The Structure of Board Games

Joseph M. Abdou *

April 14, 2013

We introduce a variant of extensive games called board games. In the standard extensive model (Kuhn model) a game is described by a rooted tree. The position of the player determines the history. A board game is described by a rooted directed acyclic graph. A position may have multiple histories. Any board game can be represented by a stategically equivalent standard game (of imperfect information) such that on each information set the future potential plays do not depend on the actual position in that information set. This motivates the introduction of the class of extensive games with effective perfect information. We study rectangularity of strategic game forms associated to board game forms. We prove that any board game form is interactively equivalent to some standard extensive game form of perfect information and - strangely enough - it may fail to be strategically equivalent to that extensive game form.

1 Extensive game forms and board game forms

If $\varpi : X \rightrightarrows Y$ is a correspondence, then the inverse of ϖ is the correspndence $\varpi^{-1} : Y \rightrightarrows X$ defined by $\varpi^{-1}(y) = \{x \in X | y \in \varpi(x)\}$. If $\varpi : X \rightrightarrows X$ we define $\varpi^0 = \mathbf{id}_X$ and $\varpi^{k+1}(x) := \varpi \circ \varpi^k(x) := \{\varpi(y) | \, y \in \varpi^k(x)\}$

1.1 Board: definition

A board is a rooted acyclic directed graph. Precisely a *board* $\mathcal{B}$ is defined by a triple (X, ϖ, r), where X a finite set the elements of which are called *nodes* (or *positions* or *vertices*), $r \in X$ is a distinguished node called the *root* and $\varpi : X \rightrightarrows X$ is a correspondence, called the *predecessor correspondence*, and that satisfies the following: $\varpi(r) = \emptyset$ and for any $x \in X - \{r\}$, $\varpi(x) \neq \emptyset$ and there exists some $k \geq 1$ such that

*Ecole d'Economie de Paris, Paris 1, CNRS, 106-112 boulevard de l'Hôpital 75647 Paris Cedex 13 - France; email: abdou@univ-paris1.fr

$\varpi^k(x) = \{r\}$. We put $\varphi(x) = \{y \in X \mid x \in \varpi(y)\}$ or equivalently $\varphi = \varpi^{-1}$ Thus $\varphi(x)$ is the set of immediate successors of x.

A *path* is non-empty sequence of nodes $(x_0, \cdots, x_l)$ where $l \in \mathbb{N}$ with the following property: if $l \geq 1$ and $k \in \{0, \cdots, l-1\}$: $x_k \in \varpi(x_{k+1})$. x_0 is the initial node of c, x_l is the final node of c and is denoted $e(c)$; the integer l is the length of c and is denoted $l(c)$. If c is a path of length $l \geq 1$ we denote by $d(c)$ the path obtained from c by removing the final node $e(c)$ (By convention for $c = (x_0)$, $d(c) = \emptyset$. We put $e_1 := e$ and by induction ($k \geq 1$) for $e_{k+1} := e_k \circ d = e \circ d^k$ as long as the righthand member is well defined. Let $c = (x_0, \cdots, x_k)$ and $c' = (y_0, \cdots, y_l)$ be two paths such that $x_k = y_0$, the concatenation of c, c' written $c + c'$ is the path $(x_0, \cdots, x_{k-1}, y_0, \cdots, y_l))$. A node x is *terminal* if $\varphi(x) = \emptyset$. A path c is a *history* if the initial node of c is the root r. A history c is called a *play* if the final node of c is a terminal node. We denote by Z the set of terminal nodes, P the set of plays and H the set of histories.

One important property of a board is the absence of cycles: there is no path $(x_0, \ldots, x_l)$ with $l \geq 1$ such that $x_0 = x_l$. Indeed if such a path existed then $x_0 \in \varpi^k(x_0)$ for an infinity of integers k, and that would contradict the main axiom defining the board.

An alternative way to obtain a board is to define it as an directed acyclic graph with a root (X, E, r). Precisely X is the set of vertices, $E \subset X \times X$ is the set of edges, and $r \in X$. We assume that they satisfy the following properties: (i) There is no $x \in X$ such $(x, r) \in E$ (ii) for any $y \in X - \{r\}$ there exists some $x \in X$ such that $(x, y) \in E$ (iii) there is no sequence of nodes $(x_0, \ldots, x_l)$ where $l \geq 1$ such that $(x_k, x_{k+1}) \in E$, $k = 0, \ldots, l-1$, and $x_0 = x_l$. The relation between this definition and the original one is clear: In order to go from the original definition to the second one, we have to put $E := \{(x, y) \in X \times X \mid x \in \varpi(y)\}$, and conversely in order to go from the second definition to the original one, we put, for any $y \in X$: $\varpi(y) := \{x \in X \mid (x, y) \in E\}$.

For any path $c = (x_0, \cdots, x_p)$, we denote by $\bar{c}$ the set $\{x_0, \cdots, x_p\}$. By a slight abuse of notations we sometimes write $x \in c$ for $x \in \bar{c}$. There is a partial order on X denoted $\leq$ defined as follows: $x \leq y$ if either $x = y$ or there exists some path c such that x is the initial node and y is the final node of c. Reflexivity and transitivity are straightforward. Antisymmetry results from the absence of cycles. We shall denote by $<$ the strict part of $\leq$. If $x, y \in \bar{c}$, $x \leq y$, $c(x, y)$ will denote the sub-path of c starting at x and ending at y.

A board $\mathcal{B} = (X, \varpi, r)$ is said to be a *tree* if $\varpi(x)$ is a singleton for any $x \in X - \{r\}$.

1.2 Board game forms

A *board game form* is defined by $(N, \mathcal{B}, (X_i)_{i \in N})$ where $N = \{1, \cdots, n\}$ is a set of players, $\mathcal{B}$ is a board, and $(X_i)_{i \in N}$ is a partition of non terminal nodes (empty sets X_i

are allowed). The game is played as follows: The play starts at $x_0 = r$. Generally if the play is at x_p, two cases are possible: either the node x_p is terminal and this is the end of the play or the unique player i such that $X_i \ni x_p$ learns his position and takes an action that is some successor x_{p+1} of x_p, then the play evolves to x_{p+1}. Since the set of nodes is finite, the play reaches eventually a terminal node z. We shall consider two versions : (1) Γ where the outcome is the play and (2) Γ_e where the outcome is the terminal node of the play. Note that the position does not determine a unique history.

A board game form is said to be *alternating* if, whenever $x = \varpi(y)$, then x and y do not belong to the same player. The *canonical game form* associated to $\mathcal{B}$ is the one where $N = X - Z$ and $X_i = \{i\}$ for all $i \in X - Z$.

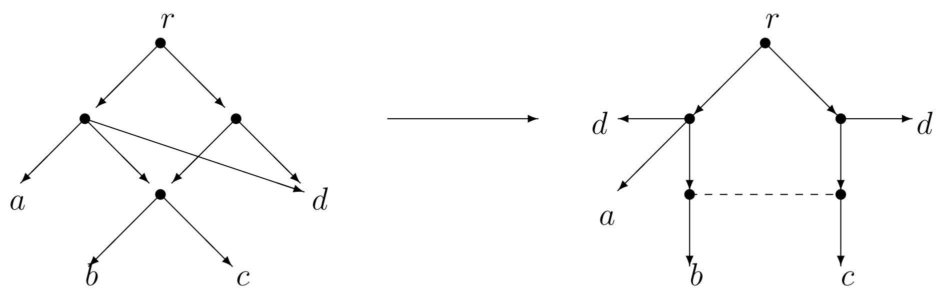

Figure 1: On the left a board. On the right the derived decision structure

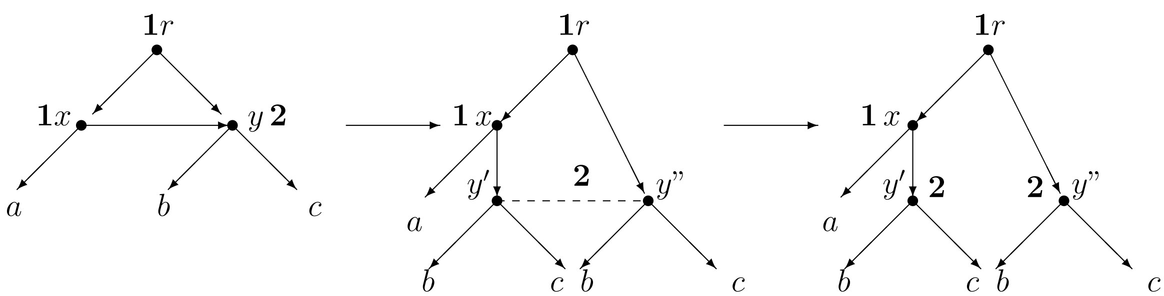

Figure 2: On the left a board game form Γ_e. In the middle the derived extensive game form Γ'_e. On the right the perfect ingormation game form Λ'_e associated to Γ'_e.

1.3 Extensive game forms

An *extensive decision structure* is defined by: $\Delta_f = (\mathcal{T}, \mathcal{U}, (A_u, u \in \mathcal{U}), \xi, A, f)$ where $\mathcal{T} = (X, \varpi, r)$ is a tree. Let Z denote the set of terminal nodes, H the set of histories, Π the set of plays, $\varphi = \varpi^{-1}$. $\mathcal{U}$ is a partition of $X - Z$, the elements of which are called information sets. To every $u \in \mathcal{U}$ is attached a set A_u called the set of actions available at u and a family of bijective maps $\xi_x : A_u \to \varphi(x)$. Define $\xi(x, a) = \xi_x(a)$ $(x \in u \in \mathcal{U}, a \in A_u)$. The map ξ is called the *transition function.* $f : Z \to A$ is the *oucome function*, $f(z)$ is the outcome of z. Put $\mathcal{Z} = Z/f$, the quotient set of Z by f and $\dot{f} : \mathcal{Z} \to A$ the quotient map. There is a binary $\prec$ relation defined on $\mathcal{U} \cup \mathcal{Z}$ as follows: $u \prec v$ if and only if: $\exists x \in u$, $\exists y \in v : x < y$. We define $u \preceq v$ if and only if $u = v$ or $u \prec v$.

Δ_f is an *effective perfect information* (EPI in the sequel) decision structure if for any $u \in \mathcal{U}$, $x, y \in \mathcal{U}$, $a, b \in A_u$: $\xi(x, a)$ and $\xi(y, b)$ have the same information set or the same outcome if and only if $a = b$.

Δ_f is a *perfect information* (PI in the sequel) decision structure if any $u \in \mathcal{U}$ in $\mathcal{U}$ is a singleton

Δ_f is a *free* decision structure if f is injective.

An *extensive* game form is defined by $\Gamma_f = [N, \Delta_f, (\mathcal{U}_i, i \in N)]$ where N is the set of players, Δ is an extensive decision structure, $(\mathcal{U}_i, i \in N)$ is a partition of $\mathcal{U}$. $\mathcal{U}_i$ is the set of information sets of player i. We put $X_i = \cup_{u \in \mathcal{U}_i} u$. $(X_i, i \in N)$ is a partition of $X - Z$. Note that neither perfect recall nor even linearity is assumed. f is a mapping from Z to A. The game is played as follows: The play starts at $x_0 = r$. If the play is at x_p and x_p is a terminal node then x_p is the end of the play , otherwise the unique player i such that $x \in X_i$ learns his information set u and takes an action $a_k \in A_u$; then the play evolves to $x_{k+1} = \xi(x_k, a_k)$. Since the set of nodes is finite, the play reaches eventually a terminal node z, the outcome is $f(z)$. Finally Γ_f is said to be *free* if f is injective and of *perfect information* if every information set is a singleton.

In this model of extensive games, the position determines a unique history, namely the unique path starting at the root and ending at that position. The player learns his information set. In particular each terminal node determines a unique play.

To any board $\mathcal{B} = (X, r, \varpi)$ we associate two *derived extensive decision structures* $\mathcal{B}' = (\mathcal{T}', \mathcal{U}', (A'_u), \xi', A')$ and $\mathcal{B}'_e = (\mathcal{T}', \mathcal{U}', (A'_u), \xi', A', e)$ that differ only by their outcome function, as follows: $\mathcal{T}' = (X', r', \varpi')$ is the *derived tree* where: $X' := H$ the set of all histories of $\mathcal{B}$, $r' := (r)$ and if $c \in H - \{r'\}$, $\varpi'(c) = d(c)$. The set of terminal nodes of $\mathcal{T}'$ is precisely the set of plays of $\mathcal{B}$: $Z' = \Pi$. Recall that $e : H \to X$. Let $\dot{e}$ be the quotient map. $\dot{e}$ is a bijection of H/e onto X. Let $\mathcal{U}' := \dot{e}^{-1}(X - Z)$, $\mathcal{Z}' = \dot{e}^{-1}(Z)$. We take $\mathcal{U}'$ as the set of information sets of $\mathcal{B}'$. Define the action set by $A'_u = \varphi(\dot{e}(u))$ and the transition function by $\xi'_c(a) := (c, a)$ $(c \in u, a \in A'_u)$. In $\mathcal{B}'$ the outcome function is the identity: $f(c) = c$ while in $\mathcal{B}'_e$, the outcome function is defined by: $f(c) := e(c)$ $(c \in Z' \equiv \Pi)$. Put $\mathcal{Z}' = Z'/f'$.

Depending on the choice of the outcome set we associate to $(N, \mathcal{B}, (X_i)_{i \in N})$, two game forms:

- $\Gamma \equiv \Gamma(N, \mathcal{B}, (X_i)_{i \in N})$ where the outcome is the play, and
- $\Gamma_e \equiv \Gamma_e(N, \mathcal{B}, (X_i)_{i \in N})$ where the outcome is the terminal node.

Moreover we associate to $(N, \mathcal{B}, (X_i)_{i \in N})$ the *derived extensive* game forms based on the derived decision structures $\mathcal{B}'$ and $\mathcal{B}'_e$ as defined above:

- $\Gamma' := (N, \mathcal{B}', (\mathcal{U}'_i)_{i \in N})$: the free extensive GF associated to $(N, \mathcal{B}, (X_i)_{i \in N})$ and
- $\Gamma'_e := (N, \mathcal{B}'_e, (\mathcal{U}'_i)_{i \in N})$: the extensive GF associated to $(N, \mathcal{B}, (X_i)_{i \in N})$

and the perfect information extensive game forms

- Λ'_e, the perfect information extensive GF associated to Γ'_e (i.e. obtained by taking $\{c\}$ as the information set containing c $(c \in H)$ and $A_c = A_u$ $(c \in u \in \mathcal{U}')$
- Λ' the perfect information extensive GF associated to Γ'.

Proposition 1.1 *An extensive decision structure $\Delta_f = (\mathcal{T}, \mathcal{U}, (A_u, u \in \mathcal{U}), \xi, A, f)$ is of effective perfect information if and only if it is derived from some board.*

Proof. The "only if" part if trivial. In order to prove the "if" part, we construct a board $\mathcal{B} = (X, r, \varpi)$ as follows: $X = \mathcal{U}' \cup \mathcal{Z}'$, $r = \{r'\}$, $Z = \mathcal{Z}'$, forall $y \in X$, $\varpi(y) := \{x \in X - Z | \exists x' \in x, a' \in A'_x : \xi'(x', a') \in y\}$ or equivalently for all $x \in X - Z$, $\varphi(x) := \{y \in X | \exists x' \in x, a' \in A'_x : \xi'(x', a') \in y\}$

2 Strategic game forms

A *strategic* game form G is defined by $\langle (S_i, i \in N), A, g \rangle$, where $N = \{1, \cdots, n\}$ is the set of *players*, S_i is the *strategy set* of player i, $(i \in N)$, A is the set of *alternatives* and $g : \prod_{i \in N} S_i \to A$ is the *outcome function*. We assume that A is finite and g is onto. Let $\mathcal{G}(N, A)$ be the set of all game forms with N and A fixed.

Equivalent game forms. We define an equivalence relation on the set $\mathcal{G}(N, A)$. Let $G \in \mathcal{G}(N, A)$, $i \in N$, $x_i \in X_i$ and $y_i \in X_i$. We say that x_i and y_i are *duplicates in G* if : $g(x_i, x_{-i}) = g(y_i, x_{-i}), \forall x_{-i} \in X_{-i}$. Clearly being duplicates in G is an equivalence relation on X_i that will be denoted $\sim_i$. The *reduced* game form associated to G is the

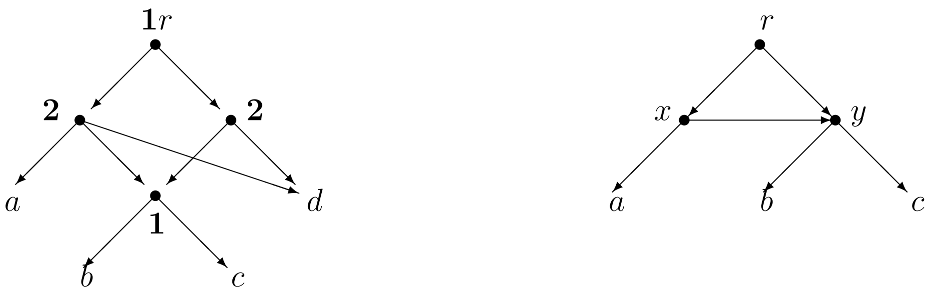

Figure 3: On the left, a two-player alternating board game form. On the right a board that cannot support a two-player alternating game form.

game form $\bar{G} = (\bar{X}_1, \cdots, \bar{X}_n, A, \bar{g})$ where $\bar{X}_i$ is the quotient of X_i by the equivalence relation $\sim_i$ and $\bar{g}$ defined as the quotient mapping of g that is : $\bar{g}(\bar{x}_1, \cdots, \bar{x}_n) = g(x_1, \cdots, x_n)$ if $x_1 \in \bar{x}_1, \cdots, x_n \in \bar{x}_n$. One obtains the reduced game form of G by eliminating all but one strategy in each equivalence class of $\sim_i$. It is easy to see that such an elimination does not depend on the order of elimination by which one proceeds. Two game forms G and G' in $\mathcal{G}(N, A)$ are said to be *strategically equivalent* if $\bar{G}$ and $\bar{G}'$ are equal up to relabeling respective strategy sets.

2.1 Rectangular game forms

A game form G is said to be rectangular if for every $a \in A$ the inverse image of a by g is a direct product. Formally: $G = (X_1, \cdots, X_n, A, g)$ is *rectangular* if for every $a \in A$ there exist $Y_1 \subset X_1, \cdots, Y_1 \subset X_1$ such that : $g^-1(a) = \prod_{i \in N} Y_i$

Fact 2.1 *Each of the following properties is equivalent to rectangularity :*

(i) $\forall S \in P(N),\ \forall x_S \in X_S,\ \forall y_S \in X_S,\ \forall x_{S^c} \in X_{S^c},\ \forall y_{S^c} \in X_{S^c}$*:*
$$g(y_S, x_{S^c}) = g(x_S, y_{S^c}) = a \Rightarrow g(x_S, x_{S^c}) = a,$$

(ii) $\forall S \in P(N),\ \forall x_S \in X_S,\ \forall x_{S^c} \in X_{S^c}$*:* $g(x_S, X_{S^c}) \cap g(x_{S^c}, X_S) = \{g(x_S, x_{S^c})\}$,

(iii) $\forall S \in P(N),\ \forall Y_S \subset X_S,\ \forall Y_{S^c} \subset Y_{S^c}$*:* $g(Y_S, X_{S^c}) \cap g(Y_{S^c}, X_S) = g(Y_S, Y_{S^c})$,

(iv) For any partition $(S_1, \cdots, S_p)$ *of* N *and any* $Y_{S_k} \subset X_{S_k}, k = 1, \cdots, p$ *one has :*
$$\cap_{k=1}^{p} g(Y_{S_k}, X_{-S_k}) = g(Y_{S_1} \times, \cdots, \times Y_{S_p}).$$

PROOF. Straightforward. □

To any board game form Γ, we associate the strategic game form $N(\Gamma) := \langle (S_i, i \in N), A, \pi \rangle$ as follows: $S_i = \prod_{x \in X_i} \varphi(x)$. Define $\pi(s)$ as the unique play generated by s. Similarly $N(\Gamma_e) := \langle (S_i, i \in N), A, g \rangle$ and $g = e \circ \pi$.

To any extensive game form Γ_f, we associate the strategic game form $N(\Gamma_f) := \langle (S_i, i \in N), A, g\rangle$ as follows: $S_i = \prod_{u \in \mathcal{U}_i} A(u)$. Define $\gamma(s)$ as the unique terminal node reached by s and $g = f \circ \gamma$. We also put $N(\Gamma) =:= \langle (S_i, i \in N), Z, \gamma\rangle$

Remark 2.2 Let Γ and Γ_e be the two versions of a game board and let Γ' and Γ'_e be their respective derived extensive GF. Let $N(\Gamma) \equiv \langle (S_i, i \in N), A, \pi\rangle$ and $N(\Gamma') \equiv \langle (S'_i, i \in N), A', \gamma'\rangle$. One has the identification : $N(\Gamma) = N(\Gamma')$ and $N(\Gamma_e) = N(\Gamma'_e)$. The map $\dot{e}$ induces an identification of S'_i and S_i. Using this identification, one has $\pi = \gamma'$ and consequently $g = e \circ \pi = e \circ \gamma' = g'$.

3 Rectangularity of Board game forms

Theorem 3.1 *For any free extensive game form Γ the associated strategic form $N(\Gamma)$ is rectangular.*

When an extensive form (of perfect information or not) is not free, then it normal form may fail to be rectangular. In this section we shall characterize all Board game forms that have a rectangular normal form.

Let $\mathcal{B} = (X, \varpi, r)$ be a board. An edge (x, y) is said to be *redundant* if $\varphi(x) = \{y\}$ (that is y is the unique successor of x). A Board is said to be *non redundant* if it has no redundant edge.

When the edge (x, y) is redundant we take $x_0 = x$ and $Y = \{x, y\}$. We write $I_{x,y}$ the operation that transforms the board $\mathcal{B} = (X, \varpi, r)$ into a board $\mathcal{B}' = (X', \varpi', r')$ as follows: $X' = X/\{x\}$, $r' = r$ anf for all $(s', t') \in X' - \{y\} \times X' - \{y\}$, $s' \in \varpi'(t')$ if and only if and $s' \in \varpi(t')$, $\varpi'(y) = \varpi(x)$ $\varphi'(y) = \varphi(y)$. Remark that if y is a terminal node in $\mathcal{B}$, then it is a terminal node in $\mathcal{B}'$.

If we start by a board game $\Gamma(\mathcal{B}, (X_i)_{i \in N}))$, then the operation $I_{x,y}$ transforms Γ into $\Gamma' \equiv \Gamma(\mathcal{B}', (X'_i)_{i \in N})$ where $X'_i = X_i \cap X'$. By successive application of such operations on any board $\mathcal{B}$ we obtain a non redundant board, called the non redundant version of $\mathcal{B}$.

When writing the normal form $N(\Gamma')$, a player such that $X'_i = \emptyset$ has strategy set X_i that is some singleton.

Proposition 3.2 *A board game form and its non redundant version have the same normal form.*

Example 3.3 Let $\mathcal{B}$ be the board on the right of figure 1.3. Let Γ be the game form obtained by letting player 1 act on nodes r and x and player 2 on node y; then $N(\Gamma)$ is rectangular. Let Γ' be the game form obtained by letting player 1 act on nodes r

and y and player 2 on node x; then $N(\Gamma')$ is not rectangular. If the board is played by 3 players then again the game form is not rectangular.

Theorem 3.4 *Let $\mathcal{B}$ be a board and let $\mathcal{B}'$ be its non redundant version. Then the following statements are equivalent:*

(i) $\mathcal{B}'$ is a tree,

(ii) The canonical game form of $\mathcal{B}$ (i.e. the game where each non terminal node is a distinct player) is rectangular,

(iii) All game forms $(\mathcal{B}, (X_i)_{i\in N})$ are rectangular,

(iv) All two-player game forms $(\mathcal{B}, X_1, X_2)$ are rectangular,

Proof. $(i) \Rightarrow (ii)$ follows from Theorem 3.1. $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$. In order to prove $(iv) \Rightarrow (i)$ we shall apply Lemma 3.6.

Theorem 3.5 *An alternating game form with board $\mathcal{B}$ is rectangular if and only if the non redundant version of $\mathcal{B}$ is a tree.*

Proof. The if part follows from Theorem 3.1. The only if part is an application of Lemma 3.6.

Lemma 3.6 *Assume that the non redundant board $\mathcal{B}$ is not a tree then there exist:*

(i) two plays c, d leading to the same terminal node z,

(ii) $x^* \in c \cap d$,

(iii) y^* is a successor of x^*, $y^* \in c$, $y^* \notin d$, y^* belongs to a some path leading to some terminal node $z' \neq z$.

Proof. Assume that $\mathcal{B}$ is not a tree. This amounts to the existence of some terminal node z such that $C(z)$ contains two or more plays. Let $J = \cup_{c\in C(z)}\bar{c}$, $I = \cap_{c\in C(z)}\bar{c}$. I is totally ordered by the $\leq$. I contains at least two elements, namely: r and z.

a) There exists a unique node x_0 such that (1) $x_0 \in I$, (2) x_0 has at least two predecessors and (3) there is a unique path v_0 starting at x_0 and ending at z all the nodes of which are in I. Note that x_0 may be equal to z.

b) There exists a unique node x_1 such that (1′) $x_1 \in I$, (2′) $x_1 < x_0$, (3′) there exists no node $x \in I$ such that $x_1 < x < x_0$. Note that x_1 may be equal to r.

c) The set of paths starting at x_1 and ending at x_0, denoted $C(x_1, x_0)$, contains at least two elements. It follows that there is at least one element of $C(x_1, x_0)$ with length ≥ 2. Let w_0 be an element of $C(x_1, x_0)$ with maximal length, and let $y_0 = e_2(w_0)$. Clearly $y_0 \in \varpi(x_0)$, $y_0 \in J - I$. In particular $y_0 \notin \{x_0, x_1\}$ and is not a terminal node.

.

Claim. There exists y_1 a successor of y_0 such that $y_1 \notin J$

By assumption any non terminal node has at least two successors. Let $y_1 \neq x_0$ be a successor of y_0. If $y_1 \in J$ then there would exist a path, say w_1, starting at y_1 and ending at x_0, thus with length ≥ 1. The path $w = d(w_0) + (y_0, y_1) + w_1$ would belong $C(x_1, x_0)$ and be strictly longer than w_0; this contradicts the maximality of w_0. We conclude that $y_1 \notin J$.

Let c be a play such that $c(x_1, x_0) = w_0$ let Y be the set of all nodes $y \in c(x_1, x_0) - \{x_0, x_1\}$ that have some successor in $X - J$. We have just proved that $y_0 \in Y$. Let y^* be the node of Y that is the closest to x_1. It follows that y^* is not a terminal node and that on $c(x_1, y^*) - \{x_1, y^*\}$ (possibly empty) all nodes have all their successors in J. Let x^* be the predecessor of y^* on $c(x_1, x_0)$. Note that x^* may be equal to x_1.

Claim. There exists some successor, say y_2, of x^* such that $y_2 \in J$ and $y_2 \neq y^*$.

In the case $x^* = x_1$, this follows from the definition of x_1. If $x^* \neq x_1$ then, by construction, all successors of x^* are in J and each node have at least two successors.

Claim. There exist a play $d \in C(z)$ that coincide with c on $c(r, x^*)$ and not containing y^*.

If $x^* = x_1$ this follows from the definition of x_1. If $x^* \neq x_1$ then let d consists of the concatenation of $c(r, x^*)$ followed by the edge (x^*, y_2) followed by an arbitrary path starting at y_2 and ending at z. If d were to contain y^*, then $c(x_1, x_0) \equiv w_0$ would not be the longest path in $C(x_1, x_0)$ since the concatenation of $c(x_1, x^*)$, $d(x^*, y^*)$ and $c(y^*, x_0)$, given that $d(x^*, y^*)$ contains y_1, would be strictly longer than $c(x_1, x_0)$. We conclude that d does not contain y^* □

End of the proof of theorem 3.1. Let Γ be a two-player game on $\mathcal{B}$ such that $x^* \in X_1$, $y^* \in X_2$. Let $(s_1, s_2) \in S_1 \times S_2$ and $(t_1, t_2) \in S_1 \times S_2$ such that $\gamma(s_1, s_2) = c$ and $\gamma(t_1, t_2) = d$. Note that y^* is not on d. Therefore, by modifying if necessary t_2 on node y^* one can assume that $t_2(y^*) \in X - J$. It is then clear that $\gamma(s_1, t_2)$ is not in J, hence is different from z. □

Let Γ_f be an extensive game form with perfect information and outcome map f. We shall define two elementary operations of reduction on Γ_f.

(a) Merger of terminal nodes z and z'. Here z and z' are two distinct terminal nodes with the same predecessor x and the same outcome that is $\varpi(z) = \varpi(z') = x$ and $f(z) = f(z') = a$. The merging consists in replacing z and z' by one terminal node with the same predecessor and the same outcome.

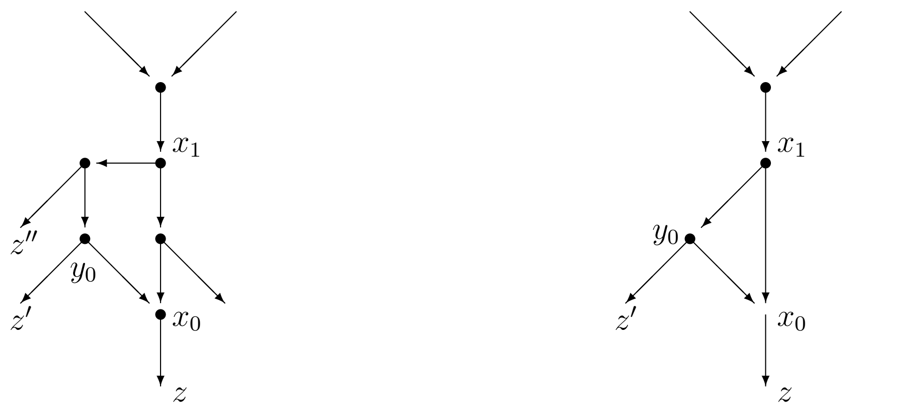

Figure 4: aaa

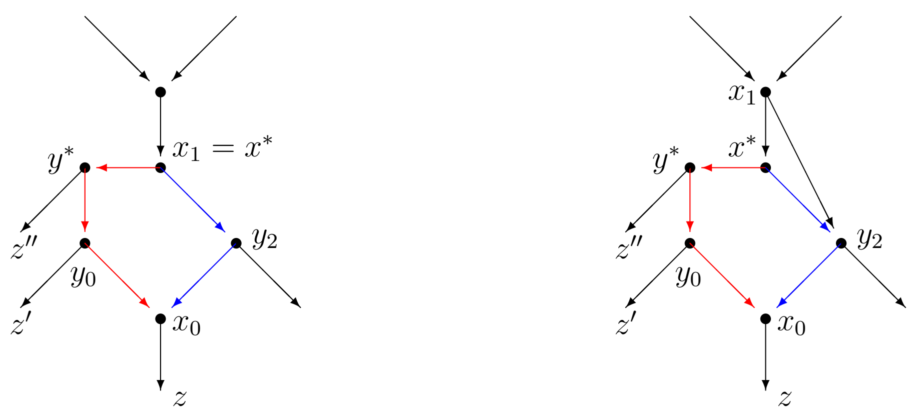

Figure 5: bbb

(b) Removal of a redundant edge (x, y). This is the operation $I_{x,y}$. If y is a node of player i, then it remains a node of the same player. If y is a terminal node with outcome a, then it remains a terminal node with the same outcome.

(c) Removal of successive moves of the same player: If y is a successor of x and belongs to the same player, then this operation is defined similarly to the merger of a redundant edge (x, y) : Remove x. In the new game the set of outgoing edges of y is the union of the set of old outgoing edges of x (less (x, y)) and the set of old edges outgoing from y. The predecessor of y in the new game is the old predecessor of x. The player of y remains the same.

A reduction of Γ_f is any sequence of elementary operations starting by Γ_f. A reduction is complete if no further reduction is possible. If no reduction of Γ_f is possible then we say that Γ_f is irreducible. The irreducible version of Γ_f is the extensive game form with perfect information $\bar{\Gamma}_{\bar{f}}$ obtained after some complete reduction. The order of elementary operations in a complete reduction is not relevant.

Proposition 3.7 *Let Γ_f be an extensive game form with perfect information and outcome map f. Any of the operations described above keeps unchanged the equivalence class of the associated normal game form.*

Let Γ_f be any extensive game form with perfect information and outcome map f. If its irreducible version is free than its normal form $N(\Gamma_f)$ is rectangular. Conversely since $N(\Gamma_f)$ is tight, in view of theorem [], rectangularity of $N(\Gamma_f)$ implies the existence of a free extensive form of perfect information Λ such that $N(\Lambda)$ is equivalent to $N(\Gamma_f)$. The following theorem tells us how we can find such a Λ.

Theorem 3.8 *Let Γ_f be any extensive game form with perfect information and outcome map f. The associated strategic form $N(\Gamma_f)$ is rectangular if and only if its irreducible version is free.*

Proof. If the irreducible version is free then by Theorem 3.1 and Proposition 3.7 $N(\Gamma_f)$ is rectangular. Conversely assume that $N(\Gamma_f)$ is rectangular. Let Γ'_f be the irreducible version of Γ_f. We associate to Γ'_f the board game form Δ obtained by identifying all terminal nodes that have the same outcome: If $Z(a) = f^{-1}(a)$ then the incoming edges of a is the union of all incoming edges to $Z(a)$. Clearly $N(\Delta) = N(\Gamma'_f)$ so that $N(\Delta)$ is rectangular. Moreover Δ is non redundant and alternating. In view of Theorem 3.4 Δ is a tree. It follows that for any $a \in A$, $Z(a)$ is a singleton. This is equivalent to say that Γ'_f is free. □

4 The Interaction class of Board game forms

4.1 Interaction bundle

A *strategic game* is an array $\Gamma = (X_1, \ldots, X_n; Q_1, \ldots, Q_n)$, where for each $i \in N = \{1, \ldots, n\}$, X_i is a non-empty set of strategies of player i, and Q_i is a quasi-order (complete, transitive, reflexive binary relation) on $X_N = \prod_{i \in N} X_i$. We denote by Q_i° the strict binary relation induced by Q_i For every coalition $S \in \mathcal{P}_0(N)$, the product $\prod_{i \in S} X_i$ is denoted X_S (by convention $X_\emptyset$ is the singleton $\{\emptyset\}$). Let $\mathcal{M} \subset \mathcal{P}_0(N)$. A strategy array $x_N \in X_N$ is an $\mathcal{M}$*-equilibrium* of the game Γ if there is no coalition $S \in \mathcal{S}$ and $y_S \in X_S$ such that $(y_S, x_{S^c})\ Q_i^\circ\ x_N$ for all $i \in S$.

Let $G = (X_1, \ldots, X_n, A, g)$ be game form. For each preference profile $R_N \in L(A^N)$, the game form G induces a game $(X_1, \ldots, X_n; Q_1, \ldots, Q_n)$ with the same strategy spaces as in G and with the Q_i defined by: $x_N\, Q_i\, y_N$ if and only if $g(x_N)\, R_i\, g(y_N)$ for $x_N, y_N \in X_N$. We denote this game by (G, R_N).

We say that $a \in A$ is an $\mathcal{M}$*-equilibrium outcome* of (G, R_N) if there is an $\mathcal{M}$-equilibrium of (G, R_N) $x_N \in X_N$ with $g(x_N) = a$. The game form G is said to be *solvable in* $\mathcal{M}$*-equilibrium* or $\mathcal{M}$*-solvable*, if for each preference profile $R_N \in Q(A)^N$, the game (G, R_N) has an $\mathcal{M}$-equilibrium. In particular, when $\mathcal{S} = \mathcal{N} = \{\{1\}, \ldots, \{n\}\}$, an $\mathcal{M}$-equilibrium is a Nash equilibrium. Similarly, when $\mathcal{M} = \mathcal{P}_0(N)$, an $\mathcal{S}$-equilibrium is a strong Nash equilibrium.

If $y_N \in X_N$, the notation $g(y_{S^c}, X_S)$ stands for $\{g(y_{S^c}, x_S) \mid x_S \in X_S\}$ if $S \neq N$ and for $g(X_N)$ if $S = N$.

Given the game form $G = (X_1, \ldots, X_n, A, g)$ the β*-interaction form* (over (N, A)) associated with G is the interaction bundle $\mathcal{E}_\beta^G$ defined as follows: For $a \in A$:

$$\mathcal{E}_\beta^{G,\mathcal{M}}[a] =$$

$$\{\psi \in \Phi(N, A) \mid \forall y_N \in g^{-1}(a), \exists S \in \mathcal{M}, \exists x_S \in X_S \colon g(x_S, y_{S^c}) \in \psi(S)\} \tag{1}$$

$$\mathcal{E}_\alpha^{G,\mathcal{M}}[a] =$$

$$\{\varphi \in \Phi(N, A) \mid \exists y_N \in g^{-1}(a), \forall S \in \mathcal{M}, \forall x_S \in X_S \colon g(x_S, y_{S^c}) \in \varphi(S)\} \tag{2}$$

If $\mathcal{M} = \mathcal{P}_0(N)$ then we drop the superscript $\mathcal{M}$ from the notation of $\mathcal{E}_\alpha^{G,\mathcal{M}}[a]$ and $\mathcal{E}_\beta^{G,\mathcal{M}}[a]$

Proposition 4.1 *a is an $\mathcal{M}$-equilibrium outcome of (G, R_N) if and only if $S \to P^c(a, S, R_N) \in \mathcal{E}_\alpha^{G,\mathcal{M}}[a]$*

For $\varphi \in \Phi(N, A)$ define φ^* by $\varphi^*(S) = (\varphi(S))^c$ $(S \in \mathcal{P}_0(N))$. One has:

$$\varphi \in \mathcal{E}_\alpha^{G,\mathcal{M}}[a] \Leftrightarrow \varphi^* \notin \mathcal{E}_\beta^{G,\mathcal{M}}[a]$$

$$\mathcal{E}_\beta^{G,\mathcal{M}}[a] = \{\varphi \in \Phi(N, A) | \exists \varphi' \in \mathcal{E}_\beta^{G}[a], \varphi'(S) = \emptyset \ (S \notin \mathcal{M}), \varphi'(T) = \varphi(T) \ (T \in \mathcal{M})\}$$

It follows that $\mathcal{E}_\beta^G$ (or $\mathcal{E}_\alpha^G$) contains all the information relevant to the determination of all the $\mathcal{M}$- equilibria correspondences of the game form G. Two game forms G and G' in $\mathcal{G}(N, A)$ are said to be *Interactively equivalent* if $\mathcal{E}_\alpha^G = \mathcal{E}_\alpha^{G'}$

Proposition 4.2 *Let $G = (X_1, \ldots, X_n, A, g)$ and $G' = (X'_1, \ldots, X'_n, A, g)$ be two game forms with the same set of players and the same set of alternatives. G and G' are interactively equivalent if and only if:*

(a) For any $x \in X$ there exists $x' \in X'$ such that $g'(x') = g(x)$ and for all $T \in \mathcal{P}_0(N)$ $g'(x'_{T^c}, X'_T) \subset g(x_{T^c}, X_T)$

(b) For any $x' \in X'$ there exists $x \in X$ such that $g(x) = g'(x')$ and for all $T \in \mathcal{P}_0(N)$ $g({}_xT^c, X_T) \subset g(x'_{T^c}, X'_T)$

Proposition 4.3 *If G and G' are interactively equivalent and if $f : A \to A'$, then $f \circ G$ and $f \circ G'$ are interactively equivalent*

Let $(\mathcal{B}, (X_i)_{i \in N})$ be a board game form. In what follows we use the notations introduced... We already know that $N(\Gamma) = N(\Gamma')$ and $N(\Gamma_e) = N(\Gamma'_e)$. On the other hand, it is clear that $N(\Gamma_e)$ and $N(\Lambda'_e)$ may fail to be strategically non equivalent. Nevertheless we have the following:

Theorem 4.4 *Let $(\mathcal{B}, (X_i)_{i \in N})$ be a board game form. $N(\Gamma_e)$ and $N(\Lambda'_e)$ are interactively equivalent.*

Lemma 4.5 *Let Γ be an any EPI game form and let Λ be its associated PI game form. Then Γ and Λ are interactively equivalent.*

Proof. Let $N(\Gamma) = (S_1, \ldots, S_n, A, g)$ and let $N(\Lambda) = (\tilde{S}_1, \ldots, \tilde{S}_n, A, \tilde{g})$. We recall that for any $x \in u \in \mathcal{U}$, one has $A_x = A_u$. $S_i = \prod_{u \in U_i} A_u$ and $\tilde{S}_i = \prod_{x \in X_i} A_x$. $g = e \circ \pi$ and $\tilde{g} = e \circ \tilde{\pi}$. If in Γ all information sets are singletons, then $\Gamma = \Lambda$, therefore the result is obvious. If some information set u of some player, say player 1, is such that $u = \{x_1, \ldots, x_p\}$ with $p \geq 2$, we shall consider the extensive game form Γ' obtained from Γ by removing u and replacing it by the p information sets $\{x_1\}, \ldots, \{x_p\}$. While $N(\Gamma)$ and $N(\Gamma')$ may not be strategically equivalent, we are going to prove that they are interactively equivalent. W.l.o.g we shall assume that for any $i \in N$, X_i is a singleton, that is, any information set can be identified to some player. Thus we have $S'_1 = S_1^u$ (the set of maps $\tau : u \to S_1$) (or $A_{x_1} \times \cdots \times A_{x_p}$) and for $i \neq 1$, $S'_i = S_i$.

The maps ξ and τ: Let $D := \{t_{-1} \in S_{-1} | \exists a \in S_1 : \pi(a, t_{-1}) \cap u \neq \emptyset\}$. Since Γ is linear (that is, no path intersects twice any information set), one can define $\xi : D \to u$ by $\xi(t_{-1})$ as the unique node in $\pi(a, t_{-1}) \cap u$, and extend ξ arbitrarily on $S_{-1} - D$. Remark that ξ depends only on the structure of the game. We also define $\tau : S_1 \to S'_1$ by $\tau(t_1)(x) = t_1$ for all $x \in u$.

In order to prove $\mathcal{E}_\alpha^{N(\Gamma)} \leq \mathcal{E}_\alpha^{N(\Gamma')}$, we are going to associate to any $s' \in S'$, $s' = (s'_1, s'_{-1}) \in S'_1 \times S'_{-1}$, the strategy $s \in S$ of $N(\Gamma)$ where $s_i = s'_i$ $(i \neq 1)$ and $s_1 = s'_1 \circ \xi(s_{-1})$ if $s_{-1} \in D$ and $s_1 = \bar{s}_1$, where $\bar{s}_1$ is a fixed element of S_1. With these notations we have the:

Claim 1. Let $T \in P_0(N)$:

$$\pi(s) = \pi'(s') \tag{3}$$

$$\pi(S_T, s_{T^c}) = \quad \pi'(S'_T, s'_{T^c}) \ (T \ni 1) \tag{4}$$

$$g(S_T, s_{T^c}) \subset \quad g'(S'_T, s'_{T^c}) \ (T^c \ni 1) \tag{5}$$

Proof of the claim. First remark that by definition of σ we have: $\pi(s'_1 \circ \xi(s_{-1}), s_{-1}) = \pi'(s'_1, s_{-1})$ This proves (3). Let $T \ni 1$, for all $t'_1 \in S'_1$ and all $t_{-1} \in S_{-1}$: $\pi(t'_1 \circ \xi(t_{-1}), t_{-1}) = \pi'(t'_1, t_{-1})$ so that $\pi'(S'_T, s_{T^c}) \subset \pi(S_T, s_{T^c})$. Conversely for any $t_1 \in S_1$, $t_{-1} \in S_{-1}$: $\pi(t_1, t_{-1}) = \pi'(\tau(t_1), t_{-1})$. so that we have $\pi(S_T, s_{T^c}) \subset \pi'(S'_T, s_{T^c})$. This proves (4). In order to prove (5), let $t_T \in S_T$. Let I be the set of players in T that precede some node in u and let $J = T - I$. We distinguish two cases:

Case 1: The strategy s hits u or equivalently $s_{-1} \in D$. If the strategy (t_T, s_{T^c}) does not hit u, we put $t'_T = t_T$. Clearly $\pi'(t'_T, s'_1, s_{T^c - \{1\}}) = \pi(t_T, s_1, s_{T^c - \{1\}})$. If the strategy (t_T, s_{T^c}) hits u we define t'_T by setting $t'_i = s_i$ for all $i \in I$ and $t'_i = t_i$ for all $i \in J$. Since the action of player 1 on u is $s_1 = s'_1 \circ \xi(s_{-1})$, it follows that in both strategies, on all information sets, players take the same actions. Given the game is of effective perfect information we have: $g'(t'_T, s'_1, s_{T^c - \{1\}}) = g(t_T, s_1, s_{T^c - \{1\}})$.

Case 2. The strategy s does not hit u or equivalently $s_{-1} \notin D$. We define $t'_T = t_T$. Note that we have $s_1 = \bar{s}_1$. Clearly $\pi'(t'_T, s'_1, s_{T^c - \{1\}}) = \pi(t_T, s_1, s_{T^c - \{1\}})$.

In fact the relation $\preceq$ on $\mathcal{U} \cup \mathcal{Z}$ is transitive and antisymmetric. We just take u as some minimal element for $\preceq$ in the set $\{v \in \mathcal{U} | \, |u| \geq 2\}$.

End of the proof. Repeat the operation untill all information sets become singletons.