Absolutely Stable Roommate Problems

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Abstract

In this paper we consider roommate problems with strict preferences. Stability concepts for these problems (the core, the largest consistent set, the von Neumann Morgenstern stable set, ...) can be defined using a notion of direct or indirect dominance. This choice leads to striking differences in terms of which matchings are expected to be stable. In this paper we adopt, and slightly adapt, the notion of absolute stability introduced by Harsanyi (1974): a roommate problem is absolutely stable if indirect dominance implies direct dominance. We then fully characterize absolutely stable roommate problems. Our main result is that a roommate problem is absolutely stable if and only if two conditions on the preferences are satisfied. We also show that absolute stability does not guarantee the solvability of a roommate problem. We then concentrate on solvable roommate problems and show that an absolutely stable roommate problem is solvable when there does not exist "ring" of three agents such that the members of this ring prefer each other above any other agent. In fact, the core of a solvable absolutely stable roommate problem is unique: all agents who mutually 'top rank' each other are matched to each other and all other agents are single.

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1 Introduction

In many social situations, agents may team up in pairs or remain on their own. Gale and Shapley (1962) coined such situations as roommate problems. Roommate problems are a generalization of the well-known marriage problem, for which Gale and Shapley (1962) showed that there always exists a stable matching: all agents are matched to an acceptable partner and no two agents prefer being matched to one another over being matched to their current partner. It is well known that the set of stable matchings is equivalent to the core of a matching problem. Even though the concept of the core has attractive properties and has been used widely in the literature, it has some important drawbacks and limitations. First and foremost, unlike in the marriage problem, the core may not exist in roommate problems. Second, the core may not satisfy external stability: a matching can be outside the core and not be blocked by a stable matching. This has been pointed out by Ehlers (2007) in the case of the marriage problem. Third, when deviating, the core relies on a direct dominance concept: one matching *directly dominates* another if some coalition of agents can find and enforce another matching which is preferred by all members of that coalition. Agents do not consider that their action may trigger other deviations, that is, the deviation is myopic. In order to take further deviations into account, Harsanyi (1974) introduced the notion of *indirect dominance*, which was later formalized by Chwe (1994). In a roommate problem[1], an end matching *indirectly dominates* an initial matching if the end matching can replace the initial matching through a sequence of matchings, such that, at each matching along the sequence, all deviators are strictly better off at the end matching compared to the status-quo they face. Indirect dominance thus captures the idea that farsighted agents consider the end matching that their matching(s) may lead to. It is immediate that direct dominance implies indirect dominance but not vice versa.

These drawbacks have led to the literature to expand in two directions by introducing alternative solution concepts and by introducing farsightedness through indirect dominance. The various solution or stability concepts (Core, Stable Sets, Largest Consistent Set, ...) can be defined using either a direct or an indirect dominance relation. The main lesson to be drawn from the literature is that this choice very often yields striking differences in terms of which matchings are expected to be

[1]More generally farsightedness has been introduced in the study of hedonic games, to which roommate problems belong (Diamantoudi and Xue, 2003).

stable. Regarding the marriage problem, Ehlers (2007) characterized von Neumann Morgenstern stable sets using the direct dominance relation, if such sets exist. He shows that these can be larger than the core. Mauleon et al (2011), using Chwe's (1994) definition of indirect dominance, showed the existence of and completely characterized the von Neumann Morgenstern farsightedly stable sets: a set of matchings is a von Neumann Morgenstern farsightedly stable set if and only if it is a singleton and belongs to the core. They also showed that the Farsighted Core can be empty. Klaus et al. (2011) investigated von Neumann Morgenstern farsightedly stable sets in the roommate problem and showed that these need not always exist or that they can contain more than one element.

A natural question to ask is if we can characterize the domain of preferences of the agents such that indirect dominance implies direct dominance in a matching problem[2]. Harsanyi (1974), in a cooperative game theory setting and using somewhat different dominance definitions[3], was the first to introduce this idea. He defined the relation "x indirectly dominates y" to be *trivial* if "at the same time x directly dominates y" and defined a game to be *absolutely stable* if every possible indirect dominance relation is also trivial. Weber (1976), using the Harsanyi's dominance definitions, provided a full characterization of absolutely stable games for the class of normalized monotonic games.

To the best of our knowledge, no characterization of absolute stability is available for other settings. In this paper we fully characterize the absolutely stable roommate problems when agents have strict preferences. Our main result (Proposition 2) is that a roommate problem is absolutely stable if and only if two conditions are satisfied. We say that when two agents prefer to be matched to one another than being on their own, then these two agents are mutually acceptable. The first condition is akin to 'reciprocity' as it states that mutually acceptable agents must prefer each other to agents that are acceptable to them but not vice versa. Now

[2]In a similar vein, but in the context of social choice, Barberá et al. (2010) studied restrictions of preferences such that individual and group strategy-proofness become equivalent concepts.

[3]Harsanyi (1974) introduced two different notions of indirect dominance. The first is based on the idea of a monotone chain: "x indirectly dominates y" if there exists a monotone chain connecting "x and y". This means that along the sequence connecting "x and y", deviating agents do not only prefer x to the status quo but in addition their deviation must also be preferred to the status quo. The second definition is the one we have introduced above and formalized by Chwe (1994): matchings along the sequence are not required to directly dominate each other.

take any agent i and let us rank the set of his mutually acceptable agents according to agent $i's$ preferences. The second condition states that if any agent k of this set, different from the lowest ranked one, has a mutually acceptable agent l he prefers to agent i, then agent l must be $i's$ lowest ranked mutually acceptable agent and agent k must be $i's$ second worst mutually acceptable agent. Suppose for instance that agents i and k are matched. This condition implies that anytime agent k looks for and finds a better partner, then either i can find a better partner than k or she prefers to be on her own.

We then describe some features of agents' preferences in a roommate problem which is absolutely stable. These features depend on the cardinality of the sets of mutually acceptable agents in the problem.

We subsequently show (Proposition 3) that a roommate problem with three agents who prefer being matched to being single is always absolutely stable. Such a roommate problem may have an empty core from which we conclude that the notion of absolute stability has little in common with well known restrictions on preferences guaranteeing existence and/or uniqueness in the roommate problem such as α-reducibility (Alcalde, 1995) or more generally, the weak top coalition property (Banerjee et al., 2001).

Next, we focus on and characterize solvable absolutely stable roommate problems. We show in Proposition 4 that an absolutely stable roommate problem is solvable when there does not exist a structure in the preference profile called "ring", formed by three agents such that the members of this ring prefer each other above any other agent. This allows us to state (Proposition 5) that, if it exists, the core of an absolutely stable roommate problem is unique. In fact, in the core all agents who mutually 'top rank' each other are matched to each other and all other agents are single.

The rest of the paper is organized as follows. Section 2 introduces roommate problems. Section 3 defines absolute stability and contains our main results. Section 4 concludes.

2 Roommate problems

A roommate problem, is a pair (N, P) where N is a finite set of agents and P is a preference profile specifying for each agent $i \in N$ a strict preference ordering over

N. That is, $P = \{P(1), ..., P(i), ..., P(n)\}$, where $P(i)$ is agent i's strict preference ordering over the agents in N, including herself which can be interpreted as the prospect of being alone. For instance, $P(i) = 1, 3, i, 2, ...$ indicates that agent i prefers agent 1 to agent 3 and she prefers to remain alone rather than to get matched to anyone else. We denote by R the weak orders associated with P. We write $j \succ_i k$ if agent i strictly prefers j to k, $j \backsim_i k$ if i is indifferent between j and k, and $j \succcurlyeq_i k$ if $j \succ_i k$ or $j \backsim_i k$.

A *matching* μ is a function $\mu : N \to N$ such that for all $i \in N$, if $\mu(i) = j$, then $\mu(j) = i$. Agent $\mu(i)$ is agent i's *mate* at μ; i.e., the agent with whom he is matched to share a room (possibly herself). We denote by $\mathcal{M}$ the set of all matchings. A matching μ is *individually rational* if each agent is acceptable to his or her partner, i.e. $\mu(i) \succcurlyeq_i i$ for all $i \in N$. We denote the *set of individually rational matchings* for a roommate problem (N, P) by $I(N, P)$. For a given matching μ, a pair $\{i, j\}$ (possibly $i = j$) is said to form a *blocking pair* if they are not matched to one another but prefer one another to their partner at μ, i.e. $j \succ_i \mu(i)$ and $i \succ_j \mu(j)$. A matching μ is *stable* if it is not blocked by any individual or any pair of agents. We denote the *set of stable matchings* for a roommate problem (N, P) by $S(N, P)$. A roommate problem (N, P) is *solvable* if $S(N, P) \neq \varnothing$. Otherwise, it is called *unsolvable*.

We extend each agent's preference over her potential partners to the set of matchings in the following way. We say that agent i prefers μ' to μ, if and only if agent i prefers her partner at μ' to her partner at μ, $\mu'(i) \succ_i \mu(i)$. Abusing notation, we write this as $\mu' \succ_i \mu$. A coalition S is a subset of the set of agents N.[4] For $S \subseteq N$, $\mu(S) = \{\mu(i) : i \in S\}$ denotes the set of mates of agents in S at μ. A matching μ is *blocked by a coalition* $S \subseteq N$ if there exists a matching μ' such that $\mu'(S) = S$ and for all $i \in S$, $\mu' \succ_i \mu$. If S blocks μ, then S is called a *blocking coalition* for μ. Note that if a coalition $S \subseteq N$ blocks a matching μ, then there exists a pair $\{i, j\}$ (possibly $i = j$) that blocks μ. The *core* of a roommate problem consists of all matchings which are not blocked by any coalition. Note that for any roommate problem the set of stable matchings equals the core.

Definition 1. Given a matching μ, a coalition $S \subseteq N$ is said to be able to enforce a matching μ' over μ if the following conditions hold: (i) $\mu'(i) \notin \{\mu(i), i\}$ implies $\{i, \mu'(i)\} \subseteq S$ and (ii) $\mu'(i) = i \neq \mu(i)$ implies $\{i, \mu(i)\} \cap S \neq \varnothing$.

[4]Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion.

In other words, this enforceability condition[5] implies both that any new pair in μ' that does not exist in μ should be between players in S, and that in order to destroy an existing pair in μ, one of the two players involved in that pair should belong to coalition S.[6] Notice that the concept of enforceability is independent of preferences. Furthermore, the fact that coalition $S \subseteq N$ can enforce a matching μ' over μ implies that there exists a sequence of matchings $\mu^0, \mu^1, ..., \mu^K$ (where $\mu^0 = \mu$ and $\mu^K = \mu'$) and a sequence of disjoint pairs $\{i_0, j_0\}, ..., \{i_{K-1}, j_{K-1}\}$ (possibly for some $k \in \{0, 1, ..., K-1\}$, $i_k = j_k$) such that for any $k \in \{1, ..., K\}$, the pair $\{i_{k-1}, j_{k-1}\} \in S$ can enforce the matching μ^k over μ^{k-1}.

Definition 2. A matching μ is directly dominated by μ', or $\mu' > \mu$, if there exists a coalition $S \subseteq N$ of agents such that $\mu' \succ_i \mu \; \forall i \in S$ and S can enforce μ' over μ.

The direct dominance relation is denoted by $>$. An alternative way of defining the core of a roommate problem is by means of the domination relation. A matching μ is in the core if there is no subset of agents who, by rearranging their partnerships only among themselves, possibly dissolving some partnerships of μ, can all obtain a strictly preferred set of partners. Formally, a matching μ is in the core if μ is not directly dominated by any other matching $\mu' \in \mathcal{M}$. Given a profile P, we denote the set of matchings in the core by $C(>)$. Even though the core may be empty in roommate problems, as Gale and Shapley (1962) showed, several papers are devoted to analyze the core as solution for this matching problem. See for instance Tan (1991), Chung (2000), Diamantoudi et al. (2004) and Iñarra et al. (2010).

We now introduce the *indirect dominance* relation. A matching μ' *indirectly dominates* μ if μ' can replace μ in a sequence of matchings, such that at each matching along the sequence all deviators are strictly better off at the end matching μ' compared to the status-quo they face. Formally, indirect dominance is defined as follows.

[5]This enforceability condition has also been used in Mauleon et al. (2011) and in Klaus et al. (2011).

[6]Notice that this enforceability condition is similar to the enforceability condition defined in Roth and Sotomayor (1990). That is, a coalition S can enforce the set of pairs in the matching μ' that concerns its members if and only if every agent in S is matched to an agent in S and vice versa.

Definition 3. A matching μ is indirectly dominated by μ', or $\mu \ll \mu'$, if there exists a sequence of matchings $\mu^0, \mu^1, ..., \mu^K$ (where $\mu^0 = \mu$ and $\mu^K = \mu'$) and a sequence of coalitions $S^0, S^1, ..., S^{K-1}$ such that for any $k \in \{1, ..., K\}$,

(i) $\mu^K \succ_i \mu^{k-1}$ $\forall i \in S^{k-1}$, and

(ii) coalition S^{k-1} can enforce the matching μ^k over μ^{k-1}.

The indirect dominance relation is denoted by $\gg$. It is clear that direct dominance implies direct dominance, if $\mu < \mu'$ then $\mu \ll \mu'$, since direct dominance can be obtained by setting $K = 1$ in Definition 3. Recently, Mauleon et al. (2011) have shown that, in marriage problems (a particular case of the roommate problem where agents are partitioned in two sets), an individually rational matching μ indirectly dominates μ' if and only if there does not exist a pair $\{i, \mu'(i)\}$ that blocks μ. Klaus et al. (2011) have generalized this result for roommate problems, and they have shown that an individually rational matching μ indirectly dominates another individually rational matching μ' if and only if there does not exist a pair $\{i, \mu'(i)\}$ that blocks μ. We refer to these papers for a proof.

Proposition 1. *(Klaus et al. (2011)) Let (N, P) be a roommate problem and $\mu, \mu' \in I(N, P)$. Then, $\mu \gg \mu'$ if and only if there does not exist a pair $\{i, \mu'(i)\}$ that blocks μ.*

Diamantoudi and Xue (2003), showed that if a matching belongs to the core, then it indirectly dominates any other matching.

Lemma 1. *If $\mu \in C(>)$, then $\forall \mu' \neq \mu$, it holds that $\mu' \ll \mu$.*

3 Absolutely Stable Roommate Problems

Let (N, P) be a roommate problem. Following Harsanyi (1974)'s definition of absolutely stable games, we define a roommate problem to be absolutely stable if and only if indirect dominance implies direct dominance.

Definition 4. A roommate problem (N, P) is **absolutely stable** if the following condition holds:

$$\mu' \gg \mu \Leftrightarrow \mu' > \mu, \ \forall \mu, \mu' \in \mathcal{M}.$$

In order to find the restrictions on P such that the problem is absolutely stable, we will introduce some definitions.

Let $i \in N$. We denote by $t(i)$ the most preferred partner for agent i. That is, $t(i) \succcurlyeq_i j$ for any $j \in N$.

Definition 5. Let (N, P) be a roommate problem. T_P denotes the set of agents who are ranked as top choice by their top choice; i.e.,

$$T_P = \{i \in N : \exists j \in N \text{ such that } j = t(i) \text{ and } i = t(j)\}.$$

Notice that if $i \in T_P$, then $t(t(i)) = i$.

Definition 6. Given the problem (N, P), the set MA_P^i denotes the set of **mutually acceptable** agents for i, that is $MA_P^i = \{j \in N : j \succ_i i \text{ and } i \succ_j j\}$. Let $\omega(i) \in MA_P^i$ denote the least preferred partner for i in this set; i.e., $\forall k \in MA_P^i : k \succcurlyeq_i \omega_{(i)}$. Let $MA_P^{i,k}$ denote the set of mutually acceptable agents of i who are **less preferred** than k, that is $MA_P^{i,k} = \{j \in MA_P^i : k \succ_i j\}$.

Definition 7. Given the problem (N, P), the set A_P^i denotes the set of agents who are acceptable to i, but not mutually acceptable; i.e., $A_P^i = \{j \in N : j \succ_i i \text{ and } j \succ_j i\}$.

We extend each agent's preferences over potential partners to sets of agents in the following way. We say that agent i prefers a set MA_P^i to a set A_P^i, if and only if agent i prefers every agent in MA_P^i to any agent in A_P^i. Abusing notation, we write this as $MA_P^i \succ_i A_P^i$.

The following concept is key for the existence of stable matchings in roommate problems.

Definition 8. Let (N, P) be a roommate problem. A ring $S = \{s_1, ..., s_k\} \subseteq N$ is an ordered set of agents such that $k \geq 3$ and for all $i \in \{1, ..., k\}$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} s_i$ (subscript modulo k).

The existence of odd rings in the preference profile is a necessary condition for the emptiness of the core in a roommate problem, as the following lemma shows:

Lemma 2. *Let (N, P) be a roommate problem such that $C(>) = \emptyset$. Then, there exists a ring $S = \{s_1, \ldots, s_k\}$, where k is odd.*

This lemma is straightforward from the necessary and sufficient condition provided by Tan (1991). We refer the reader to Appendix A for a compilation of definition and results about solvability of roommate problems.

Our main result characterizes the absolutely stable roommate problems.

Proposition 2. *A roommate problem (N, P) is absolutely stable if and only if the preference relation P satisfies the following two conditions:*

(i) $\forall i \in N$, $MA_P^i \succ_i A_P^i$,

(ii) $\forall i \in N$, *if* $\exists k \in MA_P^i \setminus \{\omega(i)\}$ *and* $\exists l \in MA_P^k$ *such that* $l \succ_k i$ *then* $MA_P^{i,k} = \{l\}$.[7]

The first condition can be interpreted as "reciprocity", in the sense that those agents to whom like agent i more are also more preferred by agent i. The second condition says that if two agents i, j are mutually acceptable but j prefers another mutually acceptable agent k more that i. Then, there cannot be any agent mutually acceptable for i less preferred than j, different from k. In other words, k is the least preferred potential partner for i among the mutually acceptable, and there are no agents in agent i's preferences less preferred than j but more preferred than k.

Example 1. *The following example shows a roommate problem which is absolutely stable.*

$P(1)$	$P(2)$	$P(3)$	$P(4)$	$P(5)$	$P(6)$	$P(7)$	$P(8)$
2	1	1	5	4	7	8	6
3	3	5	1	1	8	6	7
4	5	3	3	5	6	7	8
5	2	2	4	2			
1	4	4	2	3			

In this problem, the set of mutually acceptable agents are $MA_P^1 = \{2, 3, 4, 5\}$, $MA_P^2 = MA_P^3 = \{1\}$, $MA_P^4 = \{5, 1\}$ *and* $MA_P^5 = \{4, 1\}$, $MA_P^6 = \{7, 8\}$, $MA_P^7 = \{6, 8\}$ *and* $MA_P^8 = \{6, 7\}$. *Notice that first condition is satisfied since these agents are in the first rows of each agent's preferences. Consider for instance agent* 1*'s preferences,* $P(1)$. *Notice that agents* 1 *and* 4 *are mutually acceptable and* 4 *is not*

[7]Notice that l equals $\omega(i)$.

the worse agent in MA^1_P, however, $5 \succ_4 1$. Then, by condition (ii) of Proposition 2, agent 5 must be the immediate less preferred agent than 4 for agent i. Notice that {6,7,8} form an odd ring in the preferences.

Now, we present a Remark describing some features of agents' preferences in a roommate problem which is absolutely stable. These features depend on the cardinality of the sets of mutually acceptable agents in the problem.

Remark 1. *Let (N, P) be an absolutely stable roommate problem,*

1. *Let i be an agent such that $|MA^i_P| > 2$ and assume, without loss of generality, that $MA^i_P = \{j_1, \ldots, j_k, \omega(i)\}$ such that $j_m \succ_i j_{m+1}$, $\forall m \in \{1, \ldots, k-1\}$ and $j_k \succ_i \omega(i)$.*

   a. *$\forall j \in MA^i_P \setminus \{j_k, \omega(i)\}$, $t(j) = i$,*

   b. *$t(j) \in \{i, \omega(i)\}$*

      b.1 *If $t(j_k) = i$ then either $\omega(i) \in T_P$ or $t(\omega(i)) \in \{i, t(i)\}$, and*

      b.2 *If $t(j_k) = \omega(i)$ then $t(\omega(i)) = j_k$*

2. *Let i be an agent such that $|MA^i_P| \leq 2$. Then either $t(i) \in T_P$ or $i \in S$ where S is a ring in P such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$.*

3. *For all $i \notin T_P$, there is no agent $j \notin T_P$ such that $i \in MA^j_P$, except from those belonging to a ring S in P such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$.*

Example 2 ((Example 1 continued))**.** *In this example, the only agent satisfying $|MA^i_P| \geq 2$ is agent 1 with $MA^1_P = \{2, 3, 4, 5\}$ and $2 \succ_1 3 \succ_1 4 \succ_1 5$. We can see that $\forall j \in \{2, 3\}$, $t(j) = i$ (condition (a)). Moreover, it must happen that $t(4) \in \{1, 5\}$ (condition (b)). In this case, $t(4) = 5$ and therefore $t(5) = 4$ (condition (b.2)).*

On the other hand, all the other agents satisfy $|MA^i_P| \leq 2$. For all agent $i \in \{2, 3, 4, 5\}$, we can check that $t(i) \in T_P$. Agents in the set $\{6, 7, 8\}$ form a ring satisfying that $\forall s_i \in \{6, 7, 8\}$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} k$ for all $k \in N \setminus \{s_{i+1}, s_{i-1}\}$.

Notice also that among agents from 1 to 5, there is no pair of agents who do not belong to T_P such that there are mutually acceptable. In our example, the only agent *who is not in T_P is agent* 3*, and there is no agent j in $P(3)$ such that $t(3) \succ_3 j \succ_3 3$ and $j \in MA^3_P$.*

The following result shows that all roommate problems such that $|N| = 3$ in which all players prefer to be matched to being unmatched are absolutely stable.

Proposition 3. *Let (N, P) be a roommate problem such that $|N| = 3$ and $\forall i \in N$: $j \succ_i i$ if $j \neq i$. Then (N, P) is absolutely stable.*

Note that this class of roommate problems can have an empty core when the three players form an odd ring in P. This then implies that the notion of absolute stability has little in common with restrictions on preferences which guarantee the existence of stable matching and/or the uniqueness of stable matchings [e.g. α-reducibility (Alcalde, 1995) or more generally, the weak top coalition property (Banerjee et al., 2001)].

The following proposition characterizes the absolutely stable roommate problems with a non-empty core.

Proposition 4. *Let (N, P) be an absolutely stable roommate problem. $C(>) \neq \emptyset$ if and only if there is no ring S in P such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$.*

The following result, derived from the previous one, states that if a matching problem is absolutely stable and the core is non-empty, it has a unique stable matching, which consists of the matching in which all agents who mutually top rank each other are matched to one another and all other agents remain single

Proposition 5. *Let (N, P) be an absolutely stable problem. Then, (i) the core is unique, $C(<) = \{\mu_C\}$, and (ii) for all $i \in T_P$, $\mu_C(i) = t(i)$, and for all $j \notin T_P$, $\mu_C(j) = j$.*

Example 3 ((Example 1 continued))**.** *In this example, we have already seen that there is a ring $S = \{6, 7, 8\}$ in P such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$. Therefore this roommate problem is unsolvable, that is there is no stable matching.*

Consider the roommate problem derived from the previous one such that $N = \{1, 2, 3, 4, 5\}$ and $P = \{P(1), P(2), P(3), P(4), P(5)\}$. In this case, there is no ring in preferences satisfying the conditions above and therefore the problem is solvable. The core, in this case, is formed by the matching $\mu^* = \{\{1, 2\}, \{3\}, \{4, 5\}\}$.

4 Conclusion

We have characterized absolute stable roommate problems, when preferences are strict. That is, we have obtained under which conditions on preference profiles indirect dominance implies direct dominance in roommate problems.

Furthermore, we have characterized when these problems have a non-empty core. This characterization has allowed us to state that the core, when it is not empty, is unique and each pair is formed either by two agents who are ranked top choice each other or by a singleton.

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Appendix A

Tan (1991) establishes a necessary and sufficient condition for the solvability of roommate problems with strict preferences in terms of stable partitions. This notion, which is crucial in the investigation of the core for these problems, can be formally defined as follows:

Let $A = \{a_1, ..., a_k\} \subseteq N$ be an ordered set of agents. The set A is a *ring* if $k \geq 3$ and for all $i \in \{1, ..., k\}$, $a_{i+1} \succ_{a_i} a_{i-1} \succ_{a_i} a_i$ (subscript modulo k). The set A is a *pair of mutually acceptable agents* if $k = 2$ and for all $i \in \{1, 2\}$, $a_{i-1} \succ_{a_i} a_i$ (subscript modulo 2).[8] The set A is a *singleton* if $k = 1$.

A *stable partition* is a partition P of N such that:
(i) For all $A \in P$, the set A is a ring, a mutually acceptable pair of agents or a singleton, and
(ii) For any sets $A = \{a_1, ..., a_k\}$ and $B = \{b_1, ..., b_l\}$ of P (possibly $A = B$), the following condition holds:

$$\text{if } \; b_j \succ_{a_i} a_{i-1} \text{ then } b_{j-1} \succ_{b_j} a_i,$$

for all $i \in \{1, ..., k\}$ and $j \in \{1, ..., l\}$ such that $b_j \neq a_{i+1}$.

Condition (i) specifies the sets in a stable partition, and condition (ii) contains the notion of stability to be applied between these sets (and also inside each set).

Note that a stable partition is a generalization of a stable matching. To see this, consider a matching μ and a partition P formed by pairs of agents and/or singletons. Let $A = \{a_1, a_2 = \mu(a_1)\}$ and $B = \{b_1, b_2 = \mu(b_1)\}$ be sets of P. If P is a stable partition then Condition (ii) implies that if $b_1 \succ_{a_1} a_2$ then $b_2 \succ_{b_1} a_2$, which is the usual notion of stability. Hence μ is a stable matching.

The following assertions are proven by Tan (1991).

Remark 2. *(Iñarra et. al. (2010)) (i) A roommate problem $(N, (\succ_x)_{x \in N})$ has no stable matchings if and only if there exists a stable partition with an odd ring.*[9] *(ii) Any two stable partitions have exactly the same odd rings.(iii) Every even ring in a stable partition can be broken into pairs of mutually acceptable agents preserving stability.*

[8]Hereafter we omit subscript modulo k.

[9]A ring is odd (even) if its cardinality is odd (even).

Appendix B

Proof of Proposition 2. ($\Rightarrow$) By contradiction, we will show that if one of the conditions (i) or (ii) is not satisfied, then $\mu \gg \mu' \nRightarrow \mu > \mu'$.

- Suppose that condition (i) is not satisfied. Then there exists an agent $i \in N$ such that $k \succ_i j$ for some $k \in A_P^i$ and some $j \in MA_P^i$. Let μ_2 be a matching such that $\mu_2(i) = k$ and $\mu_2(s) = s$ for every $s \neq i, k$, and let μ_1 be a matching such that $\mu_1(i) = j$ and $\mu_1(s) = s$ for every $s \neq i, j$. Then $\mu_1 \gg \mu_2$ (since $k \succ_k i$, agent k enforces the matching in which every agents is alone, and this matching is blocked by $\{i, j\}$ enforcing μ_1). However, $\mu_1 \not> \mu_2$ since $\mu_2(i) \succ_i \mu_1(i)$.

- Suppose that condition (ii) is not satisfied. Then there exists an agent $k \in MA_P^i \setminus \{\omega(i)\}$ and an agent $l \in MA_P^k$ such that $l \succ_k i$ and $\{l\} \neq MA_P^{i,k}$. Then it must be the case that there exists some agent $j \neq l$ such that $j \in MA_P^{i,k}$. Let μ_2 be a matching such that $\mu_2(i) = k$ and $\mu_2(s) = s$ for every $s \neq i, k$, and let μ_1 be a matching such that $\mu_1(k) = l$, $\mu_1(i) = j$ and where $\mu_1(s) = s$ for every $s \neq i, k, l, j$. Then $\mu_1 \gg \mu_2$ (since $\{k, l\}$ block μ_2 enforcing a matching in which i and j are alone, and this matching is blocked by $\{i, j\}$ enforcing μ_1). However, $\mu_1 \not> \mu_2$ since $\mu_2(i) \succ_i \mu_1(i)$.

($\Leftarrow$) Now we will prove that if $\mu_1 \gg \mu_2$ and conditions (i) and (ii) are satisfied, then $\mu_1 > \mu_2$.
Given that $\mu_1 \gg \mu_2$, we define the set of agents who have a different partner in both matchings. Let $D = \{i \in N : \mu_1(i) \neq \mu_2(i)\}$.

First, we prove that for any agent $i \in D$ such that $\mu_1(i) \neq i$, $\mu_1(i) \succ_i \mu_2(i)$. By contradiction, let $\mu_1(i) = j$ and let $\mu_2(i) = k$ and assume that $k \succ_i j$ (which implies that $k \neq j$). Notice that $j \in MA_P^i$ because otherwise $\{i, j\}$ will never be formed contradicting $\mu_1 \gg \mu_2$. Since $\mu_1 \gg \mu_2$ and i prefers μ_2 to μ_1, we must have that k prefers μ_1 to μ_2 because, otherwise, $\{i, k\}$ would be a blocking pair of μ_1 contradicting that $\mu_1 \gg \mu_2$ [see Proposition 1 of Klaus et al. (2011)] . By condition (i), we have that $k \in MA_P^i$. Then, the new partner of k at μ_1, $\mu_1(k) = l$ ($l \neq k, j$), and such that $l \succ_k i$, also belongs to the set of mutually acceptable agents of player k, $l \in MA_P^k$. But then, according to (ii), it must be that $MA_P^{i,k} = \{l\}$. But this is a contradiction, since $j \in MA_P^{i,k}$. Hence, player i should also prefer $\mu_1(i)$ to $\mu_2(i)$.

Second, consider any agent i in D such that $\mu_1(i) = i$ and $\mu_2(i) = k$. Since $\mu_1 \gg \mu_2$, then either $\mu_1(k) \succ_k i$ and k deviates leaving agent i unmatched (with $\mu_1(k)$ also preferring μ_1 to μ_2), or $i \succ_i k$ and agent i individually deviates.

Let $D' = \{i \in D : \mu_1(i) \succ_i \mu_2(i)\}$. Then the coalition D' deviates from μ_2 enforcing μ_1 (agents in $D \setminus D'$ are singletons at μ_1) and $\mu_1 > \mu_2$ as we wanted to prove. $\square$

Proof of Remark 1. We will show the different parts of this remark by contradiction.

1. Assume that (**a**) is not satisfied. That is, there exists an agent $j \in MA_P^i \setminus \{j_k, \omega(i)\}$ such that $t(j) \neq i$. This implies that $\exists k \in N$ such that $t(j) = k$ and $k \succ_j i$. By condition (i) of Proposition 2, $k \in MA_P^j$ and then by condition (ii) of Proposition 2, $MA_P^{i,j} = \{k\}$. However, this contradicts that $\{j_k, \omega(i)\} \subseteq MA_P^{i,j}$.

Now we will show that (**b**) must be satisfied as well. The fact that $t(j_k) \in \{i, \omega(i)\}$ is straightforward from condition (ii) of Proposition 2.

In order to prove (**b.1**), let $t(j_k) = i$. First, we will show that if $\omega(i) \notin T_P$ then either $t(\omega(i)) = i$ or $t(\omega(i)) = t(i)$. Let $\omega(i) \notin T_P$. Then, there exists and agent k such that $t(\omega(i)) = k$ and $t(k) = l \neq \omega(i)$ so $l \succ_k \omega(i)$ and by condition (i) of Proposition 2, $l \in MA_P^k$. If $k = i$ we are done, so assume that $k \neq i$. Since $k \succ_{\omega(i)} i$ and $i \in MA_P^{\omega(i)}$, by condition (i) of Proposition 2, $k \in MA_P^{\omega(i)}$. Thus, $\exists k \in MA_P^{\omega(i)} \setminus \{\omega(\omega(i))\}$ and $\exists l \in MA_P^k$ such that $l \succ_k \omega(i)$. Then, by condition (ii) of Proposition 2, it holds that $\{l\} = MA_P^{\omega(i),k}$. Since $k \succ_{\omega(i)} i$ and $i \in MA_P^{\omega(i)}$, we have that $l = i$. Given that $l \in MA_P^k$ and $l = i$, it holds that $k \in MA_P^i$. Let $k \neq t(i)$, otherwise we are done. Then since $i \in MA_P^k \setminus \{\omega(k)\}$ and there exists an agent $k' \in MA_P^i$ such that $k' \succ_i k$ (remember that $k \neq t(i)$), by condition (ii) of Proposition 2, $\{k'\} = MA_P^{k,i}$. But this implies that $k' = \omega(i)$ and this is a contradiction since $\omega(i) \not\succ_i k$. So we have proved that when $\omega(i) \notin T_P$ either $t(\omega(i)) = i$ or $t(\omega(i)) = t(i)$.
Now we will show that if $t(\omega(i)) \notin \{i, t(i)\}$, then $\omega(i) \in T_P$. Let $t(\omega(i)) = k$ with $k \neq i, t(i)$. By condition (ii) of Proposition 2, either $t(k) = \omega(i)$ and we are done, or there exists an agent $l \in MA_P^k$ such that $l \succ_k \omega(i)$ and $\{l\} = MA_P^{\omega(i),k}$, which implies that $l = i$. Following the previous reasoning,

we achieve the same contradiction ($\omega(i) \not\succ_i k$) and this proves that $\omega(i) \in T_P$ as desired.

Next, we proceed to prove (**b.2**). Let $t(j_k) = \omega(i)$. We will prove that in this case $t(\omega(i)) = j_k$. Since $i \in MA_P^{j_k}$, we have that $\omega(i) \in MA_P^{j_k} \setminus \{\omega(j_k)\}$. By condition (ii) of Proposition 2, either $t(\omega(i)) = j_k$ and we are done, or there exists an agent $k \in MA_P^{\omega(i)}$ such that $k \succ_{\omega(i)} j_k$ and $\{k\} = MA_P^{j_k,\omega(i)}$. Then, $k = i$, with $i \in MA_P^{\omega(i)} \setminus \{\omega(\omega(i))\}$. Hence, by condition (ii) of Proposition 2, we have that for any $j \in MA_P^i \setminus \{\omega(i)\}$, $j \succ_i \omega(i)$, then $\{j\} = MA_P^{\omega(i),i}$. But $|MA_P^i \setminus \{\omega(i)\}| > 1$, and then $\{j\} = MA_P^{\omega(i),i}$ for all $j \in MA_P^i \setminus \{\omega(i)\}$, contradicting the uniqueness of $MA_P^{\omega(i),i}$.

2. Let i be an agent such that $|MA_P^i| \leq 2$. We will prove that either $t(i) \in T_P$ or agent i belongs to a ring S such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} t$ for any $t \in N \setminus \{s_{i+1}, s_{i-1}\}$.

Consider first that $MA_P^i = \{j\}$ and assume that $t(j) = k$ with $k \neq i$. By the reasoning in [**1.**], if $|MA_P^j| > 2$, then $t(k) = j$ and we are done. So let $|MA_P^j| \leq 2$. Since $k \in MA_P^j \setminus \{\omega(j)\}$, by condition (ii) of Proposition 2, either $t(k) = j$ or there exists an agent $l \in MA_P^k$ such that $l \succ_k j$ and $\{l\} = MA_P^{j,k}$. However, this implies $l = i$ (since $MA_P^i = \{j\}$ and $|MA_P^j| \leq 2$). And this is a contradiction since $i \in MA_P^k$ but $k \notin MA_P^i$. Hence, if $MA_P^i = \{j\}$, then either $t(j) = i$ or $t(j) = k$ with $t(k) = j$.

Without loss of generality, let $MA_P^i = \{j, k\}$ with $j \succ_i k$. Since $j \in MA_P^i \setminus \{\omega(i)\}$ and by condition (ii) of Proposition 2, we deduce (following the same reasoning as before) that $t(j) \in \{i, k\}$. Let assume that $t(j) = k$, otherwise we are done. We will show that then $t(k) = j$.

Assume that there exists an agent $s \in MA_P^j \setminus \{i, k\}$ such that $s \succ_j i$. Since $j \in MA_P^i \setminus \omega(i)$, by condition (ii) of Proposition 2, $\{s\} = MA_P^{i,j}$, which implies $s = k$. Therefore, there cannot be any agent s between k and i in agent j's preferences (with $k \succ_j s \succ_j i$).

Consider now the case such that there exists an agent $s \in MA_P^j$ such that $i \succ_j s$. Then $|MA_P^j| > 2$ and by the reasoning of [**1.**], $t(j) = k$ implies $t(k) = j$.

Let $MA^j_P = \{k, i\}$ with $k \succ_j i$. Then, $k \in MA^j_P \setminus \omega(j)$ and by condition (ii) of Proposition 2, either $t(k) = j$ and we are done or there exists and agent $l \in MA^k_P$ such that $l \succ_k j$ and $\{l\} = MA^{j,k}_P$, which implies $l = i$. Given that there cannot be any agent between i and j in agent k's preferences, we have that $S = \{i, j, k\}$ form a ring in P such that $\forall s_i \in S$, $s_{i+1} \succ_i s_{i-1} \succ_i t$ for any $t \in N \setminus \{s_{i+1}, s_{i-1}\}$. Therefore if $t(j) = k$, then either $t(k) = j$ or $i \in S$ where S is a ring in preferences with $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_i s_{i-1} \succ_i t$ for any $t \in N \setminus \{s_{i+1}, s_{i-1}\}$.

3. Now, we will prove, by contradiction, that there is no pair of agents not belonging to T_P who are mutually acceptable among themselves, except from those belonging to an odd ring S such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_i s_{i-1} \succ_i t$ for any $t \in N \setminus \{s_{i+1}, s_{i-1}\}$. Assume that there are two agents $i, j \notin T_P$ such that $i \in MA^j_P$ and they do not belong to a ring with the features mentioned above. First, notice that $|MA^i_P| \leq 2$, otherwise by [**1.**], $i \in T_P$ and this is a contradiction. We know by [**2.**] that $t(i) \in T_P$, which implies that $t(i) \neq j$, since $j \notin T_P$. Let $t(i) = k$ (with $k \neq j$). Since $k \succ_i j$, by condition (i) of Proposition 2, we have that $k \in MA^i_P \setminus \{\omega(i)\}$. Since $i \notin T_P$, it follows that $t(k) \neq i$, and there is at least one agent, l, with $l = t(k)$. By condition (i) of Proposition 2, $l \in MA^k_P \setminus \{\omega(k)\}$ with $l \succ_k i$. By condition (ii) of Proposition 2, $\{l\} = MA^{i,k}_P$, which implies $l = j$ and therefore $t(k) = j$. However, this contradicts $j \notin T_P$ given that, as mentioned above, $t(i) = k$ with $k \in T_P$.

□

Proof of Proposition 3. Suppose that $\forall i \in N$, $j \succ_i i$ if $j \neq i$ but (N, P) is not absolutely stable.. Then there exist μ_1 and μ_2 such that $\mu_1 \gg \mu_2$ but $\mu_1 \not> \mu_2$. First note that neither μ_1 nor μ_2 can be the matching in which every agent is a singleton, since this matching is directly dominated by all other matchings (because all agents prefer to be matched over being unmatched). There are three possible matchings: $\mu_i = \{\{i\}, \{j, k\}\}$, $\mu_j = \{\{j\}, \{i, k\}\}$ and $\mu_i = \{\{k\}, \{i, j\}\}$. Assume, without loss of generality, that $\mu_2 = \mu_k$ and $\mu_1 = \mu_j$. The same reasoning could be applied for any other pair of matchings satisfying $\mu_1 \gg \mu_2$. Since $\mu_1 \gg \mu_2$ it must be [by Proposition 1] that i is better off in μ_1 (since j is worse off being unmatched). Note that k is also better off in μ_1 since she is unmatched in μ_2. But then i and k can enforce μ_1 over μ_2 and they are both better off, contradicting that $\mu_1 \not> \mu_2$. □

Proof of Proposition 4. ($\Rightarrow$) The existence of a ring S in the preferences with $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$, for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$, is a sufficient condition for non-existence of stable matchings in any stable matching (absolutely stable or not). We prove it as follows:
Let μ be a matching such that $\mu(s_i) = j$ for some $s_i \in S$ and some $j \notin S$. This matching is blocked by the pair $\{s_i, s_{i-1}\}$. Therefore any matching containing a pair formed by an agent in the ring and an agent outside the ring is not stable. Consider then a matching μ' satisfying that $\mu'(s_i) = s_{i+1}$ and $\mu'(s_{i-1}) = s_{i-1}$ (given that $|S| = 3$, maximizing the number of agents in the ring matched among themselves, there is always one agent in the ring who is alone at μ'). This matching is blocked by the pair $\{s_{i-1}, s_{i+1}\}$. Therefore any matching in which agents in S are matched among themselves is not stable. Hence, there is no matching stable as we wanted to prove.

($\Leftarrow$) Now, we will show that if a roommate problem is absolutely stable and unsolvable then there exists a ring S in P satisfying that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$.

Let (N, P) be unsolvable and absolutely stable. Since (N, P) is unsolvable there exists a ring $S = \{s_1, ..., s_k\} \subseteq N$ where k is odd (an odd ring). See appendix A.

- We first show that it must be that $|S| = 3$. Suppose not, then consider $\{s_1, ..., s_4\} \subset S$. Let μ_2 be a matching such that $\mu_2(s_2) = s_3$ and $\mu_2(s) = s$ for every $s \notin \{s_2, s_3\}$, and let μ_1 be a matching such that $\mu_1(s_1) = s_2$, $\mu_1(s_3) = s_4$, and $\mu_1(s) = s$ for every $s \notin \{s_1, ..., s_4\}$. Then $\mu_1 \gg \mu_2$. However, $\mu_1 \not\succ \mu_2$ since $\mu_2(s_2) \succ_{s_2} \mu_1(s_2)$. This contradicts absolute stability of (N, P).

- We now show that for any $s_i \in S$ there cannot exist an agent $j \notin S$ such that $j \in MA_P^{s_i}$. Suppose first that $j \in MA_P^{s_i}$ and $s_{i+1} \succ_{s_i} j$. Let μ_2 be a matching such that $\mu_2(s_i) = s_{i+1}$ and $\mu_2(s) = s$ for every $s \notin \{s_i, s_{i+1}\}$, and let μ_1 be a matching such that $\mu_1(s_{i+1}) = s_{i-1}$, $\mu_1(s_i) = j$, and $\mu_1(s) = s$ for every $s \notin S \cup \{j\}$. Then $\mu_1 \gg \mu_2$. However, $\mu_1 \not\succ \mu_2$ since $\mu_2(s_i) \succ_{s_i} \mu_1(s_i)$. This contradicts absolute stability of (N, P). Suppose instead that $j \in MA_P^{s_i}$ and $j \succ_{s_i} s_{i+1}$. Let μ_2 be a matching such that $\mu_2(s_i) = s_{i-1}$ and $\mu_2(s) = s$ for every $s \notin \{s_i, s_{i-1}\}$, and let μ_1 be a matching such that $\mu_1(s_i) = j$, $\mu_1(s_{i-1}) = s_{i+1}$, and $\mu_1(s) = s$ for every $s \notin S \cup \{j\}$. Then $\mu_1 \gg \mu_2$. However, $\mu_1 \not\succ \mu_2$ since $\mu_2(s_{i-1}) \succ_{s_{i-1}} \mu_1(s_{i-1})$. This contradicts absolute stability of (N, P).

- We finally show that for any $s_i \in S$ there cannot exist an agent $j \notin S$ such that $j \succ_{s_i} s_{i-1}$. Suppose not, then from the argument developed in the paragraph above we must have that $j \in A_P^{s_i}$. Then let μ_2 be a matching such that $\mu_2(s_i) = j$ and $\mu_2(s) = s$ for every $s \notin \{s_i, j\}$, and let μ_1 be a matching such that $\mu_1(s_i) = s_{i-1}$ and $\mu_1(s) = s$ for every $s \notin \{s_i, s_{i-1}\}$. Then $\mu_1 \gg \mu_2$. Note in particular that, since $j \in A_P^{s_i}$, agent j is better off in μ_1. However, $\mu_1 \not> \mu_2$ since $\mu_2(s_i) \succ_{s_i} \mu_1(s_i)$. This contradicts absolute stability of (N, P).

But then we have found a ring S in P such that $|S| = 3$ and $\forall s_i \in S$, $s_{i+1} \succ_{s_i} s_{i-1} \succ_{s_i} j$ for any $j \in N \setminus \{s_{i+1}, s_{i-1}\}$.

$\square$

Proof of Proposition 5. To prove (i) consider an absolutely stable roommate problem, (N, P), with more than one matching. By Lemma 1, we know that the matchings in the core dominate each other indirectly. However, by Proposition 2, they do not dominate each other directly, contradicting that the problem is absolutely stable. Hence the core must be unique.

To prove (ii) suppose then that there is an agent i who is not matched in the core to her most preferred partner. Let $j = \mu_C(i)$ such that $t(i) \succ_i j$. Since $j \neq t(i)$, there exists an agent k such that $k \succ_i j$. Let μ_2 be a matching such that $\mu_2(i) = k$ and the rest of the individuals are unmatched. By Lemma 1, $\mu_C \gg \mu_2$. Assume that $i \succ_k k$, then $\mu_C \not> \mu_2$ since $k \succ_i j$ and the pair $\{i, j\}$ does not block μ_2 and enforce μ_C. Therefore, $k \succ_k i$. Then $\{k\}$ blocks μ_2 enforcing a new matching in which agent i is unmatched. So if $\mu_C > \mu_2$, then $i = j$ proving that either $\mu_C(i) = t(i)$ or $\mu_C(i) = i$.

$\square$