Auction Design without Quasilinear Preferences

Brian Baisa *†

November 20, 2012

Job Market Paper

Abstract

I analyze auction design in a private value setting where I remove the standard restriction that bidders' preferences are linear in money. Instead, I only assume that bidders' have positive wealth effects and declining marginal utility of money. While most research in auction design restricts bidders' preferences to be quasilinear, there are many economic environments in which this restriction is violated. I show that removing the quasilinearity restriction leads to qualitatively different solutions to the auction design problem whether we are concerned with efficiency or revenue maximization.

On efficiency, I show that probabilistic allocations of the good can Pareto dominate the second price auction; and there is no symmetric mechanism that is both Pareto efficient and dominant-strategy implementable. On revenue, I construct a probability demand mechanism with greater expected revenues than standard auction formats when there are sufficiently many bidders.

Keywords: Auctions; Mechanism Design; Vickrey Auctions; Wealth Effects.
JEL Classification: C70, D44, D82.

---

*brian.baisa@yale.edu, Yale University, Department of Economics.

†I gratefully acknowledge financial support received from the Cowles Foundation Fellowship, the Yale School of Management, the International Center for Finance and the Whitebox Advisors Fellowship. I am especially grateful to Dirk Bergemann, Benjamin Polak, Larry Samuelson and Johannes Hörner for numerous conversations and encouragement while advising me with this project. I would also like to thank Cihan Artunç, Yeon-Koo Che, Eduardo Faingold, Amanda Gregg, Yingni Guo, Vitor Farinha Luz, Drew Fudenberg, Adam Kapor, Phil Haile, Sofia Moroni, David Rappoport, Peter Troyan and Kieran Walsh for helpful comments and conversations regarding this project.

1 Introduction

In the auction design literature, it is standard to assume that bidders have quasilinear preferences. Yet there are many well-known environments in which this restriction is violated: bidders may be risk averse, have wealth effects, face financing constraints or be budget constrained. In this paper, I study a canonical private value auction setting for a single good and relax the quasilinearity restriction by assuming only that bidders have declining marginal utility of money and positive wealth effects; i.e., the indivisible good for sale is a normal good. I show that the auction design problem leads to qualitatively different prescriptions relative to those of the quasilinear benchmark. Instead of using standard formats where the good is given to the highest bidder with probability one, the auctioneer prefers mechanisms where she can allocate the good to one of many different bidders, each with strictly positive probability. For auctioneers concerned with efficiency, such probabilistic allocations can Pareto dominate the second price auction. And for auctioneers concerned with maximizing expected revenues, I construct a probability demand mechanism that generates strictly greater expected revenues than standard auction formats when there are sufficiently many bidders.

The setting I study is very general. The only restrictions I impose on a bidder's preferences are declining marginal utility of money and positive wealth effects. I allow for multidimensional heterogeneity across bidders,[1] including cases with heterogeneity across risk preferences, initial wealth levels, wealth effects, financial/budget constraints and/or valuations. I place no functional form restrictions on bidders' utility functions.

I first study the problem of an auctioneer concerned with efficiency. In a private value setting with quasilinear preferences, the Vickrey-Clarke-Groves mechanism implements a Pareto efficient allocation in dominant strategies. In the single good environment this is the second price auction. The second price auction has the additional desirable property that it is symmetric. In other words, the outcome of the auction is invariant to relabeling bidders. I show that without the quasilinearity restriction, there are probabilistic allocations of the good that can Pareto dominate the dominant strategy equilibrium outcome of the second price auction (Proposition 2). This leads to a natural question: while the dominant strategy equilibrium outcome of the second price auction is not Pareto efficient, is there another symmetric mechanism that satisfies these two desirable properties in this more general environment? In answering this question, I obtain a negative result: I show there is no symmetric mechanism that implements a Pareto efficient allocation in dominant strategies (Proposition 3).

While I obtain an impossibility result for the problem of an auctioneer concerned with

[1] The benchmark model only allows for heterogeneity across a single dimension, a bidder's valuation.

efficiency, I achieve more positive results for the problem of an auctioneer concerned with maximizing expected revenues. I show that the auctioneer can use probabilistic allocations of the good to increase revenues relative to standard auction formats. In order to show this, I first study bidders' preferences towards gambles. I begin by studying an example where a bidder is indifferent between buying and not buying the good at a price of 64. I show that rather than buying the good at a price of 64, the bidder prefers to enter a carefully structured gamble where she wins with good with probability $\frac{1}{2}$ and pays strictly greater than 32 in expectation. Thus, there are gambles she prefers where she wins the good with a probability less than one and pays a price per unit of probability that strictly exceeds her willingness to pay for the good. The gambles are preferred to buying the good at a price of 64. Proposition 1 generalizes this result for any bidder who has positive wealth effects and declining marginal utility of money.

I use the intuition from this proposition to construct a mechanism that I call the probability demand mechanism. In the mechanism, the auctioneer exploits this feature of bidder preferences by selling probabilities of winning the good like a divisible good that is in net supply one. Bidders report a demand curve over probabilities of winning. The curve gives the total probability of winning the bidder demands (Q) for a given price per unit price of probability (P). I use intuition developed in my example to show that a bidder demands a strictly positive probability of winning, even at per unit prices above her willingness to pay for the good.[2] Thus, the auctioneer can increase her revenues by selling probabilities of winning the good to many different bidders. My probability demand mechanism does this. In the mechanism, bidders report their probability demand curves and the auctioneer then uses an algorithm similar to that of the Vickrey auction for a divisible good to determine each bidder's probability of winning and expected transfer.

I make revenue comparisons between the probability demand mechanism and other standard auction formats under the assumption that bidders' preferences are independently and identically distributed. With sufficiently many bidders, the expected revenues from the probability demand mechanism exceed the expected revenues from any auction in a large class of standard formats (Propositions 8 and 9). This class of standard formats includes the first price, second price and all pay auctions, as well as modifications of these formats that allow for entry fees and/or reserve prices.

I illustrate the practical applicability of the probability demand mechanism by considering an example with financially constrained bidders. I show that the probability demand mechanism has greater expected revenues than commonly studied auction formats, even

[2] Her willingness to pay for the good is the price that makes her indifferent between accepting or rejecting a take-it-or-leave-it offer for the good.

when there are few bidders.

The rest of the paper proceeds as follows. The remainder of the introduction relates my work to the current literature on auction design. Section 2 describes the model and specifies the assumptions I place on bidders' preferences. Section 3 motivates the use of probabilistic allocations and provides an example in which the dominant strategy equilibrium outcome of the second price auction is Pareto dominated by a probabilistic allocation of the good. Section 4 shows there is no symmetric mechanism that respects individual rationality and implements a Pareto efficient allocation in dominant strategies. Section 5 outlines the construction of the probability demand mechanism. Section 6 focuses on revenue comparisons between the probability demand mechanism and standard auction formats. Section 7 provides a numerical example illustrating the practical applicability of my results. Section 8 shows that some of my results can be extended to give new insights into the problem of auctions for divisible goods. Section 9 concludes.

Related literature

This paper builds on the private value auction literature pioneered by Vickrey (1961) and later by Myerson (1981) and Riley and Samuelson (1981). Each of these papers addresses an auction design problem under the restriction that bidders' preferences are quasilinear. Myerson studies the problem of an expected revenue maximizing auctioneer and Vickrey studies the problem of an auctioneer concerned with efficiency. Their results show the auction design problem is solved by a mechanism where the good is assigned to a bidder only if she has the highest valuation among all bidders.[3] My results show there are qualitative differences in the solution to the auction design problem when we drop the quasilinearity restriction. Instead, I show the solution to the auction design problem implies probabilistic allocations of the good that give many different bidders a positive probability of winning including bidders who do not have the highest 'valuation'.

While most research in auction design focuses on the quasilinear environment, there is a smaller literature that considers auction models without the assumption of quasilinearity. This was first discussed by Maskin and Riley (1984). Their paper characterizes certain properties of expected revenue maximizing auctions when a bidder's type is a single dimensional variable, $\theta \in [0, 1]$. They show that the exact construction depends on the common prior over the distribution of types and the functional form of the bidders' utility functions and describe some basic properties of the revenue-maximizing auction.

Matthews (1983, 1987) and Hu, Matthews and Zhou (2010) also study auctions without

[3]In Myerson (1981), the good is assigned to the bidder with the highest 'virtual' valuation. Under the monotone likelihood ratio property, this corresponds to the bidder with the highest valuation.

quasilinearity. However, they focus on a more structured setting where all bidders have identical risk preferences and there are no wealth effects. Within this framework, they make comparisons between auction formats.

Che and Gale (2006) consider a payoff environment that is closer to the one studied here. They assume that bidders have multidimensional types and use this to show that when bidders are risk averse, the first price auction generates greater revenues than the second price auction. This builds on their earlier work, which studies standard auction formats with more specific departures from the quasilinear environment (see Che and Gale (1996, 1998, 2000)).

My work differs from this prior work in two respects: (1) I consider a very general setting that allows for heterogeneity in bidders across many dimensions; and (2) I study the problem from an auction design perspective. This contrasts with prior work that focuses on comparisons between standard auction formats.

My solution to the auction design problem uses probabilistic allocations of the good. This approach has been advocated in other auction design settings. For the case of budget-constrained bidders, Pai and Vohra (2010) show that the expected revenue-maximizing mechanism may employ probabilistic allocations. Baisa (2012) studies an auction design problem in which bidders use a non-linear probability weighting function to evaluate risks (as described by Kahneman and Tversky (1979, 1992)) and finds that the solution to the auction design problem uses probabilistic allocations of the good to exploit features of the bidder's probability weighting function.

Outside of auction design, Morgan (2000) advocates using lotteries to raise money for public goods. The intuition for using lotteries in Morgan's model is distinct from the intuition used here. Morgan shows that lottery tickets can help to overcome the free rider problem when people's preferences are linear in money. No such free rider problem is present in my model.

2 The model

2.1 The payoff environment

I consider a private value auction setting with a single risk neutral seller and $N \geq 2$ buyers, indexed by $i \in \{1, \ldots, N\}$. There is a single indivisible object for sale. Bidder i's preferences are given by u_i, where

$$u_i : \{0, 1\} \times \mathbb{R} \to \mathbb{R}.$$

I let $u_i(1, w_i)$ denote bidder i's utility when she owns the object and has wealth w_i. Similarly, $u_i(0, w_i)$ denotes bidder i's utility when she does not own the object and has wealth w_i. The object is a "good" and not a "bad"; it is better to have the object than to not have the object for any fixed wealth level, i.e.

$$u_i(1, w) > u_i(0, w), \ \forall w.$$

This environment differs from the quasilinear case where a bidder's preferences are completely described by a one-dimensional signal, her 'valuation' of the object. This valuation is the most i is willing to pay for the object. In this setting, I let $k(u_i, w_i)$ denote bidder i's willingness to pay for the good when she has an initial wealth w_i and a utility function u_i. This is the highest price bidder i will accept in a take-it-or-leave-it offer for the good. Given i's preferences, u_i, and initial wealth level, w_i, her willingness to pay is given by $k(w_i, u_i)$, where k is such that

$$u_i(1, w_i - k) = u_i(0, w_i). \tag{2.1}$$

Note that bidder i's willingness to pay provides only a partial description of her preferences in this environment.

With this, I define the notion of positive wealth effects in the indivisible good setting. The notion of positive wealth effects is analogous to the familiar notion in the divisible goods case, where a bidder's demand for the good increases as her wealth increases for a constant price level.

Definition 1. (Positive wealth effects)
Bidder i with preferences u_i has positive wealth effects if

$$\frac{\partial k(w, u_i)}{\partial w} > 0,$$

for all w.

Definition 2 states that increasing a bidder's wealth increases her willingness to pay for the good. In this paper, I will only impose two assumptions on a bidder's preferences. I state them below.

Assumption 1. (Declining marginal utility of money)
Bidder i's utility is twice continuously differentiable and strictly increasing in money,

$$\frac{\partial u_i(x, w)}{\partial w} > 0, \ \textit{and} \ \frac{\partial^2 u_i(x, w)}{\partial w^2} < 0 \ \textit{for any} \ x = 0, 1.$$

Assumption 2. *Bidders have positive wealth effects.*

I define $\mathcal{U}$ as the set of all utility functions which satisfy Assumptions 1 and 2.

2.2 Allocations and mechanisms

By the revelation principle, I can limit attention to direct revelation mechanisms. A mechanism describes how the good is allocated and how transfers are made. I define $\mathbb{A}$ as the set of all feasible assignments, where

$$\mathbb{A} := \{a|\ a \in \{0,1\}^N \text{ and } \sum_{i=1}^{N} a_i \leq 1\},$$

where $a_i = 1$ if bidder i is given the object. A feasible outcome ϕ specifies both transfers and a feasible assignment: $\phi \in \mathbb{A} \times \mathbb{R}^N$. I define $\Phi := \mathbb{A} \times \mathbb{R}^N$ as the set of feasible outcomes. A (probabilistic) allocation is a distribution over feasible outcomes. Thus, an allocation α is an element of $\Delta(\Phi)$.

I use the notation $\mathbb{E}_\alpha [u_i|w_i]$ to denote the expected utility of bidder i under allocation $\alpha \in \Delta(\Phi)$ when she has preferences u_i and initial wealth w_i. I refer to the auctioneer as person 0 and use $\mathbb{E}_\alpha [u_0]$ to denote the expected transfers she receives under the allocation α.

I will be particularly interested in Pareto efficient allocations. An allocation is ex-post Pareto efficient if for a given profile of preferences, increasing one person's expected utility necessarily decreases another person's expected utility (including the auctioneer).

Definition 2. (Ex-post Pareto efficient allocations)
Assume bidders have preferences $u = (u_1, \ldots, u_N) \in \mathcal{U}^N$ and initial wealth levels, $w = (w_1, \ldots, w_N)$. A feasible allocation α is ex-post Pareto efficient if there does not exist a $\alpha' \in \Delta(\Phi)$ s.t.

$$\mathbb{E}_{\alpha'} [u_i|w_i] \geq \mathbb{E}_\alpha [u_i|w_i] \ \forall i = 0, 1 \ldots N.$$

where the above inequality holds strictly for at least one i.

This is equivalent to the notion of an ex-post Pareto superior (efficient) allocation given in Holmström and Myerson (1983) and Myerson (1991).

In the quasilinear environment, all Pareto efficient allocations assign the good to the bidder(s) who has the highest 'valuation' with probability 1. Without quasilinearity, Pareto efficiency places restrictions on both how the object is allocated and also how transfers are made.

A direct revelation mechanism M maps a profile of reported preferences and initial wealth levels to an allocation. That is,

$$M : \mathcal{U}^N \times \mathbb{R}^N \to \Delta(\Phi).$$

The direct revelation mechanism implements an ex-post Pareto efficient allocation if for any reported preferences $u = (u_1 \ldots u_N)$ and initial wealth levels $w = (w_1 \ldots w_N)$, the allocation provided by M is Pareto efficient.

Thus, a mechanism implements a Pareto efficient allocation if it is able to overcome the incomplete information problem. That is, the mechanism specifies an allocation that would be Pareto efficient in the complete information environment where all bidders' private information is known.

With non-quasilinear preferences, it is with loss of generality to consider only expected transfers. The Pareto efficiency requirement restricts the way payments must be structured.

A mechanism is implementable in dominant strategies if it is a best response for all bidders to truthfully report their private information to the auctioneer, independent of their opponents' reports.

Definition 3. (Dominant strategy incentive compatibility)
A direct revelation mechanism M is dominant strategy incentive compatible if for any profile of preferences and initial wealth levels $u = (u_i, u_{-i}) \in \mathcal{U}^N$ and $w = (w_i, w_{-i}) \in \mathbb{R}^N$,

$$\mathbb{E}_{M(u,w)} \left[u_i | w_i \right] \geq \mathbb{E}_{M((u',u_{-i}),(w',w_{-i}))} \left[u_i | w_i \right],$$

for any $(u', w') \in \mathcal{U} \times \mathbb{R}$ and $i = 1 \ldots N$.

3 Probabilistic allocations

The auction design problem has been well studied under the restriction that bidders have quasilinear preferences. An auctioneer concerned with implementing a Pareto efficient allocation can do so in dominant strategies by using a second price auction. Myerson (1981) shows that the revenue-maximizing auction assigns the good to the bidder with the highest (non-negative) virtual valuation. Dropping the quasilinearity restriction gives qualitatively different solutions to each problem. In particular, the auctioneer prefers to use mechanisms which allow for probabilistic allocations of the good. Such probabilistic allocations give many bidders a positive probability of winning the good. This contrasts with standard mechanisms that assign the good to a bidder with the highest willingness to pay with certainty.

In this section, I motivate the use of probabilistic allocations by studying a bidder's preferences towards gambles. I first consider an example of a bidder who is indifferent between accepting and not accepting a take-it-or-leave-it offer for the good at a price of 64 (i.e. her willingness to pay for the good is 64). I show that rather than buying the good at a price of 64, the bidder prefers instead to enter a carefully structured gamble where she wins with good with probability $q \in (0, 1)$ and pays strictly greater than $64q$ to the auctioneer in expectation. When preferences are restricted to be quasilinear, such gambles violate bidders' individual rationality constraints. Proposition 1 shows that this holds more generally, for any bidder who has positive wealth effects and declining marginal utility of money. This result illustrates a contrast between this environment and the quasilinear environment.

I use this result to show that there exist probabilistic allocations of the good that Pareto dominate the dominant strategy equilibrium outcome of the second price auction (Proposition 2). Thus, while the second price auction still has a dominant strategy equilibrium without the quasilinearity restriction, this equilibrium does not necessarily implement a Pareto efficient allocation.

I begin with the example of a bidder with an initial wealth of w_i and preferences described by the utility function u, where

$$u(x, w) = 4\mathbb{I}_{x=1} + \sqrt{w}.$$

She receives 4 utils from being given the object and $\sqrt{w}$ utils from having final wealth w. Her willingness to pay for the good k is the price where she is indifferent between buying and not buying, $k:\ u(1, w_i - k) = u(0, w_i)$. Her willingness to pay is the horizontal distance between the curves $u_i(1, w)$ and $u_i(0, w)$. Figure 3.1 illustrates that her willingness to pay increases in her initial wealth level. In addition, each curve is concave in wealth. Thus, her preferences display positive wealth effects and declining marginal utility of money (Assumptions 1 and 2).

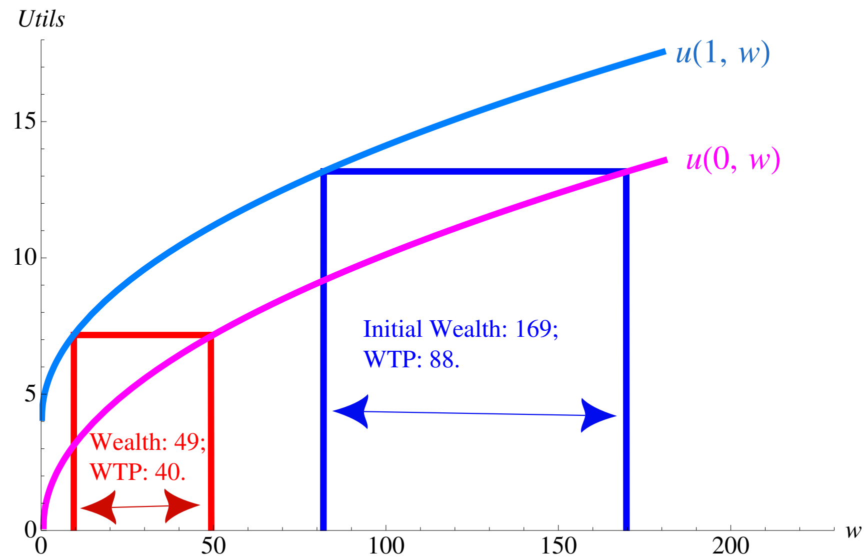

Figure 3.1: $u(x, w) = 4\mathbb{I}_{x=1} + \sqrt{w}$

Assume the bidder is given a take-it-or-leave-it offer to buy the good at a price of 64. If she has initial wealth w_i, her utility from accepting the offer is $u(1, w_i - 64)$. If she rejects the offer and remains at her status quo, her utility is $u(0, w_i)$. When the bidder has an initial wealth of 100, she is indifferent between accepting and rejecting the take-it-or-leave-it offer for the good at the price of 64. If her initial wealth is lower than 100, she prefers to reject the offer and conversely, if her initial wealth is greater than 100, she prefers to accept the offer. Figure 3.2 shows her indirect utility as a function of her initial wealth.

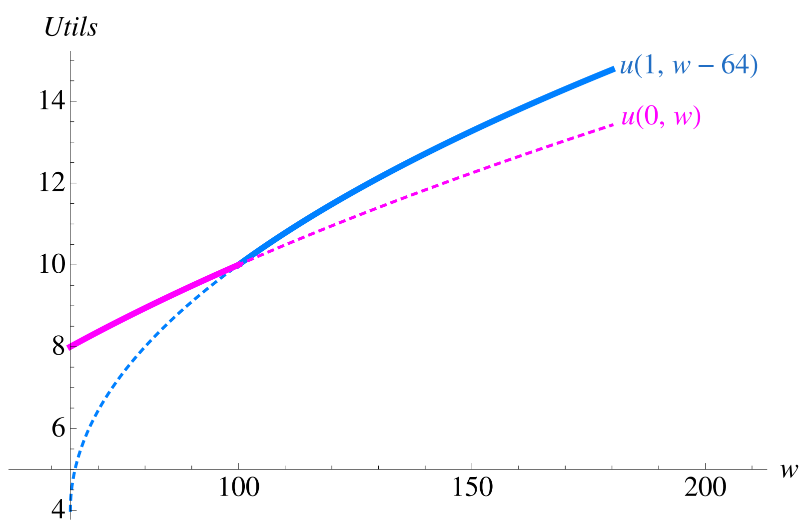

Figure 3.2: Indirect utility function for a fixed price of 64.

Notice there is a convex kink in the bidders indirect utility function at the wealth level $w = 100$, the point where she is indifferent between accepting and rejecting the take-it-or-leave-it offer. This is clearly seen by zooming in on the point of intersection.

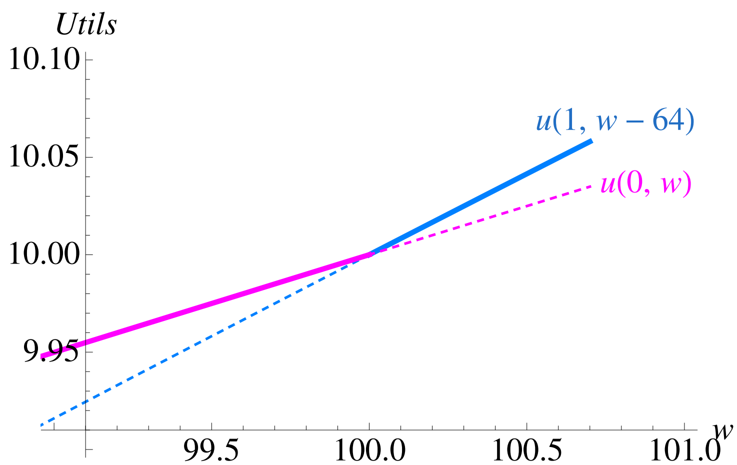

Figure 3.3: Indirect utility function for wealth between 99 and 101.

The intuition for the convexity in the indirect utility follows from positive wealth effects. For low wealth levels, the utility of remaining at the status quo exceeds the utility of accepting the offer and paying 64 for the object. As wealth increases, there is a natural complementarity between owning the good and having higher wealth levels that causes the utility from purchasing the good ($u(1, w - 64)$) to increases faster in w than the utility from not purchasing the good ($u(0, w)$). At the point where the two curves cross, there is a convexity because one curve is increasing faster than the other. That is, the curves $u(1, w - 64)$ and $u(0, w)$ cross once (in w), with the former crossing the latter from below. When the two curves cross at $w = 100$, it follows that $\frac{\partial}{\partial w}u(1, w - 64) > \frac{\partial}{\partial w}u(0, w)$.

Consider a bidder with initial wealth of 100. Since she is indifferent between accepting or rejecting a take-it-or-leave-it offer to buy the good at a price of 64, she is also indifferent between entering a gamble where she wins with probability $\frac{1}{2}$ and pays 64 when she wins and 0 when she does not. The convexity in the bidder's indirect utility function implies that she instead (strictly) prefers to enter a gamble of the form: with probability $\frac{1}{2}$ get the good and pay $63 + \epsilon$; and with probability $\frac{1}{2}$ do not get the good and pay $1 + \epsilon$, where $\epsilon > 0$ is sufficiently small. Or equivalently,

$$\frac{1}{2}u(1, 100 - 63 - \epsilon) + \frac{1}{2}u(0, 99 - \epsilon) > u(0, 100) = u(1, 100 - 64).$$

Since $\frac{\partial}{\partial w}u(1, w - 64) > \frac{\partial}{\partial w}u(0, w)$, the marginal utility gained from paying one less dollar when getting the good and pay 64 is outweighs marginal disutility from paying one additional dollar when not getting the object and paying 0.

Thus, the bidder will strictly prefer to enter a gamble of the form: win with probability $\frac{1}{2}$ and pay $63 + \epsilon$; lose with probability $\frac{1}{2}$ and pay $1 + \epsilon$, for a sufficiently small $\epsilon > 0$. The above inequality shows the bidder strictly prefers this gamble relative to her status quo (i.e. no object and wealth 100). While the bidder is willing to pay at most 64 for the object (with certainty), there are gambles this bidder is willing to accept where she wins the good with probability $\frac{1}{2}$ and pays the auctioneer $32 + \epsilon$ in expectation.[4]

Thus, the bidder strictly prefers a gamble where, in expectation, she pays a price per unit of probability that strictly exceeds her willingness to pay for the good. Proposition 1 shows that this result holds can be generalized for any bidder who has positive wealth effects and declining marginal utility of money.

Proposition 1.
Consider a bidder with preferences $u \in \mathcal{U}$, initial wealth w, and willingness to pay for the good k. There exist gambles the bidder strictly prefers to her status quo in which she wins the good with probability $q \in (0, 1)$ and pays strictly greater than qk in expectation.

In other words, Proposition 1 implies that a bidder with declining marginal utility of money and positive wealth effects is willing to accept a gamble where she wins the good with a probability less than one, yet she pays a *price per unit of probability* that exceeds her willingness to pay for the good (with certainty). The proof of the proposition follows similarly to that in the example.

This proposition highlights a crucial difference from the quasilinear environment. If we restrict a bidder's preferences to be quasilinear, and she has a willingness to pay (i.e. valuation) of k, she will always strictly prefer her status quo over any gamble where she wins the good with probability $q \in (0, 1)$ and pays strictly greater than qk in expectation. Any such gamble violates her individual rationality constraint.

Using the insights of Proposition 1, I show that there are probabilistic allocations of the good which Pareto dominate the dominant strategy equilibrium outcome of the second price auction. To see this, consider a case where there are two identical bidders, each with initial wealth w and utility function $u \in \mathcal{U}$. Thus, each has an equal willingness to pay for the good, which we call k.

In the second price auction, it is remains a dominant strategy for each bidder to bid her willingness to pay, even without the quasilinearity restriction. Thus, in the dominant

[4]This preference for lotteries is similar to that shown by Kwang (1965) who shows people can have a preference for lotteries when purchasing an indivisible good.

strategy equilibrium outcome of the second price auction, each bidder submits a bid of k. The winner is selected randomly and pays k. Since the winner pays a price that is exactly equal to her willingness to pay for the good, she is indifferent between winning and losing (i.e. remaining at her status quo wealth level).

Using Proposition 1, I construct a probabilistic allocation of the good which Pareto dominates the dominant strategy equilibrium outcome of the second price auction. I construct a gamble where each bidder wins the good with probability $\frac{1}{2}$ and pays strictly greater than $\frac{1}{2}k$ in expectation. Proposition 1 implies that there are gambles of this form that the bidder strictly prefers over her status quo. The auctioneer can allow both (identical) bidders to take one such gamble.[5] Each bidder strictly prefers the probabilistic allocation over her status quo. Thus, each bidder receives a higher expected payoff from the probabilistic allocation of the good than she would in the dominant strategy equilibrium outcome of the second price auction.[6] The auctioneer also gets higher expected revenues from the probabilistic allocation than she does from the dominant strategy equilibrium outcome of the second price auction. Thus, the probabilistic allocation is a Pareto improvement from the dominant strategy equilibrium outcome of the second price auction.[7]

Proposition 2.
There exist probabilistic allocations of the good which Pareto dominate the dominant strategy equilibrium outcome of the second price auction.

The proposition stands in contrast to work by Sakai (2008) and Saitoh and Serizawa (2008). These authors claim the second price auction is Pareto efficient in environments where agents have positive wealth effects. These authors obtain their results by implicitly restricting themselves to deterministic mechanisms and using notions of envy-freeness in place of ex-ante Pareto efficiency. This example shows that mechanisms which use probabilistic allocations rules can (ex-ante) Pareto dominate the second price auction.

Proposition 2 illustrates the first significant qualitative difference from prescriptions of the quasilinear benchmark. When bidders' preferences are restricted to be quasilinear, the second price auction implements a Pareto efficient allocation in dominant strategies. Without the restriction of quasilinearity, the auctioneer can construct a probabilistic allocation

[5]Note since each bidder wins with probability $\frac{1}{2}$, this is feasible.

[6]For this example, in the dominant strategy equilibrium outcome of the second price auction, a bidder pays her willingness to pay for the good if she is given the object. That is, the payoff from winning equals the payoff from remaining at her status quo.

[7]This result is not driven by the fact that both bidders have identical preferences. It is possible to use a similar argument in cases where bidders have different preferences or initial wealth levels. The auctioneer can also create Pareto improving probabilistic allocations where she receives higher revenues than the SPA with certainty.

of the good which Pareto dominates this allocation. In addition, this same allocation provides the auctioneer with revenues that exceed any bidder's willingness to pay for the good. Importantly, it does so without violating either bidder's individual rationality constraint.

In the remainder of the paper, I study two questions which naturally emerge from Propositions 1 and 2.

Question 1 *(efficiency):* Without the quasilinearity restriction, Proposition 2 shows that the dominant strategy equilibrium outcome of the second price auction can be Pareto dominated by a probabilistic allocation of the good. This result leads to a natural question. Is it possible to construct a mechanism that retains the desirable properties of the second price auction when we drop the quasilinearity restriction? I address this question in Section 4 and I show the answer is no.

Question 2 *(revenue):* Proposition 1 shows that there are gambles that do not violate a bidder's individual rationality constraint, where she wins the good with a probability less than one and pays a price per unit of probability that exceeds her willingness to pay for the good. This leads me to ask if the auctioneer construct a mechanism that uses probabilistic allocations to exploit this feature of bidders' preferences? Here I obtain a positive result and construct a probability demand mechanism that outperforms standard auction formats when there are sufficiently many bidders. I address this in Sections 5 and 6.

4 Pareto efficient mechanisms

In the quasilinear environment, the second price auction is the unique format that implements an individually rational and Pareto efficient allocation in dominant strategies. In addition, the second price auction is symmetric. That is, the mechanism delivers the same outcome if we relabel bidders. Such mechanisms are sometimes also referred to as being anonymous. Without the quasilinearity restriction, Proposition 2 shows that the dominant strategy equilibrium outcome of the second price auction can be Pareto dominated by probabilistic allocations of the good. This leads to the question addressed in this section: Is there a symmetric mechanism that implements a Pareto efficient allocation in dominant strategies and respects individual rationality? I show the answer to this question is no.

Without the quasilinearity assumption, dominant strategy incentive compatibility imposes more restrictions on the problem. With the quasilinearity restriction, a bidder can only misreport her type by submitting another (single-dimensional) valuation. Without quasilinearity, dominant strategy incentive compatibility requires that each bidder has no incentive to misreport her preferences for any set of reports by other bidders. She can misreport her preferences by reporting another (infinite-dimensional) utility function. Since each bidder has a much larger set of possible deviations, this makes dominant strategy implementation more difficult to achieve.

Without the quasilinearity assumption, requiring Pareto efficiency also imposes more restrictions on the allocation. When bidders' preferences are restricted to be quasilinear, Pareto efficiency requires that the bidder with the highest valuation is given the object. Proposition 2 illustrates that this is not true when we drop the quasilinearity restriction. In the relaxed environment, the relevant notion of Pareto efficiency places restrictions both on how transfers are made and with what probability each bidder receives the good.

With the quasilinearity restriction, there is a unique symmetric mechanism that is individual rational and implements a Pareto efficient allocation in dominant strategies. However, without quasilinearity, it becomes more difficult to design such a mechanism. This leads to the impossibility result I state below.

Proposition 3.
There is no symmetric direct revelation mechanism M that is individually rational and implements a Pareto efficient allocation in dominant strategies.

The proof of the proposition is by contradiction and is developed in the appendix. I show that the necessary conditions for both Pareto efficiency and dominant strategy incentive compatibility form a contradiction. I prove this result by considering only a subset of the incentive constraints required for dominant strategy implementation. I show that there is no mechanism that both satisfies this subset of incentive constraints and implements a Pareto efficient allocation. Thus, there is no mechanism that satisfies all dominant strategy incentive constraints and implements a Pareto efficient outcome.

In particular, I consider a subset that includes a continuum of types that are close to quasilinear. Additionally, I consider a single type that is more non-linear.

I use the necessary conditions for Pareto efficiency and dominant strategy incentive compatibility to show that if all players report types that are almost quasilinear, the mechanism must specify an allocation that is close to the allocation specified by the second price auction. In particular, the bidder with the highest type wins the object with certainty and pays an amount approximately equal to the second highest bidder's willingness to pay for the good.

However, when one bidder is of the non-linear type, the necessary conditions for Pareto efficiency places strict restrictions on the structure of the allocation. Under these necessary conditions, a bidder whose type is almost quasilinear can increase her payoff by underreporting. This violates the dominant strategy incentive compatibility condition.

The disconnect between the necessary conditions for dominant strategy incentive compatibility and implementation of a Pareto efficient allocation are similar to those seen in

Jehiel and Moldovanu (2001). Their results show that in an interdependent values setting where bidder types are multidimensional, it is generally impossible to construct a mechanism that is ex-post incentive compatible and also implements a Pareto efficient allocation. Proposition 3 show that there is a similar results are obtained in the private values case when we allow for multidimensional types and relax the quasilinearity restriction.

5 The probability demand mechanism

The previous section illustrated the importance of probabilistic allocations for an auctioneer who is concerned with efficiency. Proposition 2 shows that probabilistic allocations of the good can improve on efficiency relative to the prescriptions of the benchmark model, though Proposition 3 provides a more negative result.

In this section I study my second question, the problem of an auctioneer concerned with maximizing expected revenues. I obtain more positive results and show that an auctioneer can use probabilistic allocations of the good to increase revenues. I construct a probability demand mechanism that generates strictly greater expected revenues than standard auction formats when there are sufficiently many bidders.

I obtain the results in this section by building off of the intuition developed in Section 3. Recall that Proposition 1 shows that if a bidder is indifferent between accepting or rejecting a take-it-or-leave-it offer for the good at the price of k, there are instead gambles she strictly prefers to enter where she wins the good with probability q and pays strictly greater than qk in expectation. This is not true when bidders' preferences are restricted to be quasilinear. Any such gamble violates a bidder's individual rationality constraint. Most auction formats studied in the quasilinear benchmark setting do not exploit this feature of bidders' preferences. In the first and second price auctions, it is a dominated strategy for a bidder to submit a bid above her willingness to pay for the good.

I construct a probability demand mechanism that uses this property of bidder preferences to generate greater revenues than standard auction formats. In the mechanism, the auctioneer sells the bidders probabilities of winning the good. The probabilities can be thought of as a divisible good that is in net supply one. Instead of submitting a single dimensional bid, a bidder reports a probability demand curve to the auctioneer. A bidder's probability demand curve states how much probability of winning she demands when she pays a price of p per unit (in expectation). This construction exploits the fact that a bidder is willing to pay a higher per unit price for a $q \in (0, 1)$ probability of winning the good than she is willing to pay for winning the good with certainty.

The auctioneer uses the reported probability demand curves to calculate each bidder's expected transfer and probability of winning. This illustrates the connection between the probability demand mechanism and the Vickrey auction for a divisible good. In the Vickrey auction for a divisible good, bidders report demand curves for quantities of the good. The auctioneer uses the reported demand curves to calculate each bidder's transfer and quantity allocation. In the probability demand mechanism, I use a similar method to determine each bidder's expected transfers and probability of winning the good.

While there are similarities between my probability demand mechanism and the Vickrey auction for a divisible good, removing the quasilinearity restriction poses challenges not seen in the former. First, it is *with* loss of generality to only consider expected transfers when preferences are non-quasilinear. Thus, I must also precisely specify how payments are structured in the probability demand mechanism. And secondly, truth-telling is not a dominant strategy in the probability demand mechanism. Instead, I am able to place a lower bound on a bidder's report to the auctioneer. I do this by showing that it is a dominated strategy for a bidder to underreport her probability demand curve to the auctioneer. I show that this lower bound on bidders' reports is sufficient to obtain results regarding revenue comparisons between the probability demand mechanism and standard auction formats.

In this section I discuss the construction of the probability demand mechanism. Section 6 discusses revenue comparisons between the probability demand mechanism and standard auction formats.

5.1 The probability demand curve

In the probability demand mechanism, a bidder submits a probability demand curve to the auctioneer. This contrasts with standard auction formats where bidders submit single-dimensional bids.

A probability demand curve states the probability of winning the good a bidder demands when she pays a price of p per unit of probability. Since it is with loss of generality to restrict attention to expected transfers when preferences are non-quasilinear, I first specify how payments are structured. Given a bidder's expected transfers and probability of winning the good, I construct payments to maximize her expected utility. That is, payments are made efficiently.

To understand this, first consider a specific probabilistic allocation where bidder i wins the good with probability q and pays x to the auctioneer in expectation. The bidder has preferences $u_i \in \mathcal{U}$ and initial wealth w_i. The efficient payment scheme maximizes her expected utility given that she wins with probability q and pays x in expectation. Since the bidder has declining marginal utility of money (both when she she wins and loses) in the efficient payment scheme, she pays a fixed amount conditional on winning or losing. Assume that she pays p_w and p_l conditional on winning or losing, respectively. In the efficient payment scheme, p_w^* and p_l^* are such that,

$$(p_w^*, p_l^*) = \arg\max_{p_w, p_l} q u_i(1, w_i - p_w) + (1-q) u_i(0, w_i - p_l).$$

$$s.t.\ x = q p_w + (1-q) p_l.$$

The payments p_w^* and p_l^* maximize her expected utility under the constraint that she wins the good with probability q and pays x in expectation. This induces an indirect utility function V_i that is a function of her expected payments and the probability she wins the good.

$$V_i(q, -x) := q u_i(1, w_i - p_w^*) + (1-q) u_i(0, w_i - p_l^*).$$

The indirect utility function gives the maximal expected utility for bidder i when she wins with object with probability q and pays x in expectation.

If bidder i makes an efficient payment, her expected utility from winning the good with probability q and paying a price of p per unit of probability is $V_i(q, -qp)$. I form bidder i's probability demand curve by assuming that she makes efficient payment.

$$q(p, u_i, w_i) := \arg\max_{q \in [0,1]} V_i(q, -qp). \tag{5.1}$$

Bidder i's probability demand curve states the probability of winning she demands when she pays a price of p per unit of probability in expectation. I economize notation by writing bidder i's probability demand curve as $q_i(p) = q(p, u_i, w_i)$.

A bidder's probability demand curve is continuous and weakly decreasing in the price p. In addition, she demands a strictly positive probability of winning the good at a per unit price that exceed her willingness to pay for the entire good. Specifically, if her willingness to pay is k_i,[8] there exists a $\delta > 0$ such that $q_i(k_i + \delta) > 0$.

Proposition 4.
Consider bidder i with preferences $\hat{u}_i \in \mathcal{U}$ and willingness to pay k_i. Her probability demand curve $q_i(p)$ is continuous and weakly decreasing in p. In addition, there exists a $\delta > 0$ such that $q_i(k_i + \delta) > 0$.

The proposition shows a bidder is willing to buy a strictly positive probability of winning the good at a per unit price that exceeds her willingness to pay for the good. As a basis for comparison, if a bidder has quasilinear preferences, with a valuation of v for the object,

[8] That is, k_i is such that $u_i(1, w_i - k_i) = u_i(0, w_i)$.

her probability demand curve is a step function where, $q(p) = \mathbb{I}_{p \leq v}$. Notice that this is discontinuous and $q(p) = 0$ for all prices above the bidders willingness to pay for the good v.

I call highest per unit price of probability where bidder i demands a positive probability of winning $\overline{p}_i$. Formally,

$$\overline{p}_i := \sup_p \{p : q_i(p) > 0\}.$$

Figure 5.1 shows the probability demand curve for a bidder who has initial wealth of 100 and preferences $u(x, w) = 4\mathbb{I}_{x=1} + \sqrt{w}$. This is the same example used in Section 3. It illustrates the features stated in Proposition 4. For the example in figure 5.1, $\overline{p} = 80$. The bidder has willingness to pay of 64 for the (entire) good. Yet, if the auctioneer sells probabilities of winning the good at a per unit price less than 80, the bidder demands a strictly positive probability of winning the good.

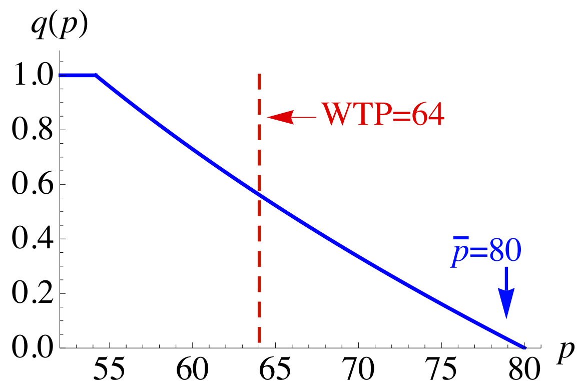

Figure 5.1: A probability demand curve

5.2 The probability demand mechanism

In the probability demand mechanism, a bidder reports a probability demand curve $q(p, u_i, w_i)$, as well as her preferences u_i and initial wealth level w_i. The auctioneer uses the reported demand curves to calculate each bidder's probability of winning and expected payment. Given a bidder's probability of winning and expected payments, her payments are then structured efficiently.

It is equivalent to assume that each bidder reports only her preferences and initial wealth levels and that the auctioneer derives her probability demand curve directly. I work with the indirect mechanism, where bidders also report probability demand curves, to illustrate the connection to the divisible goods setting. For economy of notation I write $q_i(p) = q(p, u_i, w_i)$.

The probability that bidder i wins the object is calculated from the reported probability demand curves. Given the reported demand curves the auctioneer calculates the (lowest) price for probabilities of winning the good that 'clears the market.' That is, she finds the (lowest) price p^* where the total reported demand for probabilities of winning the good equals one.

$$p^* := \inf_p \{p : \; 1 = \sum_{i=1}^{N} q_i(p)\}. \tag{5.2}$$

Given p^*, the probability bidder i wins the good is $q_i(p^*)$. This is the (reported) probability of winning that she demands when she pays a price of p^* per unit of probability and payments are made efficiently.

The market clearing price p^* is not the per unit price bidders pay for probabilities of winning the good. Instead each bidder faces a residual probability supply curve that represents her marginal price curve for probabilities of winning the good. The residual probability supply curve is based on other bidders reported demand curves and is analogous to the residual supply curve in Vickrey auction for a divisible goods auction.

Given a price p, a bidder's residual probability supply curve $S_i(p)$ states the amount of probability of winning the good that is not demanded by the $N-1$ other bidders.

$$S_i(p) = \begin{cases} 1 - \sum_{j \neq i} q_j(p) & if \; 1 - \sum_{j \neq i} q_j(p) > 0 \\ 0 & otherwise \end{cases} \tag{5.3}$$

Bidder i's (reported) probability demand curve equals her residual probability supply curve at the price p^*. Thus, the market clearing price sets each bidder's probability demand curve equal to her residual probability supply curve.

Bidder i's expected payments are determined by treating her residual probability supply curve as her (expected) marginal price curve. Her expected payment to the auctioneer is X_i, where,

$$X_i = \int_0^{p*} t dS_i(t). \tag{5.4}$$

Note that I suppress notation in writing X_i; it is a function of the complete profile of reported demand curves $(q_1 \ldots q_N)$. Figure 5.2 illustrates this graphically.

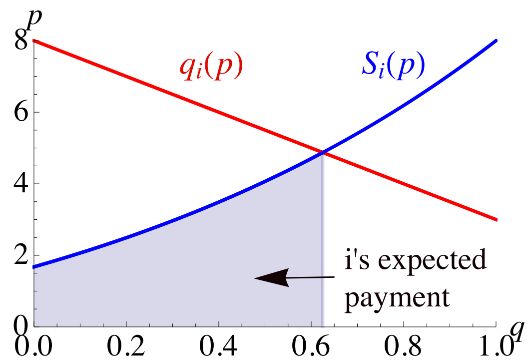

Figure 5.2: Expected transfers from the probability demand mechanism.

Thus, the reported probability demand curves determine each bidder's expected transfers and probability of winning the good. Notice that bidder i pays a marginal price for a unit of probability that is lower than p^* for any amount of probability less than $q_i(p^*)$. Thus, $X_i \leq p^* q_i(p^*)$.

Without the quasilinearity restriction on bidders' preferences, it is with loss of generality to consider only expected transfers. Thus, to complete the description of the mechanism I specify precisely how payments are structured.

Each bidder's payment to the auctioneer is structured efficiently given her probability of winning the object, expected transfers and reported preferences. Assume that for a given profile of reported demand curves $(q_1 \ldots q_N)$, bidder i wins with probability $q_i(p^*)$ and pays the auctioneer X_i in expectation. If bidder i reports her preferences to be u_i and her initial wealth to be w_i, she then pays $p^*_{i,w}$ when she wins and $p^*_{i,l}$ when she loses where,

$$(p^*_{i,w}, p^*_{i,l}) = \arg\max_{p_w, p_l} q_i(p^*) u_i(1, w_i - p_w) + (1 - q_i(p^*))\, u_i(0, w_i - p_l). \tag{5.5}$$
$$s.t.\ X_i = q_i(p^*) p_w + (1 - q_i(p^*)) p_l.$$

Definition 4. (The probability demand mechanism)
The probability demand mechanism maps reported demand curves $(q_1, \ldots, q_N)$, preferences $(u_1 \ldots u_N)$, and initial wealth levels $(w_1 \ldots w_N)$ to a probabilistic allocation described by (5.2) − (5.5).

The probability demand mechanism can be modified so bidders only submit a probability demand curve to the auctioneer. This is done using a two-step procedure where bidders first submit probability demand curves and each bidder's probability of winning and expected payments are calculated as discussed above. Then, the auctioneer tells each bidder her probability of winning $q_i(p^*)$ and the expected payment she will make X_i. Before the actual winner is selected, each bidder decides how to structure her payments, given that her probability of winning is $q_i(p^*)$ and her expected payment is X_i.

5.3 An analogy: the Vickrey auction for a divisible good

In this subsection I develop the intuition for my probability demand mechanism by discussing its connection to the standard Vickrey auction for a divisible good. The probability demand auction treats probabilities of winning the good like quantities of a divisible good. Bidders submit a probability demand curve to the auctioneer. The auctioneer uses the reported probability demand curves to calculate each bidder's probability of winning the good (Equation 5.1) and expected transfers (Equation 5.3). In the standard quasilinear divisible good environment, bidders' preferences admit an analogous downward sloping demand curve for quantities of the good. The Vickrey auction for a divisible good uses the reported demand curve to calculate each bidder's payment and quantity allocation. While this is a departure from my indivisible goods setting, the methodology developed in this literature is useful in developing the intuition behind the construction of the probability demand mechanism.[9]

Thus in this subsection, I depart from the indivisible goods model and review the divisible goods case to build the intuition for my probability demand mechanism. I assume the seller owns a divisible good in net supply 1 and faces N bidders. Here, in the standard divisible goods setting, bidders' preferences are restricted to be quasilinear. Thus, there are no include wealth effects or declining marginal utility of wealth. Bidder i's preferences over quantities (q) and money (m) are given by U_i, where

$$U_i(q, m) = g_i(q) + m,$$

and $g_i : [0, 1] \to \mathbb{R}$ is an increasing, concave and continuously differentiable function. It is without loss of generality to set i's initial wealth to be $M_i = 0$. Bidder i's preferences are used to construct her demand curve. Her demand curve $D_i(p)$ states her quantity demanded at price p, where

$$D_i(p) := \arg\max_{q\in[0,1]} U_i(q, -qp) = \arg\max_{q} g_i(q) - qp. \tag{5.6}$$

Since g_i is continuously differentiable and concave, D_i is decreasing in p. The demand curve is analogous to my probability demand curve described in Equation 5.1.

In the mechanism, each bidder reports her demand curve to the auctioneer. Given the reported demand curves $(D_1, \ldots, D_N)$, the auctioneer finds the lowest per unit price p^* that clears the market (i.e. total demand is exactly one). The market clearing price is found using an equation which is analogous to Equation 5.2 in the description of the probability

[9] For a more detailed explanation, see Chapter 12 of Auction Design by Vijay Krishna (2002).

demand mechanism.

$$p^* := \inf_p \{p : \; 1 = \sum_{i=1}^{N} D_i(p)\}. \tag{5.7}$$

The market clearing price p^* to determines each bidder's allocation of the good. Given p^*, bidder i receives a quantity $D_i(p^*)$ of the good. Notice that at price p^*, bidder i's demand equals her residual supply, $D_i(p^*) = S_i(p^*)$.

However, the market clearing price p^* does not determine the per unit price the bidder pays. Instead, each bidder faces a residual supply curve S_i that represents her marginal price curve. This residual supply curve is analogous to the residual probability supply curve (Equation 5.3) used in the probability demand mechanism. A residual supply curve $S_i(p)$ states the remaining supply of the good available to bidder i if the auctioneer first satisfies the demands of the $N-1$ other bidders at price, p.

$$S_i(p) := \begin{cases} 1 - \sum_{j \neq i} D_j(p) & if \; 1 - \sum_{j \neq i} D_j(p) > 0 \\ 0 & otherwise \end{cases}. \tag{5.8}$$

Note that this is independent of i's preferences.

She purchases quantities of the good up until the point where her marginal price is p^*. Thus, bidder i's payment X_i is a function of her reported demand curve and her residual supply curve,[10] where

$$X_i = \int_0^{p^*} x dS_i(x). \tag{5.9}$$

This is analogous to equation 5.4 in the description of the probability demand mechanism. Thus, she wins a quantity of $D_i(p^*)$ and makes a payment of X_i. Since bidder preferences are restricted to be quasilinear, it is without loss of generality to focus on expected transfers. Thus there is no need to discuss an analog to Equation 5.5 seen in the description of the probability demand mechanism.

Notice that she would buy the same quantity and make the same payments if she was a first degree price discriminating monopsonist facing a residual supply curve S_i. Figure 5.2 illustrates this.

[10] Note that I suppress notation for both p^* and X_i. Each is a function of all reported demand curves.

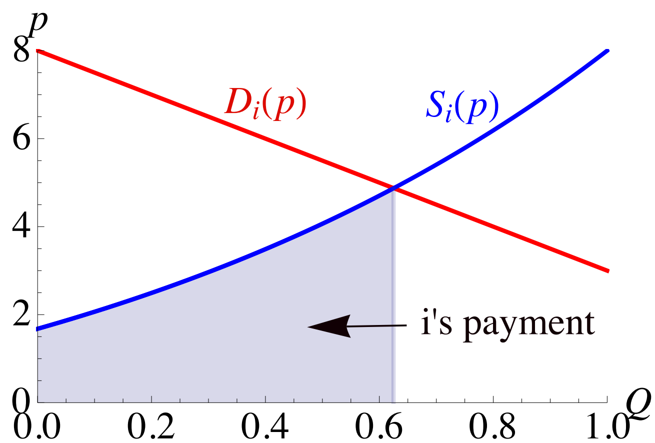

Figure 5.3: Transfers from the Vickrey auction for a divisible good.

Bidder i's payoff only depends on p^* and S_i. The residual supply curve represents her marginal price curve and her demand curve represents her marginal benefit curve. If she takes the residual supply curve as given, she maximizes her payoff by setting marginal benefit equal to the marginal price. By truthfully reporting her demand curve, her demand curve (marginal benefit) intersects her residual supply curve (marginal price) when the two are equal. Thus, for any possible preferences reported by bidders $1, \ldots i-1, i+1, \ldots N$, it is a best response for bidder i to truthfully report her demand curve.

Proposition 5.
In the Vickrey auction for a divisible good, it is a dominant strategy for a bidder to truthfully reveal her demand curve to the auctioneer.

5.4 Behavior in the probability demand mechanism

The construction of the probability demand mechanism illustrates its similarities to the Vickrey auction for a divisible good. Yet, in the benchmark divisible good model, bidders' preferences are restricted to be quasilinear. Under this restriction on preferences, the Vickrey auction for a divisible good has a dominant strategy equilibrium (Proposition 5). However, in the indivisible good setting that I study, I drop the quasilinearity restriction on bidders' preferences.

Removing the quasilinearity restriction adds an additional layer of complexity not seen in the benchmark divisible good setting. In particular, the probability demand mechanism does not have a dominant strategy equilibrium. Instead, I show it is a dominated strategy for a bidder to report a demand curve which underreports her demand for winning probabilities. I can then use a bidder's probability demand curve as a lower bound on her report to the auctioneer. This lower bound on a bidder's report is sufficient for forming revenue comparisons between the probability demand mechanism and standard auction formats.

In the benchmark divisible good setting, a bidder's marginal willingness to pay for an additional unit of quantity is unaffected by changes in her wealth; there are no wealth effects when her preferences are quasilinear. She prefers to buy an additional unit of the good if and only if her marginal willingness to pay for the unit is above the marginal price (the residual supply curve). Thus, for a given residual supply curve, a bidder's optimal choice is given by the point where her demand curve intersects her residual supply curve. Beyond this, it is unaffected by the precise shape of the supply curve curve.

In my indivisible good setting studied here, a bidder's willingness to pay for a marginal increase in her probability of winning is decreasing in the amount of money she spends. This is a direct implication of positive wealth effects. I use this observation to argue that it is a dominated strategy for a bidder to underreport her probability demand curve in the probability demand mechanism.

I first consider a case where it is a best response for a bidder to truthfully reveal her demand curve. If bidder i faces a perfectly elastic residual probability supply curve, she pays a constant marginal price for a unit of probability of winning the good. Assume this price is p_E. In this case, it is a best response to truthfully report her probability demand curve; it states the probability of winning the good that she desires when she pays price of p_E per unit of probability. By truthfully reporting her probability demand curve, she wins the good with probability $q_i(p_E)$ and pays $p_E q_i(p_E)$ in expectation.

Next, I alternatively consider a case where bidder i faces a more inelastic (relative to perfectly elastic) residual probability supply curve. Assume her residual probability supply curve still passes through the point $(p_E, q_i(p_E))$. If bidder i truthfully reports her probability demand curve she wins the good with the probability $q_i(p_E)$. Thus, her probability of winning the good is the same as it is when she faces the perfectly elastic supply curve. However, she pays less when she faces the more inelastic supply curve. The marginal price she pays for all but the final unit of probability she acquires is less than p_E. Thus, she pays less than $p_E q_i(p_E)$ in expectation for a $q_i(p_E)$ probability of winning.

Positive wealth effects imply that she has a higher willingness to pay for a marginal increase in her probability of winning the good relative to the case where she faces the perfectly elastic residual supply curve. If she had paid exactly $p_E q_i(p_E)$ in expectation for a $q_i(p_E)$ probability of winning, her marginal willingness to pay for an additional unit is p_E. Here she pays strictly less than $p_E q_i(p_E)$, thus her marginal willingness to pay for increasing her probability of winning is greater than p_E. Yet the marginal price of an additional unit of probability is p_E. By marginally increasing her reported demand curve, she increases her probability of winning the good and pays a marginal price that is below her marginal willingness to pay for the additional unit of probability.

The differences in expected transfers that emerge from differences in the shape of the residual probability supply curve are illustrated by Figure 5.3. The figure shows a bidder's expected transfers when she faces either a relatively elastic residual probability supply curve (S_a) or a relatively inelastic probability supply curve (S_b).

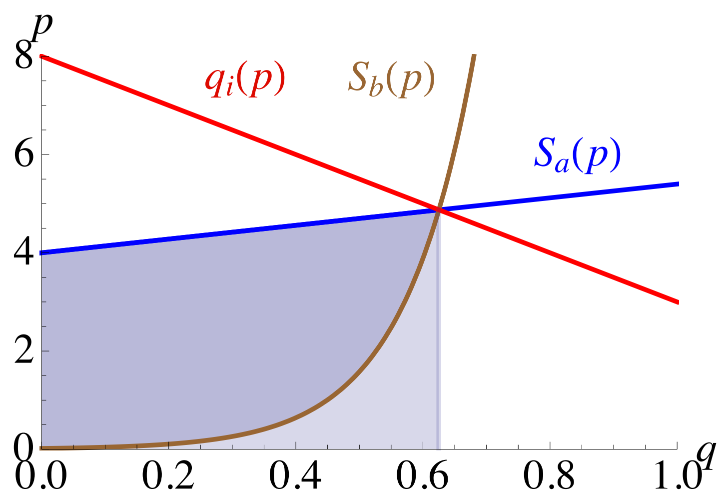

Figure 5.4: Expected transfers and the shape of the residual probability supply curve.

Proposition 6.
Assume bidder i has probability demand curve $q_i(p)$. It is a weakly dominated strategy to report a probability demand curve $\tilde{q}_i$ where $\tilde{q}_i(p) < q_i(p)$ for some $p \in \mathbb{R}_+$. It is dominated by the strategy

$$\overline{q_i}(p) = \max\{q_i(p), \tilde{q}_i(p)\}.$$

While it is not a dominant strategy for bidders to truthfully report their probability demand curves, Proposition 6 shows that truthful reporting can serve as a lower bound on a bidder's possible report. This lower bound on a bidder's report enables revenue comparisons between the probability demand mechanism and other auction formats.

6 Revenue comparisons

The probability demand mechanism has clear contrasts with standard auction formats studied in the quasilinear setting. In these formats, bidders submitted a non-negative real number bid to the auctioneer. Thus, the bids are single dimensional. A bidder wins the object only if she has the highest bid. This description is true of the first price, the second price and all pay auctions, as well as each of these formats modified to included reserve prices or entry fees.

I make revenue comparisons with these standard auction formats when bidder preferences are independent and identically distributed. With sufficiently many bidders, the expected revenues from the probability demand mechanism will exceed the expected revenues of a large class of standard auction formats. Proposition 6 shows that it is a dominated strategy for a bidder to underreport her probability demand curve. In this section, I use this result to bound bidder reports and obtain revenue comparisons.

I show this in three steps. First, I derive an upper bound on the expected revenues from any mechanism that respects bidders' interim individual rationality constraints. Next, I show that as $N \to \infty$, a lower bound on the expected revenue from the probability demand mechanism approaches the revenue upper bound for any individually rational mechanism. Lastly, I define a broad class of 'highest-bid' mechanisms. This class includes most commonly studied auction formats. I show that even when there are many bidders the expected revenues in the equilibrium of a highest-bid mechanism do not approach the revenue upper bound.

6.1 A revenue upper bound for all mechanisms

To make revenue comparisons, I assume that bidders' preferences are independently and identically distributed. That is, I assume there is a distribution over possible 'types' of bidder preferences. I describe bidder i's preferences by a type t_i, where $t_i \in \mathbb{R}^M$ and M is finite. I let the first element represent bidder i's initial wealth level w_i. A bidder with type t_i has preferences described by the utility function $u(x, w, t_i)$ when her type is t_i. I assume that all bidders preferences have declining marginal utility of money and positive wealth effects. That is, for any $t_i \in \mathbb{R}^M$, $u(x, w, t_i) \in \mathcal{U}$. At the same time, this setup allows for heterogeneity across multiple dimensions. I do not place restrictions on the functional forms of bidders' utility functions. While the standard quasilinear model allows for heterogeneity in 'valuations', this setup allows for heterogeneity across risk preferences, initial wealth levels and financing constraints.

If two types $t, t' \in \mathbb{R}^M$ are relatively close, they have similar preferences. Specifically, a bidders utility function $u(x, w, t)$ is uniformly continuous in t.

Each bidder's preference is an i.i.d. draw of a random variable with density f. The distribution f has full support over T, a compact subset of $\mathbb{R}^M$. The density maps types to a non-negative number, $f : T \to \mathbb{R}_+$.

For a given distribution over types, I can define an upper bound on revenues from any mechanism that is interim individually rational. To do so, I first consider the highest per unit price where a bidder demands a positive probability of winning the good. Recall that $\overline{p}_i$ is defined as the highest per unit price where bidder i demands a positive probability of winning. Note that $\overline{p}_i$ is a function of bidder i's type, t_i. Thus I write it as $\overline{p}(t_i)$, where

$$\overline{p}(t_i) := \sup_p \{p : q(p, t_i) > 0\}.$$

Since preferences u are continuous in t, it follows that $\overline{p}(t_i)$ is continuous in t_i. Given the distribution of preferences f, I let $\overline{P}$ denote the highest possible per unit price where a bidder demands a positive probability of winning the good.

$$\overline{P} := \sup\{\overline{p}(t) : t \in supp(f)\}.$$

I assume that $\overline{P} < \infty$. Thus, for a sufficiently high per unit price, no bidder demands any probability of winning the good. The price $\overline{P}$ serves as an upper bound on the expected revenues from any interim individually rational mechanism. Since no bidder is ever willing to pay a per unit price that exceeds $\overline{P}$ for any positive probability of winning the good, a mechanism with expected revenues that exceed $\overline{P}$ necessarily violates some bidder's interim individual rationality constraint.

Proposition 7.
All mechanisms which are interim individually rational generate expected revenues less than or equal to $\overline{P}$.

This is a generous upper bound on revenues. It only assumes that a mechanism respects interim individual rationality constraints and holds for any number of bidders.

6.2 Revenues from the probability demand mechanism

Proposition 6 states that in the probability demand mechanism, it is a weakly dominated strategy for a bidder to submit a demand curve where she underreports her demand. By assuming that bidders play only undominated strategies, I show that as $N \to \infty$ a lower bound on expected revenue from the probability demand mechanism approaches the upper bound on expected revenues from any interim individually rational mechanism $\overline{P}$.

Proposition 8.
As $N \to \infty$, if all bidders play undominated strategies, the expected revenue from the probability demand mechanism approaches $\overline{P}$.

The formal proof of the above result is in the appendix, though I discuss the intuition here. Using the definition of $\overline{P}$ and the continuity of $\overline{p}$ in t, there is a positive probability that a randomly drawn bidder has a demand curve where $q_i(\overline{P} - \epsilon) > 0$ for some $\epsilon > 0$. In other words, there is a positive probability that a randomly drawn bidder demands a positive probability of winning the good when she pays a per unit price that is slightly under $\overline{P}$.

With sufficiently many people, there are many bidders who demand a positive probability of winning when the per unit price is $\overline{P} - \epsilon$. Thus, as the number of bidders increases to infinity, there is no residual supply at the price of $\overline{P} - \epsilon$. If a bidder does win the good with a positive probability, she pays a per unit price that exceeds $\overline{P} - \epsilon$. Thus, the expected revenues will exceed $\overline{P} - \epsilon$ as the number of bidders increases.

I illustrate this for a special case where bidders all have the same initial wealth of 100 and preferences u given by,

$$u(x, w) = 4\mathbb{I}_{x=1} + \sqrt{w}.$$

By assuming that bidders truthfully report their probability demand curves, I obtain a lower bound on a bidder's residual probability supply. As N increases, the lower bound on the residual probability supply curves approaches the revenue upper bound of $\overline{P} = 80$.

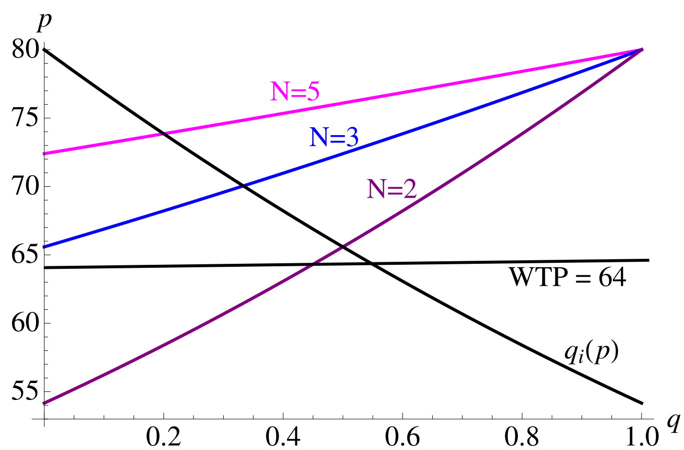

Figure 6.1: Bounds on residual probability supply curves.

6.3 Revenue comparisons with standard auction formats

While the probability demand mechanism approaches the expected revenue upper bound of any individually rational mechanism, this alone is not an interesting result. In the benchmark quasilinear environment there are many mechanisms that approach an analogously defined bound. Here, this is not the case. I show expected revenues from standard auction formats do not approach this upper bound, even when there are many bidders.

I begin by looking at the two most commonly studied formats in the auction literature; the first and second price auctions. In each format, it is a dominated strategy for a bidder to submit a bid that exceeds her willingness to pay for the good. Thus, expected revenues are bounded by the highest willingness to pay of any bidder. However, Proposition 1 shows that each bidder is willing to purchase a positive amount of probability of winning at a price per unit that exceeds her willingness to pay for the (entire) good. That is, if bidder i has preferences such that, $u_i(1, w_i - k_i) = u_i(0, w_i)$, then $\overline{p}_i > k_i$.

The probability demand mechanism exploits this feature. As $N \to \infty$, expected revenues from the probability demand mechanism approach the highest price any agent is willing to pay for positive probability of winning the good $\overline{P}$. Thus, the expected revenues from the probability demand mechanism exceed the expected revenues from the first price or second price auction when there are sufficiently many bidders.

Corollary 1.
Assume that bidders play undominated strategies. When N is sufficiently large, expected revenues from the probability demand mechanism exceed expected revenues of the first or second price auctions.

This result generalizes beyond the first and second price auction. It extends to a broad class of indirect mechanisms where bidders submit single dimensional bids. In particular, I focus on 'highest bid wins' mechanisms, where a bidder receives the object only if she submits the highest bid. Additionally, the bidder is able to leave the auction at no cost by placing a bid of 0.

Thus, I limit attention to mechanisms where each bidder reports a message $m_i \in \mathbb{R}_+$. The indirect mechanism M maps the N messages to a distribution over feasible outcomes:

$$M : \mathbb{R}_+^N \to \Delta(\Phi).$$

If $m_i = 0$, bidder i makes no transfers and wins the good with 0 probability. This ensures interim individual rationality by allowing the bidder to exit the auction by bidding 0.

Definition 5. (Highest bid mechanism)
The indirect mechanism Γ is a highest bid mechanism if i is given the object only if she submits the highest bid,

$$\max_{j \neq i} m_j > m_i \implies y_i = 0.$$

And if $m_i = 0$, i pays 0 transfers and $y_i = 0$.

This class of mechanisms includes many familiar mechanisms such as the first price, second price and all pay auctions. It also includes each of these formats with entry fees or reserve prices.

Most commonly studied auction formats have the property that along the equilibrium path, the probability of a tie is zero. With a sufficient amount of heterogeneity in preferences, this is to be expected. For example, this property is always true of any equilibrium of an all pay auction. I will say an equilibrium of a highest bid mechanism is a "no-tie" equilibrium if in equilibrium there is a zero probability of a tie along the equilibrium path.

Definition 6. (No-tie equilibrium)
A Bayesian Nash Equilibrium of a highest bid mechanism is a "no-tie" equilibrium if in equilibrium:

$$P(m_i = \max_{j \neq i} m_j | m_i > 0) = 0 \ \forall i.$$

Whether or not a mechanism has a "no-tie" equilibrium depends on the underlying distribution of preferences f and how the mechanism Γ structures payments.

For a given distribution of preferences, there is an upper bound on the expected revenues in any "no-tie" equilibrium of a highest bid mechanism. The upper bound is independent of the number of bidders and is strictly less than the revenue upper bound derived for any interim individually rational mechanism. This shows that any "no-tie" equilibrium of a highest bid mechanism will generate strictly less revenues than the probability demand mechanism when N is sufficiently large.

Proposition 9.
Given a fixed distribution of preferences f, there exists an $\alpha > 0$ such that for any N, the expected revenues from any no-tie Bayesian Nash Equilibrium of a highest bid mechanism are less than $\overline{P} - \alpha$.

A direct corollary of this result is that when N is sufficiently large, expected revenues from the probability demand mechanism are greater than the expected revenues from any "no-tie" equilibrium of a highest bid mechanism.

Corollary 2.
Assume bidders play only undominated strategies. For a fixed distribution of preferences f, there exists an $\mathcal{N}$ such that for all $N > \mathcal{N}$, expected revenues of the probability demand mechanism exceed the expected revenues of any no-tie Bayesian Nash Equilibrium of a highest bid mechanism.

The results of this section show that when there are sufficiently many bidders, an auctioneer generates higher revenue from the probability demand mechanism than from most commonly studied auction formats. These formats include first price, second price and all pay auctions. Additionally, this includes any of the three formats with reserve prices or entry fees. For the first price or second price auction, this holds by Corollary 1, and for the all pay this holds from Corollary 2 by noting that any Bayesian Nash Equilibrium of an all-pay auction is a no-tie equilibrium.

In the next section, I use a numerical example to compare the expected revenues of the probability demand mechanism with the expected revenues of the first and second price auctions. In particular, I consider a setting where bidders are financially constrained. The example shows that in practice, the probability demand mechanism can generate revenues that exceed the revenues of the first or second price auction even when there are relatively few bidders.

The results illustrate that sharp qualitative differences emerge in the auction design problem when we relax the quasilinearity restriction. With sufficiently many bidders, the probability demand mechanism generates greater revenues than commonly studied auction formats. The probability demand mechanism exploits features of bidder preferences that standard auctions do not. In particular, a bidder is willing to pay a higher price per unit of probability for a low probability of winning versus a high probability of winning.

Standard auction formats prescribed in the quasilinear benchmark do not exploit this feature of bidders' preferences. For example, in the first or second price auctions, it is a dominated strategy for a bidder to submit a bid above her willingness to pay for the good. Thus, revenues are bounded by the highest bidder's willingness to pay for the good. Corollary 2 shows that this result extends to a broader class of highest bid mechanisms.

7 A numerical example

The results from the previous section show that expected revenues from the probability demand mechanism exceed the expected revenues from standard auction formats when there are sufficiently many bidders. However, when considering implementing the probability demand mechanism, two natural questions emerge. First, how many bidders are needed for the probability demand mechanism to generate greater revenues than standard auction formats? And second, how much greater are the revenues from the probability demand mechanism than other auction formats?

The answers to both questions depend on the assumed distribution of preferences. In this section, I consider a particular setting that is embedded in my model: a setting with financially constrained bidders. Each bidder must borrow money to finance her payments to the auctioneer and the more money she borrows, the higher an interest rate she must pay on her loan. This is a similar setting to that studied by Che and Gale (1998). I find that revenues from the probability demand mechanism exceed those from standard auction formats, even when there is a small number of bidders. In addition, the differences in revenues are non-negligible.

7.1 The example

Bidders are financially constrained and have private values for the good for sale. Each bidder has a valuation of the good v_i, where $v_i \sim \mathcal{U}[5, 15]$. However, I depart from the quasilinear environment by assuming that bidders are financially constrained. In order to make payments to the auctioneer, a bidder loans money from the bank. The interest rate paid on a loan of m dollars is $r(m)$, where

$$r(m) = \frac{m}{100}.$$

Thus, a bidder who pays the auctioneer 10 must pay an additional 1 in interest. The cost of borrowing is marginally increasing in the amount borrowed and the total amount the bidder pays (with interest), $mr(m)$, is convex and continuously increasing in m. A bidder's payoff is given by:

$$u_i(x, -m) = v_i \mathbb{I}_{x=1} - m(1 + r(m)).$$

The bidder's utility function satisfies Assumptions 1 and 2. That is, she has declining marginal utility of money and positive wealth effects. The financing constraint is the only departure from the quasilinear environment. For context, we can view the auctioneer as a person selling the house, and the bidders as prospective buyers. If prices are in units of $\$10,000$ or $\$100,000$ this can be seen as a rough approximation such a situation.

I show that in this setting, the probability demand mechanism has expected revenues that exceed the revenues of standard auction formats, even when there are few bidders. I write a bidder's probability demand curve using the methodology developed in section 5.1.

$$q_i(p) = \begin{cases} 1 & \frac{50}{p}\left(\frac{v_i - p}{p}\right) > 1 \\ \frac{50}{p}\left(\frac{v_i - p}{p}\right) & 1 > \frac{50}{p}\left(\frac{v_i - p}{p}\right) > 0 \\ 0 & p > v_i \end{cases}.$$

This is illustrated in Figure 7.1.

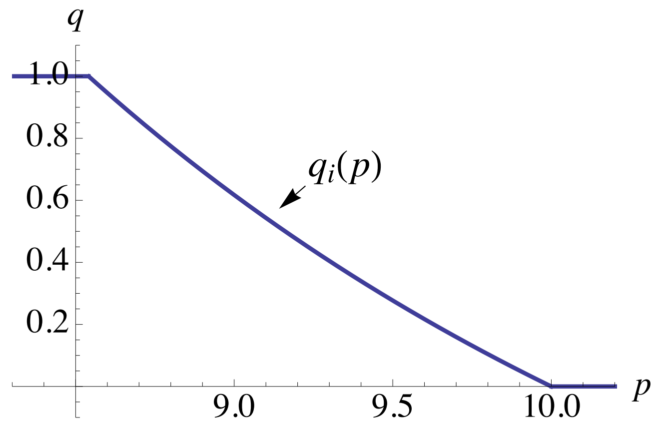

Figure 7.1: Probability demand curve for bidder i when $v_i = 10$.

7.2 Revenue comparisons

Proposition 6 shows that in the probability demand mechanism, it is a dominated strategy for a bidder to underreport its probability demand curve to the auctioneer. When restricting attention to undominated strategies, truthful reporting then serves as a lower bound on a bidder's report. Thus, the case where all bidders truthfully report their probability demand curves serves as a lower bound on expected revenue from the probability demand mechanism.

I compare the expected revenues from the probability demand mechanism to the expected revenues of the first and second price auctions. Applying results of Che and Gale (1998) gives the equilibrium bidding function for the first price auction,

$$b_f(v_i) = 10\sqrt{25 + \frac{5}{N} + \frac{N-1}{N}v_i} - 50.$$

In the second price auction, it is a dominant strategy for a bidder to bid her willingness to pay for the good. That is, bidder i bids b^s such that $u_i(1, -b_s) = u_i(0, 0)$. This is given by

$$b_s(v_i) = 10\sqrt{25 + v_i} - 50.$$

Figure 7.2 illustrates the revenue comparisons between the three formats using a Monte Carlo simulation.

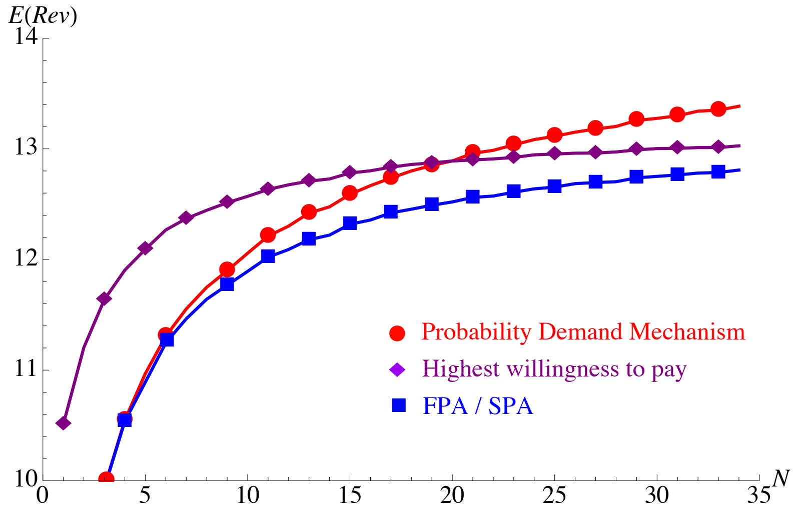

Figure 7.2: Revenue comparisons between formats

The line marked with circles is the revenue lower bound for the probability demand mechanism. Note that the line market with squares represents the revenues of both the first and second price auction. The results of Che and Gale (1998) show the first price has greater expected revenues than the second price auction when bidders face financing constraints. In this environment, the difference in expected revenues between first and second price auctions are relatively small when compared to the expected revenue difference between either format and the probability demand mechanism. When there are 4 or more bidders, the lower bound on expected revenues from the probability demand mechanism exceeds the expected revenues of both the first and second price auctions.

The diamond marked line is the expected value of the highest (lowest private cost) bidder's willingness to pay for the good. When there are 21 or more bidders, the lower bound on revenues from the probability demand mechanism will actually exceed any bidder's willingness to pay for the (entire) good in expectation.

The difference between the lower bound on expected revenues from the probability demand mechanism and the expected revenues from the first price auction is illustrated in Figure 7.3. The figure illustrates that the expected revenue difference between the probability demand mechanism is positive even when N is small. Also the difference in expected revenues between the two formats grows as the number of bidders increases. As the number of bidders increases, expected revenue from the probability demand mechanism approaches 15. This contrasts with the first and second price auctions. As the number of bidders increases, the expected revenues approach the highest possible willingness to pay of any bidder. Here, this is 13.24.

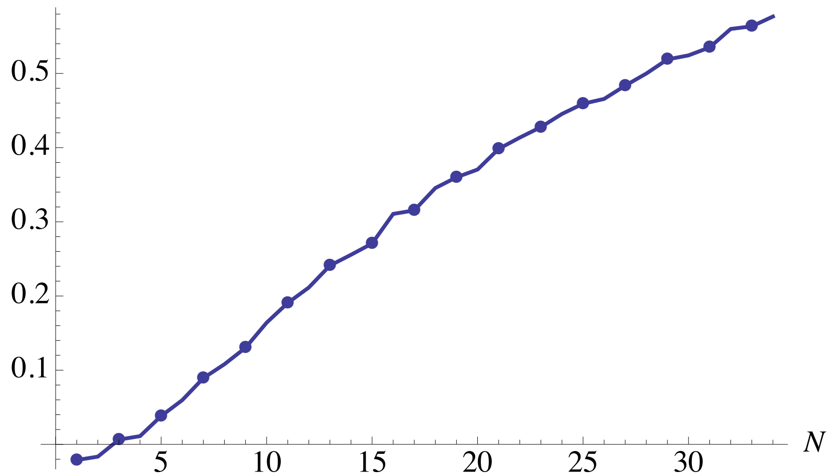

Figure 7.3: Revenue differential between PDM and FPA

8 Divisible goods with non-quasilinear preferences

The construction of the probability demand mechanism illustrates the connection between the indivisible goods setting and the perfectly divisible good setting. However, the indivisible goods problem is complicated by the non-quasilinearity of bidders preferences. The benchmark divisible goods auction setting also restricts bidders to have quasilinear preferences. In this section, I analyze the problem of an auctioneer selling a divisible good and relax the restriction that bidders have quasilinear preferences. I use the insights I develop in the indivisible goods case to obtain novel results for the case of an auctioneer selling a divisible good to bidders with non-quasilinear preferences.

In the indivisible good setting, when preferences are quasilinear, the second price auction implements a Pareto efficient allocation in dominant strategies. Proposition 2 in Section 3 shows this does not hold when we drop the quasilinearity restriction. In fact, there is no mechanism which implements a Pareto efficient allocation in dominant strategies (Proposition 3).

This is analogous to the results obtained in the divisible goods case. When bidders' preferences for the divisible good are quasilinear the Vickrey auction for a divisible good is Pareto efficient in dominant strategies. When I drop the quasilinearity restriction and assume only that bidders have declining marginal utility of money and positive wealth effects, it is no longer a dominant strategy for a bidder to truthfully report her demand curve. Proposition 11 is the divisible goods analog of Proposition 3 and shows that there is no symmetric mechanism that implements a Pareto efficient allocation in dominant strategies.

8.1 The divisible goods setting

In the standard setting for the Vickrey auction for a divisible good, bidders have quasilinear preferences with utility functions of the form $U_i(q, m) = g_i(q) + m$. This is described in Subsection 5.3. I now relax quasilinearity and allow bidders to have preferences which exhibit positive wealth effects.

I assume bidder i starts with an initial wealth of $w_i \in \mathbb{R}$. I assume that U_i is twice continuously differentiable and strictly increasing in Q and M. In addition, bidders have declining marginal utility of money. That is,

$$\frac{\partial^2}{\partial m^2} U_i(q, m) < 0. \tag{8.1}$$

I also assume bidders have positive wealth effects. I say a bidder has positive wealth effects if, for a fixed per unit price, increasing her wealth increases her marginal willingness to pay for another unit of the good. Thus a bidder has positive wealth effects if for any fixed per unit price, $p > 0$

$$\frac{d^2}{dmdq} U_i(q, m - qp) > 0 \ for \ m \in \mathbb{R}, \ q \in (0, 1).$$

In terms of partial derivatives, this is written as

$$\frac{\partial^2}{\partial m \partial q} U_i(q, m - qp) - p \frac{\partial^2}{\partial^2 m} U_i(q, m - qp) > 0 \ for \ m \in \mathbb{R}, \ q \in (0, 1). \tag{8.2}$$

This condition is less strict than imposing that U_i be supermodular. Supermodularity is a sufficient, but not necessary, condition for positive wealth effects, due to the declining marginal utility of money, as seen in the last term of the above expression. Increasing a bidder's wealth also decreases her disutility from spending money on an additional unit of the good. Thus, purchasing an incremental unit becomes relatively more attractive.

8.2 The relation to the indivisible good setting

Recall that in the indivisible good setting I defined the function $V_i(q, x)$ to be bidder i's indirect utility. Since V_i satisfies Equations 8.1 and 8.2, I can use the results from the discussion of the probability demand mechanism to show it is not a dominant strategy for a bidder to truthfully reveal her demand curve in the Vickrey auction for a divisible good when she has positive wealth effects. However, it is a dominated strategy for a bidder to underreport her demand curve.

Proposition 10.

In the Vickrey auction for a divisible good, when bidders have positive wealth effects, it is a dominated strategy for a bidder to underreport her demand. Reporting a demand curve $\overline{D}$ *with* $\overline{D}(p) < D_i(p)$ *for some* $p \in \mathbb{R}_+$ *is weakly dominated by reporting* $\tilde{D}$ *defined by* $\tilde{D}(p) = \max\{D_i(p), \overline{D}(p)\}$.

The proof of this result is analogous to the proof of Proposition 6. The intuition is also similar. A bidder's demand curve gives the amount of the good she demands for a fixed price p. However, she pays a marginal price strictly less than p for all but the final unit purchased. Thus, for a given market clearing price p^*, if i truthfully states her demand, she wins a quantity of $D_i(p^*)$ and pays strictly less than $p^* D_i(p^*)$. Consider a case where $D_i(p^*) > 0$. Her marginal willingness to pay for an additional unit of quantity is p^* when she wins quantity $D_i(p^*)$ and pays a constant marginal rate of p^*. Since i faces an upward sloping residual supply curve, in the Vickrey auction she pays a marginal price which is less than p^* for all but her final marginal unit. Thus, positive wealth effects imply her marginal willingness to pay for an additional unit is greater than p^*. By increasing her reported demand curve above D_i, she is able to purchase a marginal unit of quantity at a rate that is strictly below her marginal willingness to pay. Thus, she has an incentive to overreport her demand but never to underreport.

Proposition 10 generalizes recent work done by Hafalir, Ravi and Sayedi (2012) to a more general preference domain. It shows that their result is due to the fact that budget constrained agents have positive wealth effects.

In the indivisible good case, I showed that while the second price auction was Pareto efficient in dominant strategies with the quasilinearity restriction, this result fails when we drop quasilinearity. I then obtained an impossibility result that there does not exist a mechanism which simultaneously satisfies both properties.

In the divisible good case, the Vickrey auction is Pareto efficient in dominant strategies when bidders' preferences are quasilinear. Without the quasilinearity restriction, I obtain a similar impossibility here in the divisible goods case. That is, I show there is no mechanism which is Pareto efficient in dominant strategies when bidders have positive wealth effects.

Proposition 11.
There does not exists a symmetric and individually rational mechanism that implements a Pareto efficient allocation in dominant strategies.

The proof is identical to the analogous proposition in the indivisible good case. I use the proof of Proposition 3 to prove Proposition 11, treating quantities as probabilities.

Proposition 11 can be seen as generalizing the results of recent work by Dobzinski, Lavi and Nisan (2012), who obtain similar results for budget constrained bidders. Proposition 11 shows that their impossibility result need not depend on the assumption that bidders have budget constraints.

9 Conclusion

In the auction literature, it is standard to restrict bidders' preferences to be quasilinear. The quasilinearity restriction allows for tractable analysis and provides specific prescriptions for auctioneers concerned with efficiency or maximizing revenue. However, there many economic environments where the quasilinearity restriction is violated. This includes cases where bidders have risk aversion, wealth effects or financing constraints. In this paper, I show that there are significant qualitative differences in the auction design problem when we drop the quasilinearity restriction. I replace the quasilinearity restriction with only two assumptions on bidder preferences, (1) bidders have declining marginal utility of money, and (2) bidders have positive wealth effects (equivalently, the good for sale is a normal good). I do not place specific functional form restrictions on bidder preferences. I show that instead of using standard formats where the good is given to the highest bidder with probability one, the auctioneer prefers mechanisms where she can allocate the good to one of many different bidders, each with strictly positive probability.

For efficiency, *with* the quasilinearity restriction, the dominant strategy equilibrium outcome of the second price auction is Pareto efficient. *Without* this restriction, I show there are probabilistic allocations that can Pareto dominate the dominant strategy equilibrium outcome of the second price auction, and there is no symmetric mechanism that implements a Pareto efficient allocation in dominant strategies.

For revenue maximization, *with* the quasilinearity restriction, the optimal auction is a sealed bid auction with a suitably chosen reserve price. *Without* this restriction, I show that my probability demand mechanism has greater expected revenues than any standard auction format when there are sufficiently many bidders. I consider an example with financially constrained bidders to illustrate that in practice, the probability demand mechanism can outperform standard formats even when there are few bidders.

The results in this paper study the canonical private values auction setting, though a natural extension would extend the methodology developed here to an interdependent values setting. The properties of bidder preferences that motivate the use of probabilistic allocations in the private values case remain present in the interdependent values case; thus, similar results should be expected.

10 Appendix

Proof of Proposition 1. I fix u_i and suppress all notation involving u_i. Recall I define the function:

$$k_i(w) := \{k : u_i(1, w-k) = u_i(0, w)\}.$$

For simplicity, I use the following notation: $G(w) := u_i(1, w)$ and $B(w) := u_i(0, w)$. Since $u_i(x, w)$ is strictly increasing in w, both B and G are invertible. Recall $k_i(w)$ is defined as, $G(w - k_i(w)) = B(w) \implies k_i(w) = w - G^{-1}(B(w))$. Since G^{-1} and B are continuously differentiable, k_i is continuously differentiable. By the inverse function theorem, $k_i'(w)$ is,

$$k_i'(w) = 1 - \frac{1}{G(G^{-1}(B(w))}B'(w).$$

Positive wealth effects imply $k_i'(w) > 0$. Thus,

$$G'(G^{-1}(B(w)) > B'(w).$$

Recall:

$$k_i(w) = w - G^{-1}(B(w)) \implies G^{-1}(B(w)) = w - k_i(w).$$

Substituting this into the above expression gives:

$$G'(w - k_i(w)) > B'(w).$$

This is the convexity in the bidder's indirect utility. I next construct a gamble of the form: bidder i wins the good with probability q and pays $k_i(w_i) - (1-q)\delta$ when she wins. When she does not win she pays $q\delta + \epsilon$. I show that when $\epsilon, \delta > 0$ are sufficiently small, bidder i prefers this gamble over her status quo. Note that in the gamble, she wins with probability q and pays strictly greater than $qk_i(w_i)$ in expectation. Her expected utility from entering this gamble is

$$qG(w_i - k_i(w_i) + (1-q)\delta) + (1-q)B(w_i - q\delta - \epsilon).$$

Since both G and B are twice continuously differentiable, I can approximate G and B by a first-order Taylor expansion. For ϵ, δ sufficiently small, this implies

$$G(w_i - k_i(w_i) + (1-q)\delta) \approx G(w_i - k_i(w_i)) + ((1-q)\delta)\, G'(w_i - k_i(w_i))$$

and

$$B(w_i - q\delta - \epsilon) \approx B(w_i) - (q\delta + \epsilon)\, B'(w_i).$$

Thus, for ϵ, δ sufficiently small, the expected utility of the gamble is approximated by,

$$q\left(G(w_i - k_i(w_i)) + (1-q)\delta G'(w_i - k_i(w_i))\right) + (1-q)\left(B(w_i) - (q\delta + \epsilon)\, B'(w_i)\right).$$

Noting that $G(w_i - k_i(w_i)) = B(w_i)$, by the definition of k_i, the above expression simplifies to

$$B(w_i) + q(1-q)\delta\left(G'(w_i - k_i(w_i)) - B'(w_i)\right) - \epsilon(1-q)B'(w_i).$$

Since $G'(w - k_i(w)) > B'(w)$, when ϵ is sufficiently small, the above expression is greater than $B(w_i)$. Thus, i strictly prefers a gamble (relative to her status quo) where wins with probability q and pays $> qk_i(w_i)$ in expectation. $\square$

Proof of Proposition 3. I need only consider a case with 2 bidders. The proof can be easily adapted for N bidders. It is without loss of generality to normalize each bidder to have an initial wealth of 0.[11]

I only consider a subset of the incentive constraints necessary for dominant strategy implementation. I show that there is no individually rational mechanism that implements a Pareto efficient allocation and respects this limited set of incentive constraints. Thus, there is no individually rational mechanism which implements a Pareto efficient allocation and respects all possible incentive constraints.

There is a continuum of types indexed by v, where $v \in \mathbb{R}_+$. A bidder who is type $v \in \mathbb{R}_+$ has preferences described by the utility function u, where

$$u(x, w, v) = v\mathbb{I}_{x=1} + \epsilon f(w) + w,$$

where $\epsilon > 0$ is sufficiently small. I assume that f is strictly increasing, twice continuously differentiable and $f'' < 0$. A bidder who is type $v \in \mathbb{R}_+$ has preferences that are almost quasilinear.

Additionally, there is a single other type, b. A bidder who is type b has preferences described by the utility function u, where

$$u(x, w, b) = B\mathbb{I}_{x=1} + f(w),$$

[11]This is without loss of generality. If a bidder has initial wealth w and preferences u, her preferences over lotteries are equivalent to those of a bidder with initial wealth w and preferences $\hat{u}$, where $\hat{u}(x, d) = u(x, w + d)\ \forall d \in \mathbb{R}, x \in \{0, 1\}$.

where $B > 0$ is fixed. A bidder who is type b receives $B > 0$ utils from being given the object and $f(w)$ utils from having a final wealth of w.

The space of possible preferences is T, where $T = \mathbb{R}_+ \cup b$. Bidder i has type $t_i \in T$. M represents a direct revelation mechanism which maps reported types to a distribution over feasible outcomes. Let $q_i(t_1, t_2)$ be the probability i is given the good when reported types are $t_1, t_2 \in \mathbb{R}_+ \cup b$. Similarly, $x_i(t_1, t_2)$ gives i's expected transfers. Assume that M is Pareto efficient and dominant strategy implementable on this limited domain. I then draw a contradiction.

Necessary conditions for Pareto efficiency

First I state a few necessary conditions on M to implement a Pareto efficient allocation which help to simplify the problem.

1. The object is always allocated to a bidder. That is $q_1(t_1, t_2) + q_2(t_1, t_2) = 1$, for any $t_1, t_2 \in T$.

2. Given $t_1, t_2 \in T$, bidder i pays a constant transfer, whether or not she is given the object. This holds via Jensen's inequality.

3. The mechanism M is then fully described by $q_1(t_1, t_2)$, $x_1(t_1, t_2)$ and $x_2(t_1, t_2)$, where $t_1, t_2 \in T$. I can then use V_i to represent the indirect utility of i who wins with probability q and pays x with certainty.

$$V(q, -x, t_i) = \begin{cases} qv + \epsilon f(-x) - x & \text{if } t_i = v \in \mathbb{R}_+ \\ qb + f(-x) & \text{if } t_i = B. \end{cases}$$

For a fixed level of payments to the auctioneer, X, Pareto efficiency requires that for a fixed level of transfers, the allocation maximizes a weighted sum of expected utilities. That is, there are $m_1, m_2 > 0$ such that:

$$(q_1(t_1, t_2), x_1(t_1, t_2), x_2(t_1, t_2)) \in \arg \max_{q \in [0,1], x_1, x_2} m_1 V(q, x_1, t_1) + m_2 V(1 - q, X - x_1, t_2).$$

The solution to this maximization problem ensures that if each agent receives a positive level of probability, each has an equal marginal willingness to pay for an additional unit of probability. A bidder's marginal willingness to pay is given by:

$$G(q, -x, t_i) := \frac{\frac{\partial}{\partial q} V(q, -x, t_i)}{\frac{\partial}{\partial x} V(q, -x, t_i)}.$$

This gives how much a bidder is willing to pay for a marginal increase in her probability of winning, given that she wins with probability q and pays x with certainty. The necessary conditions for Pareto efficiency from the maximization problem imply:

$$q(t_1,t_2)=1 \implies G(1,-x_1(t_1,t_2),t_1) \geq G(0,0,t_2).$$

$$q(t_1,t_2)\in(0,1) \implies G(q(t_1,t_2),-x_1(t_1,t_2),t_1)=G(1-q(t_1,t_2),-x_2(t_1,t_2),t_2)$$

$$q(t_1,t_2)=0 \implies G(0,0,t_1) \leq G(1,-x_2(t_1,t_2),t_2).$$

The case when both players are almost quasilinear

If both players report types that are almost quasilinear, the necessary conditions for both Pareto efficiency and dominant strategy incentive compatibility require that the outcome of the mechanism is close to that of the second price auction. That is, the highest bidder wins with probability 1 and pays an amount that is approximately the willingness to pay of the second highest bidder.

$$G(q,-x,t_i)=\frac{\frac{\partial}{\partial q}V(q,-x,t_i)}{\frac{\partial}{\partial x}V(q,-x,t_i)}=\frac{v}{1+\epsilon f'(-x)}\approx v.$$

Thus if $v_i>v_j$, then

$$v_i \approx G(1,-x,v_i) > G(0,0,v_j) \approx v_j,$$

for any x that does not violate i's individual rationality condition. Thus if $v_1>v_2$ and ϵ is sufficiently small,[12] Pareto efficiency condition requires $q(v_1,v_2)=1$.

If $q(v_1,v_2)=0$, then individual rationality requires that $x_1(v_1,v_2)=0$. If $q(v_1,v_2)=1$, individual rationality requires that $x(v_1,v_2)<v_1$. The dominant strategy incentive constraints then require that $x_1(v_1,v_2)\approx v_2$. If $x_1(v_1,v_2)>v_2$, 1 has an incentive to deviate to be a type that wins with probability 1 yet pays less than $x_1(v_1,v_2)$. If $x_1(v_1,v_2)<v_2$, then if 1 was instead type v', where $x_1(v_1,v_2)<v'<v_2$, she has an incentive to state her type to be v_1.

Similarly, if both bidders report the same type v, the dominant strategy incentive constraints imply that both $q(v,v)=\frac{1}{2}$ and $x_1(v,v)=x_2(v,v)=\frac{v}{2}$.

Necessary conditions when a bidder has non-linear preferences

If both bidders report type b, symmetry requires $x_1(b,b)=x_2(b,b)$ and $q(b,b)=\frac{1}{2}$. If instead bidder 1 reports type $v\in\mathbb{R}_+$ and 2 reports type b, both $q_1(v,b)$ and $x_1(v,b)$ are weakly increasing in v by standard incentive arguments. When v is sufficiently large, Pareto efficiency requires $q(v,b)=1$, and when v is sufficiently small, $q(v,b)=0$. Additionally,

[12] For a given functional form of f, we can write a function $\delta(\epsilon)$ such that when $v_1>v_2+\delta(\epsilon)$, then $G(1,-x,v_i)>G(0,0,v_j)$, where $\delta(\epsilon)\to 0$ as $\epsilon\to 0$.

there is an intermediate range where each bidder wins with a strictly positive probability. The following lemma shows there is a v such that it is equivalent for 1 to report v or b when 2 reports b.

Lemma. *There exists a type $v_{Eq} \in \mathbb{R}_+$ such that $x(.\ ,b)$ and $q(.\ ,b)$ are continuous at v_{Eq} and $x(v_{Eq}, b) = x(b,b)$ and $q(v_{Eq}, b) = q(b,b) = \frac{1}{2}$.*

Proof. Assume there is no such type v_{Eq}. Recall that $q(v,b)$ is weakly increasing in v. Then there exists a $v_c \in \mathbb{R}_+$ such that

$$\lim_{v \nearrow v_c} q(v_c, b) < \frac{1}{2} < \lim_{v \searrow v_c} q(v_c, b).$$

That is, there is a discontinuity at point v_c. Consider bidder 2 reporting a type of approximately v_c. Note that the Pareto efficiency condition requires that

$$\frac{B}{f'(-x(b,v))} \approx v.$$

Thus

$$\lim_{v \nearrow v_c} \frac{B}{f'(-x(b,v))} \approx v_c \approx \lim_{v \searrow v_c} \frac{B}{f'(-x(b,v))}.$$

If both bidders report type $v_c - \delta$ where $\delta > 0$ is sufficiently small, then each wins with probability of $\frac{1}{2}$ and pays approximately $\frac{1}{2}(v_c - \delta)$. Dominant strategy incentive compatibility implies that no bidder has an incentive to report her type to be b:

$$\frac{1}{2}(v_c - \delta) + \epsilon f(x_1(v_c - \delta, v_c - \delta)) - x_1(v_c - \delta, v_c - \delta) \approx 0 \geq q(b, v_c - \delta)(v_c - \delta) + \epsilon f(x_1(b, v_c - \delta)) - x_1(b, v_c - \delta).$$

Since this holds for all $\delta > 0$,

$$\lim_{v \nearrow v_c} q(b,v)v \leq \lim_{v \nearrow v_c} x(b,v).$$

Now consider a case where 1 is type b and 2 is type $v_c + \delta$, where $\delta > 0$ is sufficiently small. If 1 truthfully reports her type as b, she wins with probability $q(b, v_c + \delta) < \frac{1}{2}$ (since we assume symmetry and that $q(v_c + \delta, b) > \frac{1}{2}$) and pays $x(b, v + \delta)$, where

$$\lim_{\delta \to 0^+} x_1(b, v + \delta) \approx \lim_{\delta \to 0^+} x_1(b, v - \delta) \geq \lim_{\delta \to 0^+} q(b, v - \delta)(v - \delta).$$

Thus, $x_1(b, v_c + \delta) \geq \lim_{\delta \to 0^+} q(b, v_c - \delta)(v_c - \delta) > \frac{1}{2}v$. If she deviates and reports her type as $v_c + \delta$, she wins with probability $\frac{1}{2}$ and pays approximately $\frac{1}{2}(v_c + \delta)$. This violates the incentive constraint of type b and gives a contradiction. □

Using this lemma, I show there is a violation of the incentive compatibility conditions. By the lemma, $x_1(v_{Eq,}b) = x_1(b,b)$, and $q(v_{Eq}, b) = q(b,b) = \frac{1}{2}$. Pareto efficiency requires that:

$$v_{Eq} \approx \frac{v_{Eq}}{1 + \epsilon f'(-x_1(v_{Eq,}b))} = \frac{b}{f'(-x_1(v_{Eq,}b))}.$$

But $v_{Eq} \approx \frac{b}{f'(-x(v_{Eq,}b))}$ implies that $x(v_{Eq}, b) \approx \frac{1}{2}v_{Eq}$. To see this consider the maximization problem:

$$\max_{q\in[0,1]} qB + f(-qP)$$

The necessary first order condition give:

$$P = \frac{B}{f'(-qP)}.$$

If $P = v_{Eq}$ and $q = \frac{1}{2}$, then

$$\frac{B}{f'(-\frac{1}{2}v_{Eq})} = v_{Eq}.$$

Thus, type b has a marginal willingness to pay which equals v_{Eq} only if she pays $\frac{1}{2}v_{Eq}$. Thus, $x_1(v_{Eq,}b) = \frac{1}{2}v_{Eq}$ and $q(v_{Eq}, b) = \frac{1}{2}$. Thus, if bidder 1 is type v_{Eq}, she receives a payoff of ≈ 0 by truthfully reporting if 2 is type b.

A bidder with type v_{Eq} can deviate and state her type is $v_{Eq} - c$ where c is such that $q(v_{Eq} - c, b) > 0$. The individual rationality condition for type $v_{Eq} - c$ requires $q(v_{Eq} - c, b)(v_{Eq} - c) + \epsilon f(-x_1(v_{Eq} - c, b)) - x_1(v_{Eq} - c, b) \geq 0$. Thus, $q(v_{Eq} - c, b)(v_{Eq} - c) \geq x_1(v_{Eq} - c, b)$., yet by stating type $v_{Eq} - c$ she receives a positive payoff. This violates her incentive compatibility condition yielding our contradiction. □

Proof of Proposition 4. First I show that there exists a $\delta > 0$ such that $q_i(k_i+\delta) > 0$. By Proposition 1, there exists a gamble in which i strictly prefers to her status quo where she wins the good with probability $q \in (0,1)$ and pays $(k_i + \delta)q$ in expectation for a sufficiently small $\delta > 0$. Thus,

$$qu_i(1, w_i - x_w) + (1-q)u_i(0, w_i - x_l) > u_i(0, w_i).$$

such that $qx_w + (1-q)x_l > (k_i + \delta)\, q$. Recall,

$$V_i(q, -x) = \max_{x_w, x_l} qu_i(1, w_i - x_w) + (1-q)u_i(0, w_i - x_l)$$

$$s.t.\ x = qx_w + (1-q)x_l.$$

Thus $V_i(q, -(k_i+\delta)q) \geq qu_i(1, w_i - x_w) + (1-q)u_i(0, w_i - x_l) > u_i(0, w_i) = V_i(0,0)$. Recall the probability demand curve is defined as,

$$q_i(k_i+\delta) = \max_q V_i(q, -(k_i+\delta)q).$$

Since it is shown above that $V_i(q, -(k_i+\delta)q) > V_i(0,0)$, then $q_i(k_i+\delta) > 0$.

Next I show that $q_i(p)$ is weakly decreasing. I prove this by contradiction. If this does not hold, there exists $p_H > p_L > 0$ and $q_i(p_H) > q_i(p_L)$. For simplicity of notation, let $q_H = q_i(p_H)$ and $q_L = q_i(p_L)$. Note that $V_i(q,x)$ is increasing in x. Thus,

$$V_i(q_H, -q_H p_H) \geq V_i(q_L, -q_L p_H) \geq V_i(q_L, -q_L p_L) \geq V_i(q_H, -q_H p_L).$$

This implies that, $V_i(q_H, -q_H p_H) \geq V_i(q_H, -q_H p_L)$, which contradicts that V_i is strictly increasing in the second argument.

Finally I show that $q_i(p)$ is continuous in p. First note that $V_i(q,x)$ is continuous in both q and x. If q_i is not continuous in p, there exists a $\hat{p}$ such that $\lim_{p\to\hat{p}^+} q_i(p) < \lim_{p\to\hat{p}^-} q_i(p)$. Let $q_l := \lim_{p\to\hat{p}^+} q_i(p)$ and $q_h = \lim_{p\to\hat{p}^-} q_i(p)$. Since V_i is continuous in both arguments then, $V_i(q_l, -\hat{p}q_l) = V_i(q_h, -\hat{p}q_h)$. Let x_w and x_l be the efficient payments when i wins with probability q_l and pays $\hat{p}q_l$ in expectation. Similarly, let y_w and y_l be the efficient payments when i wins with probability q_h and pays $\hat{p}q_h$ in expectation. Thus,

$$V_i(q_l, -\hat{p}q_l) = q_l u_i(1, w_i - x_w) + (1-q_l)u_i(0, w_i - x_l)$$

$$V_i(q_h, -\hat{p}q_h) = q_h u_i(1, w_i - y_w) + (1-q_h)u_i(0, w_i - y_l)$$

Since $V_i(q_l, -\hat{p}q_l) = V_i(q_h, -\hat{p}q_h) = \frac{1}{2}\left(V_i(q_l, -\hat{p}q_l) + V_i(q_h, -\hat{p}q_h)\right)$. This equals,

$$\frac{1}{2}\left(q_l u_i(1, w_i - x_w) + (1-q_l)u_i(0, w_i - x_l) + q_h u_i(1, w_i - y_w) + (1-q_h)u_i(0, w_i - y_l)\right),$$

Since $\frac{\partial^2 u_i(x,w)}{\partial^2 w} < 0$ for $x = 0, 1$, then by Jensen's inequality,

$$\frac{1}{2}\left(q_l u_i(1, w_i - x_w) + q_h u_i(1, w_i - y_w)\right) < \frac{q_l+q_h}{2} u_i(1, w_i - \frac{q_l x_w + q_h y_w}{q_l + q_h})$$

$$\frac{1}{2}\left((1-q_l)u_i(0, w_i - x_l) + (1-q_h)u_i(0, w_i - y_l)\right) < \left(\frac{1-q_l+1-q_h}{2}\right) u_i(0, w_i - \frac{(1-q_l)x_l + (1-q_h)y_l}{1-q_l+1-q_h}).$$

Let $q_m = \frac{q_l+q_h}{2}$, $z_w = \frac{q_l x_w + q_h y_w}{q_l+q_h}$ and $z_l = \frac{(1-q_l)x_l+(1-q_h)y_l}{1-q_l+1-q_h}$. Note that $\hat{p}q_m = q_m z_m + (1-q_m)z_l$

and

$$V_i(q_m, -q_m\hat{p}) \geq q_m u_i(1, w_i - z_m) + (1 - q_m)u_i(0, w_i - z_l) > \frac{1}{2}\left(V_i(q_l, -\hat{p}q_l) + V_i(q_h, -\hat{p}q_h)\right).$$

However, we assumed that $\lim_{p\to\hat{p}^+} q(p) = q_l$, but $V(q_m, -q_m\hat{p}) > V(q_l, -q_l\hat{p})$, yielding a contradiction. $\square$

Proof of Proposition 6. I prove this by showing that if stating the demand curve $\tilde{q}_i$ instead of $\overline{q}_i$ does change bidder i's payoff, she receives a lower payoff. Thus, I need to only focus on cases where her payoff is changed by reporting $\tilde{q}_i$ instead of $\overline{q}_i$. I formalize the intuition discussed in the text.

First consider a case where i faces a perfectly elastic residual probability supply curve (i.e. a constant marginal price per unit of probability). It is a best response for her to truthfully reveal her demand curve. Let $p_E > 0$ be the constant marginal price for units of probability. If her payoff is changed by reporting $\tilde{q}_i$, then $\tilde{q}_i(p_E) < \overline{q}_i(p_E)$. By the definition of $\overline{q}_i$, then $\overline{q}_i(p_E) = q_i(p_E)$. That is, if reporting $\tilde{q}_i$ does change her payoff, it is the case that, $\tilde{q}_i$ is strictly below her demand for probability at p_E. By the construction of the probability demand curve, truthful reporting is a best response to a perfectly elastic supply curve. Thus, she decreases her payoff by reporting type $\tilde{q}_i$. Recalling that V_i is her indirect utility function under efficient payments, it follows that

$$V_i(q_i(p_E), -p_E q_i(p_E)) \geq V_i(q, -p_E q) \text{ for any } q \in (0, 1).$$

Now consider instead that bidder i faces a more inelastic (relatively to perfectly elastic) residual probability supply curve. Assume that the supply curve is such that $S_i(p_E) = q_i(p_E)$. Once again if her payoff is changed by reporting $\tilde{q}_i$, then $\tilde{q}_i(p_E) < \overline{q}_i(p_E) = q_i(p_E)$, using the same argument as before. Thus, she wins with a lower probability by reporting $\tilde{q}$. Assume that if she reports $\tilde{q}_i$, she pays $\tilde{X}$ in expectation and wins with probability $\tilde{q}(\tilde{p})$, where $\tilde{q}_i(\tilde{p}) = S_i(\tilde{p})$. Since her marginal price is strictly below p_E, $\tilde{X} < p_E\tilde{q}_i(\tilde{p}^*)$. If she instead reports $\overline{q}_i$, she wins with probability $q_i(p_E)$ and pays a marginal price below p_E for the incremental probability of winning gained by reporting $\overline{q}_i$. Thus, she pays $X \leq \tilde{X} + p_E\left(q_i(p_E) - \tilde{q}_i(\tilde{p}^*)\right)$. I now want to show that reporting $\tilde{q}_i$ decreases her payoff relative to reporting $\overline{q}_i$. Thus I want to show:

$$V_i\left(q_i(p_E), -X\right) \geq V_i(\tilde{q}_i(\tilde{p}), -\tilde{X}).$$

Since $X \leq \tilde{X} + p_E\left(q_i(p_E) - \tilde{q}_i(\tilde{p}^*)\right)$, it is sufficient to show

$$V_i\left(q_i(p_E), -\left(\tilde{X} + p_E\left(q_i(p_E) - \tilde{q}_i(\tilde{p}^*)\right)\right)\right) \geq V_i(\tilde{q}_i(\tilde{p}), -\tilde{X}).$$

Recall $\tilde{X} < p_E\tilde{q}_i(\tilde{p}^*)$. Let $Y = p_E\tilde{q}_i(\tilde{p}^*) - \tilde{X} > 0$. Thus, the above expression can be rewritten as:

$$V_i(q_i(p_E), -p_E q_i(p_E) + Y) \geq V_i(\tilde{q}_i(\tilde{p}), -p_E\tilde{q}_i(\tilde{p}^*) + Y).$$

I have already shown that

$$V_i(q_i(p_E), -p_E q_i(p_E)) \geq V_i(\tilde{q}_i(\tilde{p}), -p_E\tilde{q}_i(\tilde{p}^*)),$$

where $q_i(p_E) > \tilde{q}_i(\tilde{p})$. Notice also that $-p_E q_i(p_E) + Y > -p_E\tilde{q}_i(\tilde{p}^*) + Y$. Thus it suffices to show that

$$\frac{\partial}{\partial Y}V_i(q_i(p_E), -p_E q_i(p_E) + Y) \geq \frac{\partial}{\partial Y}V_i(\tilde{q}_i(\tilde{p}), -p_E\tilde{q}_i(\tilde{p}^*) + Y).$$

I do this by using the envelope theorem. Recall that the indirect utility function is the maximal expected utility conditional on winning with probability q and paying x in expectation. Thus the marginal (expected) utility from an increase in Y equals

$$\frac{\partial}{\partial Y}V_i(q, -X + Y) = \frac{\partial}{\partial w}u_i(0, w_i - x_l)$$

where x_l is the losing payment in the efficient payment scheme where the bidder wins with probability q and pays $-X + Y$ in expectation. Notice that bidder i pays less when losing by reporting $\tilde{q}_i$ instead of $\overline{q}_i$. This is because she pays a greater amount in the latter and wins with higher probability. The concavity of an agent's utility function then implies

$$\frac{\partial}{\partial w}u_i(0, w_i - \overline{x_l}) = \frac{\partial}{\partial Y}V_i(q_i(p_E), -p_E q_i(p_E) + Y) \geq \frac{\partial}{\partial Y}V_i(\tilde{q}_i(\tilde{x}_l\tilde{p}), -p_E\tilde{q}_i(\tilde{p}^*) + Y) = \frac{\partial}{\partial w}u_i(0, w_i - \tilde{x}_l),$$

where $\overline{x}_l$ and $\tilde{x}_l$ are the efficient payments made by i conditional on losing and reporting the demand curve $\overline{q}_i$ and $\tilde{q}_i$, respectively. The above expression holds as $\overline{x}_l \geq \tilde{x}_l$ and u_i is concave in w. □

Proof of Proposition 7. Consider a bidder i with type $t_i \in supp(f)$. Consider a gamble where i wins the object with probability $q > 0$ and pays the auctioneer x in expectation. Assume that $\frac{x}{q} > \overline{P}$. I show that bidder i obtains a strictly higher payoff by remaining at her status quo. Assume this result did not hold. Then, there exists a gamble i strictly prefers over her status quo where she wins with probability $q > 0$ and pays x in expectation. Note that i's probability demand curve is decreasing and that $q_i(\overline{P}+\epsilon) = 0$ for any $\epsilon > 0$. Thus, $q_i(\frac{x}{q}) = 0$. Thus, when i is able to purchase probability at a rate of $\overline{P}+\epsilon > \frac{x}{q}$ in expectation, she always prefers to win with 0 probability. She then pays nothing and receives a payoff of $u_i(0, w_i)$. Recall $V_i(q, -x)$ represents the highest expected payoff for i from a lottery where she wins with probability q and pays x in expectation. Thus:

$$u_i(0, w_i) = V_i(0, 0) \geq V_i(q, -q\overline{P}) > V_i(q, -x).$$

This implies that i strictly prefers her status quo over any lottery in which she wins the object with probability $q > 0$ and pays the auctioneer $x > q\overline{P}$ in expectation.

Now consider a mechanism Γ. Let $Q_i(t_i, t_{-i})$ represent the probability i is given the object given she reports type t_i and all other bidders report types t_{-i}. Similarly $X_i(t_i, t_{-i})$ is i's expected payment given all other bidders report types t_{-i}. Let $\mathbb{Q}_i(t_i)$ be i's interim probability of winning. Thus, $\mathbb{Q}_i(t_i) = \int_{t_{-i}} Q_i(t_i, t_{-i}) f(t_i, t_{-i}) dt_{-i}$. I analogously define $\mathbb{X}_i(t_i)$. Interim individual rationality requires $\overline{P}\mathbb{Q}_i(t_i) \leq \mathbb{X}_i(t_i)$.

Thus the expected revenues from bidder i ex-ante are:

$$\int_{t \in T} \mathbb{X}_i(t) f(t) dt \leq \overline{P} \int_{t \in T} \mathbb{Q}_i(t) f(t) dt.$$

Feasibility requires the object is allocated to any bidder with probability less than or equal to 1, ex-ante.

$$\sum_{i=1}^{N} \int_{t \in T} \mathbb{Q}_i(t) f(t) dt \leq 1.$$

Ex-ante expected revenues are:

$$\sum_{i=1}^{N} \int_{t \in T} \mathbb{X}_i(t) f(t) dt \leq \overline{P} \sum_{i=1}^{N} \int_{t \in T} \mathbb{Q}_i(t) f(t) dt. \leq \overline{P}. \ \square$$

Proof of Proposition 8. First, I assume all bidders truthfully report their probability demand curves. In the section I take f, the distribution over types, as given. Recall that

$$\overline{P} := \sup\{\overline{p}(t) : \ t \in supp(f)\}.$$

Recall in the body of the paper I suppress notation and write $q_i(p)$ as the probability demand curve for a bidder with type t_i. Here, I use the more formal notation, $q_i(p, t_i)$. Thus, $q(p, t)$ represents the probability demand curve of a bidder who has type $t \in T$.

Fix $\epsilon > 0$. Let $\tau(\epsilon, \delta) := \{t : \ q(\overline{P} - \epsilon, t) > \delta, \ t \in supp(f)\}$. The set $\tau(\epsilon, \delta)$ is non-empty when $\delta > 0$ is sufficiently small. In addition, there is a strictly positive probability a randomly drawn bidder has type $t \in \tau(\epsilon, \delta)$. Let $\nu(\epsilon, \delta)$ represent this probability:

$$\nu(\epsilon, \delta) = \int_{t \in \tau(\epsilon,\delta)} f(t)dt.$$

Consider a fixed $\delta > 0$ such that $\nu(\epsilon, \delta) > 0$. That is, there is a positive probability that a randomly drawn bidder demands at least a δ probability of winning the good when she pays a price of $\overline{P} - \epsilon$. Note that:

$$\sum_{i=1}^{N} q_i(\overline{P} - \epsilon, t_i) \geq \delta \sum_{1=i}^{N} \mathbb{I}_{t_i \in v(\epsilon, \delta)} \ .$$

This simply states that the total demand at price $\overline{P} - \epsilon$ is greater than the δ times the number of bidders whose demand strictly exceeds δ. By the law of large numbers, as $N \to \infty$ the probability that the sum $\delta \sum_{1=i}^{N} \mathbb{I}_{t_i \in v(\epsilon,\delta)}$ exceeds 2 approaches 1. I use $\mathbb{P}$ to denote the probability operator:

$$\lim_{N \to \infty} \mathbb{P} \left(\delta \sum_{1=i}^{N} \mathbb{I}_{t_i \in v(\epsilon,\delta)} \ > 2 \right) = 1.$$

The intuition for this is straightforward. There is a $\nu(\epsilon, \delta) > 0$ chance a randomly drawn bidder i has type t_i such that $q_i(\overline{P} - \epsilon, t_i) > \delta$. Thus, there is a positive probability a given bidder demands greater than a $\delta > 0$ probability of winning at the expected price $\overline{P} - \epsilon$. With sufficiently many bidders, the probability that there are at least $\frac{2}{\delta} < \infty$ such bidders approaches 1.

Consider $N(\alpha)$ such that $\mathbb{P}\left(\delta \sum_{1=i}^{N} \mathbb{I}_{t_i \in v(\epsilon,\delta)} \ > 2\right) > 1 - \alpha$ where $\alpha > 0$. $N(\alpha)$ is the number of bidders needed to ensure that there is a $1 - \alpha$ probability there are $\frac{2}{\delta}$ bidders who demand at least a $\delta > 0$ probability of winning at the expected price $\overline{P} - \epsilon$. If

$$\delta \sum_{1=i}^{N} \mathbb{I}_{t_i \in v(\epsilon,\delta)} \ > 2$$

then,

$$\mathbb{S}_i(\overline{P} - \epsilon) = 0 \ \forall i.$$

This holds because for any i, $q_i(\overline{P}-\epsilon) \in [0,1]$. Thus,

$$\sum_{j\neq i} q_j(\overline{P}-\epsilon, t_j) \geq \delta \sum_{j\neq i} \mathbb{I}_{t_i \in v(\epsilon,\delta)} \geq \delta \sum_{1=i}^{N} \mathbb{I}_{t_i \in v(\epsilon,\delta)} - 1 > 1.$$

Since

$$\sum_{j\neq i} q_j(\overline{P}-\epsilon, t_j) > 1 \implies \mathbb{S}_i(\overline{P}-\epsilon) = 0.$$

The total probability demand of all bidders that do not include i exceeds 1 at price $\overline{P}-\epsilon$. Thus at price $\overline{P}-\epsilon$, i's residual probability supply curve is 0. Recall

$$\mathcal{X}_i(q_1 \dots q_N) = \int_0^{p^*(q_1 \dots q_N)} t d\mathbb{S}_i(t).$$

If bidder i wins the good with positive probability, she pays a price per unit which strictly exceeds $\overline{P}-\epsilon$. That is,

$$\mathcal{X}_i(q_1 \dots q_N) > q_i(p^*)\left(\overline{P}-\epsilon\right).$$

Since $\sum q_i(p^*) = 1$, this implies that total expected transfers exceed $\overline{P}-\epsilon$:

$$\sum_{i=1}^{N} \mathcal{X}_i(q_1 \dots q_N) > \overline{P}-\epsilon,$$

when

$$\delta \sum_{1=i}^{N} \mathbb{I}_{t_i \in v(\epsilon,\delta)} > 2.$$

Since α and ϵ are arbitrary, let them be arbitrarily close to 0 when N is sufficiently large. Thus, total transfers exceed $\overline{P}-\epsilon$ with probability $1-\alpha$, where both α and ϵ are arbitrarily small when N is sufficiently large. This yields the desired result. □

Proof of Proposition 9. To prove this, consider a fixed distribution of preferences f. By assumption, f has full support over the compact set $T \subset \mathbb{R}^M$.

First I consider a bidder's preferences over gambles. I show that there exists a fixed $\epsilon > 0$ such that any bidder with type $t_i \in T$ strictly prefers her status quo to any gamble where she wins the object with probability $q > .5$ and pays the auctioneer greater than $\left(\overline{P}-\epsilon\right) q$ in expectation.

Assume bidder i has type $t_i \in T$. Consider a gamble $\mathcal{A}$ where she wins the object with probability $q > .5$ and she pays the auctioneer qx in expectation. More specifically, she pays the auctioneer x^w and x^l in expectation conditional on winning and losing, respectively. Since I assume that u_i is strictly concave in money for a fixed state, either win or lose, the bidder prefers a gamble where she wins with probability q and pays the auctioneer x^w and x^l each with certainty conditional upon winning and losing. Assume that she is indifferent between her status quo and this gamble. That is:

$$q\hat{u}_i(1, -x^w) + (1-q)\hat{u}_i(0, -x^l) = \hat{u}_i(0, 0),$$

where

$$qx = qx^w + (1-q)x^l.$$

Thus it holds that:

$$r\left(q\hat{u}_i(1, -x^w) + (1-q)\hat{u}_i(0, -x^l)\right) + (1-r)\hat{u}_i(0, 0) = \hat{u}_i(0, 0),$$

for any $r \in (0, 1)$. In other words, i is in between her status quo and a lottery where she enters the gamble with probability r and remains at her status quo with probability $1 - r$. Since her utility is strictly concave in money for the state where she does not own the good, it follows that:

$$rq\hat{u}_i(1, -x^w) + (1-rq)\hat{u}_i(0, -rx^l) > r\left(q\hat{u}_i(1, -x^w) + (1-q)\hat{u}_i(0, -x^l)\right) + (1-r)\hat{u}_i(0, 0).$$

Or equivalently,

$$rq\hat{u}_i(1, -x^w) + (1-rq)\hat{u}_i(0, -(1-q)x^l) > \hat{u}_i(0, 0).$$

By continuity, this implies that there exists an $\epsilon > 0$ such that

$$rq\hat{u}_i(1, -x^w - \epsilon) + (1-rq)\hat{u}_i(0, -(1-q)x^l - \epsilon) > \hat{u}_i(0, 0).$$

Thus, I have constructed a gamble where i wins with probability rq and pays $x^w + \epsilon$ conditional on winning and $(1-q)x^l + \epsilon$ conditional on losing, where $\epsilon > 0$. Thus she wins with probability rq yet pays the auctioneer $rqx + \epsilon$ in expectation. This is preferred to her status quo.

Recall that for any $t_i \in T$, a bidder strictly prefers her status quo to any gamble of the form: "win the object with probability $c \in (0, 1)$ and pay the auctioneer greater than $c\overline{P}$ in expectation." This was shown in the proof of Proposition 7. This implies that in the gamble where i wins with probability rq, pays $x^w + \epsilon$ conditional on winning and $(1-q)x^l + \epsilon$

conditional on losing, we have $rqx + \epsilon \leq rq\overline{P}$.

Thus for bidder i to even be indifferent to gamble $\mathcal{A}$ versus her status quo, it must be the case that $qx \leq \left(\overline{P} - \epsilon\right) q$.

Thus for any fixed $t_i \in T$, there exists an $\epsilon > 0$ such that i strictly prefers her status quo to any gamble where she wins the object with probability $q > .5$ and pays the auctioneer greater than $\left(\overline{P} - \epsilon\right) q$ in expectation. For the fixed type $t_i \in T$, I denote such an ϵ by $\epsilon(t_i)$. That is, $\epsilon(t_i)$ states that a bidder with type t_i strictly prefers her status quo to any gamble where she wins the object with probability $q > .5$ and pays the auctioneer greater than $\left(\overline{P} - \epsilon(t_i)\right) q$ in expectation.

Let $\underline{\epsilon} := \inf\{\epsilon(t_i) : \ t_i \in T\}$. Note that $\epsilon(t_i) > 0$ for all $t_i \in T$. Thus $\underline{\epsilon} > 0$, as T is a compact set.

I now use this result to place a revenue upper bound on any no-tie Bayes Nash equilibrium of a highest bid mechanism.

Let $\sigma_i(t_i)$ represent i's strategy in a highest bid mechanism. That is, $\sigma_i : T \to \Delta(\mathbb{R}_+)$. Let σ^* denote a no-tie Bayes Nash equilibrium of this highest bid mechanism, $\sigma^* = \{\sigma_1^* \ldots \sigma_N^*\}$. For a given σ^*, there is an induced winning bid. Given i's equilibrium strategy, there exists an ex-ante distribution over her bids. Let $\mathcal{F}_i(b)$ denote the cumulative distribution function of i's bids from the ex-ante point of view, given that she plays strategy σ_i^*.

Let G denote the ex-ante distribution of the highest bid submitted. Note that

$$G(b) = \prod_{i=1}^{N} \mathcal{F}_i(b).$$

For a given bid b, the probability i wins is given by $\prod_{j \neq i} \mathcal{F}_j(b)$. Note that $\prod_{j \neq i} \mathcal{F}_j(b) = \frac{G(b)}{\mathcal{F}_i(b)}$. If i submits a bid of b to the auctioneer, where $\prod_{j \neq i} \mathcal{F}_j(b) > .5$, the above result shows that individual rationality require that she pay the auctioneer no more than $\left(\overline{P} - \underline{\epsilon}\right) \prod_{j \neq i} \mathcal{F}_j(b)$. That is, she will never pay a price for probability which exceeds $\overline{P} - \underline{\epsilon}$.

More generally, for individual rationality, I also require that if i submits a bid of b and she wins with probability $\prod_{j \neq i} \mathcal{F}_j(b)$, she pays the auctioneer no more than $\overline{P} \prod_{j \neq i} \mathcal{F}_j(b)$ in expectation. That is, she will never pay a rate per unit rate for winning probability which exceeds $\overline{P}$. I can combine both of these results to put a bound on expected revenues for the auctioneer. Recall that

$$G(b) = \prod_{i=1}^{N} \mathcal{F}_i(b).$$

By the no-tie restriction, I have that G is continuous and increasing. Thus, it is differentiable almost everywhere and feasibility requires the object is never given away with probability that exceeds 1 ex-ante. That is,

$$\int_{b\in\mathbb{R}_+} G'(b)db \leq 1,$$

where

$$G'(b) = \sum_{i=1}^{N} \left(\mathcal{F}_i'(b) \prod_{j\neq i} \mathcal{F}_j(b) \right).$$

Thus the feasibility condition can be rewritten as:

$$\sum_{i=1}^{N} \int_{b\in\mathbb{R}_+} \left(\mathcal{F}_i'(b) \prod_{j\neq i} \mathcal{F}_j(b) \right) db \leq 1.$$

Note i's ex-ante probability of winning is:

$$\int_{b\in\mathbb{R}_+} \left(\mathcal{F}_i'(b) \prod_{j\neq i} \mathcal{F}_j(b) \right) db.$$

If bidder i submits a bid b such that $G(b) > .5$, she has a $\prod_{j\neq i} \mathcal{F}_j(b) = \frac{G(b)}{\mathcal{F}_i(b)} > .5$ probability of winning the object. I use this to bound the expected revenues from player i. Let $X_i(b)$ represent i's expected payment given she submits bid b. Define b^c as $b^c = \inf\{b : \ G(b) = .5\}$. That is, b^c is the median highest bid the auctioneer should expect. If i submits a bid greater than or equal to b^c, she has a $\geq .5$ probability of winning. Thus, the expected transfers from i are:

$$\int_{b\in\mathbb{R}_+} X_i(b)\mathcal{F}_i'(b)db$$

The individual rationality restrictions require $X_i(b) \leq \overline{P} \prod_{j\neq i} \mathcal{F}_j(b)$ and $X_i(b) \leq \left(\overline{P} - \epsilon\right) \prod_{j\neq i} \mathcal{F}_j(b)$ if $b > b^c$. I can then bound i's expected transfers:

$$\int_{b\in\mathbb{R}_+} X_i(b)\mathcal{F}_i'(b)db \leq \int_{0\leq b<b^c} \overline{P} \left(\mathcal{F}_i'(b) \prod_{j\neq i} \mathcal{F}_j(b) \right) db + \int_{b\geq b^c} \left(\overline{P} - \underline{\epsilon}\right) \left(\mathcal{F}_i'(b) \prod_{j\neq i} \mathcal{F}_j(b) \right) db.$$

Total transfers are given by:

$$\sum_{i=1}^{N} \int_{b\in\mathbb{R}_+} X_i(b)\mathcal{F}_i'(b)db = \sum_{i=1}^{N} \int_{b<b^c} X_i(b)\mathcal{F}_i'(b)db + \sum_{i=1}^{N} \int_{b\geq b^c} X_i(b)\mathcal{F}_i'(b)db.$$

Imposing the bound on i's expected transfers gives:

$$\sum_{i=1}^{N}\int_{b\in\mathbb{R}_+} X_i(b)\mathcal{F}_i'(b)db \leq \overline{P}\sum_{i=1}^{N}\int_{b<b^c}\mathcal{F}_i'(b)\prod_{j\neq i}\mathcal{F}_j(b)db + \left(\overline{P}-\underline{\epsilon}\right)\sum_{i=1}^{N}\int_{b\geq b^c}\mathcal{F}_i'(b)\prod_{j\neq i}\mathcal{F}_j(b)db.$$

Recall the definition of G and b^c:

$$G(b^c) = .5 = \int_0^{b^c} G'(b)db = \int_0^{b^c} G'(b)db = \sum_{i=1}^{N}\int_0^{c}\left(\mathcal{F}_i'(b)\prod_{j\neq i}\mathcal{F}_j(b)\right)db.$$

Thus,

$$\int_{b>b^c} G'(b)db = \sum_{i=1}^{N}\int_{b\geq b^c}\mathcal{F}_i'(b)\prod_{j\neq i}\mathcal{F}_j(b)db. \leq .5.$$

Feasibility implies

$$\int_{b\geq 0} G'(b)db \leq 1.$$

Recall

$$\sum_{i=1}^{N}\int_{b\in\mathbb{R}_+} X_i(b)\mathcal{F}_i'(b)db \leq \overline{P}\sum_{i=1}^{N}\int_{b<b^c}\mathcal{F}_i'(b)\prod_{j\neq i}\mathcal{F}_j(b)db + \left(\overline{P}-\underline{\epsilon}\right)\sum_{i=1}^{N}\int_{b\geq b^c}\mathcal{F}_i'(b)\prod_{j\neq i}\mathcal{F}_j(b)db.$$

Imposing the two conditions on each sum of integrals on the righthand side, I find

$$\sum_{i=1}^{N}\int_{b\in\mathbb{R}_+} X_i(b)\mathcal{F}_i'(b)db \leq .5\overline{P} + .5\left(\overline{P}-\underline{\epsilon}\right) = \overline{P} - \frac{1}{2}\underline{\epsilon}.$$

Since $\underline{\epsilon} > 0$, let $\alpha = \frac{1}{2}\underline{\epsilon}$. Thus, the expected transfers from any no-tie Bayes Nash equilibrium of a highest bid auction are bounded by $\overline{P} - \alpha$, where $\alpha > 0$. $\square$

References

Augenblick, N. (2011): "Consumer and Producer Behavior in the Market for Penny Auctions: A Theoretical and Empirical Analysis," *Working Paper*.

Ausubel, L. M. (2004): "An Efficient Ascending-Bid Auction for Multiple Objects," *American Economic Review*, 94(5), 1452–1475.

Baisa, B. (2012): "Probability Weighting and Auction Design," *Working Paper*.

Bergemann, D., and S. Morris (2005): "Robust Mechanism Design," *Econometrica*, 73(6), 1771–1813.

Che, Y.-K., and I. Gale (1996): "Expected revenue of all-pay auctions and first-price sealed-bid auctions with budget constraints," *Economic Letters*, 50, 373–379.

——— (1998): "Standard Auctions with Financially Constrained Standard Auctions with Financially Constrained Bidders," *Review of Economic Studies*, 65, 1–21.

——— (2000): "The Optimal Mechanism for Selling to a Budget-Constrained Buyer," *Journal of Economic Theory*, 92, 198–233.

——— (2006): "Revenue comparisons for auctions when bidders have arbitrary types," *Theoretical Economics*, pp. 95–118.

Crémer, J., and R. P. McLean (1988): "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica*, 56(6), 1988.

Dobzinski, S., R. Lavi, and N. Nisan (2012): "Multi-unit auctions with budget limits," *Games and Economic Behavior*, 74(2), 486–503.

Esö, P., and L. White (2004): "Precautionary Bidding in Auctions," *Econometrica*, 72(1), 77–92.

Hafalir, I. E., R. Ravi, and A. Sayedi (2012): "A near Pareto optimal auction with budget constraints," *Games and Economic Behavior*, 74, 699–708.

Hart, S., and N. Nisan (2012): "Approximate Revenue Maximization with Multiple Items," *Working Paper*.

Hu, A., S. A. Matthews, and L. Zou (2010): "Risk aversion and optimal reserve prices in the first- and second-price auctions," *Journal of Economic Theory*, 145, 1188–1202.

Hu, A., T. Offerman, and L. Zou (2012): “How risk sharing may enhance efficiency in English Auctions,” *Working Paper*.

Jehiel, P., and B. Moldavanu (2001): “Efficient Design with Interdependent Valuations,” *Econometrica*, 69(5), 1237–1259.

Kotowski, M. H. (2012): “First-Price Auctions with Budget Constraints,” *Working Paper*.

Krishna, V. (2002): *Auction Theory*. Academic Press.

Kwang, N. (1965): “Why do People Buy Lottery Tickets? Choices Involving Risk and the Indivisibility of Expenditure,” *Journal of Political Economy*, 73(5), 530–535.

Laffont, J.-J., and J. Robert (1996): “Optimal auction with financially constrained buyers,” *Economic Letters*, 52, 181–186.

Maskin, E. (2000): “Auctions, development and privatization: Efficient auctions with liquidity-constrained buyers,” *European Economic Review*, 44, 667–681.

Maskin, E., and J. Riley (1984): “Optimal Auctions with Risk Averse Buyers,” *Econometrica*, 52(6), 1473–1518.

Matthews, S. A. (1983): “Selling to risk averse buyers with unobservable tastes,” *Journal of Economic Theory*, 30(2), 370–400.

——— (1987): “Comparing Auctions for Risk Averse Buyers: A Buyer’s Point of View,” *Econometrica*, 55(3), 633–646.

Milgrom, P. (2004): *Putting Auction Theory to Work*. Cambridge University Press.

Myerson, R. B. (1981): “Optimal Auction Design,” *Mathematics of Operations Research*, 6(1).

——— (1991): *Game Theory: Analysis of Conflict*. Harvard University Press.

Myerson, R. B., and B. Hölmstrom (1983): “Efficient and Durable Decision Rules with Incomplete Information,” *Econometrica*, 51(6), 1799–1819.

Pai, M. M., and R. Vohra (2010): “Optimal Auctions with Financially Constrained Bidders,” *Working Paper*.

Riley, J. G., and W. F. Samuelson (1981): “Optimal Auctions,” *American Economic Review*, 71(3), 381–392.

Saitoh, H., and S. Serizawa (2008): "Vickrey allocation rule with income effect," *Economic Theory*, 35(2), 391–401.

Sakai, T. (2008): "Second price auctions on general preferenc domains: two characterizations," *Economic Theory*, 37, 347–356.

Vickrey, W. (1961): "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, 16(1), 8–37.

Wilson, R. (1987): "Game-Theoretic Analyses of Trading Processes," *In Advances in Economic Theory: Fifth World Congress, ed. Truman Bewley*, pp. 33–70.