Entry Deterrence in Dynamic Second-Price Auctions*

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Abstract

We examine a dynamic second-price auction with sequential and costly entry. We show that collusive equilibria exist in which placing a low early bid has a signaling effect that deters entry by subsequent bidders. As a result, fewer bidders enter on expectation, and the bidders who do enter obtain a higher expected payoff in equilibrium, compared to the benchmark equilibrium where all bidders submit their true values. A special case of this equilibrium is one with incremental bidding (i.e., after having submitted low opening bids, buyers raise their bids by a small incremental amount each period). Computations show that the social effect of collusion is primarily a transfer from the seller to the buyers, while efficiency losses are relatively small.

Keywords: English auctions, second price auctions, online auctions, bidder collusion, entry deterrence, early bidding, late bidding, incremental bidding.

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1 Introduction

Internet auctions provide a rich platform to observe competitive bidding in real life and to test wether observed bidding behavior is consistent with the predictions of auction theory.[1] Three types of bidding behavior, in particular, have been documented by the empirical literature on internet auctions: *Early bidding*, *incremental bidding*, and *late bidding*. Early bidding occurs when bids are placed shortly after opening of an internet auction (which typically runs for several days), while late bidding occurs when bids are placed in the final seconds. Incremental bidding occurs when a bidder places multiple bids over the course of the auction, with most of these bids being equal to the minimum required increment.[2] Together, these three patterns represent a large fraction of submitted bids in online auctions.[3]

Yet, these observations are puzzling in light of standard auction theory. Virtually all internet auctions are fundamentally second-price auctions. If valuations are private and all bidders are rational, then any profile of bidding strategies in which bidders bid their true valuations before the end of the auction is a Bayesian equilibrium.[4] According to this prediction, one should not observe bunching of bids early or late in the auction, nor is it clear why bidders should be submitting multiple bids in small increments. (With common values, bidders may strategically delay their bids in an effort to learn from other bids. Similarly, in the presence of naive "adaptive" bidders, late bidding may become a best response by rational

[1]See Bajari and Hortaçsu (2004), Hasker and Sickles (2010), and Levin (2011) for surveys of research in internet auction markets.

[2]Incremental bidding is also referred to as "multiple bidding," and late bidding is also referred to as "last-minute bidding" or "sniping."

[3]See Roth and Ockenfels (2002), Bajari and Hortaçsu (2003), Ockenfels and Roth (2006). Many other empirical studies confirm these findings. By examining several thousand eBay auctions for video gaming consoles, Shah *et al.* (2002) show that early, late, and incremental bidding make up 28%, 38%, and 34% of bids, respectively. Similarly, Bapna *et al.* (2003) report that 23% of bidders place early bids, 40% submit late bids, and 37% bid incrementally, in a sample of internet auctions. Che and Katayama (2013) provide a more detailed account of these bidding strategies in eBay auctions, reporting that 30–40% of bidders submit bids twice or more, and at least 70% of incremental bids equal the minimum required increments. Further, they observe that significant portions of bidders place bids in the last few seconds of the auction, i.e., 2–11% of bids are submitted in the last 15 second of the auction time, representing 4–17% of bidders. See also Shmueli *et al.* (2004), Anwar *et al.* (2004), Hossain (2008), Wintr (2008), Ely and Hossain (2009), Engelberg and Williams (2009), and Elfenbein and McManus (2010a,b).

[4]Note that, unlike in static second-price auctions, there generally exist no weakly dominant strategies in dynamic second-price auctions (see Ockenfels and Roth 2006).

bidders. However, early and incremental bidding are still difficult to explain in these cases.)

In this paper, we provide a common, strategic explanation for all three bidding patterns. We examine a dynamic second-price auction with independent private values and risk neutral bidders. Potential bidders arrive to the auction in sequence. Upon arrival, each potential bidder observes the current auction price and then decides whether to enter the auction. If the bidder enters, he incurs a non-refundable entry cost and learns his valuation. He is then free to submit bid any number of bids at any time, until the auction closes at a predetermined time period.

We show that two basic classes of equilibrium exist in this environment, which differ in the bidders' participation decisions, the number of bids submitted by each participating bidder, and the final allocation and prices.

1. In the first, "immediate revelation" equilibrium, a bidder enters if and only if the current price is below some cutoff price p^*. Then, if the bidder's value exceeds the current auction price, he submits exactly one bid, equal to his valuation, immediately after entry. Once the auction price reaches p^*, entry ceases. This will be the case after two bidders with valuations above p^** have entered; the one with the higher valuation then wins and pays a price equal to the next highest valuation.

2. In the second, "delayed revelation" equilibrium, a bidder enters if and only if the current price is below a different cutoff price, p^{**}. If the entering bidder's value exceeds p^{**}, he submits a bid equal to p^{**} immediately after entry. Once the auction price reaches p^{**}, all further entry is deterred; this will be the case after two bidders with valuations above p^{**} have entered. These bidders will submit an additional pair of truthful late bids just prior to closing. In addition, they may periodically increase their bids between the time they enter and the final period. The bidder with the higher valuation wins and pays the next highest valuation.

The crucial result is that $p^{**} < p^*$. Thus, fewer bidders will enter in the second type equilibrium on average, and entry will cease earlier, than in the first equilibrium. By delaying the revelation of their true valuations until the final period, bidders in the delayed revelation equilibrium in effect collude to deter entry by potential rival bidders. The colluding bidders then compete against one another in a single Vickrey auction in the final period. The valuation of the winning bidder, and the price the winning bidder pays, are both lower (on expectation) in the second equilibrium. However, the bidders who do enter in the second equilibrium obtain higher expected surpluses than they would in the first equilibrium.

How does "collusion by bidding low" work? Conditional on having entered the auction, each bidder will eventually bid his true value—either immediately after entry, or in the final period. In this regard, our model is not different from other second-price auctions. Things are slightly more complicated for the bidders' entry decisions. Whether entry is worthwhile depends on the expectation a bidder holds about the valuations of competing bidders when observing the current auction price. This expectation, in turn, depends on the *particular* bidding strategies used by competing bidders, whence the multiplicity of equilibria. To see how expectations matter in the equilibria described above, note that each potential entrant cares about the distribution of the highest among his competitors' valuations. In the first (immediate revelation) equilibrium, the auction price provides a lower bound for this variable, in that exactly one of the current participants must have a valuation higher than the current price. In the second (delayed revelation) equilibrium, the auction price also provides a lower bound—but because the two highest bidders pool their bids, there are now *two* current competitors with valuations above the auction price. The price that makes a potential indifferent between entering and not entering the auction is hence lower in the second equilibrium.[5]

We also show that collusive equilibria exist in which entry is deterred by bidders adopting more sophisticated incremental bidding strategies. We argue that such incremental bidding equilibria are appealing, in the sense that bidders may more easily be able coordinate on such an equilibrium, compared to the simple delayed revelation equilibrium described earlier.

The remainder of the paper proceeds as follows. Section 2 reviews the literature related to bidding behavior in internet auctions as well as entry in auctions. In Section 3 we introduce our auction model. In Section 4 we characterize the immediate revelation equilibrium of our model. In Section 5 we explore the strategy of entry deterrence via delayed revelation, and in Section 6 we construct an equilibrium in which entry is deterred via a more sophisticated strategy of delayed revelation coupled with incremental bidding. Section 7 compares the expected buyer payoffs

[5]Even though bidders care about the distribution of their opponents' valuations, and learn about this distribution from previously submitted bids, we remark that our results do not rely on bidders' risk aversion, or on an assumption of correlated or common values. We assume risk neutrality and independent private values throughout.

and seller revenues across the different types of equilibria. Section 8 concludes with a discussion of our results. Most proofs are in the Appendix.

2 Related Literature

[To be added.]

3 Sequential Second-Price Auctions with Entry

A single indivisible object is sold to $T > 2$ risk neutral potential bidders. All potential bidders are ex ante symmetric. Bidder $i \in \{1, \ldots, T\}$ has private value v_i for the object. All v_i are independent draws from a common atomless distribution F over support $[\underline{v}, \overline{v}]$, with $0 \leq \underline{v} < \overline{v}$. Initially, a bidder does not know his own private value, but knows only the distribution F. Bidders will be able to learn their valuations during the course of the auction.

3.1 Auction format

The auction format is a sequential second-price auction, or English auction, that is open over T periods. The auction price at the end of period t is denoted $p^t \geq 0$; the final ending price is p^T. The initial price at the beginning of the auction is $p^0 = 0$.

The bidders arrive to the auction in sequence, with bidder $i \in \{1, \ldots, T\}$ arriving in period i. Upon arrival, bidder i observes the sequence of past prices $p^0, \ldots, p^{i-1}$ and decides whether to enter the auction. We denote this decision by $e_i \in \{0, 1\}$, where $e_i = 1$ means "entry" and $e_i = 0$ means "no entry." If i enters, he pays an entry cost $c > 0$ (which is the same for all bidders), learns his private value v_i, and is then free to bid in any period $t \in \{i, \ldots, T\}$. If he does not enter, he leaves the auction.[6] At the onset of the auction, the pool of participating bidders is $B^0 = \emptyset$. After potential entry in period $t \geq 1$, the pool of participating bidders becomes $B^t = \{i \leq t : e_i = 1\}$, so that $B^0 \subseteq B^1 \subseteq \ldots \subseteq B^T$.

In each period t, after potential entry, there is one round of simultaneous bidding during which all bidders in B_t submit simultaneous bids. We denote by

[6]This assumption is not crucial; even if i were to remain in the pool of potential bidders he would not enter subsequently in our equilibria.

$b_i^t \in [0, \infty)$ bidder i's bid in period t. We interpret a bid of zero as "no bid." For $t \geq 0$ and $i \notin B^t$, we automatically set $b_i^t = 0$. For $i \in B^t$, we require that $b_i^t \geq b_i^{t-1}$ for all $t \geq 1$. That is, bidders cannot revise previous bids downward during the auction. We further require that $b_i^t > p^{t-1}$ if $b_i^t > b_i^{t-1}$. That is, if a bidder revises his bid upward, he must bid more than the previous period's price.

Following submission of period-t bids, the auction price p^t will be set to the second-second-highest bid among $b_1^t, \ldots, b_T^t$. (If there is more than one highest bid, the second-highest bid is equal to the highest bid.) Since $b_i^t \geq b_i^{t-1}$ $\forall i, t$, we have $p^0 \leq p^1 \leq \ldots \leq p^T$. Figure 1 depicts the timing of events.

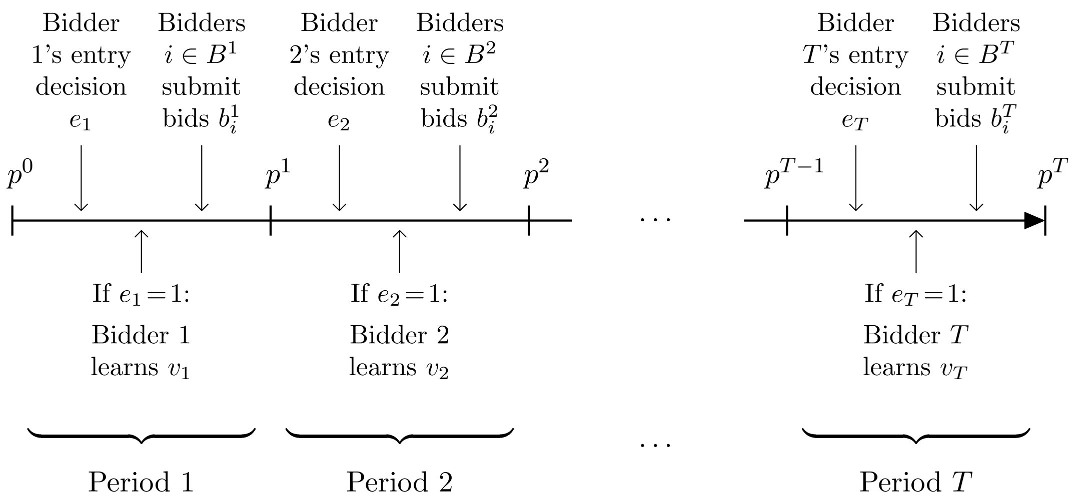

Figure 1: Timing

At the end of the final period T, if $B^T = \emptyset$ (i.e., if no bidders entered during the auction) the seller retains the object. If $B^T \neq \emptyset$, the bidder who submitted the highest bid wins the object and pays p^T. If two or more bidder submitted the highest bid, the object is awarded to the bidder who submitted the highest bid first. If there are two or more bidders who submitted the highest bid first, one of them is selected as winner by a random draw. Any bidder who does not win pays zero.

We assume that the auction rules, the entry cost c, the distribution F of values, and the arrival sequence of bidders are common knowledge. We also assume that $c < \int_{\underline{v}}^{\overline{v}} (1 - F(v))F(v)dv$. (This assumption ensures that at least two bidders can enter and obtain a positive expected surplus by bidding their valuations.)

3.2 Remarks

The entry cost c in our model has several possible interpretations. It could simply be the mental cost of introspection to determine one's willingness to pay for an item. Alternatively, c may represent the opportunity cost of the time and effort a potential bidder must spend to read and process the item description on an auction platform, in order to determine his willingness to pay. The assumption of valuations are independent and private then implies that the result of a bidder's introspection or research effort is idiosyncratic.

This interpretation is quite natural for objects such as collectibles, artwork, clothes, furniture, and the like. However, even for more "standardized" items such as electronics, some features (e.g., color) may be valued independently across buyers, or shipping costs may depend on a buyer's location (which is independent of the other location of others). Thus, the variation of the v_i should be interpreted to reflect such idiosyncratic differences. On the other hand, v_i may also contain a common value component, reflected in the level of v_i (i.e., the expectation of F). An implicit assumption in our model is that this common value component, if present, can be observed costlessly.

4 Equilibrium: Basics

In this Section, we introduce our notation for the bidders' decision rules and beliefs, and describe our immediate revelation equilibrium.

4.1 Strategies and beliefs

A bidder must make two decisions: Whether to enter the auction or not, and conditional on having entered, whether and how much to bid after entry and in each subsequent period.

An entry strategy for bidder i is, in general, a mapping from the sequence of past auction prices $p^0, p^1, \ldots, p^{i-1}$ to entry decisions (either 0 or 1):

$$e_i : [0, \infty)^i \to \{0, 1\}.$$

(Since $p^0 = p^1 = 0$, inclusion of these variables in the entry strategy is not strictly necessary.) Note that entry never depends on the bidder's valuation, since the bidder learns his valuation only after having entered the auction.

Once a bidder has entered the auction, he can submit a bid in the present bidding round and in any round thereafter. Thus, a bidding strategy for bidder i prescribes, for each period $t = i, \ldots, T$, a bid b_i^t as a function of i's information in period t. This information set includes i valuation v_i, the sequence of prices $p^0, \ldots, p^{t-1}$, and i's previous bids $b_i^i, \ldots, b_i^{t-1}$. Thus, a bidding strategy is a mapping

$$b_i^t : [\underline{v}, \overline{v}] \times [0, \infty)^t \times [0, \infty)^{t-i} \to [0, \infty)$$

that complies with the restrictions on bids imposed in Section 3.1 (that is, $b_i^t(\cdot) \geq b_i^{t-1}(\cdot)\ \forall t$ and $b_i^t(\cdot) > b_i^{t-1}(\cdot) \Rightarrow b_i^t(\cdot) > p^{t-1}$).

Finally, a bidder will also entertain beliefs about the distribution of opponents' valuations, conditional on observed information. The belief that will be relevant in our equilibria is a potential bidder's belief about the highest valuation among the currently participating bidders, conditional on observed previous prices $p^0, \ldots, p^{t-1}$. For $t \leq T$, let $w^t \equiv \max_{i \in B^t} v_i$ be the highest valuation among the bidders who have entered up to period t. In period t, the entering bidder's belief about w^{t-1} is then a conditional distribution

$$G^t(w^{t-1}|p^0, \ldots, p^{t-1}) : [\underline{v}, \overline{v}] \to [0, 1].$$

Our solution concept is, essentially, a version of sequential equilibrium (Kreps and Wilson 1982). We say that a profile of entry strategies $(e_i)_{i=1,\ldots,T}$, bidding strategies $(b_i)_{i=1,\ldots,T}$, and beliefs $G(\cdot)$, constitutes an *equilibrium* of the auction game if the following conditions hold for all $i = 1, \ldots, T$:

(i) Bidder i's bidding strategy b_i are optimal given $(b_j)_{j \neq i}$ and $(e_j)_{j \neq i}$;

(ii) bidder i's entry strategy e_i are sequentially rational given beliefs G^i for all $p^{i-1} \in [0, \overline{v}]$;

(iii) there exists a sequence of perturbed strategy profiles $(\widetilde{b}_i(\varepsilon), \widetilde{e}_i(\varepsilon)) \to (b_i, e_i)$ as $\varepsilon \to 0$ such that any weakly increasing price sequence is possible under $(\widetilde{b}_i(\varepsilon), \widetilde{e}_i(\varepsilon))$, and for every $p^0, \ldots, p^{t-1} \in [0, \overline{v}]$ the belief $G^t(\cdot\,|p^0, \ldots, p^{t-t})$ is the limit of conditional distributions derived from Bayes' Rule under the perturbed strategies, as $\varepsilon \to 0$.

Whenever a strategy or belief depends on fewer variables than the ones included above, any unnecessary arguments will be dropped.

4.2 Immediate revelation equilibrium

In this section, we focus on very simple strategies and beliefs. We call a bidding strategy an *immediate revelation strategy* if, immediately after entry, each bidder submits a bid equal to his private valuation (if it exceeds the current price) and never revises his bid thereafter:

$$b_i^t(v_i, p^{t-1}, b_i^{t-1}) \;=\; \begin{cases} 0 & \text{if } \left[t = i \text{ and } v_i \leq p^{t-1}\right], \\ v_i & \text{if } \left[t = i \text{ and } v_i > p^{t-1}\right], \\ b_i^{t-1} & \text{otherwise.} \end{cases} \tag{1}$$

We will show that an equilibrium exists where all bidders follow the immediate revelation bidding strategy.

The optimality of the bidding strategy (1) is readily established: Conditional on all other bidders following strategy (1), and conditional on a fixed set of participants, a single bidder who enters the auction clearly cannot do better than bid his true valuation at some point before the end of the auction. But since potential bidders adopt a threshold entry strategy (as will be shown below), it is optimal for every bidder who has already entered the auction to bid his valuation immediately after entry—doing so results in a weakly higher price path than delaying a truthful bid, and thus reduces the likelihood of entry by competitors.

Let us, therefore, assume a profile of immediate revelation bidding strategies in order to fully characterize the bidders' entry decisions under this hypothesis. Given bidding strategy (1) and observed price p^{t-1}, the conditional distribution of w^{t-1} at time t is

$$G(w^{t-1}|p^{t-1}) \;=\; \frac{F(w^{t-1}) - F(p^{t-1})}{1 - F(p^{t-1})}. \tag{2}$$

Now consider bidder T, who observes price p^{T-1} before deciding whether to enter in period T. The payoff relevant variables for this bidder are his own valuation, v_T, and the highest valuation among participating rival bidders, w^{T-1}. Since v_T is itself a draw from F, bidder T's expected surplus after entry, if he bids according to (1), is given by

$$U_T(p^{T-1}) \;=\; \int_{p^{T-1}}^{\overline{v}} \int_{p^{T-1}}^{\overline{v}} (v_T - w^{T-1})\, dG(w^{T-1}|p^{T-1})\, dF(v_T). \tag{3}$$

Bidder T enters if and only if $U_T(p^{T-1}) > c$.

Using the belief (2), we can express T's expected post-entry payoff (3) as

$$\begin{aligned}
U_T(p^{T-1}) &= \int_{p^{T-1}}^{\overline{v}} \left[\int_{p^{T-1}}^{v_T} (v_T - w^{T-1}) \frac{1}{1-F(p^{T-1})} dF(w^{T-1}) \right] dF(v_T) \\
&= \int_{p^{T-1}}^{\overline{v}} \frac{1}{1-F(p^{T-1})} \Bigg[-F(p^{T-1})(v_T - p^{T-1}) \\
&\qquad\qquad + \int_{p^{T-1}}^{v_T} F(w^{T-1}) dw^{T-1} \Bigg] dF(v_T) \\
&= \int_{p^{T-1}}^{\overline{v}} \frac{1}{1-F(p^{T-1})} \left[\int_{p^{T-1}}^{v_T} F(w^{T-1}) - F(p^{T-1}) dw^{T-1} \right] dF(v_T) \\
&= \int_{p^{T-1}}^{\overline{v}} \frac{1}{1-F(p^{T-1})} \left[\int_{w^{T-1}}^{\overline{v}} dF(v_T) \right] (F(w^{T-1}) - F(p^{T-1})) dw^{T-1} \\
&= \int_{p^{T-1}}^{\overline{v}} \frac{F(w^{T-1}) - F(p^{T-1})}{1-F(p^{T-1})} (1 - F(w^{T-1})) dw^{T-1}. \qquad (4)
\end{aligned}$$

(The second line is obtained by integration by parts, and the fourth line is obtained by reversing the order of integration.) The expression in (4) is strictly decreasing in p^{T-1}, larger than c at $p^{T-1} = \underline{v}$ (by our assumption on c), and zero at $p^{T-1} = \overline{v}$. Thus a unique price $p^* \in (\underline{v}, \overline{v})$ exists at which bidder T becomes indifferent between entering and not entering the auction. This price is implicitly defined by the condition

$$\int_{p^*}^{\overline{v}} \frac{F(v) - F(p^*)}{1-F(p^*)} (1 - F(v)) dv \;=\; c. \qquad (5)$$

Bidder T hence enters in period T if $p^{T-1} < p^*$, and stays out if $p^{T-1} \geq p^*$.

We show in the Appendix that, given a profile of immediate revelation strategies, p^* is the entry threshold adopted by *all* potential bidders, regardless of their position in the arrival queue. Thus, the equilibrium entry strategy for bidder $i = 1, \ldots, T$ is given by a stationary treshold strategy

$$e_i(p^{i-1}) \;=\; \begin{cases} 1 & \text{if } p^{i-1} < p^*, \\ 0 & \text{if } p^{i-1} \geq p^*. \end{cases} \qquad (6)$$

We thus obtain the following result:

Proposition 1. ***(Immediate Revelation Equilibrium)*** *There exists an equilibrium of the auction game in which the following holds for all $i = 1, \ldots, T$*

(i) Upon arrival, bidder i's belief about the distribution of the highest value among bidders $1, \ldots, i-1$ is given by (2)*;*

(ii) bidder i's entry strategy is given by (6)*; that is, bidder i enters if and only if $p^{i-1} < p^*$ where p^* is implicitly defined by* (5)*;*

(iii) bidder i's bidding strategy is an immediate revelation strategy (1)*.*

The equilibrium characterized in Proposition 1 is interim efficient, in that the participating bidder with the highest valuation wins (and pays a price equal to the second-highest value among the participants). However, the outcome is not necessarily ex-post efficient. Once the auction price reaches p^*, entry ceases. Since only participating bidders learn their valuation, it is possible for a non-participating bidder to have a higher valuation than the winning bidder.

5 Delayed Revelation and Entry Deterrence

In this section, we explore how early entrants in the auction can deter entry by later potential participants via a strategy of delayed revelation. By this, we mean a strategy of bidding below one's true value after entry and revising this bid upward later.

5.1 Preliminaries

Delayed revelation will impact bidders' beliefs about the distribution of their opponents' valuations, and thereby affect entry in the auction. We now establish a preliminary result connecting bidders' beliefs and entry decisions. This result will then be used to construct various equilibria with entry deterrence.

If at the start of period t there are exactly two bidders participating in the auction whose values are larger than p^{t-1}, the conditional distribution of w^{t-1} is

$$H(w^{t-1}|p^{t-1}) \;=\; \left[\frac{F(w^{t-1}) - F(p^{t-1})}{1 - F(p^{t-1})}\right]^2. \tag{7}$$

Now suppose that potential entrant T, after observing p^{T-1}, believes that two bidders in B^{T-1} have valuations larger than p^{T-1}, and that these two bidders will truthfully bid their valuations in period T. Bidder T's expected surplus from entering the auction and then bidding his own valuation in period T is

$$U_T(p^{T-1}) \;=\; \int_{p^{T-1}}^{\overline{v}} \int_{p^{T-1}}^{v_T} (v_T - w^{T-1})\, dH^T(w^{T-1}|p^{T-1})\, dF(v_T).$$

This is the same expression as (3) in Section 4.2, with H replacing G. Similar to our previous steps, we can thus express U_T as follows:

$$\begin{aligned}
U_T(p^{T-1}) &= \int_{p^{T-1}}^{\overline{v}} \left[\int_{p^{T-1}}^{v_T} (v_T - w^{T-1}) \frac{2(F(w^{T-1}) - F(p^{T-1}))}{(1 - F(p^{T-1}))^2} dF(w^{T-1}) \right] dF(v_T) \\
&= \int_{p^{T-1}}^{\overline{v}} \frac{1}{(1 - F(p^{T-1}))^2} \Bigg[v_T (F(v_T) - F(p^{T-1}))^2 \\
&\qquad - \int_{p^{T-1}}^{v_T} 2w^{T-1}(F(w^{T-1}) - F(p^{T-1})) dF(w^{T-1}) \Bigg] dF(v_T) \\
&= \int_{p^{T-1}}^{\overline{v}} \frac{1}{(1 - F(p^{T-1}))^2} \left[\int_{p^{T-1}}^{v_T} (F(w^{T-1}) - F(p^{T-1}))^2 dw^{T-1} \right] dF(v_T) \\
&= \int_{p^{T-1}}^{\overline{v}} \frac{1}{1 - F(p^{T-1})} \left[\int_{w^{T-1}}^{\overline{v}} dF(v^T) \right] (F(w^{T-1}) - F(p^{**}))^2 dw^{T-1} \\
&= \int_{p^{T-1}}^{\overline{v}} \frac{(F(w^{T-1}) - F(p^{T-1}))^2}{1 - F(p^{T-1})} (1 - F(w^{T-1})) dw^{T-1}.
\end{aligned}$$

A unique price $p^{**} \in (0, \overline{v})$ exists such that $U_T(p^{**}) = c$, implicitly defined by the condition

$$\int_{p^{**}}^{\overline{v}} \left[\frac{F(v) - F(p^{**})}{1 - F(p^{**})} \right]^2 (1 - F(v)) dv \;=\; c. \tag{8}$$

If $p^{T-1} < p^{**}$ bidder T enters in period T, and if $p^{T-1} \geq p^{**}$ he does not enter.

As was the case for p^* in the immediate revelation equilibrium, it can be shown that p^{**} is also the entry threshold for every bidder $i < T$ in the arrival queue. More specifically, we can show the following result:

Lemma 2. *Suppose that every bidder i adopts a bidding strategy (conditional on entry) such that (i) if $p^{i-1} < p^{**}$ and $v_i > p^{i-1}$ then $b_i^i = \min\{v_i, p^{**}\}$; and (ii) if $v_i > p^{T-1}$ then $b_i^T = v_i$. Then in any equilibrium the profile of entry strategies satisfies the following for all i: If in period i bidder i believes that exactly two bidders in B^{i-1} have valuations above p^{i-1}, then i enters the auction in period i if and only if $p^{i-1} < p^{**}$, as defined in* (8). *Moreover, $p^{**} < p^*$, as defined in* (5).

The proof of Lemma 2 is in the Appendix; however, the intuition why $p^{**} < p^*$ is straightforward: At any price p, the larger the number of participating bidders whose valuation exceeds p, the lower is the expected surplus for an additional bidder who enters at price p. Thus, there exist values of p for which entering the auction is worthwhile if only one existing bidder's valuation exceeds p, and is not worthwhile if two existing bidders' valuations exceed p. This effect can be exploited by early bidders to deter entry by later bidders. We will show how in the following section.

5.2 A simple delayed revelation equilibrium

We call a bidding strategy a *delayed revelation strategy* if, after entry, some bidder submits a bid below his valuation, and then revises this bids to reflect his true valuation in the final bidding round. In particular, we will focus on the following bidding strategy:

$$b_i^t(v_i, p^{t-1}, b_i^{t-1}) = \begin{cases} 0 & \text{if } \left[t = i \text{ and } v_i \leq p^{t-1}\right] \\ & \text{or } \left[t = i < T \text{ and } p^{t-1} = p^{**}\right], \\ v_i & \text{if } \left[t = i < T \text{ and } p^{**} > v_i > p^{t-1}\right] \\ & \text{or } \left[t = i < T \text{ and } v_i > p^{t-1} > p^{**}\right] \\ & \text{or } \left[t = T \text{ and } v_i > p^{T-1}\right], \\ p^{**} & \text{if } \left[t = i < T \text{ and } v_i \geq p^{**} > p^{t-1}\right], \\ b_i^{t-1} & \text{otherwise,} \end{cases} \tag{9}$$

where p^{**} is the price defined in (8). Strategy (9) is identical to the immediate revelation strategy (1), with two exceptions: First, a bidder whose valuation is above the threshold p^{**} does not bid his valuation upon entry if the current price is below p^{**}. Instead, this bidder submits p^{**} after entry, but will revise his bid to reflect his true valuation by the final period. Second, a bidder who enters at price p^{**} does not bid until the final period, at which time he bids his valuation.

We will show that an equilibrium exists where all bidders follow the delayed revelation bidding strategy. Note that, if all participating bidders adopt this strategy, we have $p^t \leq p^{**}$ for all $t < T$. Furthermore, the distribution of w^{t-1} conditional on price p^{t-1}—and thus an entering bidder's belief about w^{t-1}—is

$$G(w^{t-1}|p^{t-1}) \ \text{ if } \ p^{t-1} < p^{**}, \quad H(w^{t-1}|p^{t-1}) \ \text{ if } \ p^{t-1} = p^{**}, \tag{10}$$

where G and H are the beliefs given in (2) and (7), respectively. This is so because the only possibility that a price of p^{**} is observed—under the presumed bidding strategy—is for exactly two bidders to have submitted a bid of p^{**}. In this case, there will be exactly two bidders with valuations above p^{**} in B^t.

By Lemma 2, bidder t does not to enter if $p^{t-1} = p^{**}$. If all participating bidders continue to follow the strategy (9), the price will stay at p^{**} until the final round of bidding, so that entry will be deterred in all subsequent periods as well. On the other hand, if $p^{t-1} < p^{**}$ then exactly one bidder has a valuation above p^{t-1} given the presumed bidding strategy, and the analysis in Section 4.2 implies that bidder t should enter in period t. Our equilibrium entry strategy will hence remain as it were in the immediate revelation equilibrium, except when the price equals p^{**}, in which case entry is deterred:

$$e_i(p^{i-1}) \ = \begin{cases} 1 & \text{if } p^{i-1} < p^{**} \text{ or } p^{i-1} \in (p^{**}, p^*), \\ 0 & \text{if } p^{i-1} = p^{**} \text{ or } p^{i-1} \geq p^*. \end{cases} \tag{11}$$

As shown above, for $p^{t-1} \leq p^{**}$ this entry strategy is optimal assuming bidding proceeds as prescribed in (9). Unlike in the immediate revelation equilibrium, however, prices $p^{t-1} > p^{**}$ cannot be observed under the prescribed bidding strategy for any t. Yet, the equilibrium entry strategy must be sequentially rational given beliefs at such prices as well. It is straightforward to construct perturbed entry and bidding strategies that generate out-of-equilibrium beliefs to support (11) at prices above p^{**} (this will be done in the Appendix).[7] We then have the following result:

---

[7]We employ the following construction. Any bidder who observes an out-of-equilibrium price explains this observation as the result of truthful bidding by two or more players who either should have have bid the entry deterring price p^{**}, or who should not have entered the auction but did so mistakenly and then bid truthfully. In either case, the out-of-equilibrium price will be equal to the second-highest valuation among participating bidders, and as long as this is below the threshold p^* a potential bidder will enter.

Proposition 3. ***(Delayed Revelation Equilibrium)*** *There exists an equilibrium of the auction game in which the following holds for all $i = 1, \ldots, T$:*

(i) Upon arrival, bidder i's belief along the equilibrium price path about the distribution of the highest value among bidders $1, \ldots, i-1$ is given by (10)*;*

(ii) bidder i's entry strategy is given by (11)*; that is, bidder i enters if and only if $p^{i-1} < p^{**}$ or $p^{**} < p^{i-1} < p^*$, where p^* and p^{**} are implicitly defined by* (5) *and* (8)*, respectively;*

(iii) bidder i's bidding strategy is the delayed revelation strategy (9)*.*

Just like in the immediate revelation equilibrium of Section 4.2, the object will get awarded to the bidder with the highest valuation among the participating bidders, and this bidder pays the second-highest valuation among the participants. However, the pool of participants will be different across the two equilibria. In particular, in the delayed revelation equilibrium entry ceases once the auction price is p^{**} or above. As shown in Lemma 2, the entry threshold p^{**} is less than the threshold p^* in the immediate revelation equilibrium. Thus, it has a positive probability that the participants with the highest and second-highest valuations in the immediate revelation equilibrium do not enter in the delayed revelation equilibrium. In this case, the final allocation and price will be different across the two equilibria.

6 Incremental Bidding

6.1 Collusion and coordination

By delaying the revelation of their true valuations until the final period, the first and second bidder to arrive who have valuations above p^{**} in effect collude to deter entry by potential rival bidders. The two colluding bidders then compete against one another in a single Vickrey auction in the final period. Thus, the delayed revelation equilibrium described in Proposition 3 leads to larger expected surpluses for these bidders than the immediate revelation equilibrium described in Proposition 1.[8]

[8] More precisely, the distribution of surpluses received by the two colluding bidders in the delayed revelation equilibrium first-order stochastically dominates the distribution of surpluses these bidders obtain in the immediate revelation equilibrium.

The outcome of this collusive effort depends on whether two coordination attempts succeed. First, the equilibrium calls on the first two first bidders with valuations above p^{**} to initially submit matching bids p^{**}. Suppose that only one of these bidders were to bid p^{**}, and the second bidder submitted some other bid upon entry—say, his valuation, if this bidder was under the impression that the immediate revelation equilibrium was being played instead. In this event, the price would still not rise above p^{**}. Thus, a single bidder who wants to collude can attempt to do so safely, even if the bidder who he is colluding with is not aware of this attempt, as long as all future buyers react price p^{**} by staying out of the auction.

Second, potential entrants who see a price of p^{**} must interpret this price to mean that two bidders participate in the auction whose valuations exceed p^{**}. However, the same price can also occur in the immediate revelation equilibrium—namely, if the second highest bidder in some period happens to have a valuation equal to p^{**}—and in this case entry would not cease at p^{**}. Thus, the entry-deterring effect of bidding p^{**} in the delayed revelation equilibrium rests on all other bidders believing that this equilibrium, and not the immediate revelation equilibrium, is being played. Unlike coordination among the two colluding bidders, coordination among the many potential entrants may be harder to achieve. Is there a way for colluding bidders to signal to potential entrants more strongly that the price they observe was generated by a delayed revelation strategy? In other words, does a collusive bidding strategy exist that induces in a price path which would be even less likely to occur under truthful bidding, compared to a path where the price simply remains stuck at p^{**}?

Consider the following bidding strategy. Upon entry, i bids 0 if $v_i \leq p^{i-1}$, v_i if $p^{i-1} < v_i < p^{**}$, and p^{**} if $v_i \geq p^{**} = p^{i-1}$. Once two bidders i, j with values $v_i, v_j \geq p^{**}$ have entered in period t, we have $p^t = p^{**}$. In all subsequent periods $t < t' < T$, bidders i and j submit bids $b_i^{t'} = b_i^{t'-1} + \kappa$ and $b_j^{t'} = b_j^{t'-1} + \kappa$, where $\kappa > 0$ is some small increment. If $b_i^{t'-1} + \kappa > v_i$ for some t', i stops raising his bids, and similarly for j. In the final period both bidders reveal their valuations, that is, $b_i^T = v_i$ and $b_j^T = v_j$.

If no other bidders enter after t, this strategy will induce a slowly rising price path where

$$p^{t+1} = p^t + \kappa.$$

In the immediate revelation equilibrium, this sequence of prices is infinitely less likely to be observed than the price sequence

$$p^t = p^{t+1} = \ldots = p^{**}.$$

The first sequence would require that in every period a bidder enters whose valuation is exceeds the previous entrant's valuation exactly by the amount κ. On the other hand, the second sequence only requires that one bidder enters with a valuation exactly equal to p^{**}, one bidder enters with a valuation larger than p^{**}, and all other participants have valuations below p^{**}. Thus, observing the former sequence virtually guarantees that it was generated by two bidders submitting incremental bids below their true valuations. But since $p^t \geq p^{**}$ and no bidder bids above his valuation, an incremental price path signals that two bidders have valuations above p^{**}. Hence, as long as the price is slowly rising by the increment κ in every period, no new bidders will enter.

6.2 Incremental bidding equilibria

To fully formalize the ideas introduced above, let us introduce a state variable $\theta^t \in \{0, 1\}$ defined as follows:

$$\theta^t = \begin{cases} 1 & \text{if } t \geq 3 \text{ and } \Big[\big[p^{t-1} = p^{t-2} + \kappa > p^{**}\big] \text{ or} \\ & \qquad\qquad\qquad \big[p^{t-1} = p^{**} \text{ and } p^{t-2} < p^{**}\big]\Big], \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

If $\theta^t = 1$, the auction is in a state of *incremental bidding* in period t. This is the case after the price equals p^{**} for the first time, and in every period thereafter in which the last observed price is a κ-increment above the second-last price. The increment $\kappa > 0$ will be endogenous to the equilibrium; however, its value will be arbitrary.

We now assume that a potential entrant observes the last two prices, and can hence condition his entry and bidding strategies in period t on the state variable θ^t. In particular, we consider the entry strategy

$$e_i(p^{i-1}, \theta^t) = \begin{cases} 0 & \text{if } \theta^t = 1 \text{ or } p^{t-1} \geq p^*, \\ 1 & \text{otherwise.} \end{cases} \tag{13}$$

Under this strategy, a potential bidder enters the auction unless the auction is in the incremental bidding state, or the price has reached the entry threshold p^*. After entry, bidder i's plays the following bidding strategy:

$$b_i^t(v_i, p^{t-1}, b_i^{t-1}, \theta^t) = \begin{cases} 0 & \text{if } \big[t = i \text{ and } v_i \leq p^{t-1}\big] \\ & \text{or } \big[t = i < T \text{ and } p^{t-1} \geq p^*\big] \\ & \text{or } \big[t = i < T \text{ and } \theta^t = 1 \\ & \qquad\qquad \text{and } v_i < p^{t-1} + \kappa\big], \\ v_i & \text{if } \big[t = i < T \text{ and } p^{t-1} < v_i < p^{**}\big] \\ & \text{or } \big[t = T \text{ and } v_i > p^{T-1}\big], \\ p^{**} & \text{if } \big[t = i < T \text{ and } v_i \geq p^{**} > p^{t-1}\big], \\ p^{t-1} + \kappa & \text{if } \big[\theta^t = 1 \text{ and } v_i \geq p^{t-1} + \kappa\big] \\ & \text{or } \big[t = i < T \text{ and } \theta^t = 0 \\ & \qquad \text{and } v_i - \kappa \geq p^{t-1} \geq p^{**}\big], \\ b_i^{t-1} & \text{otherwise.} \end{cases} \tag{14}$$

This k*-incremental bidding strategy* is identical to the delayed revelation strategy (9), with two exceptions: First, bidders with values above the current price plus κ submit incremental bids, if the auction is in the incremental bidding state. Second, should a bidder enter at price p^{**} or higher and the auction is *not* in the incremental bidding state, the entering bidder will attempt to restart an incremental bidding phase by submitting an incremental bid.[9]

Now consider the beliefs of potential entrants. In any period t, assuming that all previous bids were generated by the k- incremental bidding strategy (14), the distribution of w^{t-1} conditional on observed prices p^{t-1} which are consistent with this strategy is given by

$$G(w^{t-1}|p^{t-1}) \text{ if } \theta^t = 0, \quad H(w^{t-1}|p^{t-1}) \text{ if } \theta^t = 1. \tag{15}$$

Under these beliefs, the arguments made in Section 4 imply that bidder t should enter the auction if $\theta^t = 0$ (as long as $p^{t-1} < p^*$), and the arguments made in Section 5 imply that bidder t should not enter the auction if $\theta^t = 1$. Entry strategy

[9]This case will occur if the auction was in the incremental bidding state but has left that state because the price increased to the valuation of one of the bidders who were submitting incremental bids.

(13) is therefore sequentially rational given Bayesian beliefs under the presumed bidding strategy.

The optimality of the bidding strategy given the entry strategy still needs to be established, and out-of-equilibrium beliefs taken care of. Again, this will be done in the Appendix. We then have:

Proposition 4. ***(κ-Incremental Bidding Equilibrium)*** *Let $\kappa > 0$. There exists an equilibrium of the auction game in which the following holds for all $i = 1, \ldots, T$:*

(i) Upon arrival, bidder i's ($i > 1$) belief along the equilibrium price path about the distribution of the highest value among bidders $1, \ldots, i-1$ is given by (15);

(ii) bidder i enters if and only if $p^{i-1} < p^*$ and $\theta^t = 0$, where p^* is implicitly defined by (5) and θ^t is the state variable given by (12);

(iii) bidder i's bidding strategy (conditional on entry) is the κ-incremental bidding strategy; that is, bidder i adopts the strategy (14) using κ as the bid increment.

Note that many incremental bidding equilibria exist which differ by the value of the increment κ. The smaller is κ, the more closely will the equilibrium price sequence under κ-incremental bidding resemble the price sequence in the delayed revelation equilibrium characterized in Proposition 3.

Furthermore, note that in an incremental bidding equilibrium, entry may resume after a phase of entry deterrence. This will be the case whenever the price increases to the valuation of one of the colluding bidders, who then stops incrementing his bid and thereby halts the progression of increasing prices. If, at this moment, the price is still below the entry threshold p^* (i.e., the threshold in the immediate revelation equilibrium), entry resumes until a bidder enters whose valuation exceeds the current price sufficiently to reinitiate the incremental bidding phase. Therefore, the final price in an incremental bidding equilibrium will be larger than that in the delayed revelation equilibrium with positive probability. The smaller κ, however, the smaller is the probability that an incremental bidding phase is interrupted, and the larger is the probability that the final allocation and price in the incremental bidding equilibrium coincides with that in the delayed revelation equilibrium.

It is also possible that the final price in the incremental bidding equilibrium exceeds that in the immediate revelation equilibrium. This happens when the bidder who wins in the immediate revelation equilibrium is deterred from entering because the auction is an an incremental bidding state, and when this state is interrupted additional bidders enter who would not enter in the immediate revelation equilibrium because that price at that point already exceeds p^*. Again, this event is very unlikely if k is small.

7 Comparison of Equilibria

In the previous three sections, we characterized three types of bidding strategies that can arise in equilibrium of our auction game: immediate revelation, delayed revelation, and incremental bidding. In the latter two cases, early bidders collude to deter entry by future competitors. In this section, we will illustrate the three equilibria with an example, and then examine the effects of bidder collusion on prices, allocations, and welfare.

7.1 Collusion and entry deterrence: An illustration

Figure 2 shows the bids, entry decisions, and price sequences that arise in all three equilibria, in an example with twenty potential buyers. The top panel depicts the bidders' valuations (shaded vertical bars) along with their bids. The middle panel shows the evolution of prices, and the bottom panel the bidders' entry decisions. The immediate revelation (IR) equilibrium is shown in blue, the delayed revelation (DR) equilibrium in red, and the incremental bidding (IB) equilibrium in green.

The IR equilibrium is very simple: Entering participants bid their valuations if they exceed the current price, and maintain these bids to the end. The price sequence under this bidding strategy is the blue line in the middle panel. The IR price first exceeds the entry threshold p^* at the end of period 13, and no further bidders enter from period 14 onward. Bidder 7 eventually wins and pays v_{13}.

Things are quite different in the two collusive equilibria. In both cases, bidders 4 and 6 (the first two bidders with valuations above p^{**}) submit coordinated bids equal to p^{**} after entering in periods 4 and 6, respectively. Thus, the price equals p^{**} at the end of period 6 and entry is deterred starting in period 7. In the DR equilibrium, nothing changes from this moment onward until period 20. In the final round, the colluding bidders reveal their valuations; bidder 6 then wins and

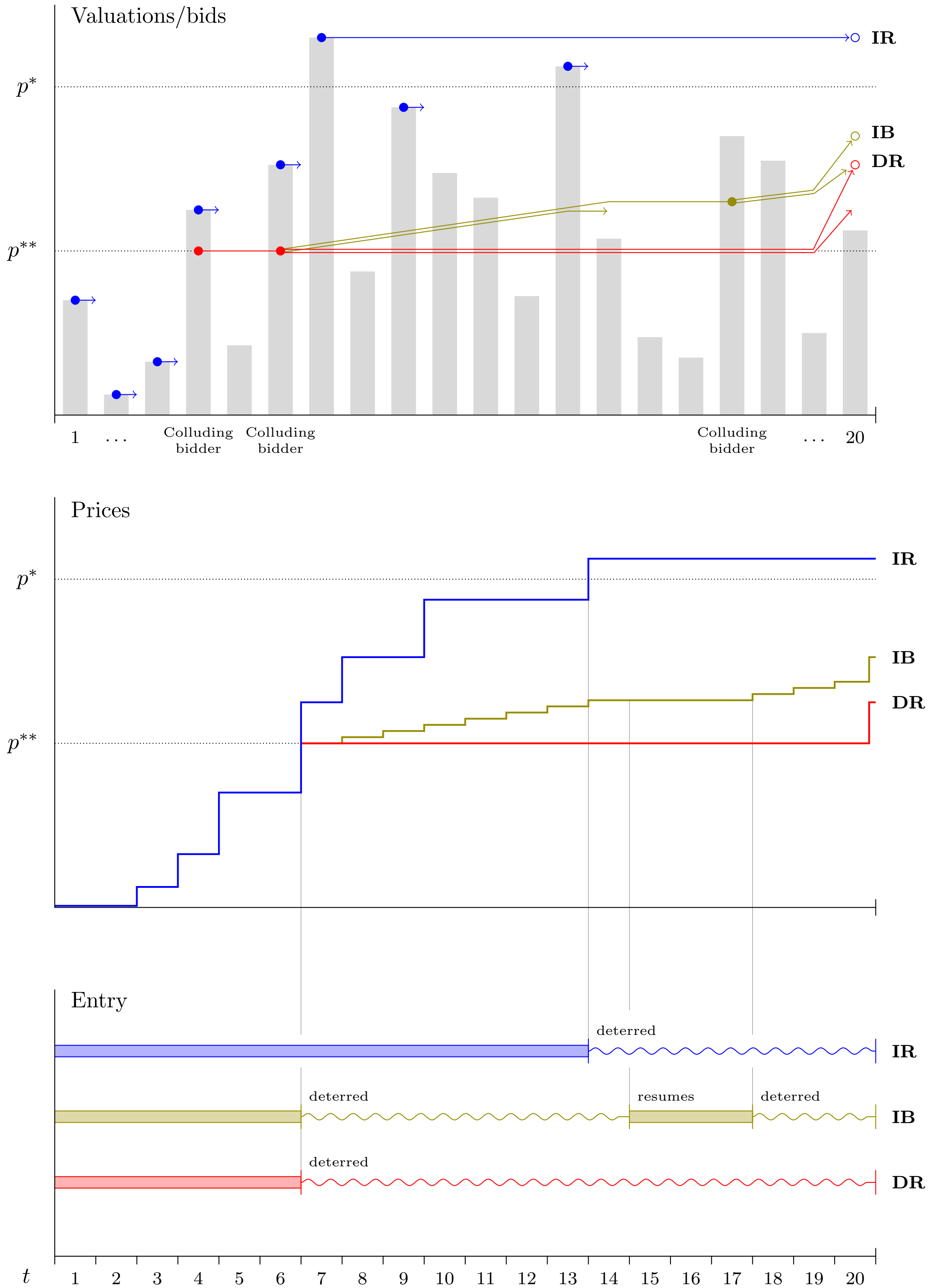

Figure 2: Bids, entry decisions, and prices in three types of equilibrium

pays v_4. In the IB equilibrium, on the other hand, the colluding bidders slowly increment their bids in lockstep, and the price gradually increases from $p^7 = p^{**}$ to $p^8 = p^{**} + \kappa$ to $p^9 = p^{**} + 2\kappa$, and so on. This process stops in period 14 as bidder 4's valuation is reached, which means that entry resumes in period 15. The incremental phase is restarted by entering bidder 17, who replaces bidder 4 in the group of colluding bidders. In the final period, these buyers submit truthful bids; bidder 17 wins and pays v_6.

The top and bottom panel of Figure 2 show that, for prices between p^{**} and p^*, entry is deterred whenever the gap between the highest and second-highest submitted bid vanishes. This is consistent with our result in Lemma 2: If exactly two bidders have valuations above the current price, and that price is at least p^{**}, entry is not profitable on expectation. But since valuations and bids are private, potential entrants can only know that this is the case by observing patterns in the price path that reveal, in effect, the same information. In the DR equilibrium, the signal to "stay out" is a price that equals p^{**} exactly. In the IB equilibrium, the signal is a price that is gradually increasing in κ-increments. As we discussed in Section 6.1, the second pattern is infinitely less likely to occur in the IR equilibrium than the first, and thus sends a clearer message to potential entrants that a collusive equilibrium is being played by the currently participating bidders. This message comes at the expense of potential intermittent periods of entry, as is the case in our example in rounds $t = 15, 16, 17$. However, the likelihood that incremental bidding breaks down is small when κ is small.

7.2 Effects on outcomes and welfare

Wether collusion by early bidders affects the final outcome of the auction depends on whether later bidders, who are deterred from entering, would have affected the identity of the winner or the price paid by the winner. In the example of Figure 2, both outcomes are different in all three equilibria. But this need not necessarily be the case: If bidders $7, \ldots, 20$ had valuations below v_4, for example, the fact that fewer bidders enter in the DR or IB equilibrium than in the IR equilibrium would be of no consequence.

We now examine the probability that the DR equilibrium (and thus the IB equilibrium with $\kappa \approx 0$) results in different allocations and prices, relative to the IR equilibrium, as well as the magnitude of the resulting changes in seller surplus and social welfare. Seller surplus is simply the price at which the auction ends,

p^T, and since fewer bidders enter in DR equilibrium p^T is lower on expectation in the DR equilibrium. The effect of the equilibrium on social welfare is a little more complicated to assess. Social welfare is the valuation of the winning bidder, which is w^T, minus any entry costs that were paid. Since fewer bidders enter in the DR equilibrium, w^T is lower on expectation this equilibrium. Thus, collusion by early bidders not only reduces the seller's surplus, but also the payoffs received by later bidders, who are deterred from entering in the DR equilibrium. However, these bidders do not pay an entry cost, and so the overall effect of collusion on welfare is ambiguous.

To gauge the likelihood that outcomes are different across equilibria, and to measure the size of this difference, let us assume that all valuations are uniformly distributed on the unit interval. Under the uniform value assumption, conditions (5) and (8) readily yield the following values for the entry thresholds in IR and DR equilibrium:

$$p^* = 1 - \sqrt{6c}, \quad p^{**} = 1 - \sqrt{12c}. \tag{16}$$

We can then characterize the differences between the IR equilibrium and the DR equilibrium asymptotically, as the number of bidders becomes large:

Proposition 5. *Suppose that $v_i \sim U[0,1]$ for all $i = 1, \ldots, T$. Comparing the immediate revelation equilibrium and the delayed revelation equilibrium of the auction game, we have the following:*

$$\begin{aligned}
\lim_{T\to\infty} Pr\big[\text{Price}_{IR} \neq \text{Price}_{DR}\big] &= \frac{1}{2}, \\
\lim_{T\to\infty} Pr\big[\text{Winner}_{IR} \neq \text{Winner}_{DR}\big] &= 1 - \sqrt{1/2}, \\
\lim_{T\to\infty} \frac{E\big[\text{Price}_{DR}\big]}{E\big[\text{Price}_{IR}\big]} &= \frac{1 - \frac{2}{3}\sqrt{12c}}{1 - \frac{2}{3}\sqrt{6c}} \\
\lim_{T\to\infty} \frac{E\big[\text{Welfare}_{DR}\big]}{E\big[\text{Welfare}_{IR}\big]} &= \frac{1 - \sqrt{3c}}{1 - \sqrt{(8/3)c}}.
\end{aligned}$$

To say more, we consider four different entry costs ($c = .001, .005, .01, .02$) and four different values for the number of potential bidders ($T = 10, 20, 50, 100$). Table 1 compares the distribution of outcomes of the immediate revelation and

				Probability of		Change in	
T	c	p^{**}	p^{*}	Different price	Different winner	Expected price	Expected welfare
10	.001	.8905	.9225	.0387	.0199	−0.16%	−0.02%
20				.1729	.0918	−0.68%	−0.10%
50				.4508	.2567	−1.93%	−0.29%
100				.4992	.2920	−2.25%	−0.33%
∞				*.5000*	*.2929*	*−2.26%*	*−0.33%*
10	.005	.7551	.8268	.2106	.1120	−2.07%	−0.31%
20				.4372	.2470	−4.46%	−0.67%
50				.4998	.2927	−5.40%	−0.79%
100				.5000	.2929	−5.41%	−0.79%
∞				*.5000*	*.2929*	*−5.41%*	*−0.79%*
10	.01	.6536	.7551	.3442	.1882	−5.14%	−0.77%
20				.4895	.2835	−7.76%	−1.15%
50				.5000	.2929	−8.08%	−1.18%
100				.5000	.2929	−8.08%	−1.18%
∞				*.5000*	*.2929*	*−8.08%*	*−1.18%*
10	.02	.5101	.6536	.4565	.2584	−10.83%	−1.61%
20				.4996	.2923	−12.40%	−1.82%
50				.5000	.2929	−12.44%	−1.82%
100				.5000	.2929	−12.44%	−1.82%
∞				*.5000*	*.2929*	*−12.44%*	*−1.82%*

Table 1: Effects of bidder collusion on outcomes ($v_i \sim U[0,1]$)

delayed revelation equilibrium for these parameter combinations.[10] The probability that the DR and IR equilibria result in different allocations increases in c and T, and so does the probability that the equilibria result in different final prices. The probability that collusion affects the outcome is not negligible, and is in fact quite substantial when the number of potential bidders is large.

The DR equilibrium reduces both seller surplus and social welfare, relative to the IR equilibrium. However, the relative difference in these variables across equilibria is less pronounced than what one might expect, given the probabilities that allocations and prices differ. This is especially true for social welfare, which

[10]The reported numbers also approximate the difference between the IR and IB equilibria for $\kappa \approx 0$. Details regarding our computations are in Appendix B.

in DR equilibrium is less than two percent below its value in IR equilibrium, in all cases reported. The reason why the effect of collusion on welfare is low is threefold. First, one effect of collusion is that it transfers some surplus from the seller to the buyers, as can be seen in the column comparing the expected price under the two equilibria. This part of the effect is welfare neutral. Second, while the DR equilibrium frequently allocates the object to the "wrong" winner, the difference between the respective winners' valuations may not be large. Third, the bidders who do not enter in DR equilibrium but enter in IR equilibrium save their entry costs.[11] Thus, the main effect of collusion on surpluses is a transfer from the seller to the buyers.

Figure 3 illustrates these three effects for uniform values and the same entry costs used in Table 1 (i.e., $c = .001, .005., .01, .02$). Each panel in the figure plots the expected difference in welfare W^T, winner valuation w^T, and price p^T, between the DR and the IR equilibrium, for a different entry cost and for time horizons $T = 2, \ldots, 20$. The difference in prices across the equilibria is the expected loss to the seller from bidder collusion. The difference between in welfare across the equilibria is the expected social loss from bidder collusion. The social loss is much less than the seller's loss, and the difference is the buyers' gain in DR equilibrium, relative to IR equilibrium. Figure 3 reveals that this gain comes from two sources: A direct collusive effect that represents the effect of collusion on price, and an indirect cost savings effect that represents the entry costs of those bidders who enter in IR equilibrium but not in DR equilibrium. Both effects are approximately equal in size and, taken together, make up for a significant share of the seller's loss. Thus, in this example, the societal effect of collusion is primarily a redistributive transfer from the seller to the buyers; the efficiency loss is of a much smaller magnitude.

8 Discussion

[To be added.]

[11] If these cost savings were ignored, "welfare" in the DR equilibrium would be up to 6% lower in DR equilibrium (in the cases examined in the Table 1).

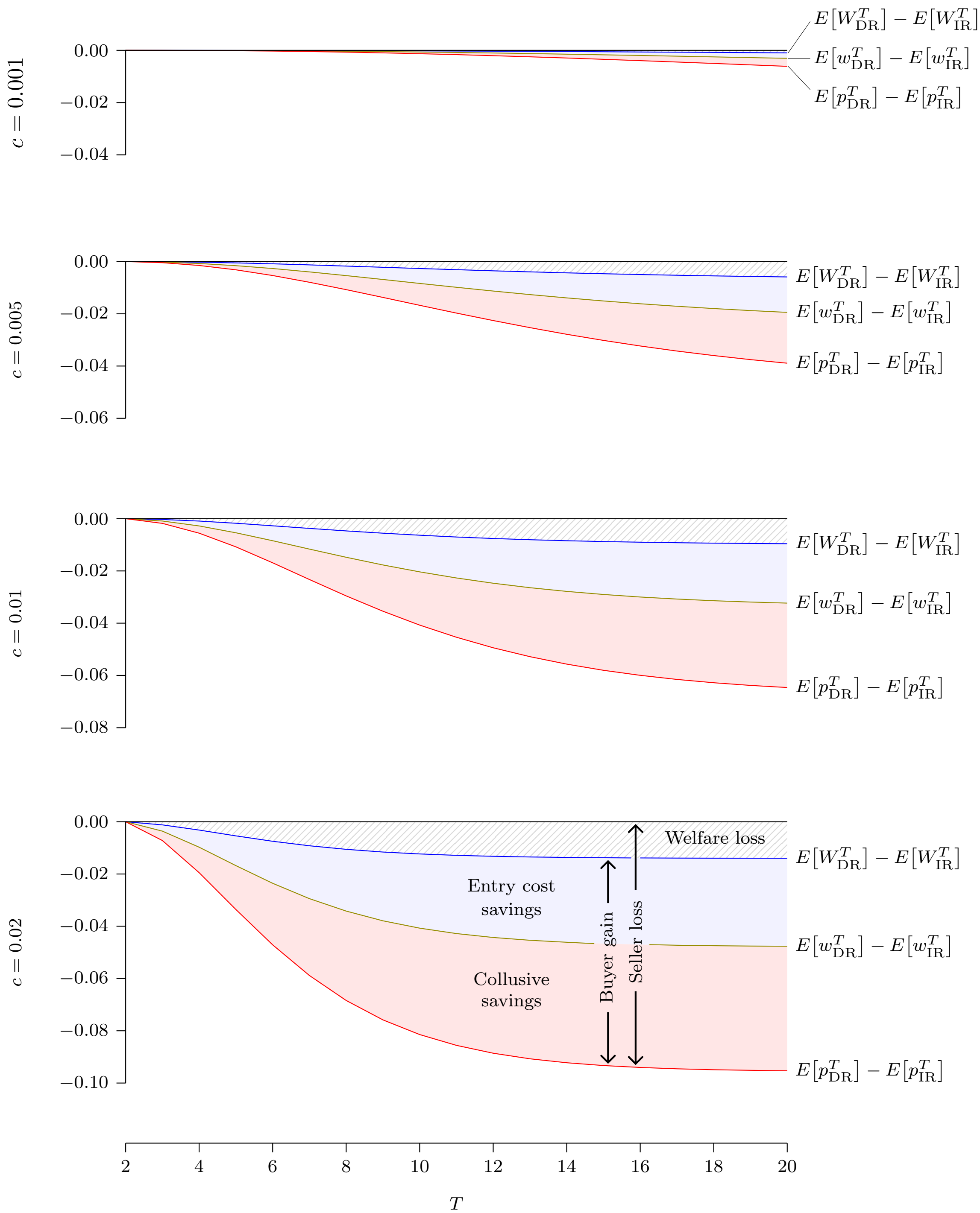

Figure 3: Effects of bidder collusion on buyer and seller surplus ($v_i \sim U[0, 1]$)

Appendix A: Proofs

Proof of Proposition 1

Part (iii) of the result was established in the text. Part (i) is straightforward: For $t \geq 3$, all $p^{t-1} \in [0, \overline{v}]$ can occur in equilibrium. In period 2, the second-price auction format implies that bidder 2 observes first-period price $p^1 = 0$ regardless of actions taken by the previous bidder. Thus, for all $t \geq 2$, the belief $G(w^{t-1}|p^{t-1})$ is given by (2) and computed from the equilibrium strategies using Bayes' Rule for all feasible p^{t-1}. To establish part (ii) of the result, we need to prove that p^* is the entry threshold not only for bidder T (which was already shown in the text), but also for all bidders $t < T$. We split the argument into two steps.

Step 1. We show that $p^{t-1} \geq p^*$ implies that bidder t does not enter. This will be done by induction. Suppose $p^{T-2} \geq p^*$; then $p^{T-1} \geq p^*$ and bidder T will not enter in period T. Knowing that bidder T will not enter, bidder $T-1$ competes against the highest bidder in B^{T-2}, whose valuation is distributed by $G(w^{T-2}|p^{T-2})$. This is the problem examined in the main text in Section 4.2, and we know that, since $p^{T-2} \geq p^*$, it is optimal for bidder $T-1$ not to enter in period $T-1$. Now suppose that $p^{T-3} \geq p^*$. Then $p^{T-1} \geq p^{T-2} \geq p^*$, so that bidders T and $T-1$ will not enter. Bidder $T-2$ therefore competes against the highest valuation bidder in B^{T-3}, whose valuation is distributed by $G(w^{T-3}|p^{T-3})$. Because $p^{T-3} \geq p^*$, it is optimal for bidder $T-2$ not to enter. Continuing in this fashion, we conclude that bidder $t \in \{1, \ldots, T\}$ does not enter in period t if $p^{t-1} \geq p^*$.

Step 2. We show that $p^{t-1} < p^*$ implies that bidder t enters. Let z^{t+1} be the highest bids submitted by bidders who will enter after period t, and let $Z(z^{t+1}|p^t)$ be the distribution of z^{t+1} conditional on period-t price p^t. Note that, if bidder t enters in period t and bids v_t, then $p^t = \min\{v_t, w^{t-1}\}$, where w^{t-1} is the highest valuation of bidders $j \in B^{t-1}$. Under the bidding strategies (1), the continuation payoff (not including the entry cost c) from entering at price $p^{t-1} < p^*$ to bidder t is thus given by

$$
\begin{aligned}
U_t(p^{t-1}) \ = \ & \int_{p^{t-1}}^{\overline{v}} \int_{p^{t-1}}^{v_t} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) \\
& \qquad\qquad dZ(z^{t+1}| \min\{v_t, w^{t-1}\})\, dG(w^{t-1}|p^{t-1})\, dF(v_t).
\end{aligned}
$$

Define

$$
\begin{aligned}
A(v_t) &= \int_{p^{t-1}}^{p^*} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) \\
&\qquad\qquad dZ(z^{t+1}| \min\{v_t, w^{t-1}\})\, dG(w^{t-1}|p^{t-1}), \\
B(v_t) &= \int_{p^*}^{v_t} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) \\
&\qquad\qquad dZ(z^{t+1}| \min\{v_t, w^{t-1}\})\, dG(w^{t-1}|p^{t-1}),
\end{aligned}
$$

and express bidder t's payoff from entering as follows:

$$
U_t(p^{t-1}) = \int_{p^{t-1}}^{\overline{v}} [A(v_t) + B(v_t)]\, dF(v_t) > \int_{p^*}^{\overline{v}} [A(v_t) + B(v_t)]\, dF(v_t). \tag{17}
$$

Now consider two cases.

1. First, suppose $v_t \geq p^*$ and $w^{t-1} \geq p^*$. Then the price at the end of period t will be $p^t = \min\{v_t, w^{t-1}\} \geq p^*$, and no entry will occur after period t as shown in Step 1. Thus, conditional on $v_t \geq p^*$ and $w^{t-1} \geq p^*$ we have $z^{t+1} = 0$, which allows us to write

$$
\begin{aligned}
B(v_t) &= \int_{p^*}^{v_t} (v_t - w^{t-1}) dG(w^{t-1}|p^{t-1}) \\
&= (1 - G(p^*|p^{t-1})) \int_{p^*}^{v_t} (v_t - w^{t-1}) dG(w^{t-1}|p^*).
\end{aligned} \tag{18}
$$

2. Second, suppose $v_t \geq p^*$ and $w^{t-1} \leq p^*$. If during some period $s > t$ a bidder enters with $v_s \geq p^*$, the price at the end of period s will be $p^s = \min\{v_t, v_s\} \geq p^*$, and no further entry will occur after period s, as shown in Step 1. In this event, $z^{t+1} = v_s > p^*$ with distribution $G(z^{t+1}|p^*)$. If no bidder with $v_s \geq p^*$ enters during any period $s > t$, we have $z^{t+1} < p^*$. Since bidder t's payoff will be lower in the first event than in the second, we can write

$$
\begin{aligned}
A(v_t) &> \int_{p^{t-1}}^{p^*} \int_{p^*}^{v_t} (v_t - z^{t+1}) dG(z^{t+1}|p^*) dG(w^{t-1}|p^{t-1}) \\
&= G(p^*|p^{t-1}) \int_{p^*}^{v_t} (v_t - z^{t+1}) dG(z^{t+1}|p^*).
\end{aligned} \tag{19}
$$

Combining (17)–(19), we have

$$U_t(p^{t-1}) \;>\; \int_{p^*}^{\overline{v}} \int_{p^*}^{v_t} (v_t - v) dG(v|p^*) \;=\; c$$

as shown in the main text in Section 4.2. Thus, when $p^{t-1} < p^*$, the expected surplus for bidder t from entering the auction in period t exceeds the entry cost c, so bidder t enters. $\square$

Proof of Lemma 2

Let p^{**} be defined by (8) and let p^* be defined by (5). To show that $p^{**} < p^*$, define

$$L_k(p) \;\equiv\; \int_p^{\overline{v}} \left[\frac{F(v) - F(p)}{1 - F(p)}\right]^k (1 - F(v))dv$$

and note that $L_1(p^*) = L_2(p^{**}) = c$ and $L_1(p) < L_2(p)$ $\forall p < \overline{v}$. Therefore, $L_2(p^*) < c$, and since L_2 is strictly decreasing we conclude that $p^* > p^{**}$. We now prove that, under the assumptions of Lemma 2, p^{**} is the entry threshold not only for bidder T (which was already shown in Section 5.1), but also for all bidders $t < T$. The argument is parallel to the one we made to prove Proposition 1 (ii), and proceeds in the same two steps.

Step 1. We show that $p^{t-1} \geq p^{**}$ implies that bidder t does not enter. Suppose $p^{T-2} \geq p^{**}$; then $p^{T-1} \geq p^{**}$ and bidder T will not enter in period T. Knowing that bidder T will not enter, bidder $T-1$ competes against the highest valuation among bidders in B^{T-2}, w^{T-2}. Furthermore, if i believes that exactly two bidders in B^{T-2} have valuations above p^{T-2}, w^{T-2} follows distribution $H(w^{T-2}|p^{T-2})$. The same argument we made for bidder T in Section 5.1 then implies that it is optimal for bidder $T-1$ not to enter in period $T-1$. Continuing inductively, we conclude that bidder $t \in \{1, \ldots, T\}$ does not enter in period t if $p^{t-1} \geq p^{**}$.

Step 2. We show that $p^{t-1} < p^{**}$ implies that bidder t enters. Define z^{t+1} and $Z(z^{t+1}|p^t)$ as in the proof of Proposition 1. Since, by assumption, all bidders $j \in B^T$ with $v_j > p^{T-1}$ will bid their valuations in period T, the continuation payoff (not including the entry cost c) from entering at price $p^{t-1} < p^*$ to bidder t is thus given by

$$U_t(p^{t-1}) \;=\; \int_{p^{t-1}}^{\overline{v}} \int_{p^{t-1}}^{v_t} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\})$$

$$dZ(z^{t+1}|\min\{v_t, w^{t-1}\})\, dH(w^{t-1}|p^{t-1})\, dF(v_t).$$

By setting

$$\begin{aligned} A(v_t) &= \int_{p^{t-1}}^{p^{**}} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) \\ &\qquad dZ(z^{t+1}|\min\{v_t, w^{t-1}\})\, dH(w^{t-1}|p^{t-1}), \\ B(v_t) &= \int_{p^{**}}^{v_t} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) \\ &\qquad dZ(z^{t+1}|\min\{v_t, w^{t-1}\})\, dH(w^{t-1}|p^{t-1}), \end{aligned}$$

we can express bidder t's payoff from entering as follows:

$$U_t(p^{t-1}) = \int_{p^{t-1}}^{\overline{v}} [A(v_t) + B(v_t)]\, dF(v_t) > \int_{p^{**}}^{\overline{v}} [A(v_t) + B(v_t)]\, dF(v_t).$$

Mirroring our proof of Proposition 1 (ii), we consider two cases.

1. First, suppose $v_t \geq p^{**}$ and $w^{t-1} \geq p^{**}$. Under the assumed bidding strategies, the price at the end of period t will be $p^t = p^{**}$ and no entry will occur after period t (as shown in Step 1), so we can write

$$\begin{aligned} B(v_t) &= \int_{p^{**}}^{v_t} (v_t - w^{t-1}) dG(w^{t-1}|p^{t-1}) \\ &= (1 - G(p^{**}|p^{t-1})) \int_{p^{**}}^{v_t} (v_t - w^{t-1}) dG(w^{t-1}|p^{**}). \end{aligned}$$

2. Second, suppose $v_t \geq p^{**}$ and $w^{t-1} \leq p^{**}$. If during some period $s > t$ a bidder enters with $v_s \geq p^{**}$, under the assumed bidding strategies the price at the end of period s will be $p^s = p^{**}$ and no further entry will occur after period s (as shown in Step 1). In this event, $z^{t+1} = v_s > p^{**}$ with distribution $G(z^{t+1}|p^{**})$. If no bidder with $v_s \geq p^{**}$ enters during any period $s > t$, we have $z^{t+1} < p^{**}$. Since bidder t's payoff will be lower in the first event than in the second, we can write

$$\begin{aligned} A(v_t) &> \int_{p^{t-1}}^{p^{**}} \int_{p^{**}}^{v_t} (v_t - z^{t+1}) dG(z^{t+1}|p^{**}) dG(w^{t-1}|p^{t-1}) \\ &= G(p^{**}|p^{t-1}) \int_{p^{**}}^{v_t} (v_t - z^{t+1}) dG(z^{t+1}|p^{**}). \end{aligned}$$

Combining the last three equations, we have

$$U_t(p^{t-1}) \;>\; \int_{p^{**}}^{\overline{v}} \int_{p^{**}}^{v_t} (v_t - v) dH(v|p^*) \;=\; c$$

as shown in Section 5.1. Thus, when $p^{t-1} < p^{**}$, the expected surplus for bidder t from entering the auction in period t exceeds the entry cost c, so bidder t enters. □

Proof of Proposition 3

Most of the result was shown already in the text in Section 5.2. What is left is to establish the optimality of the equilibrium bidding strategy given the equilibrium entry strategy (Step 1), and the optimality of the entry strategy following off-equilibrium prices; that is, prices that exceed p^{**} (Step 2).

Step 1. Clearly, in the final period a truthful bid $b_i^T = v_i$ is optimal for every bidder i. Let us therefore consider bidding in periods $t < T$. We need to consider three cases.

1. $p^{t-1} = p^{**}$. Suppose any participating bidder i deviates from the equilibrium strategy in any period t when $p^{t-1} = p^{**}$. If this deviation does not change the price in periods $s \geq t$, the deviation has no effect on the final allocation and price. If the deviation changes the price in some period $s \geq t$ from $p^s = p^{**}$ to $p^s \neq p^{**}$, then additional bidders will enter with positive probability, reducing the payoff to bidder i.

2. $p^{t-1} > p^{**}$. Once the price has surpassed p^{**} (an off-equilibrium contingency), the entry and bidding strategies are identical to those in the immediate revelation equilibrium; the bidding strategy is therefore optimal.

3. $p^{t-1} < p^{**}$. The bidding strategy calls for bidders with valuations $v_i \geq p^{**}$ to bid p^{**}, and for bidders with valuations $v_i \leq p^{**}$ to bid v_i.

   3a. $v_i \geq p^{*}*$. This case is identical to the first case: If i deviates and bids $b_i^t \neq p^{**}$, and this deviation does not change the price in periods $s \geq t$, it has no effect on the final allocation and price. If the deviation changes the price in some period $s \geq t$ from $p^s = p^{**}$ to $p^s \neq p^{**}$, then additional bidders will enter with positive probability, reducing the payoff to i.

3b. $v_i < p^{**}$. If i deviates by bidding $b_i^t < v_i$ in some period t, the deviation will not affect the final allocation and prices. If i deviates by bidding $b_i^t > v_i$ in period $t < T$, at best this deviation reduces the number of bidders who enter in periods $s > t$, namely if i submits a bid $b_i^t \geq p^{**}$ and this bid results in an entry-deterring price p^{**} earlier than what would otherwise have been the case. However, this requires that some other bidder $j \neq i$ submits bid $b_j^s \geq p^{**}$ in some period s. Under our bidding strategy this only happens if $v_j \geq p^{**}$. If such a bidder participates, he will outbid i in the final period, leaving the outcome for i unchanged. If such a bidder does not participate, then i will have the highest final bid and will win. In this case, i either pays the same price as before (if i had won without the deviation), or pays price $p^T > v_i$ (if i had lost without the deviation).

We therefore conclude that no entering bidder has an incentive to deviate from the equilibrium bidding strategies (9).

Step 2. Next, we consider the bidders' entry decisions. For sequential equilibrium we need to find a sequence of strategy profiles, converging to the equilibrium strategies, such that prices above p^{**} are possible along the sequence and the equilibrium entry strategies are sequentially rational under the limit of Bayesian beliefs generated by the sequence of perturbed strategies.

To this end, let $\varepsilon \in (0, 1)$ and consider the following perturbed strategy for every player i:

$\widetilde{e}_i$: In period i, enter with probability $(1-\varepsilon)e_i(p^{i-1}) + \varepsilon(1 - e_i(p^{i-1}))$, where $e_i(\cdot)$ is the equilibrium entry strategy.

$\widetilde{b}_i$: Conditional on having entered, bid as follows: With probability $1-\varepsilon$ play the equilibrium bidding strategy (9); with probability ε bid $b_i^i = b_i^{i+1} = \ldots = v_i$.

Note that any weakly increasing sequence of prices can occur under this strategy profile, as long as $\varepsilon > 0$. Furthermore, as $\varepsilon \to 0$ the profile converges to the equilibrium strategies.

Now suppose some potential bidder t observes out-of-equilibrium price $p^{t-1} > p^{**}$. This can only happen if at least two participating bidders did not play their equilibrium strategies, and submitted bids larger than p^{**}. Given the perturbed profile, these must be truthful bids. Thus, any observed $p^{t-1} > p^{**}$ will be the second-highest valuation of bidders in B^{t-1}, which means that the resulting Bayesian posterior distribution of w^{t-1} is $G(w^{t-1}|p^{t-1})$ (where G is defined in (2)). This distribution does not depend on ε; the limit belief as $\varepsilon \to 0$ is therefore also $G(w^{t-1}|p^{t-1})$. As shown in Section 4.2, it is then optimal for bidder t to enter the auction as long as $p^{t-1} < p^*$, as prescribed by the equilibrium entry strategy (11). □

Proof of Proposition 4

We need to establish the optimality of the equilibrium bidding strategy given the equilibrium entry strategy (Step 1), and the optimality of the entry strategy following off-equilibrium prices (Step 2).

Step 1. This step is almost identical to that in the proof of Proposition 3. By not following strategy (14) when all rivals follow (14), the best bidder i can hope for is to retard the entry process in periods when there would otherwise be entry. In parallel to our arguments above, this will entail i bidding above v_i in some period, and at least one other bidder j submitting the same bid in the same period. Since j is still following the equilibrium strategy, his valuation v_j will exceed the collusive bid and therefore exceed v_i. This means that j will outbid i in period T, guaranteeing a loss for i. On the other hand, if no such bidder j exists, then i will either lose, or win but pay a price above v_i. In all cases, i is no better off than he would be had he followed strategy (14).

Step 2. For all $2 \leq t \leq T$, prices p^{t-1} above p^{**} that are not κ-increments over p^{t-2} cannot arise under the incremental bidding strategy. The equilibrium prescribes entry in all periods t where this is the case, unless $p^{t-1} \geq p^*$. To show that, in this case, entry is sequentially rational under limit Bayesian beliefs, consider the following perturbed strategy for every player i:

$\widetilde{e}_i$: In period i, enter with probability $(1-\varepsilon)e_i(p^{i-1})+\varepsilon(1-e_i(p^{i-1}))$, where $e_i(\cdot)$ is the equilibrium entry strategy.

$\widetilde{b}_i$: Conditional on having entered, bid as follows: If $v_i \leq p^{**}$, play the equilibrium strategy (14). If $v_i > p^{**}$, then with probability $1-\varepsilon$ play the equilibrium bidding strategy (14). With probability ε, play a strategy that is identical to (14) up to a randomly and uniformly selected period $t^* \geq i$. If $p^{t-1} < v_i$, then in every period $t \geq t^*$ bid $b_i^t = v_i$.

Note that any weakly increasing sequence of prices $p^1 \le p^2 \le \dots$ can occur under this strategy profile, as long as $\varepsilon > 0$. Furthermore, as $\varepsilon \to 0$ the profile converges to the equilibrium strategies.

Now suppose some potential bidder t observes a price $p^{t-1} > p^{**}$ that is not a κ-increment over p^{t-1}. Given the perturbed strategies, this means that at least one bidder submitted a truthful bid when he should not have done so in the equilibrium. Furthermore, p^{t-1} will then be equal the second highest valuation of bidders in B^{t-1}, and the resulting Bayesian posterior distribution of w^{t-1} is $G(w^{t-1}|p^{t-1})$ (where G is defined in (2)). This distribution does not depend on ε; the limit belief as $\varepsilon \to 0$ is therefore also $G(w^{t-1}|p^{t-1})$. As shown in Section 4.2, it is then optimal for bidder t to enter the auction, as long as $p^{t-1} < p^*$, as prescribed by the equilibrium entry strategy (13). $\square$

Proof of Proposition 5

Step 1. Let us first examine the probabilities that outcomes are different across equilibria, as $T \to \infty$. Let s and $t > s$ be the first two bidders for which $v_s, v_t > p^{**}$; the probability that these bidders exist approaches one. In the DR equilibrium, entry stops after period t. As far as the IR equilibrium is concerned, there are three cases:

1. $v_s \ge p^*$ and $v_t \ge p^*$. In this case, entry stops after period t.

2. Either $v_s \ge p^*$ or $v_t \ge p^*$, but not both. In this case, entry continues until one more bidder participates whose valuation exceeds p^* (the probability that this happens approaches one).

3. $v_s < p^*$ and $v_t < p^*$. In this case, entry continues until two bidders participate whose valuations exceed p^* (the probability that this happens approaches one).

In case 1, the two equilibria result in the same final price. In case 2 and 3, the IR equilibrium results in a higher final price. With uniform values, the probability of case 1 is

$$\left(\frac{1-p^*}{1-p^{**}}\right)^2 = \left(\frac{\sqrt{6c}}{\sqrt{12c}}\right)^2 = \frac{1}{2}.$$

With the remaining probability 1/2, the final price is different in the two equilibria.

Furthermore, in the first case the two equilibria also result in the same final allocation. In the second case, the IR equilibrium results in the same final allocation if and only if $v_u \in [p^*, \max\{v_s, v_t\}]$, where $u = \min\{t' > t : v_{t'} \geq p^*\}$ is the next bidder to arrive after t with valuation above p^*. Conditional on being in case 2, and assuming uniform values, this has probability 1/2. Case 2 itself has probability

$$2\left(\frac{1-p^*}{1-p^{**}}\right)\left(\frac{p^*-p^{**}}{1-p^{**}}\right) \;=\; 2\left(\frac{\sqrt{12c}-\sqrt{6c}}{\sqrt{12c}}\right)\left(\frac{\sqrt{6c}}{\sqrt{12c}}\right) \;=\; \sqrt{2}-1.$$

In the third case, the two equilibria must result in different allocations. Thus, the overall probability of the IR and DR equilibrium resulting in the same final allocation is

$$\frac{1}{2} \;+\; \left(\sqrt{2}-1\right)\cdot\frac{1}{2} \;=\; \sqrt{1/2}.$$

With the remaining probability $1-\sqrt{1/2}$, the final allocation is different in the two equilibria.

Step 2. Next, we examine the relative changes in final price and welfare across equilibria, as $T \to \infty$. In the IR equilibrium, entry stops once two bidders have entered with valuations above p^*; the probability that this happens approaches one. Given that all bidders reveal their valuations truthfully, the winner's valuation is the higher of two values conditionally distributed on $[p^*, \overline{v}]$, and the price is the lower of two values conditionally distributed on $[p^*, \overline{v}]$. Under the uniform value assumption, this boils down to

$$\begin{aligned}
\lim_{T\to\infty} E[w_{IR}^T] \;&=\; \tfrac{2}{3}\overline{v}+\tfrac{1}{3}p^* \;=\; \tfrac{2}{3}+\tfrac{1}{3}\left(1-\sqrt{6c}\right) \;=\; 1-\tfrac{1}{3}\sqrt{6c},\\
\lim_{T\to\infty} E[p_{IR}^T] \;&=\; \tfrac{1}{3}\overline{v}+\tfrac{2}{3}p^* \;=\; \tfrac{1}{3}+\tfrac{2}{3}\left(1-\sqrt{6c}\right) \;=\; 1-\tfrac{2}{3}\sqrt{6c}.
\end{aligned}$$

In the DR equilibrium, entry stops once two bidders have entered with valuations above p^{**}. As $T \to \infty$, the probability that this happens approaches one. Given that all bidders bid truthfully in the final period, the winner's valuation is the higher of two values conditionally distributed on $[p^{**}, \overline{v}]$, and the price is the lower of two values conditionally distributed on $[p^{**}, \overline{v}]$. Under the uniform value assumption, this boils down to

$$\lim_{T\to\infty} E[w_{DR}^T] \;=\; \tfrac{2}{3}\overline{v}+\tfrac{1}{3}p^* \;=\; \tfrac{2}{3}+\tfrac{1}{3}\left(1-\sqrt{12c}\right) \;=\; 1-\tfrac{1}{3}\sqrt{12c},$$

$$\lim_{T\to\infty} E[p_{DR}^T] \;=\; \tfrac{1}{3}\overline{v} + \tfrac{2}{3}p^* \;=\; \tfrac{1}{3} + \tfrac{2}{3}\left(1 - \sqrt{12c}\right) \;=\; 1 - \tfrac{2}{3}\sqrt{12c}.$$

The expression in the result for the relative difference in seller surplus across equilibria follows immediately from the above. To examine welfare, we also need to compute the expected number of entries in both equilibria. Consider the IR equilibrium first. The expected arrival time of the first bidder with valuation above p^* is

$$(1-p^*)\cdot 1 + p^*\Big((1-p^*)\cdot 2 + p^*\Big((1-p^*)\cdot 3 + p^*\Big(\ldots\Big)\Big)\Big)$$
$$= \;(1-p^*)\left[\frac{1}{1-p^*} + p^*\frac{1}{1-p^*} + \ldots\right] \;=\; \frac{1}{1-p^*}.$$

The expected duration from the first arrival of a bidder with valuation above p^* to the next such arrival is the same. Thus, the expected number of bidders in the IR equilibrium is $2/(1-p^*) = 1/\sqrt{6c}$, and expected welfare is

$$1 - \frac{1}{3}\sqrt{6c} - \frac{2}{\sqrt{6c}}\cdot c \;=\; 1 - \sqrt{(8/3)c} \;\text{ as }\; T\to\infty.$$

For the DR equilibrium, the expected number of bidders is similarly given by $2/(1-p^{**}) = 2/\sqrt{12c}$, and expected welfare is

$$1 - \frac{1}{3}\sqrt{12c} - \frac{2}{\sqrt{12c}}\cdot c \;=\; 1 - \sqrt{3c} \;\text{ as }\; T\to\infty.$$

The expression in the result for the relative difference in expected welfare across equilibria now follows. □

Appendix B: Computations

In Section 7 we quantitatively compare the immediate revelation (IR) equilibrium and the delayed revelation (DR) equilibrium for the case of uniformly distributed values on the unit interval. In the following, we describe how the statistics reported in Table 1 are computed.

B.1 Probabilities of different outcomes

The outcome in the DR equilibrium will be different from the outcome in the IR equilibrium if two conditions hold: First, two bidders must arrive before the final period T whose valuations are above p^{**} (otherwise, entry will not be deterred in either equilibrium). Second, at least on of these two bidders must have valuation below p^* (otherwise, entry will be deterred in the same period in both equilibria). This gives us two cases:

1. The second colluding bidder arrives in period t and both colluding bidders have valuations between p^{**} and p^*. The probability of this event is

$$C^t = (t-1)F(p^{**})^{t-2}((F(p^*) - F(p^{**}))^2.$$

   Conditional on this event, the highest and second-highest valuations among the two colluding bidders are distributed as follows:

$$M_1(v) = \left(\frac{F(v) - F(p^{**})}{F(p^*) - F(p^{**})}\right)^2, \quad M_2(v) = 1 - \left(\frac{F(p^*) - F(v)}{F(p^*) - F(p^{**})}\right)^2.$$

2. The second colluding bidder arrives in period t and exactly one colluding bidder has valuation between p^{**} and p^* (and the other has valuation between p^* and $\overline{v}$). The probability of this event is

$$\widetilde{C}^t = 2(t-1)F(p^{**})^{t-2}(F(p^*) - F(p^{**}))(1 - F(p^*)).$$

   Conditional on this event, the highest and second-highest valuations among the two colluding bidders are distributed as follows:

$$\widetilde{M}_1(v) = \frac{F(v) - F(p^*)}{1 - F(p^*)}, \quad \widetilde{M}_2(v) = \frac{F(v) - F(p^{**})}{F(p^*) - F(p^{**})}.$$

The following probabilities can now be evaluated numerically.

B.1.1 Allocations

In case 1, a different bidder will win in IR equilibrium if at least one bidder is deterred in DR equilibrium whose valuation exceeds the higher of the two colluding bidders' valuations. If v denotes the higher of the colluding bidders' valuations, this has conditional probability $1-F(v)^{T-t}$. In case 2, a different bidder will win in IR equilibrium if the valuation of the first bidder in $t+1, t+2, \ldots, T$ with valuation above p^* exceeds the higher of the colluding bidders' valuations. If v denotes the higher of the colluding bidders' valuations, this has conditional probability

$$(1-F(v))\Big[1+F(p^*)+F(p^*)^2+\ \ldots\ +F(p^*)^{T-t-1}\Big] = \frac{1-F(p^*)^{T-t}}{1-F(p^*)}(1-F(v)).$$

Thus, the probability that the winner in the DR equilibrium is not the same as the winner in the IR equilibrium is

$$\sum_{t=2}^{T-1}\left(C^t\int_{p^{**}}^{p^*} 1-F(v)^{T-t}dM_1(v)+\widetilde{C}^t\int_{p^*}^{\overline{v}}(1-F(v))\frac{1-F(p^*)^{T-t}}{1-F(p^*)}d\widetilde{M}_1(v)\right).$$

For $v_i \sim U[0,1]$, this probability can be expressed as

$$2\sum_{t=2}^{T-1}(t-1)(p^{**})^{t-2}\left(\left[\frac{v^2}{2}-p^{**}v-\frac{v^{T-t+2}}{T-t+2}+p^{**}\frac{v^{T-t+1}}{T-t+1}\right]_{v=p^{**}}^{v=p^*}\right.$$
$$\left.+\ (p^*-p^{**})\frac{1-(p^*)^{T-t}}{1-p^*}\left[v-\frac{v^2}{2}\right]_{v=p^*}^{v=1}\right),$$

where $p^*=1-\sqrt{6c}$ and $p^{**}=1-\sqrt{12c}$.

B.1.2 Prices

The final price will be different in IR equilibrium if at least one bidder is deterred in DR equilibrium whose valuation exceeds the lower of the two colluding bidders' valuations. If v denotes the higher of the colluding bidders' valuations, this has conditional probability $1-F(v)^{T-t}$. Thus, the probability that the final price in the DR equilibrium is not the same as the final price in the IR equilibrium is

$$\sum_{t=2}^{T-1}\left(C^t\int_{p^{**}}^{p^*} 1-F(v)^{T-t}dM_2(v)+\widetilde{C}^t\int_{p^{**}}^{p^*} 1-F(v)^{T-t}d\widetilde{M}_2(v)\right).$$

For $v_i \sim U[0,1]$, this probability can be expressed as

$$2\sum_{t=2}^{T-1}(t-1)(p^{**})^{t-2}\left[v-\frac{v^2}{2}-\frac{v^{T-t+1}}{T-t+1}+\frac{v^{T-t+2}}{T-t+2}\right]_{v=p^{**}}^{v=p^*},$$

where $p^* = 1 - \sqrt{6c}$ and $p^{**} = 1 - \sqrt{12c}$.

B.2 Expected seller surplus and welfare

Seller surplus and welfare will depend on the following variables: The highest valuation among the bidders in B^T (w^T), the second-highest valuation among the bidders in B^T (p^T, given truthful bidding in the last period), and the entry costs paid by participating bidders, $c|B^T|$. Suppose the IR equilibrium is played, and consider the following three cases:

1. No potential bidder has valuation above p^*, and T potential bidders have valuations below p^*. In this case, all T bidders participate, so that $|B^T| = T$. This event has probability
$$C^0 = F(p^*)^T,$$
and w^T and p^T are conditionally distributed by
$$M_1^0(v) = \left(\frac{F(v)}{F(p^*)}\right)^T, \quad M_2^0(v) = \left(\frac{F(v)}{F(p^*)}\right)^T + T\frac{F(p^*) - F(v)}{F(p^*)}\left(\frac{F(v)}{F(p^*)}\right)^{T-1}.$$

2. One potential bidder has valuation above p^*, and $T - 1$ potential bidders have valuations below p^*. In this case, all T bidders participate, so that $|B^T| = T$. This event has probability
$$C^1 = T(1 - F(p^*))F(p^*)^{T-1},$$
and w^T and p^T are conditionally distributed by
$$M_1^1(v) = \frac{F(v) - F(p^*)}{1 - F(p^*)}, \quad M_2^1(v) = \left(\frac{F(v)}{F(p^*)}\right)^{T-1}.$$

3. At least two potential bidders have valuations above p^*, the second of these bidders arrives in period t ($2 \leq t \leq T$), and all other bidders who arrive before period t have valuations below p^*. In this case, the first t bidders participate, so that $|B^T| = t$. This event has probability
$$C^{2,t} = (t-1)(1 - F(p^*))^2 F(p^*)^{t-2},$$

and w^T and p^T are conditionally distributed by

$$M_1^2(v) = \left(\frac{F(v) - F(p^*)}{1 - F(p^*)}\right)^2, \quad M_2^2(v) = 1 - \left(\frac{1 - F(v)}{1 - F(p^*)}\right)^2.$$

In the DR equilibrium, we have three analogous cases, with p^{**} replacing p^*, and we use $\widetilde{C}$ and $\widetilde{M}$ to denote the corresponding probabilities and distributions. The following expectations can now be evaluated numerically:

B.2.1 Seller surplus

Seller surplus is the final price, which is the second-highest valuation among bidders in B^T. In IR equilibrium, this is

$$E[p_{IR}^T] \;=\; C^0 \int_{\underline{v}}^{p^*} v dM_2^0(v) \;+\; C^1 \int_{\underline{v}}^{p^*} v dM_2^1(v) \;+\; \sum_{t=2}^{T} C^{2,t} \int_{p^*}^{\overline{v}} v dM_2^2(v).$$

For $v_i \sim U[0,1]$, this can be expressed as

$$\frac{T-1}{T+1}(p^*)^{T+1} \;+\; (T-1)(1-p^*)(p^*)^T \\ \qquad\qquad + \; \frac{1}{3}(1-2p^*)\Big(1 - T(p^*)^{T-1} + (T-1)(p^*)^T\Big),$$

where $p^* = 1 - \sqrt{6c}$. Similarly, in DR equilibrium, we have

$$E[p_{DR}^T] \;=\; \widetilde{C}^0 \int_{\underline{v}}^{p^{**}} v d\widetilde{M}_2^0(v) \;+\; \widetilde{C}^1 \int_{\underline{v}}^{p^{**}} v d\widetilde{M}_2^1(v) \;+\; \sum_{t=2}^{T} \widetilde{C}^{2,t} \int_{p^{**}}^{\overline{v}} v d\widetilde{M}_2^2(v),$$

and for $v_i \sim U[0,1]$, this can be expressed as

$$\frac{T-1}{T+1}(p^{**})^{T+1} \;+\; (T-1)(1-p^{**})(p^{**})^T \\ \qquad\qquad + \; \frac{1}{3}(1-2p^{**})\Big(1 - T(p^{**})^{T-1} + (T-1)(p^{**})^T\Big),$$

where $p^{**} = 1 - \sqrt{12c}$.

B.2.2 Welfare

Welfare is the highest valuation among the bidders in B^T, w^T, minus the entry costs paid by these bidders, $c|B^T|$. In IR equilibrium, this is

$$E[w^T_{IR}] \;=\; C^0\left(\int_{\underline{v}}^{p^*} vdM_1^0(v) - Tc\right) \;+\; C^1\left(\int_{p^*}^{\overline{v}} vdM_1^1(v) - Tc\right) \\ +\; \sum_{t=2}^{T} C^{2,t}\left(\int_{p^*}^{\overline{v}} vdM_1^2(v) - tc\right).$$

For $v_i \sim U[0,1]$, this can be expressed as

$$\frac{T}{T+1}(p^*)^{T+1} \;+\; \frac{T}{2}\Big((p^*)^{T-1} - (p^*)^{T+1}\Big) \;+\; \frac{1}{3}(2+p^*)\Big(1 - T(p^*)^{T-1} + (T-1)(p^*)^T\Big) \\ -\; c\left[2\frac{1-(p^*)^{T-1}}{1-p^*} - (T-2)(p^*)^{T-1}\right],$$

where $p^* = 1 - \sqrt{6c}$. Similarly, in DR equilibrium, we have

$$E[w^T_{DR}] \;=\; \widetilde{C}^0\left(\int_{\underline{v}}^{p^{**}} vd\widetilde{M}_1^0(v) - Tc\right) \;+\; \widetilde{C}^1\left(\int_{p^{**}}^{\overline{v}} vd\widetilde{M}_1^1(v) - Tc\right) \\ +\; \sum_{t=2}^{T} \widetilde{C}^{2,t}\left(\int_{p^{**}}^{\overline{v}} vd\widetilde{M}_1^2(v) - tc\right).$$

For $v_i \sim U[0,1]$, this can be expressed as

$$\frac{T}{T+1}(p^{**})^{T+1} \;+\; \frac{T}{2}\Big((p^{**})^{T-1} - (p^{**})^{T+1}\Big) \;+\; \frac{1}{3}(2+p^{**})\Big(1 - T(p^{**})^{T-1} \\ +\,(T-1)(p^{**})^T\Big) \;-\; c\left[2\frac{1-(p^{**})^{T-1}}{1-p^{**}} - (T-2)(p^{**})^{T-1}\right],$$

where $p^* = 1 - \sqrt{12c}$.

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