International Game Theory Review

SOCIAL CHOICE AND COOPERATIVE GAMES: VOTING GAMES AS SOCIAL AGGREGATION FUNCTIONS

MATHIEU MARTIN
THEMA, University of Cergy-Pontoise
95000 Cergy-Pontoise, France
mathieu.martin@u-cergy.fr

MAURICE SALLES
CREM (UMR-CNRS 6211) and Institute for SCW, University of Caen
14032 Caen cedex, France
CPNSS, London School of Economics, Houghton Street
London WC2A 2AB, UK
Murat Sertel Center for Advanced Economic Studies, Bilgi University
Istanbul, Turkey
maurice.salles@unicaen.fr

Received (Day Month Year)
Revised (Day Month Year)

We consider voting games as procedures to aggregate individual preferences. We survey positive results on the non-emptiness of the core of voting games and explore other solutions concepts that are basic supersets of the core such as Rubinstein's stability set and two types of uncovered sets. We consider cases where the sets of alternatives are 'ordinary' sets, finite sets and infinite sets with possibly a topological structure.

Keywords: Social choice, Aggregation functions, voting games

Subject Classification: JEL C7, D7

1. Introduction

The rebirth of social choice theory in the 1940s due to Black (1958) and Arrow (1963), largely ignored the theory of games. It is surprising since the first edition of von Neumann and Morgenstern's *Theory of Games and Economic Behavior* was published in 1944. This major book of the last century includes a long chapter on 'simple games.' On the basis of an ordinal version of simple games, Guilbaud (1952, 1968) and later Wilson (1972) were probably the first authors to establish links between Arrovian social choice theory and voting games (see also Bloomfield and Wilson (1972)). Arrow, in the first edition of his 1963 book which appeared in 1951 was clear about this as can be checked page 7. However, one must also note that Arrow mentioned that he did ignore the strategic voting issue (which is related to non cooperative games and the concept of Nash equilibrium), an issue that was only solved twenty years later by Gibbard (1973), Pattanaik (1973, 1978) and Satterthwaite (1975).

When we consider this rebirth of social choice theory we can distinguish two sources for most of the further developments: first, the so-called axiomatic analysis stemming from Arrow's impossibility theorem and, second, formal political analysis mainly stemming from Black's analysis of majority rule and the identification of conditions of homogeneity of individual preferences that guarantee some kind of social rationality, be it the transitivity of the social preference generated by majority rule (or transitivity-like properties) or the existence of a best element (a Condorcet winner). It should be noted, however, that Arrow gave the first relation-wise (or discrete) version of the most famous of these conditions of homogeneity, viz. Black's single-peakedness. It is within the Arrovian framework that Sen (1966), Inada (1964, 1969), Sen and Pattanaik (1969), Fishburn (1972, 1973) developed a thorough analysis of these conditions of homogeneity for majority rule and some direct extensions. With other assumptions about individual rationality (for instance, in case individual preferences have a symmetric part—indifference—that is not transitive results were mainly due to Fishburn (1970), Inada (1970) and Pattanaik (1971)). A useful and excellent survey is Gaertner's book (2001). In 1975 we had a rather complete view and treatment of these conditions as far as majority rule was concerned.

Majority rule says that an option a is socially better than an option b if the number of individuals who prefer a to b is strictly greater than the number of individuals who prefer b to a. This means that if we consider individual preferences where indifference between options is possible, an individual who is indifferent between a and b is exactly similar to a non-existent person (at least regarding a and b). The voting game framework is different. Given the set of individuals, one defines a priori the groups of individuals that have power. If all individuals in such a group prefer a to b, then a is socially preferred to b. To explain simply the difference, consider a set of 10 individuals with 3 individuals who prefer a to b, 2 individuals who prefer b to a and 5 individuals who are indifferent between a and b. With majority rule, one obtains a socially preferred to b. If we consider a majority voting game with symmetry (a voting game with symmetry—or with anonymity in the social choice terminology—will be called a quota game; it is a majority voting game if the quota—the minimum number of individuals to form a powerful group—is greater than half the number of individuals), groups of at least 6 individuals are endowed with power so that one cannot say any more that a is socially better than b. In passing, this shows the power of the people who are indifferent between a and b.

With voting games, a notion related to the notion of Condorcet winner is the non-emptiness of the core. Although the notion of core was not introduced by von Neumann and Morgenstern, it became the major solution concept of cooperative games (with interesting and important variations on the so-called welfare theorems of general equilibrium theory due to Debreu and Scarf (1963), with formal extensions in Aumann (1964)). Dummett and Farquharson (1961) showed that the core of a majority voting game was non-empty when the individual preferences satisfied an extended version of Black's single-peakedness. In 1975, Nakamura extended Dummett and Farquharson's result to all proper voting games. In two papers, Salles (1975, 1976) gave conditions of homogeneity for the transitivity and a transitivity-like property of the social preference generated by a proper (and, for transitivity, proper and strong) voting game. In 1979, Nakamura demonstrated what we consider as the main result on voting games in the discrete case, viz. that the core of any voting game is always non-empty (that is whatever are the individual preferences) if and only if the number of options is strictly less than a number given by the structure of the game.

Although the interest in these questions somewhat faded away, there has been recently a renewal of this interest thanks to papers by Barberà and Ehlers (2011) and by Jain (2009).

Even though we concentrate our paper on positive results, it is well known and rather obvious that the core can be empty (rather than considering the Condorcet paradox as a paradox of majority rule, one can interpret it as a case where a majority game has an empty core). The core is defined on the basis of a strict preference/dominance relation. One can find meaningful sub-relations of this relation to define supersets of the core such as Rubinstein stability set (Rubinstein (1980)) and various uncovered sets (Gillies (1959), Miller (1980), Bordes (1983), McKelvey (1986), Penn (2006), Duggan (201?)).

The main part of the paper is devoted to the case of finite or unstructured sets of alternatives. We however allude to infinite sets of alternatives with a topological structure.

2. Aggregation Functions, Voting Rules, Voting Games, Core of Voting Games

Let X be a set of alternatives (options, social states, candidates, allocations of standard microeconomic theory etc.) and $\#X$ be the cardinal of X (the number of its elements when X is finite). A binary relation $\succeq$ over X is a set of ordered pairs (x, y) with x and y in X, i.e., $\succeq$ is a subset of the Cartesian product $X \times X$. We use the notation $x \succeq y$ rather than $(x, y) \in \succeq$. Intuitively $x \succeq y$ means in our context 'x is at least as good as y.' The asymmetric component of $\succeq$, denoted by $\succ$, meaning 'is better than,' is defined by $x \succ y$ if $x \succeq y$ and $\neg y \succeq x$ ($\neg$ is the negation symbol). The symmetric component, denoted by $\sim$, meaning 'there is an indifference between,' is defined by $x \sim y$ if $x \succeq y$ and $y \succeq x$. The binary relation $\succeq$ is reflexive if for all $x \in X$, $x \succeq x$. It is complete if for all x, $y \in X$, $x \succeq y$ or $y \succeq x$ (note that if $\succeq$ is complete, it is reflexive and that, in this case, $x \succ y \Leftrightarrow \neg y \succeq x$). It is transitive if for all x, y, $z \in X$, $x \succeq y$ and $y \succeq z \Rightarrow x \succeq z$. A binary relation which is complete and transitive is a complete preorder. A linear order is an anti-symmetric complete preorder: for all x and $y \in X$, $x \succeq y$ and $y \succeq x \Rightarrow x = y$. With X finite, a complete preorder ranked the alternatives from a most preferred to a least preferred with possible ties and a linear order ranks the alternatives without ties. When the asymmetric component, $\succ$ is transitive, we will say that $\succeq$ is quasi-transitive. The asymmetric component will be said to be acyclic if there is no finite subset of alternatives $\{x_1, ..., x_k\} \subseteq X$ such that $x_1 \succ x_2$, $x_2 \succ x_3$,...,$x_{k-1} \succ x_k$ and $x_k \succ x_1$.

Let N be a finite set of individuals (economic agents, voters) of cardinal n, i.e., $N = \{1, ..., n\}$. Each individual $i \in N$ has a preference over X given by a complete preorder $\succeq_i$. The aggregation question is about the construction of a social (collective, synthetic) preference $\succeq_S$ (or in some cases a social choice) from a list of individual preferences (one preference per individual).

Let $\mathbb{P}$ be the set of complete preorders over X, $\mathbb{LO}$ be the set of linear orders, $\mathbb{P}'$ be a subset of $\mathbb{P}$, and $\mathbb{B}$ be the set of complete binary relations over X.

Definition 1. An *aggregation function* is a function from $\mathbb{P}'^n$ into $\mathbb{B}$, where $\mathbb{P}'^n$ is the Cartesian product of $\mathbb{P}'$ n times.

This means that the aggregation function f associates a complete (social) binary relation $\succeq_S$ to a n-list (also called a *profile*) $(\succeq_1, ..., \succeq_n)$ of (individual) complete preorders:
$f : (\succeq_1, ..., \succeq_n) \mapsto \succeq_S$.

Definition 2. A *social welfare function* is an aggregation function for which $\mathbb{B} = \mathbb{P}$.

In this case, the sort of rationality required for the social preference is absolutely identical with the rationality we assumed for individuals.

As mentioned previously, the modern rebirth of social choice theory in the 1940s is due to the works of Arrow and Black. The main result due to Black concerns the majority rule and the existence of a transitive social preference generated by this rule when the individual preferences are appropriately restricted.

Definition 3. The *majority rule* is an aggregation function for which for all $x, y \in X$ and all n-list $(\succeq_1, ..., \succeq_n) \in \mathbb{P}'^n$, $x \succ_S y \Leftrightarrow \#\{i : x \succ_i y\} > \#\{i : y \succ_i x\}$ and $y \succeq_S x$ otherwise.

Suppose that there are three individuals 1, 2, and 3, and three alternatives a, b and c with the following preferences: $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$ and $c \succ_3 a \succ_3 b$. This means that individual 1 prefers a to b, b to c and a to c etc. It is obvious that the majority rule generates $a \succ_S b$, $b \succ_S c$ and $c \succ_S a$. This is the Condorcet paradox. It indicates that the majority rule is not in general, that is when the given individual preferences are permissible (in $\mathbb{P}'$), a social welfare function.

In voting games, *coalitions*, i.e., non-empty subsets of the set of individuals N are a priori endowed with power. This power can be, as in the case of symmetric (anonymous) games, defined by a number of individuals, called a quota q (for instance $q > n/2$).

Definition 4. A *voting game* is an ordered pair $G = (N, \mathbb{W})$ where $\mathbb{W} \subseteq 2^N - \emptyset$ and $\mathbb{W}$ satisfies the following monotonicity property:

$$C_1 \in \mathbb{W} \text{ and } C_1 \subseteq C_2 \Rightarrow C_2 \in \mathbb{W}.$$

A voting game G is *proper* if $C \in \mathbb{W} \Rightarrow N - C \notin \mathbb{W}$.
It is *strong* if $C \notin \mathbb{W} \Rightarrow N - C \in \mathbb{W}$.

A voting game with quota $q \leq n$ (or q-game) is defined by $C \in \mathbb{W}$ if $\#C \geq q$.

With $n = 9$, the q-game with $q = 5$ is proper and strong and the q-game with $q = 6$ is only proper.

$\mathbb{W}$ will denote the set of winning (powerful) coalitions.

Definition 5. A *voting game of aggregation* is an aggregation function f for which for all $x, y \in X$ and all n-list $(\succeq_1, ..., \succeq_n) \in \mathbb{P}'^n$, $x \succ_S y \Leftrightarrow$ there exists a coalition $C \in \mathbb{W}$ such that for all $i \in C$, $x \succ_i y$, and $y \succeq_S x$ otherwise[a].

We will consider now individual preferences given by any complete preorder. A remarkable idea of Nakamura (1979) was to link the number of alternatives to a number given by the structure of the voting game. This number will be called the Nakamura number. Given a voting game G, we will assume that the intersection of the winning coalitions is empty. (If there was an individual in all winning coalitions, this individual would have a veto power. Without her, no coalition can be powerful.) Such games will be called *non-weak*. The Nakamura number is defined as follows.

Definition 6. Given a non-weak voting game G, the *Nakamura number* $\nu(G)$ of G is

$$\nu(G) = min\{\#\mathcal{W} : \mathcal{W} \subseteq \mathbb{W} \textit{ such that } \cap\{C : C \in \mathcal{W}\} = \emptyset\}.$$

The Nakamura number is the smallest number of winning coalitions whose intersection is empty.

Given a voting game of aggregation, we can now define the core associated with a n-list $(\succeq_1, ..., \succeq_n)$.

Definition 7. The *core* of the voting game of aggregation f associated to the n-list $(\succeq_1, ..., \succeq_n) \in \mathbb{P}'^n$, denoted $Cor(f, (\succeq_1, ..., \succeq_n))$, is the set of maximal elements of X for the binary relation $\succ_S$, i.e., $Cor(f, (\succeq_1, ..., \succeq_n)) = \{x \in X : (\nexists y \in X) y \succ_S x\}$

[a]There is no need to have a *complete* social preference. It is supposed here mainly because we previously defined aggregation functions as having values that are complete binary relations.

We have the following theorem due to Nakamura (1979).

Theorem 1. *Let X be finite, G be non-weak and f be the voting game of aggregation associated to G. Then $Cor(f, (\succeq_1, ..., \succeq_n)) \neq \emptyset$ for all $(\succeq_1, ..., \succeq_n) \in \mathbb{P}(X)^n$ if and only if $\nu(G) > \#X$.*

For quota games the above inequality can be replaced by $q > \frac{n(\#X-1)}{\#X}$. Under this form, the result was already essentially in a paper of Ferejohn and Grether (1974). For a quota voting game G with quota q, $\nu(G) = \lceil \frac{n}{n-q} \rceil$ where $\lceil x \rceil$ is the smallest integer larger than x.

3. Restricted Preferences and Collective Rationality

Black introduced a condition on the set of individual preferences called *single-peakedness*. Black considered that the set of alternatives was the real line (or more exactly a part of it). Individuals had a preference represented by a curve with a unique maximum, strictly increasing up to the maximum and strictly decreasing from the maximum. He demonstrated that the median maximum was the alternative selected by the majority rule, i.e., using this rule, was a point socially preferred to every other point. This is a famous result of *Public Choice* called 'the theorem of the median voter' (not always attributed to Black!). Arrow translated this condition in his set-theoretic/relation framework.

Definition 8. Let $\{a, b, c\} \subseteq X$. A set of complete preorders $\succeq$ over X satisfies the condition of *single-peakedness over* $\{a, b, c\}$ if either $a \sim b$ and $b \sim c$ or there is one of the three alternatives, say b, such that $b \succ a$ or $b \succ c$.

Let $\mathbb{BL}$ denote the set of complete preorders over X such that the condition of single-peakedness is satisfied for all $\{x, y, z\} \subseteq X$. Black's main result is the following theorem 2.

Theorem 2. *If $\mathbb{P}' = \mathbb{BL}$ and if, for any $\{x, y, z\} \subseteq X$, the number of individuals for which $\neg(x \sim_i y$ and $y \sim_i z)$ is odd, the majority rule is a social welfare function.*

This only means that the social preference is transitive[b]. With three alternatives, there are 13 complete preorders and 8 single-peaked complete preorders. Single-

[b]Incidentally, we can avoid this condition of oddity if we only require that the asymmetric component of $\succeq_S$, $\succ_S$, be transitive (see Sen (1970) and Sen and Pattanaik (1969)).

peakedness essentially means that among the three alternatives there is one which is never the (strictly) worst. There is an interesting and intuitively meaningful geometrical representation. If the three alternatives a, b and c are on a line with b between a and c, we have the following possibilities:

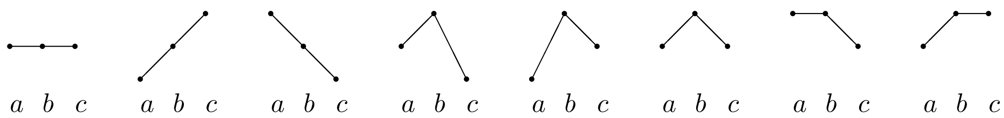

Fig. 1. Black's single-peakedness condition over $\{a, b, c\}$

When the alternatives a, b or/and c are at the same horizontal level, this means that there is an indifference between them, and when one of the alternatives $x \in \{a, b, c\}$ is vertically above $y \in \{a, b, c\}$, this means $x \succ y$. Alternatives a, b and c are linearly ordered a being on the left, b in the center and c on the right. It is then very easy to interpret the admissible (single-peaked) preferences from a political viewpoint, for instance when a, b and c are candidates to an election.

In 1961, Dummett and Farquharson introduced an extended version of single-peakedness.

Definition 9. Let $\{a, b, c\} \subseteq X$. A set of complete preorders $\succeq$ over X satisfies the condition of *extended single-peakedness over* $\{a, b, c\}$ if there is one of the three alternatives, say b, such that $b \succeq a$ or $b \succeq c$.

Let $\mathbb{DF}$ denote the set of complete preorders over X such that the condition of extended single-peakedness is satisfied for all $\{x, y, z\} \subseteq X$. Theorem 3 is due to Dummett and Farquharson (1961).

Theorem 3. *Let X be finite and f be any proper voting game of aggregation. Then for all n-list $(\succeq_1, ..., \succeq_n) \in \mathbb{DF}^n$, $Cor(f, (\succeq_1, ..., \succeq_n)) \neq \emptyset$.*[c]

[c]In fact, Dummett and Farquharson considered a majority game defined by $x \succeq_S y$ if $\#\{i \in N : x \succeq_i y\} > n/2$ or $\#\{i \in N : x \succeq_i y\} = n/2$ and $x \succeq_1 y$. In case of equality, individual 1 plays a specific rle (like a president in a committee, for instance). This defines a proper and strong voting game of aggregation with $\succeq_S$ obviously complete. Then they proved the existence of a maximum element (called a top), i.e., the existence of a x such that $x \succeq_S y$ for all $y \in X$. The formulation we give is essentially due to Nakamura (1975). Salles and Wendell (1977) used the similarity between Dummett and Farquharson's condition and quasi-concavity of utility functions in dimension 1 to extend some of Nakamura's results. On the other hand, Pattanaik (1971) used an analysis very similar to Dummett and Farquharson's analysis, with the same kind of proof method, for conditions extending Black-type of conditions—for instance look at Figure 1 upside down.

As seen in Figure 2, for a three-alternative subset, the extended single-peakedness condition adds two complete preorders to the 8 complete preorders of Figure 1. This could appear as only a slight amelioration, contrary to what Dummett and Farquharson wrote in the abstract of their paper:

A condition on the preferences, **substantially weaker** than one postulated by D. Black, is shown to be sufficient for "stability" in such games.

We can easily explain why they wrote this. First, a reduction from 13 to 8 is quite different from a reduction from 13 to 10. Second, when we try to give an intuitive meaning to single-peakedness, as in the case of the left-right political spectrum mentioned above, the elimination of these two complete preorders is totally unrealistic. These two preferences do make sense in a political context.

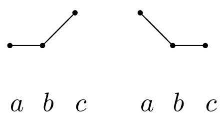

Fig. 2. Supplementary preferences for extended single-peakedness

On the other hand, the three missing complete preorders, as shown in Figure 3, do not really make sense, in this political context, unless we imagine that the voters can be totally irrational.

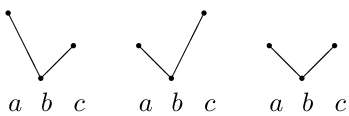

Fig. 3. Excluded preferences

Salles (1975, 1976) studied necessary and sufficient conditions restricting individual preferences to obtain a transitive social preference (strict or not) for voting games of aggregation.

First, we must define the condition of *Value Restriction* due to Sen (1966).

Definition 10. Let $\{a, b, c\} \subseteq X$. A set of complete preorders $\succeq$ over X satisfies the condition of *value restriction over* $\{a, b, c\}$ if either $a \sim b$ and $b \sim c$ or there is one of the three alternatives, say b, such that $b \succ a$ or $b \succ c$, or if either $a \sim b$

and $b \sim c$ or there is one of the three alternatives, say b, such that $a \succ b$ or $c \succ b$, or if either $a \sim b$ and $b \sim c$ or there is one of the three alternatives, say b, such that $(b \succ a$ and $b \succ c)$ or $(a \succ b$ and $c \succ b)$.

The first part of the condition is single-peakedness, the second part is *single-cavedness*, and the third part is when b is either strictly better than a and c or strictly worst than those two alternatives, i.e., b is never '*between*' a and c in terms of preference. It is well-known that the same kind of theorem as Black's theorem can be obtained for value restriction (see Gaertner, 2001).[d]

Now we can introduce another condition called *cyclical indifferences.*

Definition 11. Let $\{a, b, c\} \subseteq X$. A set of complete preorders $\succeq$ over X satisfies the condition of *cyclical indifferences over* $\{a, b, c\}$ if either the complete preorders $\succeq$ are $(a \sim b$ and $b \succ c)$ or $(b \sim c$ and $c \succ a)$ or $(c \sim a$ and $a \succ b)$, or the complete preorders $\succeq$ are $(a \succ b$ and $b \sim c)$ or $(b \succ c$ and $c \sim a)$ or $(c \succ a$ and $a \sim b)$.

Let $\mathbb{VR}$ be the set of complete preorders over X such that the condition of value restriction is satisfied for all $\{x, y, z\} \subseteq X$ and $\mathbb{CI}$ the set of complete preorders over X such that the condition of cyclical indifferences is satisfied for all $\{x, y, z\} \subseteq X$. We have then the following theorem (Salles (1975)).

Theorem 4. *Suppose that for all* $\{x, y, z\} \subseteq X$ *and all* $i \in N$, $\neg(x \sim_i y$ *and* $y \sim_i z)$, *then for any strong and proper voting game of aggregation f, the social preference* $\succeq_S$ *is transitive for all* N *and all* $(\succeq_1, ..., \succeq_n) \in \mathbb{P}'^n$ *if and only if* $\mathbb{P}' = \mathbb{VR}$.

In the case of proper voting games and transitivity of $\succ_S$, one can obtain another result. First, let us define another condition.

Definition 12. Let $\{a, b, c\} \subseteq X$. A set of complete preorders over X satisfies the condition of *cyclical dependence over* $\{a, b, c\}$ if whenever this set includes a linear order, say, $(a \succ b$ and $b \succ c)$, then it cannot include at the same time $(b \succ c$ and $c \succ a)$ and $(c \succeq a$ and $a \succeq b)$ (where at least one of the two $\succeq$s is a $\succ$), and it cannot include at the same time $(b \succeq c$ and $c \succeq a)$ (where at least one of the two $\succeq$s is a $\succ$) and $(c \succ a$ and $a \succ b)$, and it cannot include at the same time $(b \succ c$

[d]In Dasgupta and Maskin (2004), it is suggested that the case where x_j was never in an intermediate position was observed (regarding Jean-Marie Le Pen) during the French presidential elections in 2002. Although the rationale for believing this seems to be rather overwhelming, the facts are probably different. This could have been observed if the run-off would have been between Jospin and Le Pen since a significant number of voters who voted for Chirac in the first run would have voted for Le Pen, contrary to what happened in the real run-off where nearly all voters who did not vote for Le Pen in the first run did vote for Chirac in the run-off, including, of course, those who voted for Jospin. Nevertheless, this indicates that these conditions can be relevant in real-life situations; they are not only theoretical constructions of scientific wizards.

and $c \sim a$) and ($c \sim a$ and $a \succ b$).

This condition excludes some types of latin squares. It must also be observed that this condition implicitly entails the condition of *dichotomous preferences* due to Inada (1969). The condition of dichotomous preferences over $\{a, b, c\}$ restrict the admissible complete preorders to preorders having a symmetric (indifference) part as, for instance, ($a \sim b$ and $b \succ c$).

Let $\mathbb{CD}$ be the set of complete preorders such that the condition of cyclical dependence is satisfied for all $\{x, y, z\} \subseteq X$. Then the following theorem 4 is from Salles (1976).

Theorem 5. *For any proper voting game of aggregation, the social preference $\succ_S$ is transitive for all N and all $(\succeq_1, ..., \succeq_n) \in \mathbb{P}'^n$ if and only if $\mathbb{P}' = \mathbb{VR}$ or $\mathbb{P}' = \mathbb{CD}$.*

For these two theorems, some remarks are in order. The first theorem is about transitivity of the 'weak' social preference $\succeq_S$ and the voting games are both proper and strong. The second theorem is about the transitivity of the 'strict' preference $\succ_S$ (sometimes called quasi-transitivity) and the voting games are only proper. In both cases, it should be outlined that the condition of single-peakedness is a sufficient condition (as part of value restriction). However, one can easily see that the extended single-peakedness of Dummett and Farquharson is not sufficient for the transitivity of $\succ_S$. Consider a 3-person quota game with a quota of 2 with one individual having a preference $b \succ c \succ a$, one individual with a preference $a \sim b \succ c$ and one individual with a preference $c \succ a \sim b$, one gets $c \succ_S a$ and $b \succ_S c$ and $a \sim_S b$, although the extended single-peakedness is satisfied as shown in the following figure.

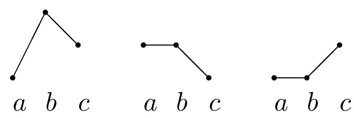

Fig. 4.

4. Uncovered Sets, Stability Sets

In this section, we will introduce sub-relations of the basic social preference/dominance relation, $\succ_S$. As indicated in the previous sections, non-existence of alternatives in the core arises because the dominance relation $\succ_S$ could be cyclical. The idea was then to find meaningful sub-relations where cycles were impossible.

The various forms of the sub-relations giving rise to the uncovered sets and, to some extend, to the stability set deal with this problem, at least when X is a finite set of alternatives. We will consider two forms of the uncovered sets: the Gillies (1959) uncovered set and the Miller (1980) uncovered set. Other forms are discussed in Penn (2006) and in the definite treatment of Duggan (201?).

Given a voting game of aggregation f (see Definition 5), we will define two sub-relations of $\succ_S$ denoted $\succ_S^G$ and $\succ_S^M$ where G stands for Gillies and M for Miller.

Definition 13. For all $x,\, y \in X$, $x \succ_S^G y$ if (1) $x \succ_S y$ and (2) $z \succ_S x \Rightarrow z \succ_S y$. For all $x,\, y \in X$, $x \succ_S^M y$ if (1) $x \succ_S y$ and (2) $y \succ_S z \Rightarrow x \succ_S z$.

These two relations are transitive (even if $\succ_S$ is not transitive). One can now define the two uncovered sets as the maximal elements of these two relations. More precisely, we will have the following definition.

Definition 14. The *G-uncovered set* of the voting game of aggregation f associated to the n-list $(\succeq_1, ..., \succeq_n) \in \mathbb{P}'^n$, denoted $Gil(f, (\succeq_1, ..., \succeq_n))$, is the set of maximal elements of X for the binary relation $\succ_S^G$, i.e., $Gil(f, (\succeq_1, ..., \succeq_n)) = \{x \in X : (\nexists y \in X) y \succ_S^M x\}$.
The *M-uncovered set* of the voting game of aggregation f associated to the n-list $(\succeq_1, ..., \succeq_n) \in \mathbb{P}'^n$, denoted $Mil(f, (\succeq_1, ..., \succeq_n))$, is the set of maximal elements of X for the binary relation $\succ_S^M$, i.e., $Mil(f, (\succeq_1, ..., \succeq_n)) = \{x \in X : (\nexists y \in X) y \succ_S^M x\}$.

Since these two sub-relations are transitive (which is easy to prove), we have the immediate following theorem for a finite X.

Theorem 6. *Let X be finite and f be any proper voting game of aggregation. Then for all n-list $(\succeq_1, ..., \succeq_n) \in \mathbb{P}^n$, $Gil(f, (\succeq_1, ..., \succeq_n)) \neq \emptyset$ and $Mil(f, (\succeq_1, ..., \succeq_n)) \neq \emptyset$.*

The definition of the uncovered sets are only based on the underlying dominance relation. For the stability set, Rubinstein's idea (see Rubinstein (1980)) was to take account of a rather sophisticated and risk-averse reasoning of individuals (voters). Rubinstein wrote:

"True I prefer b to a but if b is adopted then a situation arises where a coalition prefers c. Since c is worse than a from my point of view I will not take any chances and will not vote for b on place of a."

This leads to the following definition.

Definition 15. For all $x,\, y \in X$, $x \succ_S^R y$ if (1) there exists $C \in \mathbb{W}$ such that for all $i \in C$ $x \succ_i y$ and (2) for all $z \in X$ for which $z \succ_S x$, we have for all $i \in C$ $z \succeq_i y$.

The stability set is then the set of maximal elements of X according to $\succ^R_S$.

Definition 16. The *stability set* of the voting game of aggregation f associated to the n-list $(\succeq_1, ..., \succeq_n) \in \mathbb{P}'^n$, denoted $Rub(f, (\succeq_1, ..., \succeq_n))$, is the set of maximal elements of X for the binary relation $\succ^R_S$, i.e., $Rub(f, (\succeq_1, ..., \succeq_n)) = \{x \in X : (\not\exists y \in X) y \succ^R_S x\}$.

Rubinstein then established a possibility theorem similar to the theorems concerning the uncovered set, but he made the further assumption that individuals had preferences given by linear orderings.

Theorem 7. *Let X be finite and f be any proper voting game of aggregation. Then for all n-list $(\succ_1, ..., \succ_n) \in \mathbb{LO}^n$, $Rub(f, (\succeq_1, ..., \succeq_n)) \neq \emptyset$.*

We have already outlined that indifferences had a specific statute in voting games. This is again exemplified by the fact that this theorem is not longer true if the individual preferences are given by complete preorders. However, we have a rather imperfect theorem à la Nakamura based on Nakamura number (Le Breton and Salles (1990)).

Theorem 8. *Let f be a proper voting game of aggregation associated to game G and let ν be its Nakamura number. Suppose that $\#X \leq 2\nu - 3$. Then for all $(\succeq_1, ..., \succeq_n) \in \mathbb{P}^n$, $Rub(f, (\succeq_1, ..., \succeq_n)) \neq \emptyset$.*

For any integer ν with $\#X \geq 2\nu - 1$, there exist a set of individuals N, a proper voting game G whose Nakamura number is ν, and a n-list $(\succeq_1, ..., \succeq_n) \in \mathbb{P}^n$ such that $Rub(f, (\succeq_1, ..., \succeq_n)) = \emptyset$.

The theorem is said to be imperfect because the authors do not obtain a clear necessary and sufficient condition as Nakamura did; they do not provide a new bound which guarantees the non-emptiness of the stability set. Consider the following simple example. Let $G = (N, \mathbb{W}))$ be a quota-game with $n = 11$ and $q = 7$. The Nakamura number is equal to 3. Following Le Breton and Salles, there exists a profile of preferences such that the stability set is empty if the number of alternatives is greater or equal than 5. But the theorem does not say that the profile is built with $n = 11$ and $q = 7$, it is built with some values of n and q such that $\nu(G) = 3$. The problem is now to obtain a result which can be compared to Nakamura's result. Le Breton (1990) proposes a partial solution for quota games but, once again, a clear Nakamura's result is not obtained. Martin (2000) gives a complete solution for the quota games with the following result.

Theorem 9. *Let f be a proper quota game of aggregation associated to game G*

with $q < n-1$. $Rub(f,(\succeq_1,...,\succeq_n)) \neq \emptyset$ *for all* $(\succeq_1,...,\succeq_n) \in \mathbb{P}^n$ *if and only if* $\#X < \frac{2n}{n-q}$.

Unfortunately, this result does not hold for general games and to find a specific number for the stability set, as the Nakamura number is for the core, is an open problem. However, promising results are given in Martin and Merlin (2006) and Moyouwou and Andjiga (2006). Contrary to Le Breton and Salles (1990), these authors do not use the Nakamura number and build a specific number for the stability set.

Le Breton and Salles (1990) propose an extension of the Rubinstein concept. A greater degree of prudence of the individuals can be considered and then it is possible to explore the cases between a prudence equal to 0 (the core), 1 (the stability set) and any number $d > 1$ (the stability set of order d). A generalization of Le Breton and Salles's result is given by Li (1993) with the same weakness: a clear sufficient and necessary condition is not given. A part of the problem is solved by Martin (1998) but the proof of the necessity is still an open problem.

Other concepts originate from the stability set introduced by Rubinstein (1980) and the idea of far-sightedness. For example the largest consistent set, introduced by Chwe (1994) or the model of consistently far-sightedness introduced by Chakravorti (1999). More recently, an interesting development is given by Pongou, Diffo Lambo and Tchantcho (2008).

5. Structured Spaces of Alternatives

We will not go into details regarding the space of options with a topological and/or geometrical structure because of the manifold forms this structure can take. In standard microeconomic theory, the commodity space is often taken to be the Euclidean space, or in the specific case of exchange economies the so-called positive orthant of the Euclidean space. In this framework, individuals have preferences given by complete preorders but these preorders satisfy supplementary properties. They are possibly continuous, convex, strictly convex etc. In the case of the positive orthant of the ℓ-dimensional Euclidean space $\mathbb{R}^\ell_+$, a complete preorder $\succeq$ defined over $\mathbb{R}^\ell_+$ is continuous if for $a \in \mathbb{R}^\ell_+$, the subset $\{x \in \mathbb{R}^\ell_+ : x \succ a\}$ is open for the topology given by the metric defined by the Euclidean distance, meaning that there is a neighborhood of x all of whose points are also strictly preferred to a, and the subset $\{y \in \mathbb{R}^\ell_+ : a \succ y\}$ is also open. Arrovian social choice in this kind of framework is remarkably surveyed in Le Breton and Weymark (2011). Another framework for spaces of alternatives is considered in the so-called spatial model of politics. Although the basic space of alternatives is still the Euclidean space or a subset of it, preferences are given according to the Euclidean distance from an ideal point. This kind of framework is particularly appropriate when one considers policy spaces or political platforms (Austen-Smith and Banks (1999)).

A major result regarding the non-emptiness of the core of a quota game is due to

Greenberg (1979). Greenberg assumes that the set of alternatives is a compact and convex subset of $\mathbb{R}^{\ell}_{+}$ and individual preferences are continuous and convex (they are not necessarily complete preorders). The core of a quota game with quota q is shown to be non-empty whatever are the individual preferences if $q > (\frac{\ell}{\ell+1})n$. This result has been extended to proper voting games by Schofield (1984, 2008) and Strnad (1985). Another question regarding the core which has been much discussed is its stability. In a pioneering paper McKelvey (1979) demonstrated that, with an empty core (for a majority game), it is possible to start anywhere in the space of alternatives and to end up anywhere else by a appropriately selected sequence of majority votes. This is sometimes known as the chaos theorem. This result has been generalized by Tataru (1999) to any quota game. The stability question is also studied in relation with the dimension of the space of alternatives. The definite paper on this problem is Saari (1997).[e]

One could hope that, since the underlying relations of various forms of the uncovered sets were transitive, one would obtain non-emptiness. However things are not as simple because these underlying relations lacked an essential feature viz. sufficient continuity. Bordes and al. (1992) use another method of proof (i.e., not based on some continuity argument) to demonstrate that, for any proper voting game, $Gil(f, (\succeq_1, ..., \succeq_n)) \neq \emptyset$ when the individual complete preorders are continuous and X is a compact metric space. Unfortunately, this simple result is not true for the Miller's uncovered set. Bordes and al. obtain, however, a non-emptiness of $Mil(f, (\succeq_1, ..., \succeq_n))$ if the indifference classes of the individual preferences are negligible in a measure-theoretic sense and the voting games are proper and strong.

The stability set when the space of alternatives has some geometrical structure has not been much studied. Le Breton and Salles (1990) consider the case of a compact convex subset of the Euclidean space. The stability set is shown to be non-empty provided that a rather mild measure-theoretic condition is satisfied. Furthermore, they show, as could be expected, that, from a topological genericity point of view, one obtains a better result than with the core.

6. Conclusion

We have described some of the relations between cooperative games and social choice theory. However, since the overlapping is extremely large, there are domains that have not been covered. For other aspects we recommend Laslier (1997), Moulin (1988), Peleg (1984), and Peleg and Peters (2010). The fact that the core can be empty was at the origin of the development of other solution concepts such as the uncovered sets and the stability set which we decided to highlight. But there are other concepts such as the Banks set (Banks (1985)), the yolk (Ferejohn and al. (1984), McKelvey (1986)), the heart (Schofield (2008)), and the finesse points (Saari and Asay (2010)). A number of open questions deserve to be studied. Among

[e]See also Saari (2008) and Schofield (2008)).

these we wish to highlight the search for restrictions on individual preferences that could guarantee the existence of solution concepts for which we might have in the general case a non-emptiness result. Since these concepts are generally supersets of the core, this a priori means that we should be able to find less stringent conditions than for the core. It is also obvious that some concepts have not yet been fully explored in the case of spaces of alternatives with a geometrical and/or topological structure (including, as in the case of the core, smoothness conditions[f]).

References

Arrow, K.J. [1963] *Social Choice and Individual Values*, 2nd. Edition, Wiley, New York.

Aumann, R.J. [1964] Markets with a continuum of traders, *Econometrica* **32**, 39-50.

Austen-Smith, D. and Banks, J.S. [1999] *Positive Political Theory I*, The University of Michigan Press: Ann Arbor.

Banks, J.S. [1985] Sophisticated voting outcomes and agenda control, *Social Choice and Welfare* **1**, 295-306.

Barberà, S. and Ehlers, L. [2011] Free triples, large indifference classes and the majority rule, *Social Choice and Welfare* **37**, 559-574.

Black, D. [1958] *The Theory of Committees and Elections*, Cambridge University Press, Cambridge.

Bloomfield, S. D. and Wilson, R. [1972] The postulates of game theory, *Journal of Mathematical Sociology* **2**, 221-234.

Bordes, G. [1983] On the possibility of reasonable consistent majoritarian choice: Some positive results, *Journal of Economic Theory* **31**, 122-132.

Bordes, G., Le Breton, M. and Salles, M. [1992] Gillies and Miller's subrelations of a relation over an infinite set of alternatives: general results and applications to voting games, *Mathematics of Operations Research* **17**, 509-518.

Chakravorti, B. [1999] Far-sightedness and the voting paradox, *Journal of Economic Theory* **84**, 216-226.

Chwe, M. [1994] Farsighted coalitional stability, *Journal of Economic Theory* **64**, 299-325.

[f]This implies that the individual preferences are represented by smooth utility functions. Then powerful tools borrowed from differential topology, including singularity theory, can be used to solve fundamental problems regarding stability.

Dasgupta, P.S. and Maskin, E.S. [2004] The fairest vote of all, *Scientific American* **290(3)**, 92-97.

Debreu, G. and Scarf, H. [1963] A limit theorem on the core of an economy, *International Economic Review* **4**, 235-246.

Duggan, J. [201?] Uncovered sets, *Social Choice and Welfare* **?**, ??.

Dummett, M. and Farquharson, R. [1961] Stability in voting, *Econometrica* **29**, 33-43.

Ferejohn, J.A. and Grether, D.M. [1974] On a class of rational social decision procedures, *Journal of Economic Theory* **8**, 471-482.

Ferejohn, J.A., McKelvey, R.D. and Packel, E.W. [1984] Limiting distributions for continuous state Markov voting models, *Social Choice and Welfare* **1**, 45-67.

Fishburn, P.C. [1970] Conditions for simple majority decision functions with intransitive individual indifference, *Journal of Economic Theory* **2**, 354-367.

Fishburn, P.C. [1972] Conditions on preferences that guarantee a simple majority winner, *Journal of Mathematical Sociology* **2**, 105-112.

Fishburn, P. C. [1973] *The Theory of Social Choice*, Princeton University Press, Princeton.

Gaertner, W. [2001] *Domain Conditions in Social Choice Theory*, Cambridge University Press, Cambridge.

Gibbard, A. [1973] Manipulation of voting schemes: A general result, *Econometrica* **41**, 587-601.

Gillies, D. [1959] Solutions to general non-zero sum games, in Tucker, A. and Luce, R. (eds.), *Contributions to the Theory of Games IV*, 47-85, Princeton University Press, Princeton.

Greenberg, J. [1979] Consistent majority rules over compact sets of alternatives, *Econometrica* **47**, 627-636.

Guilbaud, G. Th. [1952] Les théories de l'intérêt général et le problème logique de l'agrégation, *Economie Appliquée* **5**, 501-551.

Guilbaud, G. Th. [1968] *Eléments de la Théorie Mathématiques des Jeux*, Dunod, Paris.

Inada, K.-I. [1964] A note on the simple majority decision rule, *Econometrica* **32**, 525-5.

Inada, K.-I. [1969] The simple majority decision rule, *Econometrica* **37**, 490-506.

Inada, K.-I. [1970] Majority rule and rationality, *Journal of Economic Theory* **2**, 27-40.

Jain, S. K. [2009] The method of majority decision and rationality conditions, in Basu, K. and Kanbur, R. (eds.), *Arguments for a Better World: Essays in Honor of Amartya Sen, Volume I, Ethics, Welfare, and Measurement*, 167-192, Oxford University Press, Oxford.

Laslier, J.-F. [1997] *Tournament Solutions and Majority Voting*, Springer, Heidelberg.

Le Breton, M. [1990] On some combinatorial problems arising in the theory of voting games, *Mathematical Social Sciences* **19**, 179-193.

Le Breton, M and Salles, M. [1990] The stability set of voting games: classification and genericity results, *International Journal of Game Theory* **19**, 111-127.

Le Breton, M. and Weymark, J.A. [2011] Arrovian social choice theory on economic domains, in Arrow, K.J., Sen, A.K. and Suzumura, K. (eds.), *Handbook of Social Choice and Welfare, Volume 2*, 191-299. North-Holland, Amsterdam.

Li, S. [1993] Stability of voting games, *Social Choice and Welfare* **10**, 51-56.

Martin, M. [1998] Quota games and stability set of order d, *Economics Letters* **59**, 145-151.

Martin, M. [2000] A note on the non-emptiness of the stability set, *Social Choice and Welfare* **17**, 559-565.

Martin, M and Merlin, V. [2006] On the characteristic number of voting games, *International Game Theory Review* **8**, 643-654.

McKelvey, R. [1979] General conditions for global intransitivities in formal voting models, *Econometrica* **47**, 1085-1112.

McKelvey, R. [1986] Covering, dominance, and institution-free properties of social choice, *American Journal of Political Science* **30**, 283-314.

Miller, N. [1980] A new solution set for tournaments and majority voting: Further graph-theoretical approaches to the theory of voting, *American Journal of Political Science* **24**, 68-96.

Moulin, H. [1988] *Axioms of Cooperative Decision Making*, Cambridge University Press, Cambridge.

Moyouwou, I. and Andjiga, N. [2006] A note on the non-emptiness of the stability set when individual preferences are weak orders, *Mathematical Social Sciences* **52**, 67-76.

Nakamura, K. [1975] The core of a simple game with ordinal preferences, *Inter-*

national Journal of Game Theory **4**, 95-104.

Nakamura, K. [1979] The vetoers in a simple game with ordinal preferences, *International Journal of Game Theory* **8**, 55-61.

Pattanaik, P.K. [1971] *Voting and Collective Choice*, Cambridge University Press, Cambridge.

Pattanaik, P.K. [1973] On the stability of sincere voting situations, *Journal of Economic Theory* **6**, 558-574.

Pattanaik, P.K. [1978] *Strategy and Group Choice*, North-Holland, Amsterdam.

Peleg, B. [1984] *Game Theoretic Analysis of Voting in Committees*, Cambridge University Press, Cambridge.

Peleg, B. and Peters, H. [2010] *Strategic Social Choice*, Springer, Heidelberg.

Penn, E. [2006] Alternative definitions of the uncovered set and their implications, *Social Choice and Welfare* **27**, 83-97.

Pongou, R., Diffo Lambo, L. and Tchantcho, B. [2008] Cooperation, stability and social welfare under majority rule, *Economic Theory* **35**, 555-574.

Rubinstein, A. [1980] Stability of decision systems under majority rule, *Journal of Economic theory* **23**, 150-159.

Saari, D.G. [1997] The generic existence of a core for q-rules, *Economic Theory* **9**, 219-260.

Saari, D.G. [2008] *Disposing Dictators, Demystifying Voting Paradoxes. Social Choice Analysis*, Cambridge University Press, Cambridge.

Saari, D.G. and Asay, G.R. [2010] Finessing a point: augmenting the core, *Social Choice and Welfare* **34**, 121-143.

Salles, M. [1975] A general possibility theorem for group decision rules with Pareto-transitivity, *Journal of Economic Theory* **11**, 110-118.

Salles, M. [1976] Characterization of transitive individual preferences for quasi-transitive collective preference under simple games, *International Economic Review* **17**, 308-318.

Salles, M. and Wendell, R. [1977] A further result on the core of voting games, *International Journal of Game Theory* **6**, 35-40.

Satterthwaite, M.A. [1975] Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedure and social welfare functions, *Journal of Economic Theory* **10**, 187-217.

Schofield, N. [1984] Social equilibrium and cycles on compact sets, *Journal of Economic Theory* **33**, 59-71.

Schofield, N. [2008] *The Spatial Model of Politics*, Routledge, Abingdon

Sen, A.K. [1966] A possibility theorem on majority decisions, *Econometrica* **34**, 491-499.

Sen, A.K. [1970] *Collective Choice and Social Welfare*, Holden-Day, San Francisco.

Sen, A.K. and Pattanaik, P. K. [1969] Necessary and sufficient conditions for rational choice under majority decisions, *Journal of Economic Theory* **1**, 178-202.

Strnad, J. [1985] The structure of continuous-valued neutral monotonic social functions, *Social Choice and Welfare* **2**, 181-195.

Tataru, M. [1999] Growth rates in multidimensional spatial voting, *Mathematical Social Sciences* **37**, 253-263.

von Neumann, J. and Morgenstern, O. (1944) *Theory of Games and Economic Behavior*, Princeton University Press, Princeton.

Wilson, R. [1972] The game-theoretic structure of Arrow's general possibility theorem, *Journal of Economic Theory* **5**, 14-20.