Ex Post Equilibria in Double Auctions of Divisible Assets*

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March 22, 2013

Abstract

We characterize ex post equilibria in uniform-price double auctions of divisible assets. Bidders receive private signals and inventories, have interdependent and linearly decreasing marginal values, and bid with demand schedules. In a static double auction we characterize a linear ex post equilibrium, in which no bidder would deviate from his strategy even if he would observe the signals and inventories of other bidders. Moreover, under certain conditions this ex post equilibrium is unique. In a dynamic market with a sequence of double auctions and stochastic arrivals of new signals, we characterize a dynamic ex post equilibrium, whose allocation path converges exponentially in time to the efficient level. We demonstrate that the socially optimal trading frequency depends on the arrival process of new information. Our ex post equilibrium aggregates dispersed private information and is robust to distributional assumptions and details of market design.

Keywords: ex post equilibrium, double auction, information aggregation, dynamic trading, trading frequency

JEL Codes: D44, D82, G14

*For helpful comments, we are grateful to Alessandro Bonatti, Bradyn Breon-Drish, Jeremy Bulow, Hui Chen, David Dicks, Darrell Duffie, Eiichiro Kazumori, Ilan Kremer, Stefano Lovo, Andrey Malenko, Konstantin Milbradt, Michael Ostrovsky, Jun Pan, Parag Pathak, Michael Peters, Paul Pfleiderer, Uday Rajan, Marzena Rostek, Gideon Saar, Andy Skrzypacz, Jiang Wang, and Bob Wilson, as well as seminar participants at Stanford University, Simon Fraser University, MIT, Copenhagen Business School, and UNC Junior Finance Faculty Roundtable. Paper URL: http://ssrn.com/abstract=2040609.

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1 Introduction

Auctions of divisible assets are common in many markets. Equity trading on exchanges, for instance, is typically organized as a batch double auction when the market opens and closes, and as continuous double auctions during the day (in the form of open limit order books). Other notable examples include the auctions of treasury bills and bonds, defaulted bonds and loans in the settlement of credit default swaps, and commodities such as milk powder, iron ore, and electricity. Analyzing the bidding behavior in these auctions helps us better understand information aggregation, allocative efficiency, and market design.

In this paper we propose an ex post equilibrium in divisible-asset double auctions with interdependent values. Interdependent values naturally arise in financial markets, as well as in goods markets where winning bidders subsequently resell part of the assets. We focus on a uniform-price double auction in which bidders bid with demand (and supply) schedules and pay for their allocations at the market-clearing price.[1] Every bidder receives a private signal and values the asset at a weighted average of his own signal and the signals of other bidders. Bidders also start with private inventories of the asset and have diminishing marginal values for owning it. Under mild conditions, we show that there exists a unique ex post equilibrium—an equilibrium in which a dealer's strategy depends only on his private information (i.e. his signal and inventory), but his strategy optimal even if he learns the private information of all other bidders (hence the "ex post" notation). That is, an ex post equilibrium implies no regret. In the ex post equilibrium of our baseline model, a bidder's demand schedule is linear in his private signal, private inventory, and the price.

The intuition for our ex post equilibrium is simple, and we now provide a heuristic description of its construction. We start by conjecturing that bidders use a symmetric demand schedule that is linear in the private signal, the private inventory, and the price. Let us consider bidder 1. Given that other bidders' demands are linear in their signals, bidder 1 can infer the sum of other bidders' signals—hence his valuation—from the sum of their equilibrium allocations, which is equal to the total supply less bidder 1's equilibrium allocation. By submitting a demand schedule, the bidder

[1] In financial markets such as stock exchanges, the demand schedules are typically represented by limit orders.

effectively selects his optimal demand at each possible market-clearing price. We show that this "price-by-price optimization" ensures the ex post optimality of each bidder's strategy, and a linear ex post equilibrium follows. We observe that this equilibrium construction relies critically on the linearity of the demand schedules (otherwise, bidder 1 cannot transform the sum of other bidders' allocations to the sum of their signals). In fact, we show that under mild conditions, only linear demand schedules can satisfy ex post optimality. Hence, the linear ex post equilibrium we have constructed is unique.

Our ex post equilibrium is tractable and can be generalized to various markets. A simultaneous auction of multiple assets, for example, also admits an ex post equilibrium and can be applied to "program trading" of multiple stocks at the NYSE and to "default management auctions" of derivative portfolios run by clearinghouses. In a separate paper, we show that the ex post equilibrium is a useful tool in analyzing price discovery in the settlement auctions of credit default swaps (Du and Zhu (2012)).

We further apply the ex post equilibrium methodology to study dynamic trading as well as the associated equilibrium price and allocative efficiency. We allow an infinite sequence of double auctions and stochastic arrivals of new signals over time. As long as each bidder's signal process is a martingale, there exists a stationary "periodic ex post equilibrium," a notion we adapt from Bergemann and Valimaki (2010). In each round of double auction, the equilibrium price reflects the average of the most recent signals possessed by bidders, and is hence a martingale. Moreover, the equilibrium allocations of assets across bidders converge exponentially to the efficient allocation over time. In markets with a finite (and potentially small) number of bidders, our result suggests that a sequence of double auctions is a simple and effective mechanism to quickly achieve allocative efficiency.

Finally, we employ the dynamic ex post equilibrium to analyze the effect of trading frequency on social welfare. We demonstrate that the socially optimal trading frequency depends critically on the arrival process of new information. For scheduled information arrival, a slow (batch) market tends to be optimal;[2] for stochastic information arrival, a fast (continuous) market tends to be optimal. Our results suggest that trading frequency affects social welfare even if everyone trades at the same speed.[3]

[2] Fuchs and Skrzypacz (2012) show that a similar result also holds in a lemons market with competitive buyers. They do not, however, explore markets for which continuous trading is optimal.

[3] Our approach differs from the small but growing theory literature that focuses on *differential*

Our ex post equilibrium has a number of desirable properties. First, it is robust to modeling details such as the probability distribution of private information and the implementation of the double auction. Such robustness is highlighted by the "Wilson criterion" (Wilson 1987) as a desirable feature for models of auctions and trading. Second, the ex post equilibrium aggregates private information with a finite number of bidders, as the market-clearing price reveals the average of dispersed signals. While information aggregation is also present in Grossman (1976), Kyle (1985), Kyle (1989), Vives (2011), Ostrovsky (2011), Rostek and Weretka (2012), and Babus and Kondor (2012), these papers study Bayesian equilibria under the normal distribution.[4] The existence and uniqueness of equilibria for non-normal distributions remains an open question in these models. Third, our ex post equilibrium is parsimonious: A bidder's one-dimensional demand schedule handles the $(n-1)$-dimensional uncertainty regarding all other bidders' valuations and inventories.[5] A one-dimensional demand schedule is the standard practice in the trading of financial securities and derivatives. Last but not least, because of its robustness, ex post optimality is a natural equilibrium selection criterion. It is particularly useful for the analysis of uniform-price auctions, which in many cases admit a continuum of Bayesian-Nash equilibria (Wilson 1979). In our static double auction, the ex post selection criterion implies the uniqueness of equilibrium under mild conditions.

1.1 Related Literature on Ex Post Equilibrium

In a static double auction, our ex post optimality condition is similar to the "uniform incentive compatible" condition of Holmström and Myerson (1983). In dynamic games, Hörner and Lovo (2009), Fudenberg and Yamamoto (2011), and Hörner, Lovo,

trading speed. For example, in Foucault, Hombert, and Rosu (2012), Pagnotta and Philippon (2012), and Biais, Foucault, and Moinas (2012), some agents can potentially trade faster than others, which has implications for adverse selection, competition, investments in technology, and welfare.

[4] We note that in Ostrovsky (2011) only the demands of the noise traders need to follow a normal distribution; the information of the strategic traders needs not be normally distributed. Rochet and Vila (1994) extend the model of Kyle (1985) to settings with arbitrary distribution of signals, under the additional assumption that the informed trader observes the demand from noise traders. Bidders in our ex post equilibrium do not have this superior information.

[5] Parsimony is one of the key features of our model. In the interdependent-value model of Dasgupta and Maskin (2000), if the number of bidders is at least three, then each bidder conditions his bids on the signals of all other bidders—a $(n-1)$-dimensional vector. In the network model of Babus and Kondor (2012), each agent submits a multi-dimensional demand schedules such that the agent's execution price with one neighbor depends on execution prices with his other neighbors. In our ex post equilibrium, each bidder's demand schedule is one-dimensional.

and Tomala (2012) have also obtained dynamic ex post equilibria. A major distinction is that the equilibria of these three studies rely on dynamic punishments to be sustained and require the discount factors to be close to 1, whereas our dynamic ex post equilibrium is stationary and imposes no restriction on the discount factor.

A number of papers study equilibria that are ex post optimal with respect to *supply shocks* when bidders have symmetric information regarding the asset value. Related papers include Klemperer and Meyer (1989), Ausubel, Cramton, Pycia, Rostek, and Weretka (2011) and Rostek and Weretka (2011), among others. Consistent with these studies, our ex post equilibrium is also ex post optimal with respect to supply shocks if bidders have purely private values. Separately, Ausubel (2004) proposes an ascending-price multi-unit auction and characterize an equilibrium in which truthful bidding is ex post optimal if bidders have purely *private* values.

Our results complement those of Perry and Reny (2005), who construct an ex post equilibrium in a multi-unit ascending-price auction with interdependent values. In their model, a bidder specifies different demands against different bidders as prices gradually rise throughout the auction; therefore, bidders' private information is naturally revealed as the auction progresses, and bidders' subsequent demands depend on this revealed information. In our ex post equilibrium of the double auction, by contrast, each bidder submits a single demand schedule against all other bidders, and no private information is revealed before the final price is determined. (Of course, our equilibrium is robust to the revelation of private information.) In addition, while Perry and Reny focus on designing an auction format that ex post implements the efficient outcome, we focus on the standard uniform-price double auction and show that multiple rounds of double auctions achieve asymptotic efficiency.

Finally, our results are related to the literature on ex post implementation. In a general setting with interdependent values and correlated signals, Crémer and McLean (1985) use bidders' beliefs to construct a revenue-maximizing mechanism in which truth-telling is an ex post equilibrium. In contrast, in our model both the equilibrium and the allocation mechanism (double auction) are independent of bidders' beliefs. Bergemann and Morris (2005) characterize a "separability" condition under which ex post implementation is equivalent to Bayesian implementation that is robust to higher order beliefs. In those separable environments, they conclude, ex post implementation/equilibrium is desirable because of its robustness to beliefs. Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006) show that if agents have *multidimen-*

sional signals, any finite, non-constant allocation rule cannot be ex post implemented for generic valuation functions. As in Perry and Reny (2005), however, we show that in many real-life markets where each bidder's signal is *one-dimensional* (i.e. a subset of $\mathbb{R}$), an ex post equilibrium exists.

2 Static Double Auction

We consider a uniform-price double auction of a divisible asset, such as commodities and financial securities and derivatives. For example, the trading of equities, futures and options are conducted in the form of continuous double auctions on exchanges. Following the Dodd-Frank Act, auction mechanisms are also gaining traction in the trading of over-the-counter derivatives such as swaps. For applications in derivatives markets, the word "asset" in our model should be read as "asset or liability."

2.1 Model

There are $n \geq 2$ bidders. Each bidder $i \in \{1, 2, \ldots, n\}$ receives a signal, $s_i \in (\underline{s}, \overline{s})$, that is observed by bidder i only.[6] Given the profile of signals $(s_1, s_2, \ldots, s_n)$, bidder i values the asset at the weighted average of all signals:

$$v_i = \alpha\, s_i + (1 - \alpha)\, \frac{1}{n-1} \sum_{j \neq i} s_j, \tag{1}$$

where $\alpha \in (0, 1]$ is a commonly known constant. Thus, bidders have interdependent values. Because other bidders' signals $\{s_j\}_{j \neq i}$ are unobservable to bidder i, v_i is also unobservable to bidder i.

An exogenous quantity S of the divisible asset is up for auction, where S can be positive, negative, or zero. Without loss of generality, we refer to S as the supply of the asset. This exogenous supply is designed to capture certain applications, such as price-independent market orders on equity exchanges or new issuance of treasury securities. Our results do not depend on this exogenous supply.

Each bidder i also has an existing inventory z_i of the asset. We let $\mathcal{Z} \subset \mathbb{R}^n$ be the set of inventory profiles. The inventory z_i is the private information of bidder i.

[6] A common support makes exposition simpler, but our results go through even if different bidders have different supports of signals, so long as each support of signals is a subset of the real line.

The total ex ante inventory

$$Z = \sum_{i=1}^{n} z_i \tag{2}$$

is constant and common knowledge. For example, in financial markets the total supply of a security (e.g., stocks or bonds) is often public information, and the net supply of a derivative contract (e.g., futures or swaps) is zero. We impose no restriction on the distribution of $\{z_i\}$, except the constraint (2). Each bidder i can buy or sell any additional quantity of the asset in the auction. We use the convention that a positive quantity $q_i \geq 0$ means buying q_i units, and a negative $q_i < 0$ means selling $-q_i$ units.

We assume that bidders have a linear-quadratic utility function. Specifically, after trading the quantity q_i at the price p in the auction, bidder i's total utility is

$$U(q_i, p; v_i, z_i) = v_i z_i + (v_i - p)q_i - \frac{\lambda}{2}(z_i + q_i)^2. \tag{3}$$

The utility specification (3) is equivalent to a linearly declining marginal value for owning the asset. For example, before the auction, bidder i's marginal value for owning the last unit of his inventory is $v_i - \lambda z_i$. In the auction, bidder i's marginal utility on the last traded unit is equal to his marginal value, $v_i - \lambda(z_i + q_i)$, minus the price paid, p:

$$\frac{\partial U(q_i, p; v_i, z_i)}{\partial q_i} = v_i - p - \lambda(z_i + q_i). \tag{4}$$

A linear-quadratic utility function is also used in Vives (2011) and Rostek and Weretka (2012), although their models have different information structures.

In the auction, each bidder i submits a downward-sloping and differentiable demand schedule $x_i(\,\cdot\,) : \mathbb{R} \to \mathbb{R}$, contingent on his signal s_i and inventory z_i. Hence, we denote bidder i's strategy as $x_i(\,\cdot\,; s_i, z_i)$. Each bidder's demand schedule is unobservable to other bidders. (As discussed shortly, our equilibrium analysis is robust to whether demand schedules are observable.) Bidder i's demand schedule specifies that bidder i wishes to buy or sell a quantity $x_i(p; s_i, z_i)$ of the asset at the price p. Given the submitted demand schedules $(x_1(\,\cdot\,; s_1, z_1), \ldots, x_n(\,\cdot\,; s_n, z_n))$, the auctioneer determines the transaction price $p^* = p^*(s_1, \ldots, s_n, z_1, \ldots, z_n)$ from the market-clearing condition

$$\sum_{i=1}^{n} x_i(p^*; s_i, z_i) = S. \tag{5}$$

In the equilibrium we state shortly, there exists a unique market clearing price. After p^* is determined, bidder i is allocated the quantity $x_i(p^*; s_i, z_i)$ of the asset and pays $x_i(p^*; s_i, z_i)p^*$.

Discussion of Model Assumptions

Before describing our equilibrium concept and solution, we discuss the motivation and interpretation of our model specification, including the valuation form (1) and the utility form (3) and (4).

First, for our results on interdependent values, it is important that each bidder i puts the same additive weight $(1-\alpha)/(n-1)$ on other bidders' signals, as shown in (1). While equal weighting is not without loss of generality, it does capture a first-order effect of dispersed information.

Second, a declining marginal utility is natural in many applications. It can be caused by risk aversion, liquidation costs, or diminishing marginal returns in production functions, among other reasons. In general, the declining marginal utility may take any form and shape, thus hard to analyze. We view a linearly declining marginal utility (4) as a first-order approximation of the general form of declining marginal utility. For our results on interdependent valuations (i.e. $\alpha < 1$), it is important that the marginal values declines in quantity at the same rate $\lambda > 0$. For pure private values (i.e., $\alpha = 1$), however, we can generalize our results to heterogeneous λ's across bidders (see Section 3.2).

Third, the general form of our model does *not* impose restrictions on parameter values. For example, our model does not require the price p or the pre-auction marginal value $v_i - \lambda z_i$ to be positive. Indeed, the market prices of many financial and commodity derivatives—including forwards, futures, and swaps—are zero upon inception and can become negative as market conditions change. Unlike stocks or bonds, derivatives positions have unlimited liability, which in our setting means that marginal values, $v_i - \lambda(z_i + q_i)$ or $v_i - \lambda z_i$, can become arbitrarily negative (e.g. AIG's large loss on credit derivatives). Importantly, derivative trades are binding contracts, signed with trading counterparties and backed by collateral posted to the clearinghouses. A unilateral break of loss-making contracts constitutes a default and leads to losses of collateral and reputation. It is for these applications that we do not impose free disposal as a necessary element of our model. In practice, it is not uncommon for investors to pay others (e.g. market makers) to dispose of loss-making derivative positions, presumably because the negative marginal values for holding these positions exceed (in absolute value) the negative price for selling them.

On the one hand, assets that have limited liabilities, such as stocks and bonds, should have nonnegative marginal values. Such restrictions can be satisfied by assuming a sufficiently small λ relative to $\{v_i\}$, among other conditions.[7] Another subtly associated with limited liability is that bidders may wish to dispose part of their initial inventory in order to affect the market price. We do not expect this practice of "burning asset" to be profitable if λ is small and $\{v_i\}$ are large. For the simplicity of the exposition and analytical solutions, in the remaining of the paper we do not explicitly state those conditions but leave them implicit. Again, such conditions need not be imposed for derivatives.

Regardless of parameter restrictions, the post-auction marginal value, $v_i - \lambda(z_i + q_i)$, is always greater than price p for buyers (with $q_i > 0$) and is always less than p for sellers (with $q_i < 0$). Otherwise, the trader would have no incentive to trade that last unit. All equilibria stated in this paper satisfy the condition $\mathrm{sign}(q_i)[v_i - p - \lambda(z_i + q_i)] \geq 0$.

Finally, the linear-quadratic utility form (3) is more restrictive than necessary. As we discuss in Section 2.4, all equilibrium prices and strategies remain the same if bidder i's utility is changed to $f(U(q_i, p; v_i, z_i))$, for any increasing function f. The robustness to scaling utilities is another desirable property of our ex post equilibrium.

2.2 Ex Post Equilibrium and Characterization

We now proceed to define our notion of ex post equilibrium and characterize an ex post equilibrium in this market.

Definition 1. An **ex post equilibrium** is a profile of strategies $(x_1, \ldots, x_n)$ such that for every profile of signals $(s_1, \ldots, s_n) \in (\underline{s}, \overline{s})^n$ and for every profile of inventories $(z_1, \ldots, z_n) \in \mathcal{Z}$, every bidder i has no incentive to deviate from x_i. That is, for any alternative strategy $\tilde{x}_i$ of bidder i,

$$U(x_i(p^*; s_i, z_i), p^*; v_i, z_i) \geq U(\tilde{x}_i(\tilde{p}; s_i, z_i), \tilde{p}; v_i, z_i),$$

[7] For example, in the case of zero external supply ($S = 0$), if we restrict the set of signals and inventories so that $v_i - \lambda z_i$ is always positive for all i, then the marginal values obtained by our equilibrium will always be positive as well.

where v_i is given by (1), p^* is the market-clearing price given x_i and $\{x_j\}_{j\neq i}$, and $\tilde{p}$ is the market-clearing price given $\tilde{x}_i$ and $\{x_j\}_{j\neq i}$.

In an ex post equilibrium, no bidder deviates from his equilibrium strategy *even if* he observes the other bidders' signals and inventories. Thus, the optimality condition in Definition 1 is written in terms of the ex post utility $U(\,\cdot\,)$, rather than the expected utility $\mathbb{E}[U(\,\cdot\,)]$. Therefore, our analysis below is valid for any joint distribution of $(s_1, \ldots, s_n)$ and $(z_1, \ldots, z_n)$, and we do not have to specify this distribution.

A modeling challenge associated with interdependent values is that the bidding strategy of bidder i must be optimal for each realization of signals $\{s_j\}_{j\neq i}$ and inventories $\{z_j\}_{j\neq i}$, but bidder i's strategy cannot depend on $\{s_j\}_{j\neq i}$ and $\{z_j\}_{j\neq i}$.

For the simplicity of exposition but at no cost of economic intuition, we first consider the case of no inventory, i.e., $\mathcal{Z} = \{(0, \ldots, 0)\}$, and suppress z_i in the strategies. We conjecture a strategy profile $(x_1, \ldots, x_n)$. For notational convenience, we define

$$\beta \equiv \frac{1-\alpha}{n-1}. \tag{6}$$

Given that all other bidders use this strategy profile and for a fixed profile of signals $(s_1, \ldots, s_n)$, the profit of bidder i at the price of p is:

$$\Pi_i(p) = \left(\alpha s_i + \beta \sum_{j\neq i} s_j - p\right)\left(S - \sum_{j\neq i} x_j(p; s_j)\right) - \frac{1}{2}\lambda\left(S - \sum_{j\neq i} x_j(p; s_j)\right)^2. \tag{7}$$

We can see that bidder i is effectively selecting an optimal price p given the residual demand $S - \sum_{j\neq i} x_j(p; s_j)$. Taking the first-order condition of $\Pi_i(p)$ at $p = p^*$, we have, for all i,

$$0 = \Pi_i'(p^*) = -x_i(p^*; s_i) + \left(\alpha s_i + \beta \sum_{j\neq i} s_j - p^* - \lambda x_i(p^*; s_i)\right)\left(-\sum_{j\neq i} \frac{\partial x_j}{\partial p}(p^*; s_j)\right). \tag{8}$$

We conjecture a symmetric linear demand schedule:

$$x_j(p; s_j) = a s_j - bp + cS, \tag{9}$$

where a, b, and c are constants. In this conjectured equilibrium, all bidders $j \neq i$ use the strategy (9). Thus, we can rewrite the each bidder j's signal s_j in terms of his demand x_j:

$$\sum_{j\neq i} s_j = \sum_{j\neq i} \frac{x_j(p^*; s_j) + bp - cS}{a} = \frac{1}{a}\left(S - x_i(p^*; s_i) + (n-1)(bp^* - cS)\right),$$

where we have also used the market clearing condition. Substituting the above equation into bidder i's first order condition (8) and rearranging, we have

$$\begin{aligned} x_i(p^*; s_i) &= \frac{\alpha(n-1)bs_i - (n-1)b\left[1 - \beta(n-1)b/a\right]p^* + S\left[1-(n-1)c\right]\beta(n-1)b/a}{1 + \lambda(n-1)b + \beta(n-1)b/a} \\ &\equiv as_i - bp^* + cS. \end{aligned}$$

Matching the coefficients and using the normalization that $\alpha + (n-1)\beta = 1$, we solve

$$a = b = \frac{1}{\lambda}\cdot\frac{n\alpha - 2}{n-1}, \quad c = \beta = \frac{1-\alpha}{n-1}.$$

It is easy to verify that under this linear strategy, $\Pi_i''(\cdot) = -n(n-1)\alpha b < 0$ if $n\alpha > 2$. Thus, we have an ex post equilibrium. Moreover, the first order condition (8) implies that $x_i(p^*; s_i) > 0$ when $v_i - p^* - \lambda x_i(p^*; s_i) > 0$, and $x_i(p^*; s_i) < 0$ when $v_i - p^* - \lambda x_i(p^*; s_i) < 0$. In other words, the bidder always (weakly) increases his total utility by trading, regardless he is buying or selling.

We now state our first main result, ex post equilibrium under private signals and inventories.

Proposition 1. *Suppose that $n\alpha > 2$. In a double auction with interdependent values and private inventories, there exists an ex post equilibrium in which bidder i submits the demand schedule*

$$x_i(p; s_i, z_i) = \frac{n\alpha - 2}{\lambda(n-1)}(s_i - p) + \frac{1-\alpha}{n-1}S - \frac{n\alpha-2}{n\alpha-1}z_i + \frac{(1-\alpha)(n\alpha-2)}{(n-1)(n\alpha-1)}Z, \tag{10}$$

and the equilibrium price is

$$p^* = \frac{1}{n}\sum_{i=1}^{n} s_i - \frac{\lambda}{n}\left(\frac{n\alpha-1}{n\alpha-2}S + Z\right). \tag{11}$$

Proof. See Section A.1. □

2.3 Uniqueness of the Ex Post Equilibrium

In this short subsection, we show that under mild conditions, ex post optimality is a sufficiently strong equilibrium selection criterion such that it implies the uniqueness of the ex post equilibrium characterized in Proposition 1.

Proposition 2. *In addition to $n\alpha > 2$, suppose that either $\alpha < 1$ and $n \geq 4$, or $\alpha = 1$ and $n \geq 3$. Let $(x_1, \ldots, x_n)$ be an arbitrary ex post equilibrium in which every x_i is continuously differentiable, $\frac{\partial x_i}{\partial p}(p; s_i, z_i) < 0$, and $\frac{\partial x_i}{\partial s_i}(p; s_i, z_i) > 0$. Then, for any $s \in (\underline{s}, \overline{s})^n$, $z \in \mathcal{Z}$ and $i \in \{1, \ldots, n\}$, at the market-clearing price $p = p^*(s, z)$, $x_i(p; s_i, z_i)$ is equal to that given by Proposition 1.*

Proof. See Section A.2. □

For any fixed s_i and z_i, the uniqueness of $x_i(p; s_i, z_i)$ in Proposition 2 applies only to market-clearing prices, i.e., $p = p^*(s_i, s_{-i}, z_i, z_{-i})$ for some $s_{-i} \in (\underline{s}, \overline{s})^{n-1}$ and $(z_i, z_{-i}) \in Z$, since the demands at non-market-clearing prices do not satisfy any optimality condition.

The proof of Proposition 2 is relatively involved, but its intuition is simple. For strategies to be ex post optimal, each bidder must be able to calculate an one-dimensional sufficient statistic of other bidders' signals from variables that he observes—the equilibrium allocation and price. Because the equilibrium allocations $\{x_i(p^*; s_i, z_i)\}$ satisfy the linear constraint $\sum_{i=1}^n x_i(p^*; s_i, z_i) = S$, and because valuations $\{v_i\}$ are linear in the signals $\{s_i\}$, it is natural to conjecture that the ex post equilibrium condition holds only if each bidder's demand is linear in his signal and the price. The main theme of the proof of Proposition 2 is to establish this linearity. As we discussed in the introduction, the uniqueness property makes the ex post equilibrium particularly appealing in uniform-price auctions, which usually admit a continuum of Bayesian-Nash equilibria (Wilson 1979).

2.4 Information Aggregation and Robustness

Information aggregation is an important property of the ex post equilibrium in Proposition 1. Equation (11) reveals that the equilibrium p^* aggregates the average signal $\sum_{i=1}^n s_i/n$, or equivalently the average valuation $\sum_{i=1}^n v_i/n$, even though the demand schedule of each bidder depends only on his own signal. In the special case of $S = 0$, i.e. if bidders only trade among themselves, the market-clearing price p^* is exactly equal to the average signal $\sum_{i=1}^{n} s_i/n$. Information aggregation in the ex post equilibrium applies to double auction with a finite number n of bidders, whereas many prior models of information aggregation rely on large markets, as in Wilson (1977), Milgrom (1979), and Kremer (2002), and Reny and Perry (2006), Kazumori (2012), among others. While Kyle (1985), Kyle (1989), Vives (2011), Ostrovsky (2011), and Rostek and Weretka (2012) also have information aggregation with a finite number of agents, their equilibria are Bayesian and rely on the normal distribution. Our ex post equilibrium, by contrast, does not hinge upon normality or any other distribution assumption of the signals.

Robustness is another key feature of the equilibrium of Proposition 1. For example, the ex post equilibrium does not require bidders to have common knowledge about the signal distributions. Nor does the ex post equilibrium rely on any particular implementation of the double auction, such as whether the bids are observable, as long as the implementation method does not change bidders' preferences.[8] Therefore, the ex post equilibrium is consistent with the Wilson criterion that desirable properties of a trading model include its robustness to common-knowledge assumptions and implementation details (Wilson, 1987).

The ex post equilibrium of Proposition 1 has yet another advantage of being less sensitive to preferences than Bayesian equilibria are. Clearly, maximizing a bidder's ex post utility U in equation (3) is equivalent—in terms of equilibrium strategies, prices and allocations—to maximizing a strictly increasing function of his ex post utility. In other words, our ex post equilibrium in a static double auction (this section and Section 3) remains an ex post equilibrium given utility function of the form $f(U(\,\cdot\,))$, where f is a strictly increasing function, and U is the original utility function (3). By contrast, in a Bayesian equilibrium and for an arbitrary increasing function f, the optimal strategy that maximizes a bidder's expected utility under the original preference, $\mathbb{E}[U(\,\cdot\,)]$, may not maximize his expected utility under the alternative preference, $\mathbb{E}[f(U(\,\cdot\,))]$, because $\mathbb{E}[U'(\,\cdot\,)f'(U(\,\cdot\,))] \neq \mathbb{E}[U'(\,\cdot\,)]\mathbb{E}[f'(U(\,\cdot\,))]$ in general (i.e., the two marginal utilities can be correlated under uncertainty). Compared with Bayesian equilibrium, therefore, an ex post equilibrium is less sensitive to assumptions on preferences and can be more appealing for practical applications.

[8] In a laboratory market, Bloomfield, O'Hara, and Saar (2011) find that market outcomes in terms of price discovery and liquidity do not vary significantly with transparency, namely whether agents observe others' demand schedules.

There are two main differences between the ex post equilibrium of Proposition 1 and rational expectation equilibria (REE) under asymmetric information (Grossman, 1976, 1981). First, strategies in the ex post equilibrium are optimal for each realization of the *n-dimensional signal* profile $(s_1, s_2, \ldots, s_n)$, whereas strategies in REE models are optimal for each realization of the *one-dimensional* equilibrium price. Because each market-clearing price corresponds to multiple possible signal profiles, the ex post optimality of this paper seems to be a sharper notation of information aggregation than the Bayesian optimality in REE models. Second, consistent with the Milgrom 1981 critique of REE models, the double-auction mechanism of this paper provides an explicit formulation of the price-formation process.

2.5 Efficiency

We now study the efficiency of the ex post equilibrium in Proposition 1. For a fixed profile of signals $(s_1, \ldots, s_n)$, the ex post efficient allocation, $\{q_i^e\}$, maximizes the social welfare:

$$\max_{\{q_i\}} \sum_{i=1}^{n} \left(v_i(z_i + q_i) - \frac{\lambda}{2}(z_i + q_i)^2 \right) \text{ subject to: } \sum_{i=1}^{n} q_i = S.$$

For each bidder i, the (ex post) efficient allocation, $\{q_i^e\}$, and the allocation in the ex post equilibrium, $\{q_i^*\}$, are given by

$$q_i^e + z_i = \frac{n\alpha - 1}{\lambda(n-1)} \left(s_i - \frac{1}{n} \sum_{j=1}^{n} s_j \right) + \frac{1}{n}(S + Z), \tag{12}$$

$$q_i^* + z_i = \frac{n\alpha - 2}{\lambda(n-1)} \left(s_i - \frac{1}{n} \sum_{j=1}^{n} s_j \right) + \frac{1}{n}(S + Z) + \frac{1}{n\alpha - 1} \left(z_i - \frac{1}{n} Z \right). \tag{13}$$

Comparing (12) and (13), we see that in both cases allocations are increasing in signals; the rate of this increase is less in the ex post equilibrium allocation. This feature is the familiar demand reduction in multi-unit auctions (see, for example, Ausubel et al. 2011). The ex post equilibrium allocation in (13) is also corrected by an extra $(z_i - Z/n)$ term, in comparison with the ex post efficient allocation in (12). This extra term indicates that the allocation in the ex post equilibrium depends not only on the heterogeneity of information, but also on the heterogeneity of existing inventories. As $n \to \infty$ and holding α constant, we see that the difference between the efficient and the ex post equilibrium allocations vanishes. However, when n is large but α is small (that is, a large weight on the common-value component), we see that the difference between the efficient and the ex post equilibrium allocations can still be substantial due to the $(z_i - Z/n)$ term.

We can define allocative inefficiency of the one-shot double auction as the difference between the total utility associated with the efficient allocation and the total utility associated with the ex post equilibrium allocation. Using Lemma 3 in Section A.7, we calculate the allocative inefficiency as:

$$\sum_{i=1}^{n}\left(v_i(z_i+q_i^e) - \frac{\lambda}{2}(z_i+q_i^e)^2\right) - \sum_{i=1}^{n}\left(v_i(z_i+q_i^*) - \frac{\lambda}{2}(z_i+q_i^*)^2\right) \tag{14}$$

$$= \frac{1}{2\lambda(n-1)^2}\sum_{i=1}^{n}\left(s_i - \frac{1}{n}\sum_{j=1}^{n}s_j\right)^2 + \frac{\lambda}{2(n\alpha-1)^2}\sum_{i=1}^{n}\left(z_i - \frac{Z}{n}\right)^2$$

$$- \frac{1}{(n-1)(n\alpha-1)}\sum_{i=1}^{n}\left(s_i - \frac{1}{n}\sum_{j=1}^{n}s_j\right)\left(z_i - \frac{Z}{n}\right) \tag{15}$$

In this calculation, we have assumed that the total revenues $p^e\sum_{i=1}^{n}q_i^e$ and $p^*\sum_{i=1}^{n}q_i^*$ enter linearly into the utility function of the agent who provides the exogenous supply S. Thus, all payments have a zero effect on total utility. Moreover, in the calculation we ignore the fixed cost of the agent for providing the supply S, which is presumably independent of bidders' allocations.

Equation (14) shows that the allocative inefficiency increases with the variance of signals and with the variance of inventories. Moreover, the allocative inefficiency decreases with the covariance of signals and inventories; this is consistent with intuition: if the higher signaled bidder tends to hold larger inventory, then the allocative inefficiency should be small. Finally, fixing α and assuming that the law of large number holds for signals and inventories, we see that the allocative inefficiency vanishes at a rate of $O(1/n)$ as n tends to infinity. This rate of convergence is same as the one in Rustichini, Satterthwaite, and Williams (1994) on double auction of a single indivisible asset. In Section 4 we show that, for a fixed number n of bidders, a sequence of double auctions achieves *exponential* convergence to the efficient allocation, as the number of auction rounds increases.

3 Extensions

3.1 Auctions of Multiple Assets

In this subsection we extend the analysis of ex post equilibrium to a simultaneous double auction of multiple assets. For example, NYSE's "program trading" allows simultaneous purchase or sale of more than 15 stocks; this trading method now accounts for about 25% of NYSE's trading volume.[9] In derivatives markets, the default of a member of a clearinghouse can often be resolved by auctioning the defaulting member's derivative portfolios to non-defaulting members.[10] In addition to bolstering the basic intuition of Proposition 1, this subsection provides additional insight regarding how the complementarity and substitutability among multiple assets affect the bidding strategies.

Suppose that there are $m \geq 2$ distinct assets. Bidder i receives a vector of private signals $\vec{s_i} \equiv (s_{i,1}, \ldots, s_{i,m})' \in (\underline{s}, \overline{s})^m$ and values asset k $(1 \leq k \leq m)$ at

$$v_{i,k} = \alpha_k \, s_{i,k} + (1 - \alpha_k) \, \frac{1}{n-1} \sum_{j \neq i} s_{j,k}, \tag{16}$$

where α_k is a known constant. Moreover, bidder i has an inventory $z_{i,k}$ of asset k. The inventory vector $\vec{z_i} = (z_{i,1}, \ldots, z_{i,m})'$ is bidder i's private information. As before, the total ex-ante vector of inventory,

$$\sum_{i=1}^{n} \vec{z_i} = \vec{Z}, \tag{17}$$

is common knowledge, where $\vec{Z} \equiv (Z_1, \ldots, Z_m)'$ is a constant vector.

Again, the joint probability distribution of $(\vec{s_1}, \ldots, \vec{s_n})'$ and $(\vec{z_1}, \ldots, \vec{z_n})'$ are inconsequential because we focus on ex post equilibrium. Let $\vec{\alpha} \equiv (\alpha_1, \ldots, \alpha_m)'$.

With multiple assets, bidder i's utility after acquiring $\vec{q_i} \equiv (q_{i,1}, \ldots, q_{i,m})'$ units of

[9] See https://usequities.nyx.com/markets/program-trading for more details. The word "program" in "program trading" does not mean that trading is done by a computer program.

[10] For example, see http://www.swapclear.com/service/default-management.html and http://www.eurexclearing.com/standalone/pdf/143818/default_management_process.pdf for details of default management processes at SwapClear and Eurex Clearing.

assets at the price vector $\vec{p} \equiv (p_1, \ldots, p_m)'$ is

$$U(\vec{q_i}, \vec{p}; \vec{v_i}, \vec{z_i}) = \sum_{k=1}^{m} v_{i,k} z_{i,k} + \sum_{k=1}^{m} (v_{i,k} - p_k) q_{i,k} - \frac{1}{2} \sum_{k=1}^{m} \sum_{l=1}^{m} (z_{i,k} + q_{i,k}) \Lambda_{k,l} (z_{i,l} + q_{i,l})$$

$$\equiv \vec{v_i} \cdot \vec{z_i} + (\vec{v_i} - \vec{p}) \cdot \vec{q_i} - \frac{1}{2} (\vec{z_i} + \vec{q_i})' \mathbf{\Lambda} (\vec{z_i} + \vec{q_i}), \tag{18}$$

where $\vec{v_i} \equiv (v_{i,1}, \ldots, v_{i,m})'$ is the vector of bidder i's valuations and $\mathbf{\Lambda} \equiv \{\Lambda_{k,l}\}$ is a symmetric, positive definite matrix. The matrix $\mathbf{\Lambda}$ captures the complementarity and substitutability among the assets. For example, a negative $\Lambda_{k,l}$ indicates that asset k and asset l are complements because holding one of them increases the marginal valuation of holding the other.

In this double auction, each bidder i simultaneously bids on all assets by submitting a demand schedule vector $\vec{x_i}(\vec{p}) \equiv (x_{i,1}(\vec{p}), \ldots, x_{i,m}(\vec{p}))'$. Bidder i's strategy is thus $\vec{x_i}(\vec{p}; \vec{s_i}, \vec{z_i})$. Due to the complementarity and substitutability among assets, bidder i's demand for any given asset can depend on the prices of all assets. The market-clearing price vector $\vec{p^*} \equiv (p_1^*, \ldots, p_m^*)'$ is determined such that, for each asset $k \in \{1, \ldots, m\}$ that has the supply S_k,

$$\sum_{i=1}^{n} x_{i,k}(\vec{p^*}; \vec{s_i}, \vec{z_i}) = S_k. \tag{19}$$

We denote by $\vec{S} \equiv (S_1, \ldots, S_m)'$ the vector of asset supplies.

In an ex post equilibrium of this multi-asset auction, bidder i's demand schedule vector $\vec{x_i}$, which depends only on his own signal vector $\vec{s_i}$ and inventory vector $\vec{z_i}$, is optimal even if he learns all other bidders' signal and inventory vectors ex post. We now characterize such an ex post equilibrium in the following proposition, where we denote by $\mathrm{Diag}(\vec{a})$ the diagonal matrix whose diagonal vector is $\vec{a}$, and by $\frac{n\vec{\alpha}-1}{n(n\vec{\alpha}-2)}$ the vector whose k-th component is $\frac{n\alpha_k - 1}{n(n\alpha_k - 2)}$.

Proposition 3. *Suppose that $n\alpha_k > 2$ for every $k \in \{1, \ldots, m\}$. In a double auction with multiple assets and interdependent values, there exists an ex post equilibrium in* *which bidder i submits the demand schedule vector*

$$\vec{x_i}(\vec{p};\vec{s_i},\vec{z_i}) = \boldsymbol{\Lambda}^{-1}\operatorname{Diag}\left(\frac{n\vec{\alpha}-2}{n-1}\right)(\vec{s_i}-\vec{p}) + \boldsymbol{\Lambda}^{-1}\operatorname{Diag}\left(\frac{1-\vec{\alpha}}{n-1}\right)\boldsymbol{\Lambda}\vec{S} \\ - \boldsymbol{\Lambda}^{-1}\operatorname{Diag}\left(\frac{n\vec{\alpha}-2}{n\vec{\alpha}-1}\right)\boldsymbol{\Lambda}\vec{z_i} + \boldsymbol{\Lambda}^{-1}\operatorname{Diag}\left(\frac{(n\vec{\alpha}-2)(1-\vec{\alpha})}{(n\vec{\alpha}-1)(n-1)}\right)\boldsymbol{\Lambda}Z, \tag{20}$$

and the equilibrium price vector is

$$\vec{p^*} = \frac{1}{n}\sum_{i=1}^{n}\vec{s_i} - \operatorname{Diag}\left(\frac{n\vec{\alpha}-1}{n(n\vec{\alpha}-2)}\right)\boldsymbol{\Lambda}\vec{S} - \frac{1}{n}\boldsymbol{\Lambda}\vec{Z}. \tag{21}$$

Proof. See Section A.3. □

Proposition 3 reveals that a bidder's equilibrium demand for any asset can depend on his signals, prices and inventories on all other assets. This interdependence of strategies is a natural consequence of the complementarity and substitutability among multiple assets. And similar to Proposition 1, the equilibrium price vector (21) aggregates bidders' dispersed information on all assets and is independent of any distributional assumption about the signals and inventories.

3.2 Heterogeneous λ's under Private Values

In this subsection we explore equilibrium bidding strategies if bidders have different declining rates of marginal valuations, i.e., different λ's. We focus on the case of pure private values (i.e. $\alpha = 1$). Private values are common in the auction literature and are reasonable in many applications. For example, the value of a commodity can be specific to a firm's production function, just as the value of a treasury security or swap contract can be specific to an investor's hedging demand.

We let λ_i be the declining rate of bidder i's marginal valuation, where the profile $\{\lambda_i\}_{i=1}^n$ is common knowledge. We can interpret the heterogenous λ_i as heterogenous risk aversion or heterogeneous diminishing returns in production functions. We work with private valuations with $\alpha = 1$ and $v_i = s_i$. Other aspects of the model is the same as the one in Section 2.1. Thus, bidder i's utility is

$$U_i(q_i, p; v_i, z_i) = v_i z_i + (v_i - p)q_i - \frac{1}{2}\lambda_i(z_i + q_i)^2. \tag{22}$$

Each bidder submits a demand schedule $x(p; v_i, z_i)$. As in Section 2, our objective is to find an ex post equilibrium.

Proposition 4. *Suppose that $n > 2$. In a double auction with private values and private inventories, there exists an ex post equilibrium in which bidder i submits the demand schedule*

$$x_i(p; v_i, z_i) = b_i(v_i - p - \lambda_i z_i), \tag{23}$$

where

$$b_i = \frac{2 + \lambda_i B - \sqrt{\lambda_i^2 B^2 + 4}}{2\lambda_i}, \tag{24}$$

and where $B \equiv \sum_{i=1}^n b_i$ is the unique positive solution to the equation

$$B = \sum_{i=1}^n \frac{2 + \lambda_i B - \sqrt{\lambda_i^2 B^2 + 4}}{2\lambda_i}. \tag{25}$$

The equilibrium price of the double auction is

$$p^* = \frac{\sum_{i=1}^n b_i(v_i - \lambda_i z_i) - S}{\sum_{i=1}^n b_i}. \tag{26}$$

Proof. See Section A.4. □

Note that the equilibrium demand schedules x_i in (23) is independent of the supply S and the total inventory Z. Therefore, Proposition 4 remains an equilibrium even if bidders face uncertainties regarding S and Z. This feature is reminiscent to Klemperer and Meyer (1989), who characterize supply function equilibria that are ex post optimal with respect to demand shocks. In their model, however, bidders's marginal values are *common knowledge*. Similarly, in a setting with a *commonly known* asset value, Ausubel, Cramton, Pycia, Rostek, and Weretka (2011) characterize an ex post equilibrium with uncertain supply.

The final price p^* with heterogenous $\{\lambda_i\}$ is the weighted average of the marginal values $v_i - \lambda z_i$, adjusted for the external supply S. The smaller is λ_i, the larger is b_i, and the more influence bidder i has on the final price.

4 Dynamic Trading

In this section we study dynamic trading in a market with stochastic arrivals of new information and an infinite sequence of uniform-price double auctions. We characterize a stationary ex post equilibrium, demonstrate its efficiency properties, and study comparative statics as we vary the frequency of trading.

The clock time is continuous. Trading occurs in repeated rounds of double auctions at each clock time in $\{0, \Delta, 2\Delta, 3\Delta, \ldots\}$, where $\Delta > 0$ is the length of clock time between consecutive rounds of trading. The smaller is Δ, the higher is the frequency of trading. (We later discuss the limiting behavior of the market as $\Delta \to 0$ and as $\Delta \to \infty$; the later case reduces to the static model of Section 2.) Bidders have a discounting factor of $e^{-r\tau}$ at the clock time τ, where $r > 0$ is the discount rate per unit of clock time. We will refer to each trading round as a "period," indexed by $t \in \{0, 1, 2, \ldots\}$, so the period-$t$ auction occurs at the clock time $t\Delta$. We will use the letter τ to denote a generic clock time.

Signals arrive stochastically. For each bidder i, his signals $\{s_{i,\tau}\}_{\tau \geq 0}$ follow a continuous-time martingale. That is, for every i and $\tau' > \tau \geq 0$,

$$\mathbb{E}[s_{i,\tau'} \mid \{s_{j,\tau''}\}_{1 \leq j \leq n, 0 \leq \tau'' \leq \tau}] = s_{i,\tau}. \tag{27}$$

Under the martingale assumption, bidder i's current signal $s_{i,\tau}$ is the best estimate of his future signals. As long as this martingale property is satisfied, the exact detail of the signal processes is inconsequential to our equilibrium analysis. For example, future signals can arrive continuously and follow a diffusion process; or, they can arrive in discrete, irregular intervals, in which case the signal process exhibits "jumps." Each bidder's signal process can have arbitrary autocorrelation and conditional variance, and any pair of signal processes, $\{s_{i,\tau}\}_{\tau \geq 0}$ and $\{s_{j,\tau}\}_{\tau \geq 0}$, for $i \neq j$, can have arbitrary conditional covariance. The realizations of bidder i's signal process are bidder i's private information.

In each period $t \geq 0$, a new uniform-price double auction is held to reallocate the asset among the bidders. At the clock time 0, which is also the trading time of the first auction, each bidder i starts with a private inventory of $z_{i,0}$ of the asset, where $z_0 \in \mathcal{Z} \subset \mathbb{R}^n$. The initial total inventory $Z = \sum_{i=1}^{n} z_{i,0}$ is common knowledge as before. The external supply S is zero in each trading period, which implies that the total inventory in each period $t \geq 1$ is also Z. In the period-t auction, bidder i starts with an inventory of $z_{i,t\Delta}$ and submits a demand schedule $x_{i,t\Delta}(p)$. The auctioneer determines the market-clearing price $p^*_{t\Delta}$ by

$$\sum_{i=1}^{n} x_{i,t\Delta}(p^*_{t\Delta}) = 0, \tag{28}$$

and bidder i receives $q_{i,t\Delta} = x_{i,t\Delta}(p^*_{t\Delta})$ units of the asset at the price of $p^*_{t\Delta}$. Inventories therefore evolve according to

$$z_{i,(t+1)\Delta} = z_{i,t\Delta} + q_{i,t\Delta}. \tag{29}$$

Bidder i's inventory history is his private information.

After describing the information structure and trading protocol, we now turn to the preferences. Given the new quantity $q_{i,t\Delta}$ in period t, bidder i's "flow" utility (not counting the price) in period t is

$$v_{i,t\Delta}(z_{i,t\Delta} + q_{i,t\Delta}) - \frac{\lambda}{2}(q_{i,t\Delta} + z_{i,t\Delta})^2,$$

where the value is interdependent:

$$v_{i,t\Delta} = \alpha\, s_{i,t\Delta} + (1-\alpha)\,\frac{1}{n-1}\sum_{j\neq i} s_{j,t\Delta}, \tag{30}$$

and $\alpha \in (0,1]$ is a constant known to all bidders. Thus, bidder i's utility in period t alone is the integral of time-discounted flow utility less the one-off payment of asset transaction, i.e.,

$$\begin{aligned}
&U(q_{i,t\Delta}, p^*_{t\Delta}; v_{i,t\Delta}, z_{i,t\Delta}) \\
&= \int_{\tau=0}^{\Delta} e^{-\tau r}\left(v_{i,t\Delta}(z_{i,t\Delta} + q_{i,t\Delta}) - \frac{1}{2}\lambda(q_{i,t\Delta} + z_{i,t\Delta})^2\right) d\tau - p^*_{t\Delta}\, q_{i,t\Delta} \\
&= \frac{1-e^{-r\Delta}}{r}\left(v_{i,t\Delta}(z_{i,t\Delta} + q_{i,t\Delta}) - \frac{1}{2}\lambda(q_{i,t\Delta} + z_{i,t\Delta})^2\right) - p^*_{t\Delta}\, q_{i,t\Delta}.
\end{aligned} \tag{31}$$

Note that bidder i's flow value in period t, $v_{i,t\Delta}$, depends only on the profile of signals at the clock time $t\Delta$, $\{s_{j,t\Delta}\}_{j=1}^{n}$. This valuation structure is natural in markets where a bidder's information about his valuation improves over time (and thus a later signal subsumes an earlier one).[11]

Bidder i's overall utility, or "continuation value," at the clock time $t\Delta$ (including the period-t auction) is

$$\begin{aligned} V_{i,t\Delta} &= \sum_{t'=t}^{\infty} e^{-r(t'-t)\Delta} U(q_{i,t'\Delta}, p^*_{t'\Delta}; v_{i,t'\Delta}, z_{i,t'\Delta}) \\ &= U(q_{i,t\Delta}, p^*_{t\Delta}; v_{i,t\Delta}, z_{i,t\Delta}) + e^{-r\Delta} V_{i,(t+1)\Delta}. \end{aligned} \tag{32}$$

We emphasize that in period t before the new auction is held, bidder i's information consists of the paths of his signals $\{s_{i,t'\Delta}\}_{t'\leq t}$ and of his inventories $\{z_{i,t'\Delta}\}_{t'\leq t}$, as well as his submitted demand schedules $\{x_{i,t'\Delta}(p)\}_{0\leq t'<t}$. For notational simplicity, we let bidder i's information set at the beginning of period t be

$$H_{i,t\Delta} = \left\{\{s_{i,t'\Delta}\}_{0\leq t'\leq t}, \{z_{i,t'\Delta}\}_{0\leq t'\leq t}, \{x_{i,t'\Delta}(p)\}_{0\leq t'<t}\right\}. \tag{33}$$

Notice that by the identity $z_{i,(t'+1)\Delta} - z_{i,t'\Delta} = q_{i,t'\Delta} = x_{i,t'\Delta}(p^*_{t'\Delta})$, a bidder can infer the previous price path $\{p^*_{t'\Delta}\}_{t'<t}$ from his history $H_{i,t\Delta}$. Bidder i's period-t strategy, $x_{i,t\Delta} = x_{i,t\Delta}(p; H_{i,t\Delta})$, is measurable with respect to $H_{i,t\Delta}$.

In this dynamic market we use the notion of periodic ex post equilibrium introduced by Bergemann and Valimaki (2010). In this notion of ex post equilibrium, for any period t each bidder's strategy is ex post optimal with respect to other bidders' histories up to period t, but is Bayesian optimal with respect to signals in the future. This equilibrium is "ex post" because, in the absence of new information immediately after the period-t auction, each bidder still has no regret.

Definition 2. A **periodic ex post equilibrium** consists of the strategy profile $\{x_{j,t\Delta}\}_{1\leq j\leq n, t\geq 0}$ such that for every bidder i and for every path of his history $H_{i,t}$, bidder i has no incentive to deviate from $\{x_{i,t'\Delta}\}_{t'\geq t}$ even if he learns the profile of other bidders' histories. That is, for every alternative strategy $\{\tilde{x}_{i,t'\Delta}\}_{t'\geq t}$ and every

[11] In principal, a new signal may arrive between two trading clock times $t\Delta$ and $(t+1)\Delta$. Given the martingale property, however,

$$\mathbb{E}[v_{i,\tau} \mid \{s_{j,\tau'}\}_{1\leq j\leq n, \tau'\leq t\Delta}] = v_{i,t\Delta}$$

for all $\tau \in (t\Delta, (t+1)\Delta)$. Thus, the specification of flow utility is almost without loss of generality.

profile of other bidders' histories $\{H_{j,t\Delta}\}_{j\neq i}$,

$$\begin{aligned}&\mathbb{E}[V_{i,t\Delta}(\{x_{i,t'\Delta}\}_{t'\geq t},\{x_{j,t\Delta}\}_{j\neq i,t'\geq t})\mid H_{i,t\Delta},\{H_{j,t\Delta}\}_{j\neq i}]\\ \geq\,&\mathbb{E}[V_{i,t\Delta}(\{\tilde{x}_{i,t'\Delta}\}_{t'\geq t},\{x_{j,t\Delta}\}_{j\neq i,t'\geq t})\mid H_{i,t\Delta},\{H_{j,t\Delta}\}_{j\neq i}],\end{aligned}$$

where the expectations are taken over all possible realizations of future signals $\{s_{j,\tau}\}_{1\leq j\leq n,\tau>t\Delta}$.

We now characterize a periodic ex post equilibrium. This equilibrium is stationary, that is, a bidder's strategy only depends on his current signal and current level of inventory, but does not depend explicitly on t.

Proposition 5. *Suppose that $n\alpha>2$, $\Delta>0$ and $r>0$. In the market with dynamic trading, there exists a stationary periodic ex post equilibrium in which bidder i submits the demand schedule*

$$x_{i,t\Delta}(p;s_{i,t\Delta},z_{i,t\Delta})=a\left(s_{i,t\Delta}-rp-\frac{\lambda(n-1)}{n\alpha-1}z_{i,t\Delta}+\frac{\lambda(1-\alpha)}{n\alpha-1}Z\right),\qquad(34)$$

where

$$a=\frac{n\alpha-1}{2(n-1)e^{-r\Delta}\lambda}\left((n\alpha-1)(1-e^{-r\Delta})+2e^{-r\Delta}-\sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2+4e^{-r\Delta}}\right)>0.\qquad(35)$$

The period-t equilibrium price is

$$p^*_{t\Delta}=\frac{1}{r}\left(\frac{1}{n}\sum_{i=1}^{n}s_{i,t\Delta}-\frac{\lambda}{n}Z\right).\qquad(36)$$

Proof. See Section A.5. □

With dynamic trading, an asset purchased in period t gives a bidder a stream of utilities during the clock time $\tau\in(t\Delta,\infty)$. Thus, the equilibrium price $p^*_{t\Delta}$ under dynamic trading is adjusted by a factor of $\int_{\tau=0}^{\infty}e^{-r\tau}\,d\tau=1/r$. In every period, the equilibrium price $p^*_{t\Delta}$ aggregates the current information on the value of the asset. Although bidders learn from $p^*_{t\Delta}$ the average signal $\sum_i s_{i,t\Delta}/n$ in period t, new information may arrive by the clock time $(t+1)\Delta$ of the next auction. Therefore, a period-$(t+1)$ strategy that depends explicitly on the lagged price $p^*_{t\Delta}$ is generally not ex post optimal. Moreover, since the signal processes are martingales, the equilibrium prices $\{p^*_{t\Delta}\}_{t\geq 0}$ also form a martingale.

The next proposition characterizes the allocative efficiency in the periodic ex post equilibrium of Proposition 5. Let us use $\{z^*_{i,t\Delta}\}$ to denote the path of inventories obtained by the periodic ex post equilibrium.

Proposition 6. *Given any* $0 \leq \underline{t} < \bar{t}$*, if* $s_{i,t\Delta} = s_{i,\underline{t}\Delta}$ *for all* i *and all* $t \in \{\underline{t}, \underline{t}+1, \ldots, \bar{t}\}$*, then*

$$z^*_{i,t\Delta} - z^e_{i,\underline{t}\Delta} = (1+d)^{t-\underline{t}}(z^*_{i,\underline{t}\Delta} - z^e_{i,\underline{t}\Delta}), \tag{37}$$

where

$$z^e_{i,\underline{t}\Delta} = \frac{n\alpha - 1}{\lambda(n-1)}\left(s_{i,\underline{t}\Delta} - \frac{1}{n}\sum_{j=1}^{n} s_{j,\underline{t}\Delta}\right) + \frac{1}{n}Z, \tag{38}$$

is the the efficient allocation in period $\underline{t}$*, and*

$$1 + d = \frac{1}{2e^{-r\Delta}}\left(\sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}} - (n\alpha-1)(1-e^{-r\Delta})\right) \in (0,1). \tag{39}$$

Moreover, let us define the rate of convergence to efficiency per unit of clock time to be $-\log[(1+d)^{1/\Delta}]$*. This convergence rate is increasing with the number* n *of bidders, the weight* α *of the private components in bidders' valuations, the discount rate* r*, and the clock-time frequency of trading* $1/\Delta$*.*

Proof. It is easy to verify that the allocation $\{z^e_{i,\underline{t}\Delta}\}$ solves

$$\max_{\{z_i\}} \sum_{i=1}^{n}\left(v_{i,\underline{t}\Delta}\, z_i - \frac{\lambda}{2}(z_i)^2\right) \text{ subject to: } \sum_{i=1}^{n} z_i = Z.$$

The convergence result and comparative statics are proved in Section A.6. ☐

Proposition 6 reveals that a sequence of double auctions serves as an effective method to dynamically achieve allocative efficiency. Allocations under the periodic ex post equilibrium converge exponentially in time to the efficient one, as determined by the most recent signals. Once new signals arrive, the efficient allocation changes accordingly, and allocations under the periodic ex post equilibrium start to converge toward the new efficient level. This convergence result complements Rustichini, Satterthwaite, and Williams (1994), Cripps and Swinkels (2006), and Reny and Perry (2006), among others, who show that allocations in a one-shot double auction converge, at a polynomial rate, to the efficient level as the number of bidders increases.

The intuition for the comparative statics of Proposition 6 is simple. A larger n makes bidders more competitive, and a larger r makes them more impatient. Both effects encourage aggressive bidding and speed up convergence. The effect of α is slightly more subtle. Intuitively, the interdependence of valuations, represented by $1-\alpha$, creates adverse selection for the bidders. To protect themselves from trading losses, bidders reduce their demand or supply relative to the fully competitive market. The higher is α, the more bidders care about the private components of their valuations, and the less they worry about adverse selection. Therefore, a higher α implies more aggressive bidding and faster convergence to the efficient allocation. Finally, a higher trading frequency increases the convergence speed in clock time, even though it makes bidders more patient and thus less aggressive in each trading period.

The dynamic ex post equilibrium of Proposition 5 differs in several aspects from those of Kyle (1985) and Kyle (1989), who model the trading behavior of informed speculator(s) in the presence of noise traders. First, while Kyle (1985, 1989) rely on the normal distribution to derive Bayesian equilibria, the equilibrium of Proposition 5 is ex post optimal and thus robust for any distribution of signals. Second, because we do not rely on noise traders to generate trades, the economic implications of allocative efficiency and welfare are more transparent in the ex post equilibrium. Third, the equilibrium price in our dynamic model immediately reflects the average current signals of all bidders, whereas prices in the Kyle (1985) model gradually reveal the information of the informed speculator over time. This last feature of Kyle (1985) can be attributed to noise traders, who provide camouflage to informed speculators.

4.1 Low and High Trading Frequency Limits

In this subsection we examine the limit of the equilibrium in Proposition 5 as $\Delta \to \infty$ (trading frequency tending to zero) and as $\Delta \to 0$ (trading frequency tending to infinity).

One can easily show that the constant a in Equation (35) of Proposition 5 tends to

$$\lim_{\Delta \to \infty} a = \frac{n\alpha - 2}{\lambda(n-1)}, \tag{40}$$

which is the coefficient in the equilibrium of Proposition 1. Thus, after we adjust s_i and λ in Proposition 1 to s_i/r and λ/r to account for the accumulation of time-discounted flow utilities, we have that as $\Delta \to \infty$, the equilibrium in Proposition 5 converges to the equilibrium in Proposition 1 with $S = 0$. This convergence is intuitive: if the interval between Δ trading is large, then trading in period 0 is essentially the only round of trading, and hence the bidding behavior in period 0 should be similar to that in static trading.

We now study the other extreme, namely as trading becomes infinitely frequent in clock time. By letting $\Delta \to 0$ in the equilibrium of Proposition 5 and using L'Hospital's rule, we obtain the following limiting equilibrium in continuous time.

Proposition 7. *Suppose that $n\alpha > 2$ and $r > 0$. As $\Delta \to 0$, the equilibrium of Proposition 5 converges to the following ex post equilibrium:*

1. *Bidder i's equilibrium strategy is represented by a process $\{x^{\infty}_{i,\tau}\}_{\tau \in \mathbb{R}^+}$. At the clock time τ, $x^{\infty}_{i,\tau}$ specifies bidder i's* rate *of order submission and is defined by*

$$x^{\infty}_{i,\tau}(p; s_{i,\tau}, z_{i,\tau}) = a^{\infty} \left(s_{i,\tau} - rp - \frac{\lambda(n-1)}{n\alpha - 1} z_{i,\tau} + \frac{\lambda(1-\alpha)}{n\alpha - 1} Z \right), \tag{41}$$

   where

$$a^{\infty} = \frac{(n\alpha - 1)(n\alpha - 2)r}{2\lambda(n-1)}. \tag{42}$$

   Given a clock time $T > 0$, in equilibrium the total amount of trading by bidder i in the clock-time interval $[0, T]$ is

$$z^*_{i,T} - z_{i,0} = \int_{\tau=0}^{T} x^{\infty}_{i,\tau}(p^*_{\tau}; s_{i,\tau}, z^*_{i,\tau})\, d\tau. \tag{43}$$

2. *The equilibrium price at any clock time τ is*

$$p^*_{\tau} = \frac{1}{r} \left(\frac{1}{n} \sum_{i=1}^{n} s_{i,\tau} - \frac{\lambda}{n} Z \right). \tag{44}$$

3. *Given any $0 \leq \underline{\tau} < \overline{\tau}$, if $s_{i,\tau} = s_{i,\underline{\tau}}$ for all i and all $\tau \in [\underline{\tau}, \overline{\tau}]$, then*

$$z^*_{i,\tau} - z^e_{i,\underline{\tau}} = e^{-\frac{1}{2} r(n\alpha - 2)(\tau - \underline{\tau})} \left(z^*_{i,\underline{\tau}} - z^e_{i,\underline{\tau}} \right), \tag{45}$$

   where

$$z^e_{i,\underline{\tau}} = \frac{n\alpha - 1}{\lambda(n-1)} \left(s_{i,\underline{\tau}} - \frac{1}{n} \sum_{j=1}^{n} s_{j,\underline{\tau}} \right) + \frac{1}{n} Z \tag{46}$$

is the the efficient allocation at clock time $\underline{\tau}$.

Proposition 7 reveals that even if all information arrives at the very beginning and if trading occurs continually, in equilibrium the efficient allocation is not reached instantaneously. The delay comes from bidders' price impact and associated demand reduction. Although submitting aggressive orders allows a bidder to achieve his desired allocation sooner, aggressive bidding also moves the price against the bidder and increases his trading cost. Facing this tradeoff, each bidders uses a finite rate of order submission in the limit. Consistent with Proposition 6, the rate of convergence to efficiency in Proposition 7, $r(n\alpha - 2)/2$, is increasing in the number of bidders n, the discount rate r, and the weight α of the private components in bidders' valuations.

4.2 Welfare and Optimal Trading Frequency

In this subsection we study the effect of trading frequency on welfare and characterize the optimal trading frequency, $1/\Delta$. We show that the optimal trading frequency depends critically on the nature of information (i.e., the signals). If new information arrives at deterministic times, then slow, batch trading (i.e., a large Δ) tends to be optimal. If new information arrives at stochastic times, then fast, continuous trading (i.e., a small Δ) tends to be optimal. Our primary objective in this subsection is to demonstrate the intuition through a simplistic but useful special case of our dynamic trading model, and our results here may serve as building blocks for future research.

We suppose that bidders enter the market at time zero with the initial inventory profile $\{z^e_{i,0}\}$, which are efficient given the time-0 signal profile $\{s_{i,0}\}$:

$$z^e_{i,0} = \frac{n\alpha - 1}{\lambda(n-1)} \left(s_{i,0} - \frac{1}{n} \sum_{j=1}^{n} s_{j,0} \right) + \frac{1}{n} Z. \tag{47}$$

Labeling the starting time to be zero is without loss of generality, and the efficient initial allocation can be interpreted as the result of previous rounds of trading. We also suppose that a new profile of signals, $\{s_i\}$, arrives at the clock time T, after which no new signals arrive. This simplistic process of information arrival is sufficient to convey the intuition. As in the main model of the dynamic market, trading can occur at clock times $\tau \in \{0, \Delta, 2\Delta, \ldots\}$, and the signals are martingales:

$$\mathbb{E}[s_i \mid \{s_{i',0}\}_{1 \le i' \le n}] = s_{i,0}. \tag{48}$$

We separately analyze two cases: $T = 0$ or T is an exponential random variable.

Information arrives at $T = 0$

Given that new information arrives at time $T = 0$, the first round of trading (at time 0) immediately reacts to this new information. By Proposition 5, the path of allocations from the periodic ex post equilibrium is:

$$z^*_{i,t\Delta} = z^e_i + (1+d)^t (z^e_{i,0} - z^e_i), \quad t \in \{1, 2, 3, \ldots\}, \tag{49}$$

where $\{z^e_{i,0}\}$ is the efficient allocation given the old signals $\{s_{i,0}\}$, and $\{z^e_i\}$ is the efficient allocation given the new signals $\{s_i\}$:

$$z^e_i = \frac{n\alpha - 1}{\lambda(n-1)} \left(s_i - \frac{1}{n} \sum_{j=1}^{n} s_j \right) + \frac{1}{n} Z. \tag{50}$$

In this case, we can define the welfare of bidders as the sum of time-discounted utilities:

$$W(\Delta) = \sum_{i=1}^{n} \sum_{t=0}^{\infty} \frac{1 - e^{-\Delta r}}{r} e^{-t\Delta r} \left(v_i z^*_{i,(t+1)\Delta} - \frac{\lambda}{2} (z^*_{i,(t+1)\Delta})^2 \right). \tag{51}$$

Proposition 8. *Suppose that $n\alpha > 2$ and $T = 0$. For any realization of the initial signals $\{s_{i,0}\}$ and any distribution of new signals $\{s_i\}$ that satisfies (48), the social welfare $W(\Delta)$ is increasing in Δ, and the optimal $\Delta^* = \infty$.*

Proof. See Section A.7. □

Proposition 8 suggests that if information arrives at the moment of trading, then slower trading (i.e., a larger Δ) is better for total welfare. The intuition for this result is simple. For a high Δ, bidders have to wait for a long time before the next round of trading. So they bid aggressively whenever they have the opportunity to trade, which leads to a relatively efficient allocation early on. For a low Δ, however, bidders know that they can trade again soon. Consequently, they bid less aggressively in each round of trading and end up paying a higher costs of holding inefficient allocations.

Although it may appear artificial that the information arrival time coincides with the trading time, in practice the trading time can adjust to meet the scheduled information announcement. Moreover, Proposition 8 provides the natural intuition that if new information repeatedly arrives at scheduled times (e.g., macroeconomic data releases or corporate earnings announcements), the optimal trading frequency should be no higher than the frequency of information arrival.

Stochastic arrival of new information

Now we turn to stochastic arrival of information. For tractability, we let T be an exponential random variable with mean $1/\nu$ and independent of all else. We let $\bar{T}$ be the clock time of the next trading period after T: $\bar{T} \equiv \min\{t\Delta : t\Delta \geq T\}$.

We also use $\{z^*_{i,t\Delta}\}$ to denote the path of allocations in the periodic ex post equilibrium of Proposition 5. Before time $\bar{T}$, we have $z^*_{i,t\Delta} = z^e_{i,0}$, and after time $\bar{T}$, the allocations start to converge toward $\{z^e_i\}$. Therefore, the social welfare is:

$$\begin{aligned} W(\Delta) =& \mathbb{E}\left[\sum_{i=1}^{n}\int_{\tau=0}^{\bar{T}} e^{-\tau r}\left(v_{i,0}z^e_{i,0} - \frac{\lambda}{2}(z^e_{i,0})^2\right) d\tau\right] \qquad (52) \\ &+ \mathbb{E}\left[e^{-r\bar{T}} \cdot \sum_{i=1}^{n}\sum_{t=0}^{\infty}\frac{1-e^{-r\Delta}}{r}e^{-t\Delta r}\left(v_i z^*_{i,\bar{T}+(t+1)\Delta} - \frac{\lambda}{2}(z^*_{i,\bar{T}+(t+1)\Delta})^2\right)\right]. \end{aligned}$$

Proposition 9. *Suppose that $n\alpha > 2$ and T is an exponential random variable. For any realization of the initial signals $\{s_{i,0}\}$ and any distribution of new signals $\{s_i\}$ that satisfies (48), $W(\Delta)$ is decreasing in Δ, and the optimal $\Delta^* = 0$.*

Proof. See Section A.8. □

Proposition 9 suggests that faster trading is better if the arrival time of new information is stochastic and unpredictable. This is because more frequent trading enables bidders to react sooner after new information arrival, which dominates the cost of lower bidding aggressiveness in the subsequent rounds of trading. As a result, a continuous market (with $\Delta^* = 0$) is optimal.

5 Conclusion

In this paper we characterize an ex post equilibrium in a uniform-price double auction with interdependent values. In the ex post equilibrium, a bidder's strategy depends only on his own private information, but he does not deviate from it even after observing the private information of other bidders. This ex post equilibrium aggregates private information dispersed across bidders, and is robust to distributional assumptions and details of auction design. Under mild conditions this ex post equilibrium is unique in the class of continuously differentiable strategy profiles. Our ex post equilibrium can be easily adapted to auctions of multiple assets.

We further generalize our ex post equilibrium to a dynamic market with an infinite sequence of double auctions and stochastic arrivals of new signals. If signals are martingales, there exists a stationary periodic ex post equilibrium, in which the equilibrium price in each auction aggregates the most recent signals, and the allocations of assets among bidders converge exponentially to the efficient level over time. A key economic implication of our analysis is that a sequence of double auctions is a simple and effective mechanism to achieve allocative efficiency. Our results also suggest that the socially optimal trading frequency is lower for scheduled information releases, but higher for information that arrives at stochastic times.

A Appendix: Proofs

A.1 Proof of Proposition 1

With inventory and given other bidders' demand schedules, bidder i's utility is

$$\Pi_i(p) = \left(S - \sum_{j\neq i} x_j(p; s_j, z_j)\right)\left(\alpha s_i + \beta \sum_{j\neq i} s_j - p\right) - \frac{1}{2}\lambda\left(z_i + S - \sum_{j\neq i} x_j(p; s_j, z_j)\right)^2,$$

where $\beta = (1-\alpha)/(n-1)$, as in Section 2. Taking the first-order condition of $\Pi_i(p)$, we obtain

$$0 = \Pi_i'(p^*) = -x_i(p^*; s_i, z_i) + \left(-\sum_{j\neq i} \frac{\partial x_j}{\partial p}(p^*; s_j, z_j)\right)\left[\alpha s_i + \beta \sum_{j\neq i} s_j - p^* - \lambda\left(z_i + x_i(p^*; s_i, z_i)\right)\right]. \tag{53}$$

As before, we conjecture a linear demand schedule

$$x_j(p; s_j, z_j) = as_j - bp + cS + dz_j + eZ,$$

and write

$$\begin{aligned}
\sum_{j\neq i} s_j &= \frac{1}{a}\left[\sum_{j\neq i} x_j(p^*; s_j, z_j) + (n-1)bp^* - (n-1)cS - d\sum_{j\neq i} z_j - (n-1)eZ\right] \\
&= \frac{1}{a}\left[S - x_i(p^*; s_i, z_i) + (n-1)bp^* - (n-1)cS - d(Z - z_i) - (n-1)eZ\right].
\end{aligned}$$

Substituting the above expression into (53) and rearranging, we have

$$\begin{aligned}
x_i(p^*; s_i, z_i) &= [1 + \lambda(n-1)b + \beta(n-1)b/a]^{-1} \cdot (n-1)b \\
&\quad \cdot \{\alpha s_i - [1 - \beta(n-1)b/a]\, p^* + S\,[1 - (n-1)c]\, \beta/a \\
&\quad + (\beta d/a - \lambda)z_i - Z[d + (n-1)e]\beta/a\} \\
&\equiv as_i - bp^* + cS + dz_i + eZ.
\end{aligned}$$

Matching the coefficients and using the normalization that $\alpha + (n-1)\beta = 1$, we solve

$$a = b = \frac{1}{\lambda} \cdot \frac{n\alpha - 2}{n-1}, \quad c = \frac{1-\alpha}{n-1}, \quad d = -\frac{n\alpha - 2}{n\alpha - 1}, \quad e = \frac{1-\alpha}{n-1} \cdot \frac{n\alpha - 2}{n\alpha - 1}.$$

A.2 Proof of Proposition 2

We fix an ex post equilibrium strategy $(x_1, \dots, x_n)$ such that for every i, x_i is continuously differentiable, $\frac{\partial x_i}{\partial p}(p; s_i, z_i) < 0$ and $\frac{\partial x_i}{\partial s_i}(p; s_i, z_i) > 0$ for every p, $(s_1, \dots, s_n) \in (\underline{s}, \overline{s})^n$ and $(z_1, \dots, z_n) \in \mathcal{Z}$.

Fix an arbitrary $s = (s_1, \dots, s_n) \in (\underline{s}, \overline{s})^n$ and $z = (z_1, \dots, z_n) \in \mathcal{Z}$. There exists a unique market-clearing price $p^*(s, z)$ [12]. We will prove that there exist a $\delta' > 0$ sufficiently small and constants A', B', D' and E' such that

$$x_i(p; s_i', z_i) = A's_i' - B'p + D'z_i + E' \tag{54}$$

holds for every $p \in (p^*(s, z) - \delta', p^*(s, z) + \delta')$, $s_i' \in (s_i - \delta', s_i + \delta')$, and $i \in \{1, \dots, n\}$,

Once (54) is established, the derivation in Section A.1 pins down the values of A', B', D' and E' to be those in Proposition 1; in particular, those values are independent of $(s_1, \dots, s_n)$, $(z_1, \dots, z_n)$, and δ'. Since $s = (s_1, \dots, s_n)$ and $z = (z_1, \dots, z_n)$ are arbitrary, the same constants A', B', D' and E' in (54) apply to any $s = (s_1, \dots, s_n) \in (\underline{s}, \overline{s})^n$, $z = (z_1, \dots, z_n) \in \mathcal{Z}$, and $p = p^*(s, z)$. This proves Proposition 2.

To prove (54), we work with the inverse function of $x_i(p; \cdot, z_i)$, to which we refer as $\tilde{s}_i(p; \cdot, z_i)$. That is, for any realized allocation $y_i \in \mathbb{R}$, we have $x_i(p; \tilde{s}_i(p; y_i, z_i), z_i) = y_i$. Because $x_i(p; s_i, z_i)$ is strictly increasing in s_i, $\tilde{s}_i(p; y_i, z_i)$ is strictly increasing y_i. Throughout the proof, we will denote dealer's realized allocation by y_i and his demand schedule by $x_i(\cdot\,; \cdot, \cdot)$. With an abuse of notation, we denote $\frac{\partial x_i}{\partial p}(p; y_i, z_i) \equiv \frac{\partial x_i}{\partial p}(p; s_i(p; y_i, z_i), z_i)$.

Fix $s = (s_1, \dots, s_n) \in (\underline{s}, \overline{s})^n$ and $z = (z_1, \dots, z_n) \in \mathcal{Z}$. Let $\bar{p} = p^*(s, z)$ and $\bar{y}_i = x_i(p^*(s); s_i, z_i)$. By continuity, there exists some $\delta > 0$ such that, for any i and any $(p, y_i) \in (\bar{p} - \delta, \bar{p} + \delta) \times (\bar{y}_i - \delta, \bar{y}_i + \delta)$, there exists some $s_i' \in (\underline{s}, \overline{s})$ such that $x_i(p; s_i', z_i) = y_i$. In other words, every price and allocation pair in $(\bar{p} - \delta, \bar{p} + \delta) \times (\bar{y}_i - \delta, \bar{y}_i + \delta)$ is "realizable" given some signal.

We will prove that there exist constants $A \neq 0$, B, D and E such that

$$\tilde{s}_i(p; y_i, z_i) = Ay_i + Bp + Dz_i + E \tag{55}$$

[12] Suppose not, i.e., for every price $p \in \mathbb{R}$, $\sum_{i=1}^n x_i(p; s_i, z_i) \neq S$, and every bidder gets $q_i = 0$. Then there exists an i such that $\sum_{j \neq i} x_j(p; s_j, z_j) \neq 0$ for every $p \in \mathbb{R}$. Suppose that $\sum_{j \neq i} x_j(p; s_j, z_j) < S$ for every $p \in \mathbb{R}$. This means that bidder i can buy $S - \sum_{j \neq i} x_j(p; s_j, z_j) > 0$ at any negative price p, which is strictly more profitable than not trading ($q_i = 0$). This contradicts the ex post optimality of $(x_1, \dots, x_n)$. Likewise when $\sum_{j \neq i} x_j(p; s_j, z_j) > S$ for every $p \in \mathbb{R}$.

for every $(p, y_i) \in (\bar{p} - \delta/n, \bar{p} + \delta/n) \times (\bar{y}_i - \delta/n, \bar{y}_i + \delta/n)$, $i \in \{1, \dots, n\}$. Clearly, this implies (54). We now proceed to prove (55). There are two cases. In Case 1, $\alpha < 1$ and $n \geq 4$. In Case 2, $\alpha = 1$ and $n \geq 3$. Since $z = (z_1, \dots, z_n) \in \mathcal{Z}$ is fixed in the rest of the proof, we omit the dependence on z_i in $\tilde{s}_i(p; y_i, z_i)$ and $\frac{\partial x_i}{\partial p}(p; y_i, z_i)$ to simplify notations.

A.2.1 Case 1: $\alpha < 1$ and $n \geq 4$

The proof for Case 1 consists of two steps.

Step 1 of Case 1: Lemma 1 and Lemma 2 below imply equation (55).

Lemma 1. *There exist functions $A(p)$, $\{B_i(p)\}$ such that*

$$\tilde{s}_i(p; y_i) = A(p) y_i + B_i(p), \tag{56}$$

holds for every $p \in (\bar{p} - \delta, \bar{p} + \delta)$ and every $y_i \in (\bar{y}_i - \delta/n, \bar{y}_i + \delta/n)$, $1 \leq i \leq n$.

Proof. This lemma is proved in Step 2 of Case 1. For this lemma we need the condition that $n \geq 4$; in the rest of the proof $n \geq 3$ suffices. □

Lemma 2. *Suppose that $l \geq 2$ and that*

$$\sum_{i=1}^{l} f_i(p; y_i) = f_{l+1}\left(p, \sum_{i=1}^{l} y_i\right), \tag{57}$$

for every $p \in P$ and $(y_1, \dots, y_l) \in \prod_{i=1}^{l} Y_i$, where Y_i is an open subset of $\mathbb{R}$, f_i is differentiable and P is an arbitrary set. Then there exist functions $G(p)$ and $\{H_i(p)\}$ such that

$$f_i(p; y_i) = G(p) y_i + H_i(p)$$

holds for every $i \in \{1, \dots, l\}$, $p \in P$ and $y_i \in Y_i$.

Proof. We differentiate (57) with respect to y_i and to y_j, where $i, j \in \{1, 2, \dots, l\}$, and obtain

$$\frac{\partial f_i}{\partial y_i}(p; y_i) = \frac{\partial f_{l+1}}{\partial y_i}\left(p; \sum_{j=1}^{l} y_j\right) = \frac{\partial f_j}{\partial y_j}(p; y_j)$$

for any $y_i \in Y_i$ and $y_j \in Y_j$. Because $(y_1, \dots, y_l)$ are arbitrary, the partial derivatives above cannot depend on any particular y_i. Thus, there exists some function $G(p)$ such that $\frac{\partial f_i}{\partial y_i}(p; y_i) = G(p)$ for all y_i. Lemma 2 then follows. □

In Step 1 of the proof of Case 1 of Proposition 2, we show that Lemma 1 and Lemma 2 imply equation (55). Let us define

$$\beta = \frac{(1-\alpha)}{n-1}, \tag{58}$$

and rewrite bidder i's ex post first-order condition as:

$$-y_i + \left(\alpha \tilde{s}_i(p; y_i) + \beta \sum_{j \neq i} \tilde{s}_j(p; y_j) - p - \lambda(z_i + y_i)\right)\left(-\sum_{j \neq i} \frac{\partial x_j}{\partial p}(p; y_j)\right) = 0, \tag{59}$$

where $y_n = S - \sum_{j=1}^{n-1} y_j$, $p \in (\bar{p} - \delta, \bar{p} + \delta)$ and $y_j \in (\bar{y}_j - \delta/n, \bar{y}_j + \delta/n)$. [13]

Our strategy is to repeatedly apply Lemma 1 and Lemma 2 to (59) in order to arrive at (55).

First, we plug the functional form of Lemma 1 into (59). Without loss of generality, we let $i = n$ and rewrite (59) as

$$\underbrace{\sum_{j=1}^{n-1} \frac{\partial x_j}{\partial p}(p; y_j)}_{\text{left-hand side of (57)}} = \underbrace{-\frac{y_n}{\alpha(A(p)y_n + B_n(p)) + \beta \sum_{j=1}^{n-1}(A(p)y_j + B_j(p)) - p - \lambda(z_n + y_n)}}_{\text{right-hand side of (57)}}.$$

Applying Lemma 2 to the above equation, we see that there exist functions $G(p)$ and $\{H_j(p)\}$ such that

$$\frac{\partial x_j}{\partial p}(p; y_j) = G(p)y_j + H_j(p), \tag{60}$$

for $j \in \{1, \ldots, n-1\}$. Note that we have used the condition $n \geq 3$ when applying Lemma 1.

By the same argument, we apply Lemma 2 to (59) for $i = 1$, and conclude that (60) holds for $j = n$ as well.

Using (56) and (60), we rewrite bidder i's ex post first-order condition as:

$$\left((\alpha - \beta)\tilde{s}_i(p; y_i) + \beta\left(A(p)S + \sum_{j=1}^{n} B_j(p)\right) - p - \lambda(z_i + y_i)\right)\left(-G(p)(S - y_i) - \sum_{j \neq i} H_j(p)\right) - y_i = 0. \tag{61}$$

[13] We restrict y_j to $(\bar{y}_j - \delta/n, \bar{y}_j + \delta/n)$ so that $y_n = S - \sum_{j=1}^{n-1} y_j \in (\bar{y}_n - \delta, \bar{y}_n + \delta)$, and as a result $\tilde{s}(p; y_n)$ and $\frac{\partial x_n}{\partial y_n}(p; y_n)$ are well-defined.

Solving for $\tilde{s}_i(p; y_i)$ in terms of p and y_i from equation (61), we see that for the solution to be consistent with (56), we must have $G(p) = 0$. Otherwise, i.e. if $G(p) \neq 0$, then (61) implies that $\tilde{s}_i(p; y_i)$ contains the term $y_i / \left(-G(p)(S - y_i) - \sum_{j \neq i} H_j(p) \right)$, contradicting the linear form of Lemma 1.

Inverting (56), we see that $x_i(p; s_i) = (s_i - B_i(p))/A(p)$. Therefore, for $\frac{\partial x_i}{\partial p}(p; s_i)$ to be independent of s_i (i.e., $G(p) = 0$), $A(p)$ must be a constant function, i.e. $A(p) = A$ for some constant $A \in \mathbb{R}$. This implies that

$$H_i(p) = -\frac{B_i'(p)}{A}, \tag{62}$$

by the definition of $H_i(p)$ in (60).

Given $G(p) = 0$ and $A(p) = A$, (61) can be rewritten as

$$(\alpha - \beta)\tilde{s}_i(p; y_i) + \beta \left(AS + \sum_{j=1}^{n} B_j(p) \right) - p - \lambda(z_i + y_i) - \frac{y_i}{-\sum_{j \neq i} H_j(p)} = 0. \tag{63}$$

For (63) to be consistent with $\tilde{s}_i(p; y_i) = Ay_i + B_i(p)$, we must have that $H_j(p) = H_j$ for some constants H_j, $j \in \{1, \ldots, n\}$, and that

$$\frac{1}{\sum_{j \neq i} H_j} = \frac{1}{\sum_{j \neq i'} H_j}, \text{ for all } i \neq i',$$

which implies that for all i, $H_i \equiv H$ for some constant H.

By (62), this means that $B_i(p) = Bp + F_i$, where $B = -HA$, and $\{F_i\}$ are some constants. Finally, (63) implies that $F_i = Dz_i + E$ for some constants D and E.

Hence, we have shown that Lemma 1 implies (55). This completes Step 1 of the proof of Case 1 of Proposition 2. In Step 2 below, we prove Lemma 1.

Step 2 of Case 1: Proof of Lemma 1.

Bidder n's ex post first order condition can be written as:

$$\sum_{j=1}^{n-1} \frac{\partial x_j}{\partial p}(p; y_j) = -\frac{y_n}{\alpha \tilde{s}_n(p; y_n) + \beta \sum_{j=1}^{n-1} \tilde{s}_j(p; y_j) - p - \lambda(z_n + y_n)}, \tag{64}$$

where $y_n = S - \sum_{j=1}^{n-1} y_j$. Differentiate (64) with respect to y_i, $i \in \{1, \ldots, n-1\}$, gives:

$$\frac{\partial}{\partial y_i}\left(\frac{\partial x_i}{\partial p}(p;y_i)\right)=\frac{\Gamma(y_1,\ldots,y_{n-1})+y_n\left(-\alpha\frac{\partial\tilde{s}_n}{\partial y_n}(p;y_n)+\beta\frac{\partial\tilde{s}_1}{\partial y_1}(p;y_1)+\lambda\right)}{\Gamma(y_1,\ldots,y_{n-1})^2}, \tag{65}$$

where

$$\Gamma(y_1,\ldots,y_{n-1})=\alpha\tilde{s}_n(p;y_n)+\beta\sum_{j=1}^{n-1}\tilde{s}_j(p;y_j)-p-\lambda(z_n+y_n). \tag{66}$$

Solving for $\Gamma(y_1,\ldots,y_{n-1})$ in (65), we get

$$\Gamma(y_1,\ldots,y_{n-1})=\rho_i\left(y_i,\sum_{j=1}^{n-1}y_j\right) \tag{67}$$

for some function ρ_i, $i\in\{1,2,\ldots,n-1\}$.

We let $\rho_{i,1}$ be the partial derivative of ρ_i with respect to its first argument, and let $\rho_{i,2}$ be the partial derivative of ρ_i with respect to its second argument. For each pair of distinct $i,k\in\{1,\ldots,n-1\}$, differentiating (67) with respect to y_i and y_k, we have

$$\frac{d\Gamma(y_1,\ldots,y_{n-1})}{dy_i}=\rho_{i,1}+\rho_{i,2}=\rho_{k,2},$$
$$\frac{d\Gamma(y_1,\ldots,y_{n-1})}{dy_k}=\rho_{k,1}+\rho_{k,2}=\rho_{i,2},$$

which imply that for all $i\neq k\in\{1,\ldots,n-1\}$,

$$\rho_{i,1}+\rho_{k,1}=0. \tag{68}$$

Clearly, (68) together with $n\geq 4$ imply that $\rho_{i,1}=-\rho_{i,1}$, i.e., $\rho_{i,1}=0$ for all $i\in\{1,\ldots,n-1\}$. That is, each ρ_i is only a function of its second argument:

$$\rho_i\left(y_i,\sum_{j=1}^{n-1}y_j\right)=\rho_i\left(\sum_{j=1}^{n-1}y_j\right). \tag{69}$$

Then, using (66), (67) and (69) for $i = 1$, we have

$$\beta \sum_{j=1}^{n-1} \tilde{s}_j(p; y_j) = \rho_1 \left(\sum_{j=1}^{n-1} y_j \right) + p + \lambda y_n - \alpha \tilde{s}_n(p; y_n). \tag{70}$$

Applying Lemma 2 to (70) (recall that $y_n = S - \sum_{j=1}^{n-1} y_j$), we conclude that, for all $j \in \{1, \ldots, n-1\}$,

$$\tilde{s}_j(p; y_j) = A(p) y_j + B_j(p). \tag{71}$$

Finally, we repeat this argument to bidder 1's ex post first-order condition and conclude that (71) holds for $j = n$ as well. This concludes the proof of Lemma 1.

A.2.2 Case 2: $\alpha = 1$ and $n \geq 3$

We now prove Case 2 of Proposition 2. Bidder n's ex post first order condition in this case is:

$$\sum_{j=1}^{n-1} \frac{\partial x_j}{\partial p}(p; y_j) = \frac{-y_n}{\tilde{s_n}(p^*; y_n) - p - \lambda(z_n + y_n)}, \tag{72}$$

for every $p \in (\bar{p} - \delta, \bar{p} + \delta)$ and $(y_1, \ldots, y_{n-1}) \in \prod_{j=1}^{n-1} (\bar{y}_j - \delta/n, \bar{y}_j + \delta/n)$, and where $y_n = S - \sum_{j=1}^{n-1} y_j$.

Applying Lemma 2 to (72) gives:

$$\frac{\partial x_j}{\partial p}(p; y_j) = G(p) y_j + H_j(p), \tag{73}$$

for $j \in \{1, \ldots, n-1\}$. Applying Lemma 2 to the ex post first-order condition of bidder 1 shows that (73) holds for $j = n$ as well.

Substituting (73) back into the first-order condition (72), we obtain:

$$(\tilde{s}_i(p; y_i) - p - \lambda(z_i + y_i)) \left(-G(p)(S - y_i) - \sum_{j \neq i} H_j(p) \right) - y_i = 0,$$

which can be rewritten as:

$$\frac{\partial x_i}{\partial p}(p; y_i) = G(p) y_i + H_i(p) = \frac{y_i}{\tilde{s}_i(p; y_i) - p - \lambda(z_i + y_i)} + G(p) S + \sum_{j=1}^{n} H_j(p). \tag{74}$$

We claim that $G(p) = 0$. Suppose for contradiction that $G(p) \neq 0$. Then matching the coefficient of y_i in (74), we must have $\tilde{s}_i(p; y_i) = \lambda y_i + B_i(p)$ for some function $B_i(p)$. But this implies that $\frac{\partial x_i}{\partial p}(p; y_i) = -B_i'(p)/\lambda$, which is independent of y_i. This implies $G(p) = 0$, a contradiction. Thus, $G(p) = 0$.

Then, (74) implies that $\tilde{s}_i(p; y_i) - p - \lambda z_i = A_i(p) y_i$ for some function $A_i(p)$. And since $\frac{\partial x_i}{\partial p}(p; y_i)$ is independent of y_i, $A_i(p)$ must be a constant function, i.e., $\tilde{s}_i(p; y_i) - p - \lambda z_i = A_i y_i$ for some $A_i \in \mathbb{R}$. Substitute this back to (74) gives:

$$\frac{\partial x_i}{\partial p}(p; y_i) = -\frac{1}{A_i} = \frac{1}{A_i - \lambda} - \sum_{j=1}^{n} \frac{1}{A_j},$$

which implies

$$\frac{1}{A_i - \lambda} - \frac{1}{A_j - \lambda} = \frac{1}{A_j} - \frac{1}{A_i}, \qquad \text{for all } i \neq j,$$

which is only possible if $A_i = A_j \equiv A \in \mathbb{R}$ for all $i \neq j$. Thus, $\tilde{s}_i(p; y_i) - p - \lambda z_i = A y_i$, which concludes the proof of this case.

A.3 Proof of Proposition 3

We define $\vec{\beta} \equiv (\beta_1, \ldots, \beta_m)'$ where, for each $k \in \{1, \ldots, m\}$,

$$\beta_k = \frac{1 - \alpha_k}{n - 1}.$$

We conjecture an ex post equilibrium in which bidder i uses the demand schedule:

$$\vec{x_i}(\vec{p}; \vec{s_i}, \vec{z_i}) = \mathbf{B}(\vec{s_i} - \vec{p}) + \mathbf{C}\vec{S} + \mathbf{D}\vec{z_i} + \mathbf{E}\vec{Z}, \tag{75}$$

where $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ are m-by-m matrices. Furthermore, we assume that $\mathbf{B}$ is symmetric and invertible.

Fix a profile of signals $(\vec{s_1}, \ldots, \vec{s_n})$ and inventories $(\vec{z_1}, \ldots, \vec{z_n})$. Bidder i's ex post first order condition at the market-clearing prices $\vec{p^*}$ is:

$$-\vec{x_i}(\vec{p^*}; \vec{s_i}, \vec{z_i}) + (n-1)\mathbf{B}\left(\mathrm{Diag}(\vec{\alpha})\vec{s_i} + \mathrm{Diag}(\vec{\beta})\sum_{j\neq i}\vec{s_j} - \vec{p^*} - \mathbf{\Lambda}(\vec{z_i} + \vec{x_i}(\vec{p^*}; \vec{s_i}, \vec{z_i}))\right) = 0. \tag{76}$$

The demand schedules in (75) yield the market-clearing price vector of

$$\vec{p^*} = \frac{1}{n}\sum_{i=1}^{n}\vec{s_i} + \mathbf{B}^{-1}\left(\mathbf{C} - \frac{1}{n}\mathbf{I}\right)\vec{S} + \mathbf{B}^{-1}\left(\frac{1}{n}\mathbf{D} + \mathbf{E}\right)\vec{Z}.$$

where $\mathbf{I}$ is the identity matrix. Substituting this expressions of $\vec{p^*}$ into (76) and rearranging, we have:

$$\begin{aligned}&\left(\mathbf{I} + (n-1)\mathbf{B}\boldsymbol{\Lambda}\right)\vec{x_i}(\vec{p^*};\vec{s_i},\vec{z_i})\\ =&(n-1)\mathbf{B}\left(\mathrm{Diag}(\vec{\alpha}-\vec{\beta})(\vec{s_i}-\vec{p^*}) - \mathrm{Diag}(n\vec{\beta})\mathbf{B}^{-1}\left(\left(\mathbf{C}-\frac{1}{n}\mathbf{I}\right)\vec{S} + \left(\frac{1}{n}\mathbf{D}+\mathbf{E}\right)\vec{Z}\right) - \boldsymbol{\Lambda}\vec{z_i}\right).\end{aligned}$$

Matching coefficients with the conjecture in (75), we obtain (where $\frac{n\vec{\alpha}-2}{n\vec{\alpha}-1}$ denotes the vector whose k-th component is $\frac{n\alpha_k-2}{n\alpha_k-1}$, etc.):

$$\begin{aligned}\mathbf{B} &= \boldsymbol{\Lambda}^{-1}\,\mathrm{Diag}\left(\frac{n\vec{\alpha}-2}{n-1}\right),\\ \mathbf{C} &= \boldsymbol{\Lambda}^{-1}\,\mathrm{Diag}\left(\frac{1-\vec{\alpha}}{n-1}\right)\boldsymbol{\Lambda},\\ \mathbf{D} &= -\boldsymbol{\Lambda}^{-1}\,\mathrm{Diag}\left(\frac{n\vec{\alpha}-2}{n\vec{\alpha}-1}\right)\boldsymbol{\Lambda},\\ \mathbf{E} &= \boldsymbol{\Lambda}^{-1}\,\mathrm{Diag}\left(\frac{(n\vec{\alpha}-2)(1-\vec{\alpha})}{(n\vec{\alpha}-1)(n-1)}\right)\boldsymbol{\Lambda}.\end{aligned}$$

A.4 Proof of Proposition 4

We conjecture that bidder i submits the demand schedule $x_i(p;v_i,z_i) = b_i(v_i - p - \lambda_i z_i)$, where $b_i > 0$. Then, bidder i's ex post first order condition is:

$$-x_i(p^*;v_i,z_i) + (v_i - p^* - \lambda_i(z_i + x_i(p^*;v_i,z_i)))\left(\sum_{j\neq i} b_j\right) = 0. \qquad (77)$$

Solving for $x_i(p^*;v_i,z_i)$ in (77) and matching coefficients with $x_i(p^*;v_i,z_i) = b_i(v_i - p^* - \lambda_i z_i)$, we obtain

$$b_i = \frac{\sum_{j\neq i} b_j}{1+\lambda_i\sum_{j\neq i} b_j} \quad\Longleftrightarrow\quad b_i + (\lambda_i b_i - 1)(B - b_i) = 0, \qquad (78)$$

where we use the fact that $\sum_{j\neq i} b_j = B - b_i$. Solving for b_i in (78), we get (24). (The quadratic equation has two solutions, but only the smaller one is the correct solution.[14]) Thus, B must solve the equation (25). To show that (25) has a unique positive solution B, we rationalize the numerators of (25) and rewrite it as

$$0 = B\left(-1 + \sum_{i=1}^{n} \frac{2}{2 + B\lambda_i + \sqrt{\lambda_i^2 B^2 + 4}}\right).$$

Under the conjecture that $B > 0$, we have

$$0 = f(B) \equiv -1 + \sum_{i=1}^{n} \frac{2}{2 + B\lambda_i + \sqrt{\lambda_i^2 B^2 + 4}}.$$

It is straightforward to see that $f'(B) < 0$, $f(0) = \frac{n}{2} - 1 > 0$, and $f(B) \to -1$ as $B \to \infty$. Thus, (25) has a unique positive solution B.

A.5 Proof of Proposition 5

We conjecture that bidders use the stationary and symmetric strategy:

$$x_{i,t\Delta}(p; s_{i,t\Delta}, z_{i,t\Delta}) = a s_{i,t\Delta} - bp + d z_{i,t\Delta} + fZ. \tag{79}$$

We let $p^*_{t\Delta}$ be the market-clearing price in period t as determined by the conjectured strategy (79):

$$p^*_{t\Delta} = \frac{a}{nb}\sum_{j=1}^{n} s_{j,t\Delta} + \frac{d+nf}{nb} Z. \tag{80}$$

For notational simplicity we write $x_{i,t\Delta}(p^*_{t\Delta}; s_{i,t\Delta}, z_{i,t\Delta})$ as $x_{i,t\Delta}$.

For a fixed period t and fixed arbitrary profiles $(s_{1,t\Delta}, \ldots, s_{n,t\Delta})$ and $(z_{1,t\Delta}, \ldots, z_{n,t\Delta})$, we want to construct the strategy in (79) so that every bidder i does not have an incentive to deviate from this strategy in period t if he anticipates that (i) others are using this strategy from period t on, and (ii) he himself will return to this strategy from period $t+1$ and onwards. Then, by the single-deviation principle, this symmetric strategy profile is a periodic ex post equilibrium.

[14] If $b_i = \frac{2+\lambda_i B + \sqrt{\lambda_i^2 B^2 + 4}}{2\lambda_i}$, then we would have $b_i > B$, which contradicts the definition of B.

Bidder i's ex post first-order condition (with respect to $p^*_{t\Delta}$) in period t is:

$$\mathbb{E}\Bigg[\Bigg(-\sum_{j\neq i}\frac{\partial x_{j,t\Delta}}{\partial p}(p^*_{t\Delta};s_{j,t\Delta},z_{j,t\Delta})\Bigg)\cdot\Bigg(\frac{1-e^{-r\Delta}}{r}\Big(v_{i,t\Delta}-\lambda(x_{i,t\Delta}+z_{i,t\Delta})$$
$$+\sum_{k=1}^{\infty}e^{-rk\Delta}\frac{\partial(z_{i,(t+k)\Delta}+x_{i,(t+k)\Delta})}{\partial x_{i,t\Delta}}(v_{i,(t+k)\Delta}-\lambda(z_{i,(t+k)\Delta}+x_{i,(t+k)\Delta}))\Big)$$
$$-p^*_{t\Delta}-\sum_{k=1}^{\infty}e^{-rk\Delta}\frac{\partial x_{i,(t+k)\Delta}}{\partial x_{i,t\Delta}}p^*_{(t+k)\Delta}\Bigg)-x_{i,t\Delta}-\sum_{k=1}^{\infty}e^{-rk\Delta}x_{i,(t+k)\Delta}\frac{\partial p^*_{(t+k)\Delta}}{\partial p^*_{t\Delta}}\;\Bigg|\;s_{i,t\Delta},\{s_{j,t\Delta}\}_{j\neq i}\Bigg]=0, \tag{81}$$

where the expectation $\mathbb{E}$ is taken over all realizations of future signals $\{s_{j,\tau}\}_{1\leq j\leq n,\tau>t\Delta}$.

If bidders follow the conjectured strategy in (79) from period $t+1$ and onwards, then we have the following evolution of inventories: for $k\geq 1$,

$$\begin{aligned}z_{i,(t+k)\Delta}+x_{i,(t+k)\Delta}=&(as_{i,(t+k)\Delta}-bp^*_{(t+k)\Delta}+fZ)+(1+d)(as_{i,(t+k-1)\Delta}-bp^*_{(t+k-1)\Delta}+fZ)\\&+\cdots+(1+d)^{k-1}(as_{i,(t+1)\Delta}-bp^*_{(t+1)\Delta}+fZ)+(1+d)^k(x_{i,t\Delta}+z_{i,t\Delta}),\end{aligned} \tag{82}$$

where $p^*_{t'\Delta}$, $t+1\leq t'\leq t+k$, is defined in (80). Equations (80) and (82) imply that

$$\frac{\partial(z_{i,(t+k)\Delta}+x_{i,(t+k)\Delta})}{\partial x_{i,t\Delta}}=(1+d)^k, \tag{83}$$
$$\frac{\partial x_{i,(t+k)\Delta}}{\partial x_{i,t\Delta}}=(1+d)^{k-1}d, \tag{84}$$
$$\frac{\partial p^*_{(t+k)\Delta}}{\partial p^*_{t\Delta}}=\frac{\partial p^*_{(t+k)\Delta}}{\partial x_{i,t\Delta}}=0. \tag{85}$$

Given the conjectured strategy in (79), the derivatives in (83), (84) and (85), and the martingale property of signals, the ex post first order condition in (81) becomes:

$$
\begin{aligned}
(n-1)b\Bigg[\frac{1-e^{-r\Delta}}{r}\Bigg(& v_{i,t\Delta}-\lambda(x_{i,t\Delta}+z_{i,t\Delta}) \\
&+\sum_{k=1}^{\infty}e^{-rk\Delta}(1+d)^k(v_{i,t\Delta}-\lambda(\mathbb{E}[z_{i,(t+k)\Delta}+x_{i,(t+k)\Delta}\mid s_{i,t\Delta},\{s_{j,t\Delta}\}_{j\neq i}]))\Bigg) \\
&-p^*_{t\Delta}-\sum_{k=1}^{\infty}e^{-rk\Delta}(1+d)^{k-1}d\,p^*_{t\Delta}\Bigg]-x_{i,t\Delta}=0,
\end{aligned} \tag{86}
$$

where, because equilibrium prices follow a martingale,

$$
\begin{aligned}
&\mathbb{E}[z_{i,(t+k)\Delta}+x_{i,(t+k)\Delta}\mid s_{i,t\Delta},\{s_{j,t\Delta}\}_{j\neq i}] \\
&=(as_{i,t\Delta}-bp^*_{t\Delta}+fZ)\left(\frac{1}{-d}-\frac{(1+d)^k}{-d}\right)+(1+d)^k(x_{i,t\Delta}+z_{i,t\Delta}).
\end{aligned} \tag{87}
$$

Averaging (86) across all bidders and using the fact that $\sum_{i=1}^n x_{i,(t+k)\Delta}=0$ and $\sum_{i=1}^n z_{i,(t+k)\Delta}=Z$, we get:

$$
p^*_{t\Delta}=\frac{1}{r}\left(\bar{s}_{t\Delta}-\frac{\lambda}{n}Z\right), \tag{88}
$$

where

$$
\bar{s}_{t\Delta}\equiv\frac{1}{n}\sum_{i=1}^{n}s_{i,t\Delta}.
$$

Therefore, in (79) we must have

$$
b=ra,\quad \frac{a\lambda}{n}+\frac{d}{n}+f=0. \tag{89}
$$

Substituting (87), (88) and (89) into the first-order condition (86), we have:

$$
\begin{aligned}
(n-1)(1-e^{-r\Delta})a\Bigg[&\frac{1}{1-e^{-r\Delta}(1+d)}\left(v_{i,t\Delta}-\bar{s}_{t\Delta}+\frac{\lambda}{n}Z\right) \\
&-\sum_{k=1}^{\infty}\lambda e^{-rk\Delta}(1+d)^k\left(\frac{1}{-d}-\frac{(1+d)^k}{-d}\right)\left(a(s_{i,t\Delta}-\bar{s}_{t\Delta})-\frac{d}{n}Z\right) \\
&-\frac{\lambda}{1-e^{-r\Delta}(1+d)^2}(x_{i,t\Delta}+z_{i,t\Delta})\Bigg]-x_{i,t\Delta}=0.
\end{aligned}
\tag{90}
$$

Rearranging the term gives:

$$
\begin{aligned}
&\left(1+\frac{(n-1)(1-e^{-r\Delta})a\lambda}{1-e^{-r\Delta}(1+d)^2}\right)x_{i,t\Delta} \\
=&\,(n-1)(1-e^{-r\Delta})a\Bigg[\frac{1}{1-e^{-r\Delta}(1+d)}\left(\alpha-\frac{1-\alpha}{n-1}\right)(s_{i,t\Delta}-\bar{s}_{t\Delta}) \\
&-\frac{\lambda e^{-r\Delta}(1+d)}{(1-(1+d)e^{-r\Delta})(1-(1+d)^2e^{-r\Delta})}\,a(s_{i,t\Delta}-\bar{s}_{t\Delta}) \\
&-\frac{\lambda}{1-e^{-r\Delta}(1+d)^2}z_{i,t\Delta}+\frac{1}{1-e^{-r\Delta}(1+d)^2}\frac{\lambda}{n}Z\Bigg].
\end{aligned}
\tag{91}
$$

On the other hand, (79) and (89) simplify the conjectured strategy to

$$
x_{i,t\Delta}=a(s_{i,t\Delta}-\bar{s}_{t\Delta})+dz_{i,t\Delta}-\frac{d}{n}Z.
$$

Matching the coefficients in the above expression with those in (91), we obtain two equations for a and d:

$$
\begin{aligned}
&\left(1+\frac{(n-1)(1-e^{-r\Delta})a\lambda}{1-e^{-r\Delta}(1+d)^2}\right)=\frac{(1-e^{-r\Delta})(n\alpha-1)}{1-e^{-r\Delta}(1+d)}-\frac{(n-1)(1-e^{-r\Delta})\lambda e^{-r\Delta}(1+d)a}{(1-(1+d)e^{-r\Delta})(1-(1+d)^2e^{-r\Delta})}, \\
&\left(1+\frac{(n-1)(1-e^{-r\Delta})a\lambda}{1-e^{-r\Delta}(1+d)^2}\right)d=-\frac{(n-1)(1-e^{-r\Delta})a\lambda}{1-e^{-r\Delta}(1+d)^2}.
\end{aligned}
$$

The solution to the above system of equations is

$$
\begin{aligned}
a &= \frac{n\alpha - 1}{2(n-1)e^{-r\Delta}\lambda}\left((n\alpha-1)(1-e^{-r\Delta}) + 2e^{-r\Delta} - \sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}\right), \\
d &= -\frac{1}{2e^{-r\Delta}}\left((n\alpha-1)(1-e^{-r\Delta}) + 2e^{-r\Delta} - \sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}\right),
\end{aligned} \tag{92}
$$

where we have $a > 0$ and $-1 < d < 0$. Finally, we have $b = ra$ and $f = -d/n - a\lambda/n$. This completes the construction of the stationary periodic ex post equilibrium.

A.6 Proof of Proposition 6

By (82) and (89), if signals do not change between period $\underline{t}$ and period $\overline{t}$, then

$$
z^*_{i,t\Delta} = \left(a\left(s_{i,\underline{t}\Delta} - \frac{1}{n}\sum_{j=1}^{n} s_{j,\underline{t}\Delta}\right) - \frac{d}{n}Z\right)\left(\frac{1}{-d} - \frac{(1+d)^{t-\underline{t}}}{-d}\right) + (1+d)^{t-\underline{t}} z^*_{i,\underline{t}\Delta}.
$$

Substituting the explicit values of a and d from Equation (92) to the above equation and noticing that $a/(-d) = (n\alpha - 1)/(\lambda(n-1))$, we obtain

$$
z^*_{i,t\Delta} = z^e_{i,\underline{t}\Delta}(1 - (1+d)^{t-\underline{t}}) + (1+d)^{t-\underline{t}} z^*_{i,\underline{t}\Delta},
$$

where

$$
z^e_{i,\underline{t}\Delta} = \frac{n\alpha - 1}{\lambda(n-1)}\left(s_{i,\underline{t}\Delta} - \frac{1}{n}\sum_{j=1}^{n} s_{j,\underline{t}\Delta}\right) + \frac{1}{n}Z.
$$

The comparative statics with respect to n, α and r follow by differentiating $1+d$ with respectively n, α and r and straightforward calculations.

For the comparative statics with respect to Δ, we find that

$$
\begin{aligned}
\frac{\partial(\log(1+d)/\Delta)}{\partial\Delta} = -\frac{1}{\Delta^2}\Bigg(& r\Delta \frac{\eta\sqrt{\eta^2(e^{r\Delta}-1)^2 + 4e^{r\Delta}} - \eta^2(e^{r\Delta}-1) - 2}{\sqrt{\eta^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}\left(\sqrt{\eta^2(e^{r\Delta}-1)^2 + 4e^{r\Delta}} - \eta(e^{r\Delta}-1)\right)} \\
& + \log\left(\frac{1}{2}\left(\sqrt{\eta^2(e^{r\Delta}-1)^2 + 4e^{r\Delta}} - \eta(e^{r\Delta}-1)\right)\right)\Bigg),
\end{aligned}
$$

where we let $\eta \equiv n\alpha - 1$. Given $\eta > 1$, it is easy to show that the two terms in the right-hand side of the above equation are both positive, which implies our conclusion.

A.7 Proof of Proposition 8

We first prove the following two lemmas.

Lemma 3.

$$\sum_{i=1}^{n}\left(v_i z_i - \frac{\lambda}{2}(z_i)^2\right) = \sum_{i=1}^{n}\left(v_i z_i^e - \frac{\lambda}{2}(z_i^e)^2\right) - \frac{\lambda}{2}\sum_{i=1}^{n}(z_i - z_i^e)^2 \tag{93}$$

Proof. We have:

$$\sum_{i=1}^{n}\left(v_i z_i - \frac{\lambda}{2}(z_i)^2\right) = \sum_{i=1}^{n}\left(v_i z_i^e - \frac{\lambda}{2}(z_i^e)^2\right) + \sum_{i=1}^{n}(v_i - \lambda z_i^e)(z_i - z_i^e) - \frac{\lambda}{2}\sum_{i=1}^{n}(z_i - z_i^e)^2. \tag{94}$$

The middle term in the right-hand side of (94) is zero because $v_i - \lambda z_i^e = p^e$ for the competitive equilibrium price p^e, and $\sum_{i=1}^{n} z_i - z_i^e = 0$. This proves the lemma. □

Lemma 4.

$$\frac{(1 - e^{-\Delta r})(1+d)^2}{1 - e^{-r\Delta}(1+d)^2} = \frac{1+d}{n\alpha - 1}. \tag{95}$$

Proof. We have:

$$\begin{aligned} e^{-r\Delta}(1+d)^2 &= \frac{2(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta} - 2(n\alpha-1)(1-e^{-r\Delta})\sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}}{4e^{-r\Delta}} \\ &= 1 - (n\alpha-1)(1-e^{-r\Delta})(1+d). \end{aligned}$$

□

Now we prove Proposition 8. By Lemma 3 and Lemma 4, we have

$$\sum_{i=1}^{n}\sum_{t=0}^{\infty}\frac{1-e^{-\Delta r}}{r}e^{-t\Delta r}\left(v_i z^*_{i,(t+1)\Delta} - \frac{\lambda}{2}(z^*_{i,(t+1)\Delta})^2\right) \tag{96}$$

$$= \frac{1}{r}\sum_{i=1}^{n}\left(v_i z_i^e - \frac{\lambda}{2}(z_i^e)^2\right) - \frac{\lambda(1-e^{-\Delta r})(1+d)^2}{2r(1-e^{-r\Delta}(1+d)^2)}\sum_{i=1}^{n}(z_{i,0}^e - z_i^e)^2 \tag{97}$$

$$= \frac{1}{r}\sum_{i=1}^{n}\left(v_i z_i^e - \frac{\lambda}{2}(z_i^e)^2\right) - \frac{\lambda(1+d)}{2r(n\alpha-1)}\sum_{i=1}^{n}(z_{i,0}^e - z_i^e)^2. \tag{98}$$

It is straightforward to show that $1+d$ is decreasing in Δ.

A.8 Proof of Proposition 9

We can rewrite the first term on the right-hand side of (52) as

$$\frac{1-\mathbb{E}[e^{-r\bar{T}}]}{r}\sum_{i=1}^{n}\left(v_{i,0}z_{i,0}^{e}-\frac{\lambda}{2}(z_{i,0}^{e})^{2}\right). \tag{99}$$

Furthermore,

$$\mathbb{E}\left[e^{-r\bar{T}}\right]=\sum_{t=0}^{\infty}e^{-(t+1)\Delta r}\left(e^{-t\Delta\nu}-e^{-(t+1)\Delta\nu}\right)=\frac{e^{-\Delta r}-e^{-\Delta(r+\nu)}}{1-e^{-\Delta(r+\nu)}}. \tag{100}$$

By Equation (96), we have

$$\mathbb{E}\left[\sum_{i=1}^{n}\sum_{t=0}^{\infty}\frac{1-e^{-\Delta r}}{r}e^{-t\Delta r}\left(v_i z^{*}_{i,\bar{T}+(t+1)\Delta}-\frac{\lambda}{2}(z^{*}_{i,\bar{T}+(t+1)\Delta})^{2}\right)\right] \tag{101}$$

$$=\frac{1}{r}\sum_{i=1}^{n}\mathbb{E}\left[v_i z_i^{e}-\frac{\lambda}{2}(z_i^{e})^{2}\right]-\frac{1+d}{n\alpha-1}\frac{\lambda}{2r}\sum_{i=1}^{n}\mathbb{E}[(z_{i,0}^{e}-z_i^{e})^{2}]. \tag{102}$$

Because $\mathbb{E}[v_i \mid \{s_{j,0}\}_{1\le j\le n}]=v_{i,0}$, applying Lemma 3 we have:

$$\sum_{i=1}^{n}\mathbb{E}\left[v_i z_i^{e}-\frac{\lambda}{2}(z_i^{e})^{2}\right]-\left(v_{i,0}z_{i,0}^{e}-\frac{\lambda}{2}(z_{i,0}^{e})^{2}\right)=\frac{\lambda}{2}\sum_{i=1}^{n}\mathbb{E}[(z_{i,0}^{e}-z_i^{e})^{2}]\equiv X. \tag{103}$$

Setting

$$Y\equiv\frac{1}{r}\sum_{i=1}^{n}\left(v_{i,0}z_{i,0}^{e}-\frac{\lambda}{2}(z_{i,0}^{e})^{2}\right), \tag{104}$$

we see that (52) is equivalent to:

$$\begin{aligned}W(\Delta)-Y&=\frac{e^{-\Delta r}-e^{-\Delta(r+\nu)}}{1-e^{-\Delta(r+\nu)}}\left(X-\frac{1+d}{n\alpha-1}X\right)\\&=\frac{e^{-\Delta r}-e^{-\Delta(r+\nu)}}{1-e^{-\Delta(r+\nu)}}\cdot\frac{(n\alpha-1)(1+e^{-r\Delta})-\sqrt{(n\alpha-1)^{2}(1-e^{-r\Delta})^{2}+4e^{-r\Delta}}}{2e^{-r\Delta}(n\alpha-1)}\cdot X.\end{aligned}$$

By taking derivatives it is easy to show that both

$$\frac{e^{-\Delta r/2}-e^{-\Delta(r/2+\nu)}}{1-e^{-\Delta(r+\nu)}} \tag{105}$$

and

$$\frac{(n\alpha - 1)(1 + e^{-r\Delta}) - \sqrt{(n\alpha - 1)^2(1 - e^{-r\Delta})^2 + 4e^{-r\Delta}}}{2e^{-r\Delta/2}(n\alpha - 1)} \tag{106}$$

are decreasing in Δ, which proves the proposition.

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