"Reverse Bayesianism": A Choice-Based Theory of Growing Awareness*

Edi Karni — Johns Hopkins University

Marie-Louise Vierø — Queen's University

March 8, 2012

Abstract

This paper invokes the axiomatic approach to explore the notion of growing awareness in the context of decision making under uncertainty. It introduces a new approach to modeling the expanding universe of a decision maker in the wake of becoming aware of new consequences, new acts, and new links between acts and consequences. New consequences or new acts represent genuine expansions of the decision maker's universe, while the discovery of new links between acts and consequences renders nonnull events that were considered null before the discovery. The expanding universe, or state space, is accompanied by extension of the set of acts. The preference relations over the expanding sets of acts are linked by a new axiom, dubbed act independence, which is motivated by the idea that decision makers have unchanging preferences over the satisfaction of basic needs. The main results are representation theorems and corresponding rules for updating beliefs over expanding state spaces and null events that have the flavor of "reverse Bayesianism."

Keywords: Awareness, unawareness, reverse Bayesianism, null events

JEL classification: D8, D81, D83

*We thank Itzhak Gilboa, Joseph Greenberg, Brian Hill, Peter Klibanoff, Salvatore Modica, Burkhard Schipper, David Schmeidler and Peter Wakker for comments and suggestions.

1 Introduction

According to the Bayesian paradigm, as new discoveries are made and new information becomes available, the universe shrinks: With the arrival of new information, events replace the prior universal state space to become the posterior state space, or universe of discourse. This process of "destruction" reflects the impossibility, in the Bayesian framework, of expanding the state space and of updating the probabilities of null events, coupled with the fact that conditioning on new information renders null events that, a-priori, were nonnull. Yet, experience and intuition alike contradict this view of the world. Becoming accustomed to things that were once inconceivable is part of history and our own life experience. There is a sense, therefore, in which our universe expands as we become aware of new opportunities.

In this paper we take a step toward modeling the process of growing awareness and expansion of the universe, or state space, in its wake.[1] To model the evolution of beliefs in response to growing awareness, we invoke the theory of choice under uncertainty; borrowing its language and structure while modifying it to fit our purpose. In particular, we allow for new consequences and feasible acts to be introduced and for new evidence to establish, in the mind of decision makers, new links between acts and consequences. The interpretation of the updating is somewhat different for the discovery of new feasible acts and consequences on the one hand and the discovery of new links between feasible acts and consequences on the other. The discovery of new feasible acts and consequences represents growing awareness and leads to genuine expansion of the decision maker's universe. By contrast, new evidence suggesting the existence of links between feasible acts and consequences that were previously considered conceivable but unfeasible, results in rendering nonnull events that prior to the discovery of the new links were null. This updating of zero probability events is part of the reverse Bayesianism nature of our model.

In this paper, a decision maker's initial perception of the state-space is determined by a primitive set of what he considers to be feasible acts and a set of feasible consequences. The *conceivable state space* consists of all the mappings from the set of feasible acts to that of the consequences.[2] Taking only those mappings from the set of feasible acts to

[1] Dekel, Lipman, and Rustichini (1998) argue that standard state spaces preclude unawareness. A choice theoretic approach therefore needs a more general point of departure than Savage (1954) and Anscombe and Aumann (1963).

[2] Here we follow the approach to defining a state space described in Schmeidler and Wakker (1987) and

the set of consequences that the decision maker actually considers possible, defines a (subjective) *feasible state space*. The discovery of new consequences and/or new feasible acts expands both the conceivable and feasible state spaces, capturing the decision maker's growing awareness of the universe. The discovery of new feasible states (that is, new links between feasible acts and consequences) that the decision maker previously believed to be impossible expands the feasible state space but not the conceivable state space. Within this framework, we axiomatize the evolution of beliefs in a way that can be described as "reverse Bayesianism."

We assume throughout that, within a given conceivable state space, decision makers' choice behavior is governed by the axioms of subjective expected utility theory. Our axioms linking preferences under different levels of awareness imply that as the state space expands, probability mass is shifted proportionally away from the nonnull events in the prior state space to events created as a result of the expansion of the state space. When new links between feasible acts and consequences are discovered, null events become nonnull, requiring the shifting of probability mass, proportionally, away from the prior nonnull events to the prior null events that have now become nonnull. We note that the same process applies in the inverse direction. The discovery that certain hypotheses about the connections between feasible acts and consequences are invalid render some events null. This requires redistributing the probability mass assigned to prior nonnull events, proportionally, among the remaining nonnull events. This process amounts to Bayesian updating.

Preference relations corresponding to different levels of awareness are defined over different domains. To link the preference relations across their corresponding domains, we introduce a new axiom, dubbed act independence, whose rational is that decision makers' well-being is determined by the satisfaction of unchanging needs, and that their choice behavior is governed by the desire to satisfy these needs. In other words, decision makers' preferences over "material" consequences of acts are derived form the ability of the consequences to satisfy these needs. A similar idea was applied to consumer theory, by Lancaster (1966) and Becker (1965), according to whom the material consequences are inputs in a "household production function" generating characteristics that determine the decision maker's well-being. In our framework, these characteristics correspond to levels of satisfaction of diverse needs. Decision makers, in our model, are supposed to be fully

Karni and Schmeidler (1991).

conscious of their needs and it is assumed that growing awareness does not alter these needs.

The main novelties of this paper are the analytical framework within which growing awareness may be formalized and its consequences analyzed, and an axiomatic depiction of the evolution of preferences as the decision maker's awareness grows. The reverse Bayesianism aspect of our approach is driven by axioms that have the flavor of Savage's (1954) sure thing principle.

The systematic evolution of beliefs depicted by our approach, makes it possible to predict, at least partially, the decision maker's behavior when something unforeseen occurs. With the discovery of a contingency that he was unaware of, the decision maker's prior conception – or "model" – of the universe is falsified. When this happens, the decision maker's prior model need not be discarded; it can still provide some guidance for behavior in the "new" expanded universe. In other words, decision makers can use their experiences and understanding of the prior state space to guide their choices when their growing awareness enables them to construct an expanded state space.

The exploration of the issue of unawareness in the literature has invoked at least three different approaches. (a) the epistemic approach (see Fagin and Halpern [1988], Modica and Rustichini [1999], Halpern [2001], Li [2009], and Hill [2010]); (b) the game-theoretic, or interactive decision making, approach (see Heifetz, Meier, and Schipper [2006], Halpern and Rego [2008], Grant and Quiggin [2011]); and (c) the choice-theoretic approach (see Kochov [2010], Schipper [2011], Li [2008], Lehrer and Teper [2011]).

Our approach falls within the third category. However, unlike other studies that take this approach, we do not take the state space as given. Instead, we construct the relevant state space from the sets of feasible acts and consequences and the perceived links between them. In so doing, we abstract from concrete interpretations of the states and treat them as abstract resolutions of uncertainty. Consequently, decision makers' unawareness concerns feasible acts, feasible consequences, and/or their links.

Kochov (2010) considers a decision maker who knows that his perception of the universe may be incomplete. He characterizes the collection of foreseen events and shows that the result of the decision maker being aware of his incomplete perception of the environment is that his beliefs are represented by a non-singleton set of priors, which he updates as his perception of the environment becomes more precise.

Schipper (2011) focuses on detecting unawareness. Taking as primitive a lattice of disjoint state spaces in the Anscombe and Aumann (1963) model,[3] he defines acts as mappings from the union of these state spaces to the set of consequences. Thus an act in Shipper's model corresponds to an equivalence class of acts in our model. Another difference between Shipper's model and the approach taken here is that he defines the preference relation on the set of all acts while we define the preference relation on the set of conceivable acts given a state space. Consequently, unlike in this work, in Schipper (2011) the decision maker may not understand how an act assigns consequences to states (because he may be unaware of some event). These differences reflect diverse motivations. While our main interest is modeling growing awareness, Shipper's main interest is the behavioral implications of unawareness of some events.

Li (2008) takes as primitives a fixed set of consequences and two, exogenously given, state spaces that correspond to a decision maker being less than fully aware and fully aware. Decision makers are characterized by preference relations, conditional on the level of awareness, over Anscombe-Aumann acts on the corresponding state spaces. Li considers two types of unawareness: "pure unawareness," depicting situations in which the decision maker's perception of the environment is coarse, and "partial unawareness," depicting situations in which the decision maker's perception of the universe is a subset of the full state space. Partial unawareness has a flavor of unawareness of consequences or links between acts and consequences. However, since the set of consequences and states are given, Li's model cannot accommodate the discovery of new consequences or new scientific links.

Lehrer and Teper (2011) model growing awareness due to the decision maker's improved ability, in the wake of information acquisition, to distinguish among events. Unlike our approach in which the state space expands by adding states that are not elements of existing events, the expansion of the state space in the model of Lehrer and Teper takes the form of finer partitions of the existing state space.

[3]A lattice in Scipper's framework could be constructed from primitives of the present model as follows: Fix finite sets of acts and consequences and consider the power sets corresponding to these sets. For each subset of acts and consequences, define a conceivable state space, as we do below. The set of all conceivable state spaces thus defined, constitute a complete lattice of spaces with partial order defined by set inclusion on acts and consequences.

2 The Meanings of Growing Awareness

The examples below illustrate the sense in which a decision maker's universe expands in the wake of his growing awareness.

2.1 Discovery of new consequences

The discovery of the New World. Columbus set out to discover a new sea route to India, presumably taking into account consequences such as ending the trip at the bottom of the ocean, having to turn back, losing some ships and crew members, reaching India, etc. He could not have included, among the set of consequences, the discovery of a new continent. This discovery expanded the universe for mankind.

The discovery of syphilis. The discovery of the New World ushered in its wake a new consequence of sexual intercourse. The risk of contracting venereal diseases was well known in the Old World. Syphilis, however, was new. Its discovery expanded the universe of the Europeans.

Discovery of a "new" consequence expands the state space and may affect the decision maker's ordinal preferences over acts. In other words, two acts that agree on the "old" state space may become distinct when associated with new consequences; as a result, one of the newly defined acts may be strictly preferred over the other.

2.2 Discovery of new feasible acts

Artificial self-sustaining nuclear chain reaction. After the discovery of nuclear fission, Szilárd and Fermi discovered neutron multiplication in uranium, proving that a nuclear chain reaction by this mechanism was possible. On December 2, 1942, Fermi created the first artificial self-sustaining nuclear chain reaction, thus making it feasible to use nuclear energy, for peaceful and military purposes.

The invention of sound recordings. By making it possible to preserve sounds, the invention of sound recording devices expanded the state space to include future replays of currently produced sounds.

The invention of new financial instruments. The invention of option trading opened up new possibilities of creating portfolios and diversifying risks.

2.3 Discovery of new links and changing beliefs

Yellow fever. To prevent ants from crawling into hospitals' beds, French doctors working in Panama during the French attempt to build the Panama Canal, placed the legs of the beds in bowls of water. These pools of water provided breeding grounds for the mosquitoes carrying yellow fever. Not being aware of the way the yellow fever was transmitted, the French did not conceive that their actions contributed to the propagation of the disease. Later, when the connection between stagnant water, mosquitoes, and yellow fever was understood, the Americans were able to eradicate yellow fever, eliminating a major stumbling point to the construction of the Panama Canal.

The velocity of light. According to Newton's mechanics the speed of light emitted by a flashlight moving in the forward direction should exceed the known speed of light by an amount equal to the speed of the flashlight. The discovery that, despite the expected boost from being emitted by a very fast source, the light is going forward at the usual speed of 186,300 miles per second, ushered in a revision of our understanding of the physical world. If we interpret the emission of light by a flashlight moving in the forward direction as an act, then the consequence, reaching a target in the direction of the movement at the speed of 186,300 miles per second, was not considered possible according to the Newtonian view of the universe. Establishing that this is not only possible but a necessary outcome, led to a revision of our beliefs about the feasible state space.

3 The Analytical Framework

We introduce a unifying framework within which the different sources of growing awareness and changing beliefs may be described and analyzed. We also illustrate how the different notions of growing awareness can be formalized in this framework.

3.1 Conceivable state spaces

States of nature, or *states* for short, are abstract representations of resolutions of uncertainty. To define the state space, we invoke the approach of Schmeidler and Wakker (1987) and Karni and Schmeidler (1991).[4] According to this approach, there is a finite, nonempty

[4] See also Gilboa (2009, Chapter 11) for a detailed discussion and an ingenious use of this approach to formulating the state space as means of resolving Newcomb's paradox.

set, F, of *feasible acts*, and a finite, nonempty set, C, of *feasible consequences.* Together these sets determine a *conceivable state space*, C^F, whose elements depict the resolutions of uncertainty. In other words, being a function on the set of feasible acts to the set of consequences, a state specifies the unique consequence that is associated with every act, thereby resolving all uncertainty.

Once the set of conceivable states is fixed, the set of acts is expanded to include what we refer to as *conceivable acts.* The notion of conceivable acts captures the idea of acts that are imaginable given the conceivable state space. In particular, we assume that the decision maker can imagine acts whose outcomes are lotteries with consequences in C as prizes. Let the set of all such lotteries be denoted by $\Delta(C)$.[5] Then the set of conceivable acts consists of the functions in the set

$$\hat{F} := \{f : C^F \to \Delta(C)\}. \tag{1}$$

We identify $c \in C$ with the degenerate lottery $\delta_c \in \Delta(C)$ that assigns c the unit probability mass. Hence, $F \subset \hat{F}$.

To illustrate these concepts we introduce the following simple example. Consider an urn containing a red ball and a black ball. A ball is drawn at random and its color is observed. Let C be a doubleton set, $\{x, y\}$, where x and y are real numbers representing dollar amounts, and $x > y$. Let there be two feasible acts: The act f_R, a "bet on red," pays off x dollars if a red ball is observed and y dollars if a black ball is observed. The act, f_B, a "bet on black," pays off x dollars if a black ball is observed and y dollars if a red ball is observed. The conceivable state space consists of four states as described below:

$F \backslash C^F$	s_1	s_2	s_3	s_4
f_R	x	x	y	y
f_B	x	y	x	y

Note that the state space is constructed using the set of feasible acts, relative to which the set of conceivable acts may seem quite abstract. Since (most of) these acts are non-feasible, there is a concern about the possibility of eliciting preferences over such acts. To

[5] To be clear, $\Delta(C) := \{p \in [0,1]^{|C|} \mid \Sigma_{c \in C} p(c) = 1\}$.

see how this is done, consider the following conceivable acts and their interpretation

$F\backslash C^F$	s_1	s_2	s_3	s_4
f_3	x	x	x	x
f_4	y	y	y	y
f_5	y	y	x	x
f_6	x	x	x	y
f_7	y	x	y	y
f_8	x	y	y	x
f_9	y	y	x	y

The constant acts f_3 and f_4 have the interpretation of "bet on $R \cup B$" and "bet against $R \cup B$", respectively. The acts f_5 and f_9 are bets on black and the act f_7 is a bet on red. The acts f_6 and f_8 are bets on and against $R \cup B$, respectively. Interpreted in this way, conceivable acts are meaningful, and preferences among such acts are not only possible to contemplate, but can be inferred from the preferences on the feasible set of acts.

"In practice, the distinction between feasible and conceivable acts is not always crucial, and in many applications the sets of states and consequences are taken as primitives." (Karni and Schmeidler (1991, p. 1766)). However, in the present context the distinction between feasible and conceivable acts is crucial. It is the set of feasible acts, together with the feasible consequences that shape the decision maker's vision of the universe.

Discovery of new consequences expand the conceivable state space. For instance, let C denote the initial set of consequences and suppose that a new consequence, $\bar{c}$, is discovered. The set of consequences of which the decision maker is aware then expands to $C' = C \cup \{\bar{c}\}$. The discovery of $\bar{c}$ requires a reformulation of the initial model, incorporating the new consequence into the range of the feasible acts. Because ranges of the feasible acts rather than the acts themselves changed, we denote the set of feasible acts with extended range by F^*. Using these notations, the expanded conceivable state space is $(C')^{F^*}$. The event $(C')^{F^*} - C^F$ represents the expansion of the decision maker's conceivable state space. The corresponding expanded set of conceivable acts is given by,

$$\hat{F}^* := \{f : (C')^{F^*} \to \Delta (C')\}. \tag{2}$$

As an illustration, let there be two feasible acts, $F = \{f_1, f_2\}$, and two consequences, $C = \{c_1, c_2\}$. The resulting conceivable state space is given by C^F and thus consists of four states as depicted in the following matrix:

$F\backslash C^F$	s_1	s_2	s_3	s_4
f_1	c_1	c_2	c_1	c_2
f_2	c_1	c_1	c_2	c_2

Suppose that a new consequence, c_3, is discovered. The new conceivable state space consists of the 9 states in the set $(C')^{F^*}$:

$F^*\backslash (C')^{F^*}$	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
f_1	c_1	c_2	c_1	c_2	c_3	c_3	c_1	c_2	c_3
f_2	c_1	c_1	c_2	c_2	c_1	c_2	c_3	c_3	c_3

Discovery of new feasible acts also expands the conceivable state space, albeit in a different way. To grasp the difference, assume again that $F = \{f_1, f_2\}$ and $C = \{c_1, c_2\}$. Suppose that a new feasible act, say f_3, is discovered. Hence, the set of feasible acts is now $F' = \{f_1, f_2, f_3\}$. The discovery of the new feasible act changes the decision maker's conception of the conceivable state space.[6] The new conceivable state space, $C^{F'}$, consists of the eight states depicted as follows

$F'\backslash C^{F'}$	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
f_1	c_1	c_2	c_1	c_2	c_1	c_2	c_1	c_2
f_2	c_1	c_1	c_2	c_2	c_1	c_1	c_2	c_2
f_3	c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_2

The elements of the expanded state space $C^{F'}$ constitute a finer partition of the original state space C^F. In other words, each state in C^F is a non-degenerate event in the expanded state space $C^{F'}$. For example, the state $s_1 := (c_1, c_1) \in C^F$ is the event $E = \{s_1, s_5\}$ in the state space $C^{F'}$. The new set of conceivable acts is

$$\hat{F}' := \{f : C^{F'} \to \Delta(C)\}.$$

[6] Ahn and Ergin (2010) model decision makers whose choice behavior depends on their perception of contingencies, represented by alternative partitions of a given state space. Unlike our work, in which the state space expands and is partitioned more finely as a result of the discovery of new acts, in Ahn and Ergin's work new acts are defined as a consequence of finer partition of the state space. These acts represent growing alertness to possibilities that were always present and were simply ignored.

Note that, unlike the discovery of new consequences, the discovery of new acts requires that the length of the vector of consequences defining each state increases. As we show later, this aspect of the evolving state space requires special treatment.

3.2 The feasible state space

The decision maker's perception of the state space is bounded by his awareness of the sets of feasible acts and consequences. However, he also entertains beliefs about the possible links between feasible acts and their potential consequences. These beliefs manifest themselves in, and may be inferred from, the decision makers' choice behavior.

To formalize this idea we consider a decision maker whose choice behavior is characterized by a preference relation, $\succcurlyeq_{\hat{F}}$ on $\hat{F}$.[7] We denote by $\succ_{\hat{F}}$ and $\sim_{\hat{F}}$ the asymmetric and symmetric parts of $\succcurlyeq_{\hat{F}}$, with the interpretations of strict preference and indifference, respectively. For any $f \in \hat{F}$, $p \in \Delta(C)$, and $s \in C^F$, let $f_{-s}p$ be the act in $\hat{F}$ obtained from f by replacing its $s-th$ coordinate with p. A state $s \in C^F$ is said to be *null* if $f_{-s}p \sim_{\hat{F}} f_{-s}q$ for all $p, q \in \Delta(C)$. A state is said to be *nonnull* if it is not null. Denote by E^N the set of null states and let $S(F,C) = C^F - E^N$ be the set of all nonnull states. Henceforth we refer to $S(F,C)$ as the *feasible state space.* The feasibility of states is a matter of beliefs about the possible links between feasible acts and consequences, and as such, it is a constituent aspect of the decision maker's subjective view of the world in which he lives. As an illustration, consider again the set of bets described, in section 3.1, by the two feasible acts, $F = \{f_R, f_B\}$, and two consequences, $C = \{x, y\}$. Given the nature of the acts involved, it is clear that the conceivable states $s_1 = (x,x)$ and $s_4 = (y,y)$ are logically impossible. Presumably, decision makers would agree and would indicate this agreement by their indifference among all acts that agree on the event $\{s_1, s_4\}$. Hence, $S(F,C) = \{s_2, s_3\}$.

New scientific evidence may change the decision maker's beliefs concerning the links between feasible acts and consequences and his perception of the feasible state space. However, unlike the discovery of new feasible consequences and/or new feasible acts, which expands both the set of conceivable and that of feasible states, changes of the decision maker's beliefs concerning the links between them changes the set of feasible states but leaves the set of conceivable states intact. Consequently, the discovery of new feasible

[7] More formally, the preference relation is a binary relation on $\hat{F}$.

consequences and/or new feasible acts represent a genuine expansion of the decision maker's universe (that is, his perception of the state space) while new evidence concerning the links from feasible acts to consequences entails updating zero probability events in the existing state space. When new links are established, the feasible acts must be extended to include the consequences $f(s)$, for some $s \in C^F - S(F, C)$. Vice versa when old links are eliminated. We denote the newly defined set of feasible acts by F^* and the corresponding feasible state space by $S(F^*, C)$.

To see how changes in the perceived links change the feasible state space, consider the case in which there are two feasible acts, $F = \{f_1, f_2\}$ and two consequences, $C = \{c_1, c_2\}$. The conceivable state space C^F consists of four states. If the decision maker does not believe that the act f_2 may result in the consequence c_2, (that is, whether f_1 results in either c_1 or c_2, the consequence c_2 is considered impossible if f_2 is chosen) then the states (c_2, c_2) and (c_1, c_2) are null. In other words, the feasible state space is $S(F, C) = \{(c_1, c_1), (c_2, c_1)\}$. Suppose that new scientific evidence establishes that f_2 may result in c_2, (independently of the consequence that is associated with the choice of f_1) and, as a result, the decision maker changes his beliefs. Then, following the discovery of the new (and final) link the feasible and conceivable state spaces coincide (that is, $S(F^*, C) = C^{F^*} = C^F$). Before the discovery of the new link, the event $\{(c_2, c_2), (c_1, c_2)\} = C^F - S(F, C)$ was null. Upon the discovery of the link, the decision maker realizes that his belief that certain states cannot possibly obtain is untenable. Hence, following the discovery of the new link, $C^F = S(F^*, C)$, and the event $\{(c_2, c_2), (c_1, c_2)\}$ becomes nonnull. By the same logic, the discovery that a link that the decision maker believed possible is, in fact, impossible, results in rendering null an event that was considered to be nonnull before the discovery.

What is a reasonable updating rule for probabilities of events that were considered impossible (null) and, as a result of scientific progress and growing understanding of the structure of the universe, become possible (nonnull)? Clearly, the Bayesian approach is useless for this purpose. Here we explore an alternative approach.

To discuss the various types of unawareness with which we are concerned, we use the following notational convention throughout. We denote by F and C, respectively, the initial sets of feasible acts and consequences, and we let $S(F, C)$ denote the corresponding feasible state space. When new elements are introduced into each of these sets we denote the corresponding new sets by F' and C'. When new acts are discovered, the new feasible state space is denoted by $S(F', C)$. When new consequences are discovered, the new feasible state space is denoted by $S(F^*, C')$, where the asterisk indicates that the range of some feasible act now includes the new consequence.

4 Growing Awareness and Choice Behavior

A decision maker's growing awareness of the feasibility of acts and consequences expands his perception of the state space and its structure. The discovery of new links between acts and consequences expands what he considers to be the feasible state space. How does the decision maker's growing awareness manifest itself in his choice behavior? In this section we address this question.

4.1 Basic preference structure

Decision makers in our model are supposed to be able to express preferences among conceivable acts. Formally, let $\mathcal{F}$ be a family of sets of conceivable acts corresponding to increasing levels of awareness from all sources (that is, from the discovery of new feasible acts, consequences, and links between them). Because the set of conceivable acts is a variable in our model, we denote the preference relation on $\hat{F}$ by $\succcurlyeq_{\hat{F}}$, and use the notation $\succ_{\hat{F}}$ and $\sim_{\hat{F}}$ to denote the asymmetric and symmetric parts of $\succcurlyeq_{\hat{F}}$, respectively. When the state space expands in the wake of discoveries of new feasible consequences, or if the set of consequences that are deemed possible under an act is expanded due to discovery of new links, the set of conceivable acts must be expanded and the preference relations must be redefined on the extended domain. For instance, if $\hat{F}^*$ is the expanded set of conceivable acts, then the corresponding preference relation is denoted by $\succcurlyeq_{\hat{F}^*}$. If the state space is expanded in the wake of the discovery of new feasible acts, then the new set of conceivable acts is denoted by $\hat{F}'$ and the expanded preference relation by $\succcurlyeq_{\hat{F}'}$.

For each $\hat{F}$ and $\alpha \in [0,1]$ define the convex combination $\alpha f + (1-\alpha) g \in \hat{F}$ by: $(\alpha f + (1-\alpha) g)(s) = \alpha f(s) + (1-\alpha) g(s)$, for all $s \in C^F$. Then, $\hat{F}$ is a convex subset in a linear space.[8]

We assume that, for each $\hat{F} \in \mathcal{F}$, $\succcurlyeq_{\hat{F}}$ abides by the axioms of expected utility theory. Formally,

[8]Throughout this paper we use Fishburn's (1970) formulation of Anscombe and Aumann (1963). According to this formulation, mixed acts, (that is, $\alpha f + (1-\alpha) f'$) are, by definition, conceivable acts.

(A.1) (**Weak order**) For all $\hat{F} \in \mathcal{F}$, the preference relation $\succcurlyeq_{\hat{F}}$ is transitive and complete.

(A.2) (**Archimedean**) For all $\hat{F} \in \mathcal{F}$ and $f, g, h \in \hat{F}$, if $f \succ_{\hat{F}} g$ and $g \succ_{\hat{F}} h$ then $\alpha f + (1 - \alpha) h \succ_{\hat{F}} g$ and $g \succ_{\hat{F}} \beta f + (1 - \beta) h$, for some $\alpha, \beta \in (0, 1)$.

(A.3) (**Independence**) For all $\hat{F} \in \mathcal{F}$, $f, g, h \in \hat{F}$, and $\alpha \in (0, 1]$, $f \succcurlyeq_{\hat{F}} g$ if and only if $\alpha f + (1 - \alpha) h \succcurlyeq_{\hat{F}} \alpha g + (1 - \alpha) h$.

In addition we suppose that, for each $\hat{F} \in \mathcal{F}$, $\succcurlyeq_{\hat{F}}$ abides by the following axioms due to Anscombe and Aumann (1963).

(A.4) (**State independence**) For all $\hat{F} \in \mathcal{F}$, $f \in \hat{F}$, $p, q \in \Delta(C)$ and nonnull $s, s' \in C^F$, $f_{-s}p \succ_{\hat{F}} f_{-s}q$ if and only if $f_{-s'}p \succ_{\hat{F}} f_{-s'}q$.

(A.5) (**Nontriviality**) For all $\hat{F} \in \mathcal{F}$, $\succ_{\hat{F}} \neq \varnothing$.

4.2 Act independence

To link the preference relations across expanding sets of conceivable acts, we introduce a new axiom, which we refer to as act independence. The essence of this axiom is that, for every given state in the original state space, the conditional preferences do not change when the state space expands. Since the expanding state space is associated with an expanding set of acts, *act independence means that the ranking of lotteries conditional on any state in the original state space is independent of the set of acts on which the preference relation is defined.*

Because the conceivable and/or feasible state space may expand as a result of growing awareness of consequences, acts, or links between them, the formulation of act independence varies according to the three ways in which the universe of discourse expands.

The first formulation of act independence concerns the case in which $C \subset C'$. It asserts that preferences over $\Delta(C)$ conditional on each state be the same regardless of the set of conceivable acts. The following definition is used to state the axiom: For all C, C' such that $C \subset C'$, and for all $p \in \Delta(C)$, define, $\tilde{p}(p) \in \Delta(C')$ by $\tilde{p}(p)(c) = p(c)$ if $c \in C$ and $\tilde{p}(p)(c) = 0$ if $c \in C' - C$.

(A.6a) (**Act independence-I**) For every given F, for all C, C' with $C \subset C'$, $p, q \in \Delta(C)$, $f \in \hat{F}$, $f^* \in \hat{F}^*$, and $s \in S(F, C) \cap S(F^*, C')$, it holds that $f_{-s}p \succcurlyeq_{\hat{F}} f_{-s}q$ if and only if $f^*_{-s}\tilde{p}(p) \succcurlyeq_{\hat{F}^*} f^*_{-s}\tilde{q}(q)$.

The second formulation of the axiom pertains to the case in which new feasible acts are discovered and, hence, $F \subset F'$. The new feasible acts increase the number of conceivable states and the number of coordinates defining each state. Hence, the newly defined states constitute a finer partition of the original state space. Thus, if $F \subset F'$ then $C^F \cap C^{F'} = \varnothing$, and for each $s \in C^F$ there corresponds an event $E(s) \subset C^{F'}$ defined by $E(s) = \{s' \in C^{F'} \mid \boldsymbol{P}_{C^F}(s') = s\}$, where $\boldsymbol{P}_{C^F}(\cdot)$ is the projection of $C^{F'}$ on C^F.[9] For $s \in C^F$, we refer to the set $E(s)$ as the projection of s on $C^{F'}$. For all $f \in \hat{F}'$, $p \in \Delta(C)$ and $s \in C^F$, define the act $f_{-E(s)}p$ by $\left(f_{-E(s)}p\right)(s') = p$ for all $s' \in E(s)$ and $\left(f_{-E(s)}p\right)(s') = f(s')$ for all $s' \in C^{F'} - E(s)$. Using these notations, we state the second formulation of the act independence axiom.

(A.6b) (**Act independence-II**) For every given C, for all F, F' with $F \subset F'$, $p, q \in \Delta(C)$, $f \in \hat{F}$, $f' \in \hat{F}'$, and $s \in S(F, C)$, it holds that $f_{-s}p \succcurlyeq_{\hat{F}} f_{-s}q$ if and only if $f'_{-E(s)}p \succcurlyeq_{\hat{F}'} f'_{-E(s)}q$.

The third formulation of the axiom applies to the discovery of new links between the original sets of acts, F, and consequences, C. Recall that when new links are established (or old links eliminated), we denote the newly defined set of feasible acts by F^*.

(A.6c) (**Act independence-III**) For every given F and C, for all $f \in \hat{F}$, $f^* \in \hat{F}^*$, $p, q \in \Delta(C)$ and $s \in S(F, C) \cap S(F^*, C)$, it holds that $f_{-s}p \succcurlyeq_{\hat{F}} f_{-s}q$ if and only if $f^*_{-s}p \succcurlyeq_{\hat{F}^*} f^*_{-s}q$.

Being a new axiom, act independence merits some explanation. The concern here is to put some structure on the evolution of the preference relation when the decision maker's state space expands in the wake of his growing awareness. To model this evolution, we invoke a variation of a model proposed by Lancaster (1966) and Becker (1965). In particular, we assume that decision makers are organisms that have needs, which they seek to satisfy by means of consumption of goods and services. Let $N = \{1, \ldots, n\}$ be a list of needs (e.g., food, shelter, clothing, entertainment, social status, etc.). The trade-offs among the satisfaction of different needs are assumed to be a matter of personal taste. Let $Z \subset \mathbb{R}^n$ be a set whose elements are levels of satisfaction of these needs. In other words, $z \in Z$ is a vector whose j-th coordinate, z_j, $j \in N$, indicates the degree to which the

[9] Suppose that $\mid F \mid = r$ and $\mid F' \mid = k > r$. Let $s = (c_1, ..., c_k) \in C^{F'}$, then $\boldsymbol{P}_{C^F}(s) = (c_1, ..., c_r) \in C^F$.

need j is satisfied. Let $\Delta(Z)$ denote the set of simple probability measures on Z, which we refer to as *need-satisfaction lotteries*.[10] A decision maker's well-being is determined by the satisfaction of his needs. Thus, at the basic level, a decision maker is characterized by an implicit preference relation, $\succsim$, on $\Delta(Z)$.

Let $C \subset \mathbb{R}^m$ be a finite, nonempty set of *feasible material consequences*. For example, $c \in C$ could be a bundle consisting of a lobster dinner, a two-bedroom apartment in an upscale neighborhood, and a James Bond movie. Let $t : C \to Z$ be a mapping representing the *technology* that generates needs satisfaction from material outcomes. Put differently, t is a "production function" that transforms material outcomes into need-satisfaction levels.[11] In our example, the dinner, the apartment, and the movie, with the appropriate input of time, allow the attainment of some levels of satisfaction of the needs for nutrition, shelter, social status and entertainment. Note that the technology is specific to the decision-maker in the sense that the same material input may produce different levels of need satisfaction for distinct decision makers. For example, reading a certain book can be entertaining for some decision makers and boring to others; a given car may elevate the social status of some decision makers while not that of others; using a certain medication may prove effective for some patients but not for others. Given a technology t, $p \in \Delta(C)$ induces a lottery l_p in $\Delta(Z)$ as follows: $l_p(z) = p\left(t^{-1}(z)\right)$, for all $z \in Z$.[12]

Growing awareness expands the sets of conceivable states and acts and thus alters the domain over which the corresponding sets of induced preference relations are defined. We postulate that *the preference relations corresponding to different levels of awareness are linked, implicitly, by a primitive, unchanging, preference relation,* $\succsim$, *on the set of need-satisfaction lotteries*. To formalize this idea, for every given $\succcurlyeq_{\hat{F}}$ on $\hat{F}$ and $s \in S(F, C)$, we define a conditional preference relation, $\succsim^s_{\hat{F}}$, on $\Delta(Z)$ induced by $\succcurlyeq_{\hat{F}}$ as follows: For all $p, q \in \Delta(X)$, if $f_{-s}p \succcurlyeq_{\hat{F}} f_{-s}q$ for all $f \in \hat{F}$, then $l_p \succsim^s_{\hat{F}} l_q$.

Act independence amounts to the requirement that the conditional preference relations, $\{\succsim^s_{\hat{F}}\}_{\hat{F} \in \mathcal{F}}$, and the unconditional preference relation, $\succsim$, on need-satisfaction lotteries agree. Put differently, *act independence captures the idea that a decision maker's preferences*

[10] A probability measure is simple if it has a finite support.

[11] In Lancaster (1966) the technology transforms material goods into "characteristics" and is linear. We do not insist on linearity and identify characteristics with needs satisfaction.

[12] Note that $t^{-1}(z)$ is the preimage of z under the technology, representing an isoquant of the "household production function." Formally, $t^{-1}(z) := \{c \in C \mid t(c) = z\}$.

regarding his basic needs and his risk attitudes toward these needs are independent of the particular process (that is, acts) by which the need-satisfaction lotteries are obtained and by the specificity of the manner by which the uncertainty is resolved. Formally, for all $\hat{F} \in \mathcal{F}$ and $s \in S(F, C)$, $\succsim_{\hat{F}}^{s} = \succsim$. It is easy to see that, by definition, act independence is a restatement of this condition. Unlike the preference relation $\succsim$, which is not directly observable, stating act independence in terms of the preference relations $\{\succcurlyeq_{\hat{F}}\}_{\hat{F} \in \mathcal{F}}$ restricts actual choice behavior and, hence, complies with the revealed preference methodology.

5 The Main Results

The analysis of the effects of growing awareness on choice behavior and the evolution of decision makers' beliefs is divided into three parts. In the first part, we explore the implications of the discovery of new consequences. In the second part we explore the implications of the discovery of new feasible acts. In the third part, we explore the implications of the discovery of new acts-consequences links. In general, the discovery of new consequences increases the number of both conceivable and feasible states, but the "dimension" of each state is unchanged. By contrast, the discovery of new feasible acts increases the number of both conceivable and feasible states and, at the same time, changes the characterization of each state in such a way that what used to be a state before the discovery of the new act, is an event in the reconstructed state space following the discovery. The discovery of new acts-consequences links increases the set of feasible states without affecting the conceivable state space.

5.1 Discovery of new consequences and its representation

The following axiom requires that, as the decision maker's awareness of consequences grows and his state space expands, his preferences conditional on the prior state space remain intact. In other words, the discovery of new consequences does not alter the preference relation conditional on the original set of feasible states.[13] To formalize this idea, let $C' \supset C$, F^*, and $S(F^*, C') \supset S(F, C)$ denote, respectively, the new set of consequences, the new set of feasible acts redefined to accommodate the new consequences, and the resulting

[13]This axiom is reminiscent of Savage's (1954) sure thing principle in that it requires that preference between acts be independent of the aspects on which they agree.

new feasible state space. For each $f \in F$ define $\tilde{f}(f) \in F^*$ as follows: For all $f(s) \in \Delta(C)$, define, $\tilde{f}(f)(s) \in \Delta(C')$ by $\tilde{f}(f)(s)(c) = f(s)(c)$ if $c \in C$ and $\tilde{f}(f)(s)(c) = 0$ if $c \in C' - C$.

(A.7) (**Awareness consistency**) For every given F, for all C, C' with $C \subset C'$, $f, g \in \hat{F}$, $f', g' \in \hat{F}^*$, $f' = \tilde{f}(f)$ and $g' = \tilde{g}(g)$ on $S(F, C)$, and $f' = g'$ on $S(F^*, C') - S(F, C)$ it holds that $f \succcurlyeq_{\hat{F}} g$ if and only if $f' \succcurlyeq_{\hat{F}^*} g'$.

Our first result describes the evolution of a decision maker's beliefs in the wake of discoveries of new consequences. Specifically, a decision maker whose preferences have the structure depicted by the axioms above is a subjective expected utility maximizer. Moreover, when he becomes aware of new consequences, the decision maker updates his beliefs in such a way that likelihood ratios of events in the original state space remain intact. That is to say, probability mass is shifted away from states in the prior state space to the posterior state space, proportionally. We refer to this property as "reverse Bayesianism."

Theorem 1 For each $\hat{F} \in \mathcal{F}$, let $\succcurlyeq_{\hat{F}}$ be a binary relation on $\hat{F}$ then, for all $\hat{F}, \hat{F}^* \in \mathcal{F}$, the following two conditions are equivalent:

(*i*) Each $\succcurlyeq_{\hat{F}}$ satisfies (A.1) - (A.5) and, jointly, $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}^*}$ satisfy (A.6a) and (A.7).

(*ii*) There exists a real-valued, non-constant, affine function, U on $\Delta(C')$ and, for any two $\hat{F}, \hat{F}^* \in \mathcal{F}$, there are probability measures, $\pi_{\hat{F}}$ on C^F and $\pi_{\hat{F}^*}$ on $(C')^{F^*}$, such that for all $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in C^F} U(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in C^F} U(g(s)) \pi_{\hat{F}}(s). \tag{3}$$

and, for all $f', g' \in \hat{F}^*$,

$$f' \succcurlyeq_{\hat{F}^*} g' \Leftrightarrow \sum_{s \in (C')^{F^*}} U\left(f'(s)\right) \pi_{\hat{F}^*}(s) \geq \sum_{s \in (C')^{F^*}} U\left(g'(s)\right) \pi_{\hat{F}^*}(s). \tag{4}$$

Moreover, U is unique up to positive linear transformations, $\pi_{\hat{F}}$ and $\pi_{\hat{F}^*}$ are unique, $\pi_{\hat{F}}(S(F, C)) = \pi_{\hat{F}^*}(S(F^*, C')) = 1$, and

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(s')}, \tag{5}$$

for all and $s, s' \in S(F, C)$.

Notice that the function U does not depend on the level of awareness.

5.2 Discovery of new feasible acts and its representation

Recall that the introduction of new feasible acts increases the number of conceivable states as well as the number of coordinates defining a state. Hence, the newly defined states constitute a finer partition of the original state space. Using the notations defined in subsection 4.2, we state the next axiom, which is analogous to axiom (A.7). The axiom requires that if two acts on the original state space disagree on two states, then the preference ranking of these acts is the same as that of two acts that disagree, in the same way, on the corresponding events in the expanded state space.

(A.8) (**Projection consistency**) For every given C, for all F, F' such that $F \subset F'$, $p, q, \bar{p}, \bar{q} \in \Delta(C)$, $h \in \hat{F}$, $h' \in \hat{F}'$, $s, s' \in S(F, C)$ and $E(s), E(s') \subset S(F', C)$, $\left((h_{-s}p)_{-s'}\bar{p}\right) \succcurlyeq_{\hat{F}} \left((h_{-s}q)_{-s'}\bar{q}\right)$ if and only if $\left(\left(h'_{-E(s)}p\right)_{-E(s')}\bar{p}\right) \succcurlyeq_{\hat{F}'} \left(\left(h'_{-E(s)}q\right)_{-E(s')}\bar{q}\right)$.

The representation theorem below describes how a decision maker's beliefs evolve as he becomes aware of new feasible acts. As before, the decision maker is a subjective expected utility maximizer. When he becomes aware of new feasible acts, the decision maker updates his beliefs in a way that the likelihood ratios of events in the original state space remain intact. Because of the difference in the evolution of the state space, probability mass is shifted from states in the prior state space to the corresponding events the posterior state space, in such a way as to preserve the likelihood ratios of the events in the posterior state space and their corresponding projected states in the prior state space.[14]

Theorem 2 For each $\hat{F} \in \mathcal{F}$, let $\succcurlyeq_{\hat{F}}$ be a binary relation on $\hat{F}$. Then for all $\hat{F}, \hat{F}' \in \mathcal{F}$, the following two conditions are equivalent:

(i) Each $\succcurlyeq_{\hat{F}}$ satisfies (A.1) - (A.5) and, jointly, $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}'}$ satisfy (A.6b) and (A.8).

(ii) There exists a real-valued, non-constant, affine function, U on $\Delta(C)$ and, for any two $\hat{F}, \hat{F}' \in \mathcal{F}$, there are probability measures, $\pi_{\hat{F}}$ on C^F and $\pi_{\hat{F}'}$ on $C^{F'}$, such that for all $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in C^F} U(f(s))\, \pi_{\hat{F}}(s) \geq \sum_{s \in C^F} U(f(s))\, \pi_{\hat{F}}(s), \tag{6}$$

[14]This is "reverse Bayesianism" applied to the present context. Li (2008) conjectures an axiomatization of the link between preferences under full awareness and those under pure unawareness and states a proposition linking the evolution of beliefs. This is in the spirit of Theorem 2. Li's axiom neither implies, nor is it implied by, our projection consistency axiom.

and, for all $f', g' \in \hat{F}'$,

$$f' \succcurlyeq_{\hat{F}'} g' \Leftrightarrow \sum_{s \in C^{F'}} U\left(f'(s)\right) \pi_{\hat{F}'}(s) \geq \sum_{s \in C^{F'}} U\left(g'(s)\right) \pi_{\hat{F}'}(s). \tag{7}$$

Moreover, U is unique up to positive linear transformations, $\pi_{\hat{F}}$ and $\pi_{\hat{F}'}$ are unique, $\pi_{\hat{F}}(S(F,C)) = \pi_{\hat{F}'}\left(S\left(F', C\right)\right) = 1$, and

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}'}(E(s))}{\pi_{\hat{F}'}(E(s'))}, \tag{8}$$

for all $s, s' \in S(F,C)$ and $E(s), E(s') \subset S(F',C)$, where $E(s)$ and $E(s')$, are the projections of s and s' on $S(F',C)$.

5.3 Discovery of new feasible states and their representation

The discovery of new acts-consequences links or the discovery that some links that were believed to exist are, in fact, nonexistent, do not affect the conceivable state space. Rather such discoveries expand or contract only the feasible state space. To model this, fix C and F, and suppose that a new link is established. Then, $S(F,C) \subset S(F^*,C)$, where F^* denotes the set of extended feasible acts as discussed in section 3.2. Using these notations we restate axiom (A.7) as follows:

(A.7a) (**Updating consistency**) For all $\hat{F}, \hat{F}^* \in \mathcal{F}$, if $S(F^*,C) \supset S(F,C)$ and $f', g' \in \hat{F}^*$, $f' = f$ and $g' = g$ on $S(F,C)$ and $f' = g'$ on $S(F^*,C) - S(F,C)$, $f \succcurlyeq_{\hat{F}} g$ if and only if $f' \succcurlyeq_{\hat{F}^*} g'$.

Similarly, if the feasible state space is contracted due to the nullification of a link that was supposed to exist, (that is, $S(F^*,C) \subset S(F,C)$), then Axiom (A.7a) can be restated as:

(A.7b) (**Bayesian updating**) For all $\hat{F}, \hat{F}^* \in \mathcal{F}$, if $S(F^*,C) \subset S(F,C)$ and $f, g \in \hat{F}$, $f = f'$ and $g = g'$ on $S(F^*,C)$ and $f = g$ on $S(F,C) - S(F^*,C)$, $f' \succcurlyeq_{\hat{F}^*} g'$ if and only if $f \succcurlyeq_{\hat{F}} g$.

Nullification of a link that was believed to hold corresponds to the shrinking of the feasible state space as new information arrives within the Bayesian paradigm.

We show next that the process of updating the zero probability events in the wake of discovery of new links between acts and consequences is the exact counterpart of Bayesian updating in the wake of discovery that some links that were presumed to exist are, in fact, non-existent.

Theorem 3 For each $\hat{F} \in \mathcal{F}$, let $\succcurlyeq_{\hat{F}}$ be a binary relation on $\hat{F}$ then, for all $\hat{F}, \hat{F}^* \in \mathcal{F}$, the following two conditions are equivalent:

(i) Each $\succcurlyeq_{\hat{F}}$ satisfies (A.1) - (A.5) and, jointly, $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}^*}$ satisfy (A.6c), (A.7a) and (A.7b).

(ii) There exists a real-valued, non-constant, affine function, U, on $\Delta(C)$ and, for any two $\hat{F}, \hat{F}^* \in \mathcal{F}$, there are probability measures, $\pi_{\hat{F}}$ on C^F and $\pi_{\hat{F}^*}$ on C^{F^*}, such that, for all $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in C^F} U\left(f\left(s\right)\right) \pi_{\hat{F}}\left(s\right) \geq \sum_{s \in C^F} U\left(g\left(s\right)\right) \pi_{\hat{F}}\left(s\right), \tag{9}$$

and, for all $f', g' \in \hat{F}^*$,

$$f' \succcurlyeq_{\hat{F}^*} g' \Leftrightarrow \sum_{s \in C^{F^*}} U\left(f'\left(s\right)\right) \pi_{\hat{F}^*}\left(s\right) \geq \sum_{s \in C^{F^*}} U\left(g'\left(s\right)\right) \pi_{\hat{F}^*}\left(s\right),$$

Moreover, U is unique up to positive linear transformations, $\pi_{\hat{F}}$ and $\pi_{\hat{F}^*}$ are unique, $\pi_{\hat{F}}\left(S\left(F,C\right)\right) = \pi_{\hat{F}^*}\left(S\left(F^*,C\right)\right) = 1$, and

$$\frac{\pi_{\hat{F}}\left(s\right)}{\pi_{\hat{F}}\left(s'\right)} = \frac{\pi_{\hat{F}^*}\left(s\right)}{\pi_{\hat{F}^*}\left(s'\right)}, \tag{10}$$

for all $s, s' \in S\left(F,C\right) \cap S\left(F^*,C\right)$.[15]

6 Concluding Remarks

The model presented in this paper predicts that, as awareness grows and the state space expands, the relative likelihoods of events in the original state space remain unchanged. The model is silent about the absolute levels of these probabilities. In other words, our theory does not predict the probability of the new events in the expanded state space. This may appear as a serious limitation of our approach. However, this appearance is

[15]Notice that $S\left(F,C\right) \cap S\left(F^*,C\right) = S\left(F,C\right)$ or $S\left(F,C\right) \cap S\left(F^*,C\right) = S\left(F^*,C\right)$.

misleading. In fact, the relation between the prior and posterior probabilities in our model is not essentially different from the Bayesian model.

To grasp this claim, consider the Bayesian model. In that model, new information shrinks the state space by rendering null events that were assigned positive prior probabilities. Furthermore, given the prior probability of an event that has been rendered null, the Bayesian model predicts the absolute levels and, consequently, the likelihood ratios, of the posterior probabilities of all the events in the original algebra. These predictions, however, are predicated on the prior, about which the Bayesian model is silent. In Savage's (1954) model, the prior is derived from a primitive preference relation over acts.

Our approach is analogous. Rather than being silent on the prior, it is silent on the posterior probabilities of the newly discovered events. If we proceed analogously to Savage (1954), the posterior is derived from a primitive preference relation on the acts defined over the expanded state space. Given the posterior, our model predicts the absolute probabilities and, consequently, the likelihood ratios, of all the events in the original algebra, including those between newly discovered and previously known events.

7 Proofs

7.1 Proof of theorem 1

(Sufficiency) Fix F and C, then, by (A.1) - (A.5), the theorem of Anscombe and Aumann (1963) and the von Neumann-Morgenstern expected utility theorem, there exists a real-valued, non-constant, function $u_{\hat{F}}$ on C such that for all $f \in \hat{F}$, $s \in S(F, C)$, and $p, q \in \Delta(C)$

$$f_{-s}p \succcurlyeq_{\hat{F}} f_{-s}q \Leftrightarrow \sum_{c \in Supp(p)} u_{\hat{F}}(c)\, p(c) \geq \sum_{c \in Supp(q)} u_{\hat{F}}(c)\, q(c). \tag{11}$$

Let $C' \supset C$, $\hat{F}^* \in \mathcal{F}$, with corresponding feasible state space $S(F^*, C') \supset S(F, C)$. Then, by the same argument as above, there exists a real-valued function $u_{\hat{F}^*}$ on C' such that for all $f \in \hat{F}^*$, $s \in S(F^*, C')$, and $p', q' \in \Delta(C')$

$$f_{-s}p' \succcurlyeq_{\hat{F}^*} f_{-s}q' \Leftrightarrow \sum_{c \in Supp(p')} u_{\hat{F}^*}(c)\, p'(c) \geq \sum_{c \in Supp(q')} u_{\hat{F}^*}(c)\, q'(c). \tag{12}$$

By (A.6a), for every $s \in S(F, C)$ and $p, q \in \Delta(C)$,

$$f_{-s}p \succcurlyeq_{\hat{F}} f_{-s}q \Leftrightarrow f_{-s}\tilde{p}(p) \succcurlyeq_{\hat{F}^*} f_{-s}\tilde{q}(q). \tag{13}$$

Hence, since $Supp(\tilde{p}(p)) = Supp(p)$, $Supp(\tilde{q}(q)) = Supp(q)$, $\tilde{p}(p)(c) = p(c)$, and $\tilde{q}(q)(c) = q(c)$ for all $c \in C$,

$$f_{-s}p \succcurlyeq_{\hat{F}} f_{-s}q \Leftrightarrow \sum_{c\in Supp(p)} u_{\hat{F}^*}(c)\, p(c) \geq \sum_{c\in Supp(q)} u_{\hat{F}^*}(c)\, q(c)\,. \tag{14}$$

The uniqueness of the von Neumann-Morgenstern utility function implies that for all $\hat{F}, \hat{F}^* \in \mathcal{F}$, $u_{\hat{F}^*}(c) = bu_{\hat{F}}(c) + a$, $b > 0$, for all $c \in C$. Hence, $u_{\hat{F}^*}$ is an extension of $u_{\hat{F}}$.

Let $u = u_{\hat{F}^*}$ and define $U(f(s)) := \sum_{c\in Supp(f(s))} u(c)\, f(s)(c)$, for all $f \in \hat{F}$ and $s \in S(F^*, C')$. Then, by Anscombe and Aumann (1963), for all $\hat{F} \in \mathcal{F}$, and $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s\in C^F} U(f(s))\, \pi_{\hat{F}}(s) \geq \sum_{s\in C^F} U(g(s))\, \pi_{\hat{F}}(s)\,, \tag{15}$$

and, for all $f', g' \in \hat{F}^*$,

$$f' \succcurlyeq_{\hat{F}^*} g' \Leftrightarrow \sum_{s\in (C')^{F^*}} U\left(f'(s)\right) \pi_{\hat{F}^*}(s) \geq \sum_{s\in (C')^{F^*}} U\left(g'(s)\right) \pi_{\hat{F}^*}(s)\,, \tag{16}$$

where $\pi_{\hat{F}}(S(F,C)) = \pi_{\hat{F}^*}(S(F^*, C')) = 1$.

Let $f', g' \in \hat{F}^*$ be as in Axiom (A.7) (that is, $f' = \tilde{f}(f)$ and $g' = \tilde{g}(g)$ on $S(F,C)$ and $f' = g'$ on $S(F^*, C') - S(F,C)$) then

$$f' \succcurlyeq_{\hat{F}^*} g' \Leftrightarrow \sum_{s\in S(F,C)} U\left(f'(s)\right) \pi_{\hat{F}^*}(s) \geq \sum_{s\in S(F,C)} U\left(g'(s)\right) \pi_{\hat{F}^*}(s)\,. \tag{17}$$

Since $\pi_{\hat{F}}(S(F,C)) = 1$, the representations (15) and (16) imply

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s\in S(F,C)} U(f(s))\, \pi_{\hat{F}}(s) \geq \sum_{s\in S(F,C)} U(g(s))\, \pi_{\hat{F}}(s)\,. \tag{18}$$

But Axiom (A.7) implies

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow f' \succcurlyeq_{\hat{F}^*} g'. \tag{19}$$

Thus the expressions in (17) and (18) are equivalent. Hence, by the uniqueness of the probabilities in Anscombe and Aumann (1963),

$$\frac{\pi_{\hat{F}^*}(s)}{\sum_{s\in S(F,C)} \pi_{\hat{F}^*}(s)} = \pi_{\hat{F}}(s)\,, \text{ for all } s \in S(F,C)\,. \tag{20}$$

(Necessity) The necessity of (A.1)-(A.5) is an implication of the Anscombe and Aumann (1963) theorem. The necessity of (A.6a) and (A.7) is immediate.

The uniqueness part is an implication of the uniqueness of the utility and probability in Anscombe and Aumann (1963). ■

7.2 Proof of theorem 2

(Sufficiency) By (A.1) - (A.5), the argument in the proof of Theorem 1, and invoking (A.6b), $u_{\hat{F}'}(c) = bu_{\hat{F}}(c) + a$, $b > 0$, for all $c \in C$ and $\hat{F}, \hat{F}' \in \mathcal{F}$. Let $u_{\hat{F}} = u$ and define $U(f(s)) := \sum_{c \in Supp(f(s))} u(c) f(s)(c)$, for all $f \in \hat{F}$ and $s \in S(F, C)$. Then, since $C^F - S(F, C)$ is a null event, by Anscombe and Aumann (1963), for all $\hat{F} \in \mathcal{F}$, and $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in S(F,C)} U(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F,C)} U(g(s)) \pi_{\hat{F}}(s). \tag{21}$$

Let $\hat{F}, \hat{F}' \in \mathcal{F}$ and, without loss of generality, suppose that $S(F', C)$ is a refinement of the states in $S(F, C)$.[16] Take $\left(\left(h'_{-E(s)}p\right)_{-E(s')}\bar{p}\right)$ and $\left(\left(h'_{-E(s)}q\right)_{-E(s')}\bar{q}\right)$ in $\hat{F}'$ as defined in Axiom (A.8). For these acts, (21) is equivalent to

$$\left(\left(h'_{-E(s)}p\right)_{-E(s')}\bar{p}\right) \succcurlyeq_{\hat{F}'} \left(\left(h'_{-E(s)}q\right)_{-E(s')}\bar{q}\right) \tag{22}$$

if and only if

$$U(p)\pi_{\hat{F}'}(E(s)) + U(\bar{p})\pi_{\hat{F}'}\left(E\left(s'\right)\right) \geq U(q)\pi_{\hat{F}'}(E(s)) + U(\bar{q})\pi_{\hat{F}'}\left(E\left(s'\right)\right). \tag{23}$$

By Axiom (A.8),

$$\left(\left(h'_{-E(s)}p\right)_{-E(s')}\bar{p}\right) \succcurlyeq_{\hat{F}'} \left(\left(h'_{-E(s)}q\right)_{-E(s')}\bar{q}\right) \Leftrightarrow \left((h_{-s}p)_{-s'}\bar{p}\right) \succcurlyeq_{\hat{F}} \left((h_{-s}q)_{-s'}\bar{q}\right). \tag{24}$$

By (21),

$$\left((h_{-s}p)_{-s'}\bar{p}\right) \succcurlyeq_{\hat{F}} \left((h_{-s}q)_{-s'}\bar{q}\right)$$

if and only if

$$\sum_{s \in S(F,C)} U\left(\left((h_{-s}p)_{-s'}\bar{p}\right)(s)\right)\pi_{\hat{F}}(s) \geq \sum_{s \in S(F,C)} U\left(\left((h_{-s}q)_{-s'}\bar{q}\right)(s)\right)\pi_{\hat{F}}(s),$$

which, since common terms cancel out, is equivalent to

$$U(p)\pi_{\hat{F}}(s) + U(\bar{p})\pi_{\hat{F}}\left(s'\right) \geq U(q)\pi_{\hat{F}}(s) + U(\bar{q})\pi_{\hat{F}}\left(s'\right). \tag{25}$$

By (24), the expressions (23) and (25) are equivalent, which holds for all $p, \bar{p}, q, \bar{q} \in \Delta(C)$, if and only if

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}'}(E(s))}{\pi_{\hat{F}'}(E(s'))}, \tag{26}$$

[16] Hence, $F \subset F'$.

for all $s, s' \in S(F, C)$ and $E(s), E(s') \subset S(F', C)$, where $E(s)$ and $E(s')$ are the projections of s and s' on $S(F', C)$.

(Necessity) The necessity of (A.1)-(A.5) is an implication of the Anscombe and Aumann (1963) theorem. The necessity of (A.6b) and (A.8) is immediate.

The uniqueness part is an implication of the uniqueness of the utility and probability in Anscombe and Aumann (1963). ■

7.3 Proof of theorem 3

(Sufficiency) By (A.1) - (A.5), the argument in the proof of Theorem 1, and invoking (A.6c), $u_{\hat{F}^*}(c) = bu_{\hat{F}}(c) + a$, $b > 0$, for all $c \in C$.

Consider the case in which $S(F, C) \subset S(F^*, C)$, that is, a new link has been discovered. Then $u_{\hat{F}^*}$ is an extension of $u_{\hat{F}}$. Let $u_{\hat{F}} = u$ and define $U(f(s)) := \sum_{c \in Supp(f(s))} u(c) f(s)(c)$. By Anscombe and Aumann (1963), for all $f, g \in \hat{F}$

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in S(F,C)} U(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F,C)} U(g(s)) \pi_{\hat{F}}(s), \tag{27}$$

and, for all $f^*, g^* \in \hat{F}^*$

$$f^* \succcurlyeq_{\hat{F}^*} g^* \Leftrightarrow \sum_{s \in S(F^*,C)} U(f^*(s)) \pi_{\hat{F}^*}(s) \geq \sum_{s \in S(F^*,C)} U(g^*(s)) \pi_{\hat{F}^*}(s). \tag{28}$$

Let $f', g' \in F^*$ be as in axiom (A.7a), then axiom (A.7a) implies that

$$\sum_{s \in S(F,C)} U(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F,C)} U(g(s)) \pi_{\hat{F}}(s), \tag{29}$$

if and only if

$$\sum_{s \in S(F,C)} U(f(s)) \pi_{\hat{F}^*}(s) \geq \sum_{s \in S(F,C)} U(g(s)) \pi_{\hat{F}^*}(s). \tag{30}$$

Hence,

$$\pi_{\hat{F}}(s) = \frac{\pi_{\hat{F}^*}(s)}{\sum_{s \in S(F,C)} \pi_{\hat{F}^*}(s')} \tag{31}$$

for all $s \in S(F, C)$. Thus, for all $s, s' \in S(F, C)$,

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(s')}. \tag{32}$$

The case in which new evidence entails the severance of existing links, and contraction of the feasible state space, is treated analogously, with $u_{\hat{F}}$ is an extension of $u_{\hat{F}^*}$ and axiom (A.7b) in place of (A.7a).

(Necessity) The necessity of (A.1)-(A.5) is an implication of Theorem 1. The necessity of (A.6c), (A.7a) and (A.7b) is immediate.

The uniqueness part is an implication of the uniqueness of the utility and probability in Anscombe and Aumann (1963). ■

References

[1] Ahn, David S. and Haluk Ergin (2010) "Framing Contingencies," *Econometrica* 78, 655-695.

[2] Anscombe, Francis J. and Robert J. Aumann (1963) "A Definition of Subjective Probability," *Annals of Mathematical Statistics* 43, 199–205.

[3] Becker, Gary S. (1965) "A Theory of the Allocation of Time," *The Economic Journal* 75, 493-517.

[4] Dekel, Eddie, Bart Lipman, and Aldo Rustichini (1998) "Standard State-Space Models Preclude Unawareness," *Econometrica* 66, 159-173.

[5] Fagin, Ronald and Joseph Y. Halpern (1988) "Belief, Awareness and Limited Reasoning, *Artificial Intelligence* 34, 39-76.

[6] Fishburn, Peter (1970) *Utility Theory for Decision Making,* Wiley, New York

[7] Gilboa, Itzhak (2009) *Theory of Decision Making under Uncertainty.* Cambridge University Press, Cambridge.

[8] Grant, Simon and John Quiggin (2011) "Inductive Reasoning About Unawareness," unpublished manuscript.

[9] Halpern, Joseph Y. (2001) "Alternative Semantics for Unawareness," *Games and Economic Behavior* 37, 321-339.

[10] Halpern, Joseph Y. and Leonardo C. Rego (2008) "Interactive Awareness Revisited," *Games and Economic Behavior* 62, 232-262.

[11] Heifetz, Aviad, Martin Meier, and Burkhard C. Schipper (2006) "Interactive Unawareness," *Journal of Economic Theory* 130, 78-94.

[12] Hill, Brian (2010) "Awareness Dynamics," *Journal of Philosophical Logic* 39, 113-137.

[13] Karni, Edi and David Schmeidler (1991) "Utility Theory with Uncertainty," in Werner Hildenbrand and Hugo Sonnenschein, eds., *Handbook of Mathematical Economics* vol. IV. Elsevier Science Publishers B.V.

[14] Kochov, Asen (2010) "A Model of Limited Foresight," working paper, University of Rochester.

[15] Lancaster, Kelvin J. (1966) "A New Approach to Consumer Theory," *Journal of Political Economy* 74, 132-157.

[16] Lehrer, Ehud and Roee Teper (2011) "Extension Rules or What Would the Wise Man Do?" unpublished manuscript.

[17] Li, Jing (2008) "A Note on Unawareness and Zero Probability," working paper, University of Pennsylvania.

[18] Li, Jing (2009) "Information Structures with Unawareness," *Journal of Economic Theory* 144, 977-993.

[19] Modica, Salvatore and Aldo Rustichini (1999) "Unawareness and Partitional Information Structures," *Games and Economic Behavior* 27, 256-298.

[20] Savage, Leonard J. (1954) *The Foundations of Statistics.* Wiley, New York.

[21] Schipper, Burkhard C. (2011) "Awareness-Dependent Subjective Expected Utility," *International Journal of Game Theory*, forthcoming.

[22] Schmeidler, David and Peter Wakker (1987) "Expected Utility and Mathematical Expectation," in John Eatwell, Murray Milgate, and Peter Newman, eds., *The New Palgrave: A Dictionary of Economics.* Macmillan Press.