Moreover, he can only change the outcome by being assigned to an exam school and that school cannot be preferable to $\mu(i)$ due to strategy-proofness.

Now consider student j who is assigned to an exam school. Given that μ is fair and non-wasteful all the seats of the exam schools that i prefers to $\mu(i)$ are filled by students with better exam score under this preference profile. We should also check whether he can be assigned to a more preferred regular school. If j deviates and participates in the second step then we should consider the subgame where students assigned to regular school in μ and j are active. We know

[20] Here, we mean the most preferred individually rational, fair and non-wasteful matching.

that truthtelling is a Nash equilibrium. We need to find the outcome of DA mechanism for the following school choice problem $(S^r, I^2, (q_s)_{s\in S^r}, (\widetilde{P}_i^2)_{i\in I^2}, \succ_{S^r})$ where $I^2 = \bigcup_{s\in S^r\cup\emptyset} \mu^{-1}(s)\cup j$. We use the sequential DA mechanism of McVitie and Wilson (1972). We consider student j after all students. Denote the matching that we have just before the turn of j with $\widetilde{\mu}$. Matching $\widetilde{\mu}$ is non-wasteful, individually rational and fair. Moreover, $\widetilde{\mu}$ is the outcome of DA mechanism for problem $(S^r, I^2\backslash\{j\}, (q_s)_{s\in S^r}, (\widetilde{P}_i^2)_{i\in I^2\backslash j}, \succ_{S^r})$. Define a new matching $\mu' : \bigcup_{s\in S^r\cup\emptyset} \mu^{-1}(s) \to S^r$ such that $\mu'(i) = \mu(i)$ for all $i \in \bigcup_{s\in S^r\cup\emptyset} \mu^{-1}(s)$. It is easy to see that μ' is individually rational, fair and non-wasteful for the problem $(S^r, I^2\backslash\{j\}, (q_s)_{s\in S^r}, P, \succ_{S^r})$ and all students in $I^2\backslash\{j\}$ weakly prefers their assignment in $\widetilde{\mu}$ to the one in μ' and μ. As a consequence of the rural hospital theorem (Roth, 1986) the number of students assigned to each regular schools is same in $\widetilde{\mu}$ to μ'.

For student j we should consider the regular schools which is preferred to $\mu(j)$. Let s' be a regular school that i prefers to $\mu(i)$. Then, all the seats of s' should be filled in $\widetilde{\mu}$. Otherwise μ cannot be non-wasteful. Also note that each student in $I^2\backslash\{j\}$ prefers his assignment in μ to the schools with unfilled seats in μ' and $\widetilde{\mu}$. Without loss of generality we change the preference profile of j by placing $\emptyset$ just before $\mu(j)$ and represent it with P_j'. Let $\widetilde{P}_j' = P_j'|S^r$. It is easy to see that if j can be assigned to a better school than $\mu(j)$ in $(S^r, I^2, (q_s)_{s\in S^r}, (\widetilde{P}_i^2)_{i\in I^2}, \succ_{S^r})$ then he will be assigned to the same school in $(S^r, I^2, (q_s)_{s\in S^r}, (\widetilde{P}_j', (\widetilde{P}_i^2)_{i\in I^2\backslash j}), \succ_{S^r})$. Define a new matching $\mu'' : \bigcup_{s\in S^r\cup\emptyset} I^2 \to S^r$ such that $\mu''(i) = \mu(i)$ for all $i \in I^2\backslash j$ and $\mu''(j) = \emptyset$. Then it is easy to see that μ'' is individually rational, fair and non-wasteful in problem $(S^r, I^2, (q_s)_{s\in S^r}, (\widetilde{P}_j', (\widetilde{P}_i^2)_{i\in I^2\backslash j}), \succ_{S^r})$. Moreover, in all stable matchings the set of students assigned to a real school will be the same (Roth, 1986). Therefore, DA mechanism will not assign j to a better school then $\mu(j)$ if he deviates and participates the second step.

Given that the strategy profile is SPNE then we need to look at the welfare comparison between ν and μ. All students assigned to an exam school in μ will be assigned to the same school in ν. Therefore, the set of students participating the second step will be the students who were not assigned to exam schools in μ. Then in the second step the outcome will be $\widetilde{\mu}$. We already mention that all students in $I^2\backslash\{j\}$ weakly prefer $\widetilde{\mu}$ to μ. Therefore, the outcome of the above strategy profile is weakly preferred to μ by all students. ■

We illustrate the result of Proposition 3 in the following example.

Example 3 *There are 1 exam school, $S^e = \{s_1\}$, two 2 regular schools, $S^r = \{s_2, s_3\}$ and three students, $I = \{i_1, i_2, i_3\}$. Priorities and preferences are given as*

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
i_1	i_2	i_3
i_2	i_i	i_2
i_3	i_3	i_1

P_{i_1}	P_{i_2}	P_{i_3}
s_2	s_3	s_2
s_1	s_2	s_3
$\emptyset$	$\emptyset$	$\emptyset$
s_3	s_1	s_1

.

We can find the student optimal individually rational, fair and non-wasteful matching by applying DA mechanism. The steps of the DA mechanism is:

	s_1	s_2	s_3
Step 1		i_1^*, i_3	i_2^*
Step 2		i_i^*	i_2, i_3^*
Step 3		i_1, i_2^*	i_3^*
Step 4	i_1^*	i_2^*	i_3^*

The final outcome of DA mechanism is $\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & i_3 \end{pmatrix}$.

Consider the following strategy profile:

- *In the first period students submit following profiles:* $s_1 \widetilde{P}^1_{i_1} \emptyset$, $\emptyset \widetilde{P}^1_{i_2} s_1$, $\emptyset \widetilde{P}^1_{i_3} s_1$.
- *All students participating the second step submit their true preference over the regular schools.*

One can verify that the strategy profile is SPNE equilibrium by checking the proof of Proposition 3. The outcome of this strategy profile is: $\mu' = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & i_3 \end{pmatrix}$ *and it Pareto dominates the student optimal individually rational, fair and non-wasteful matching* $\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & i_3 \end{pmatrix}$.

■

As a consequence of Proposition 3 we cannot say that every SPNE outcome of the preference revelation game associated with SD-DA leads to a non-wasteful, individually rational and fair matching under agents' true preferences. On the other hand we can relate every non-wasteful, individually rational and fair matching to an SPNE outcome.

Theorem 6 *Every* non-wasteful, individually rational and fair matching under agents' true preferences is led by a SPNE outcome of the preference revelation game associated with SD-DA.

Proof. We refer to the proof of Theorem 7. One can easily modify the proof of Theorem 7 and follow the same steps. ■

5.2 Subgame Perfect Nash Equilibria of TSSD

Recall that in TAP serial dictatorship mechanism is applied in both steps. Serial dictatorship mechanism is individually rational, non-wasteful, population monotonic, non-bossy and strategy-proof. Moreover, it selects a fair (mutually fair) outcome in a TAP when none of the applicants own the available schools. By following Theorem 3 one can see that every SPNE outcome of TSSD leads to a non-wasteful and individually rational matching and fair under the true preferences.

Corollary 7 Every SPNE outcome of the preference revelation game associated with TSSD leads to a non-wasteful, individually rational and fair matching under agents' true preferences.

Proof. Follows from Theorem 3, Theorem 4 and Theorem 5. ■

We illustrate this result in the following example.

Example 4 *Suppose there are 3 schools, $S = \{a, b, c\}$,1 existing teacher $I^e = \{e\}$ and 2 new graduates $I^n = \{t_1, t_2\}$. Teacher e is currently working in school a and the other two schools are tenured positions. Suppose the ranking based on the test score is given by: $c(t_1) > c(t_2) > c(e)$. The true preferences over schools and utilities of teachers are given as:*

e	t_1	t_2	U
c	a	a	3
b	b	b	2
a	c	c	1
$\emptyset$	$\emptyset$	$\emptyset$	0

Teachers can act strategically only in the first stage and they cannot benefit from a deviation in the second step. In step 1 strategies are: bc, cb, b, c and $\emptyset$. The simultaneous game induced by the current 2 stage mechanism for the teacher assignment problem is given below. In this game, t_2 is the matrix player, t_1 is the column player and e is the row player.

bc

	bc	*cb*	*b*	*c*	$\emptyset$
bc	1,2,1	1,1,2	1,2,1	1,1,2	**3,3,2**
cb	1,2,1	1,1,2	1,2,1	1,1,2	**3,3,2**
b	**1,2,1**	1,1,2	**1,2,1**	1,1,2	1,0,2
c	1,2,1	1,1,2	1,2,1	1,1,2	**3,3,2**
$\emptyset$	**1,2,1**	1,1,2	**1,2,1**	1,1,2	1,0,2

cb

	bc	*cb*	*b*	*c*	$\emptyset$
bc	1,2,1	1,1,2	1,2,1	1,1,2	2,3,1
cb	1,2,1	1,1,2	1,2,1	1,1,2	2,3,1
b	1,2,1	1,1,2	1,2,1	1,1,2	2,3,1
c	1,2,1	1,1,2	1,2,1	1,1,2	1,0,1
$\emptyset$	**1,2,1**	1,1,2	**1,2,1**	1,1,2	1,0,1

b

	bc	*cb*	*b*	*c*	$\emptyset$
bc	3,2,3	1,1,2	3,2,3	1,1,2	**3,3,2**
cb	3,2,3	1,1,2	3,2,3	1,1,2	**3,3,2**
b	1,2,0	1,1,2	1,2,0	1,1,2	1,0,2
c	3,2,3	1,1,2	3,2,3	1,1,2	**3,3,2**
$\emptyset$	1,2,0	1,1,2	1,2,0	1,1,2	1,0,2

c

	bc	*cb*	*b*	*c*	$\emptyset$
bc	1,2,1	2,1,3	1,2,1	2,1,3	2,3,1
cb	1,2,1	2,1,3	1,2,1	2,1,3	2,3,1
b	1,2,1	2,1,3	1,2,1	2,1,3	2,3,1
c	1,2,1	1,1,0	1,2,1	1,1,0	1,0,1
$\emptyset$	**1,2,1**	1,1,0	**1,2,1**	1,1,0	1,0,1

$\emptyset$

	bc	cb	b	c	$\emptyset$
bc	*3,2,3*	*2,1,3*	*3,2,3*	*2,1,3*	*2,3,0*
cb	*3,2,3*	*2,1,3*	*3,2,3*	*2,1,3*	*3,3,0*
b	*1,2,0*	*2,1,3*	*1,2,0*	*2,1,3*	*2,3,0*
c	*3,2,3*	*1,1,0*	*3,2,3*	*1,1,0*	*3,3,0*
$\emptyset$	*1,2,0*	*1,1,0*	*1,2,0*	*1,1,0*	*1,0,0*

The bold payoffs represent the NE. (1,2,1) corresponds to the payoff of the school optimal fair, non-wasteful and individually rational matching and (3,3,2) corresponds to the payoff of the agent optimal fair, non-wasteful and individually rational matching. ■

In the following theorem, we show that in TAP every non-wasteful, individually rational and fair matching can be related to an SPNE outcome of TSSD.

Theorem 7 *Every* non-wasteful, individually rational and fair matching under agents' true preferences is led by a SPNE outcome of the preference revelation game associated with TSSD.

Proof. Let μ be a non-wasteful, individually rational and fair matching. Then consider the following strategy profile $Q = (Q_i)_{i \in I}$ and $Q_i = (Q_i^1, Q_i^2)$ where Q_i^t is the submitted preference profile in step $t \in \{1, 2\}$ such that

- if $\mu(i) \in S^t$ then student i ranks $\mu(i)$ at the top of Q_i^1,
- if $\mu(i) \in S^e$ and $h(i) = \emptyset$ then student i ranks $\emptyset$ at the top of Q_i^1 and $\mu(i)$ at the top of Q_i^2,
- if $\mu(i) = h(i) \neq \emptyset$ then student i ranks $\emptyset$ at the top of Q_i^1.
- if $\mu(i) = h(i) = \emptyset$ then student i ranks $\emptyset$ at the top of Q_i^1 and Q_i^2.

It is clear that the outcome of this strategy profile is μ.

First of all any agent assigned to a school in S cannot be better off by submitting a preference profile which makes him unassigned due to individual rationality.

First consider the second step. Agent i participates in the second step if $h(i) = \emptyset$ and she ranks $\emptyset$ at the top of Q_i^1. Suppose there exists a teacher among the ones participating step 2 who can get a better allocation by deviating. Denote this teacher by j. Let he get s when he deviates. Since μ is non-wasteful all the seats of s are filled by other students. Moreover, due to the fairness all students assigned to s have higher test scores. Therefore, all seats of s are filled before student j's turn and he cannot get that school no matter what he submits. If $\mu(j) \in S$ then j cannot be better off by ranking $\emptyset$ above $\mu(j)$ since μ is individually rational. Therefore, in the subgame a Nash equilibrium is selected under preference profile Q^2.

Now consider the first step. We first show that a contractual teacher cannot be better off by deviating. Suppose there exists a contractual teacher j who can get a better allocation by deviating. Let he get s when he deviates. First note that s cannot be a contractual position. Because the system does not allow an agent with ownership to participate in the second step. Then, s is a tenured position. Since μ is non-wasteful and fair all seats of s is filled before student j's turn and she cannot get that school no matter what she submits. Now we show that a new graduate cannot be better off by deviating. First of all, a new graduate cannot increase the number of available seats when each contractual teacher $k \in I^e$ submits Q_k^1. Due to the fairness and non-wastefulness a new graduate student who is assigned to a tenured school cannot be assigned to a contractual school when he deviates. Therefore if a new graduate deviates and gets a better school s then that school should be a tenured school. However, due to the fairness and non-wastefulness all seats of that school is filled before it is her turn and she cannot get that school no matter what she submits. ■

6 A Simpler Alternative System

Theorem 1 and Theorem 2 show that the main reason behind the deficiencies observed in the current procedures may simply be due to the fact that assignments are done in sequential fashion. These impossibilities motivate us to advocate one-step assignment systems as alternative systems to sequential assignment whenever it is feasible to do so.

In the assignment systems discussed in this paper, one of the important concerns is assigning the agents to the school without violating the order in the test scores or the predetermined priorities. Additionally, decreasing the level of gaming and thereby encouraging agents to report their true preferences over the schools is another important concern. Specific to the teacher assignment system in Turkey, it would violate the laws if a contractual teacher were to be moved from her current position and assigned to a less preferred one.

Applying the agent-proposing deferred acceptance algorithm (DA) in both market will readily fulfill the concerns mentioned above. For a given assignment problem $(S, I, q, P, \succ, h)$ we can find the outcome of the DA mechanism as follows:

Step 1: *Each agent applies to her top choice school. Each school, which receives an offer, tentatively accepts all best acceptable offers up to its quota according to its priority ordering. Any unacceptable offer or any offer not honored due to the quota constraint is rejected. If an agent applies to the being unassigned option, then she is permanently assigned to it.*

In general,

Step k: *Each agent who does not have a tentatively accepted offer from the previous step makes an offer to the best school, which has not rejected him yet. If there is no such school, she is tentatively matched with the null college. Each agent, which holds tentatively accepted offers or receives new offers in this step, tentatively accepts all best acceptable offers, among the new and previously held ones, up to its quota according to its priority ordering. Any unacceptable offer or any offer not honored due to the quota constraints is rejected permanently. If an agent applies to the being unassigned option, then she is permanently assigned to it.*

The algorithm terminates when no agent is rejected any more. For any problem, DA mechanism selects a fair, non-wasteful and individually rational outcome, i.e., it is stable. Moreover, DA mechanism is immune to preference manipulation and respects the improvements in the test scores (priorities). A natural question to ask is whether or not there is another alternative which satisfies all these desirable features. The following result based on Alcalde and Barbera (1994) and Balinski and Sönmez (1999) gives a negative answer to this question and makes a strong case for DA as a remedy to the deficiencies of the systems used in the two applications we discussed.[21]

Theorem 8 *DA is the unique mechanism which is*

- *fair, individually rational, non-wasteful, strategy-proof, or*
- *fair, individually rational, non-wasteful, respects improvements in priorities.*

7 Conclusion

Although the social objectives of a standard (one-step) assignment problems and those of sequential assignment problems are quite similar, we have shown that the latter type of problems may be fundamentally different and more challenging when compared with the former type. We have shown that under sequential systems, most desirable properties are lost even though they may be satisfied stepwise. Most remarkably, sequential systems are strategically vulnerable (even if they are strategy-proof stepwise) and force participants to make hard judgment calls about what rank-order to choose in each step. As a result these systems may lead to highly inefficient and even wasteful assignments. This suggests that even though sequential systems may arguably be easier to implement in practice (e.g., in the context of school choice), such convenience may come at an important cost. The alternative use of one-step systems, such as the DA, may help avoid these costs when doing so is feasible.

References

[1] Atila Abdulkadiroğlu and Tayfun Sönmez, "House Allocation with Existing Tenants", Journal of Economic Theory 88 (1999), pp. 233-260.

[2] Atila Abdulkadiroğlu and Tayfun Sönmez, "School Choice: A Mechanism Design Approach", American Economic Review 93 (2003), pp. 729-747.

[3] Jose Alcalde and Salvador Barbera, "Top Dominance and the Possibility of Strategy-Proof Stable Solutions to Matching Problems", Economic Theory 4 (1994), pp. 417-435.

[4] Michel Balinski and Tayfun Sönmez, "A Tale of Two Mechanisms: Student Placement", Journal of Economic Theory 84 (1999), pp. 73-94.

[21] Of course, it is well-known that the outcome of DA is not necessarily Pareto efficient but it Pareto dominates any other stable matching for a given problem.

[5] Haluk Ergin and Tayfun Sönmez, "Games of School Choice under the Boston Mechanism", Journal of Public Economics 90 (2006), pp. 215-237.

[6] David Gale and Lloyd Shapley, "College Admissions and the Stability of Marriage", The American Mathematical Monthly 69 (1962), pp. 9-15.

[7] Pablo Guillen and Onur Kesten, "Matching markets with mixed ownership: the case for a real-life assignment mechanism", International Economic Review 53 (2012), pp. 1027-1046.

[8] Onur Kesten, "On Two Competing Mechanisms for Priority Based Allocation Problems", Journal of Economic Theory 127 (2006), pp 155-171.

[9] Thayer Morrill, "An Alternative Characterization of Top Trading Cycles", Economic Theory, 2012.

[10] Alvin Roth, "On the Allocation of Residents to Rural Hospitals: A General Property of Two-Sided Matching Markets," Econometrica, 54 (1986), pp 425–427.

[11] D. G. McVitie and L. B. Wilson, "The stable marriage problem", Communications of the ACM 14 (1971), pp. 486-490.

[12] Alexander Westkamp, "An Analysis of the German University Admissions System", Economic Theory, 2012.