High-dimensional Connectivity and Cooperation

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Abstract

This paper offers a modeling of "group connectivity" by proposing a generalization of the concept of a graph. The new approach captures not only binary relations between agents in a network but also high-order relations among subsets of them. The model enables to characterize the minimal structures of cooperation survival in a Spatial Prisoners' Dilemma in a Moore neighborhood and helps explain the existence of persistent "islands of cooperation" in hostile environments. The global behavior of the network is offered by some numerical simulations.

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1   Introduction

The present paper is about "group connectivity" or high-dimensional connectivity, in social and economic networks and its implications for cooperation in local interaction models.

Casual observation suggests that most economic and social interactions take place locally, within restricted subsets of a larger population. Local interaction models in economics are defined as models in which agents' preferences, information, choices or outcomes are affected by others' behavior directly rather than being mediated by markets. A common assumption in these models is that individuals interact locally, with a set of neighbors defined by a social or economic distance metric. Local interaction models explain a variety of social and economic phenomena such that the very high variance of crime rate across U.S. cities through a model in which agents' propensity to engage in criminal activities is influenced by neighbors' choices (Glaeser et al.1996), information exchange among workers (Calvo-Armengol and Jackson, 2002), information cascades (Banerjee, 1992; Bikhchandani et al., 1992), learning from neighbors (Bala and Goyal, 1998), the emergence of conformity and social norms (Young, 2001); the presence of temporal "lumping" in the spread of a given industry from a country to another (Puga and Venables, 1996); informal contacts and information networks in the job search (Montgomery, 1991), and the spread of cooperation (Eshel, Samuelson, and Shaked, 1998) among many others.

An agent's neighborhood might include family and friends, colleagues, business partners, geographic neighbors, etc. Within his neighborhood, an agent shares, exchanges and develops information, knowledge and other resources, new behaviors are learned and strategic interactions take place.

Furthermore, in most social and economic local structures, specific sets of agents

may have a "*tighter*" relation among them. Examples of these groups are families, business partners, committees, associations and lobbies among others. Group connectivity in a socio-economic system may have different effects, for instance, the members of a group may exchange information preferentially among them (i.e. partners in a company), they may imitate preferentially each other (i.e. teenagers movements), or their payoffs may depend preferentially on the payoffs of the other members of the group. The existence of group connectivity generates specific patterns of heterogeneity and externalities. .

This paper provides with a model of group connectivity where interactions may be of a high-dimensional character, i.e., the interaction of n individuals is not simply the aggregation of the interactions of $n(n-1)/2$ binary relationships. This framework is applied to the spatial Prisoners' Dilemma game in a two-dimensional lattice to analyze the minimal structures of cooperation survival, from which cooperation can spread and propagate.

In the Prisoner's Dilemma, best reply precludes cooperation regardless of how agents are matched (be it locally or globally). For many applications this is neither reasonable nor necessary. We abandon the assumption that people are rational agents choosing utility maximing actions. Instead, we assume imitation as a reasonable behavior when information is incomplete but the performance and action of others are (perhaps indirectly) observable. Imitation alone is however not enough. To protect efficient play from too much exposure to the other action (which would implied lowered payoffs), interaction also needs to be local. This can be achieved with a fixed structure like a network, or in a more fluid way through random pair (or small group) formation.

An early analysis of imitation in conjunction with local interaction is in Nowak and May (1992, 1993). Using numerical simulation, it is showed that for a substan-

tial part of the possible payoff values (the magnitude of the defection premium plays a significant role here) and most initial conditions, defectors and cooperators can coexist forever on a lattice, either in static irregular patterns or in dynamic patterns with chaotic or cyclical fluctuations around predictable long-term averages. Similar results obtain in both synchronous and asynchronous environments (Nowak et al. 1994; Hubermann and Glance 1993), when various changes are introduced in the learning procedure (noise, memory, etc.), and on a variety of network structures. Variations on these elements have been explored in numerous simulation studies. Roca et al. (2009) perform a very systematic simulation study which explores the various degrees of freedom of the problem, and emphasize the central role played by clustering (or transitivity) in sustaining efficient play. Taking an analytical approach, Eshel et al. (1998) examine the survival of cooperation in a Prisoner's Dilemma played on the circle and show that at least two thirds of cooperators exist in any stochastically stable configuration. The authors however are constrained by their methodology to stick to the circle (the one-dimensional periodic lattice) with two or four nearest neighbors. Mengel (2009) distinguishes interaction and information in an imitation-driven approach with players located on the circle and a few additional structures. Whenever agents use information beyond their interaction neighbors, the unique stable outcome is inefficient. Introducing sufficient conformism (which in effect implies that the payoffs are distorted, and not those of a Prisoner's Dilemma anymore as they include a bias towards the majority behavior) is a way of sustaining efficient play.

Grasping the meaning of "group connectivity" entails the modeling of high-dimensional connections. To cope with that we consider simplicial complexes. Roughly speaking, a simplicial complex is a generalization of a graph, in the sense that in addition to binary relations between the elements of a set, it captures high

order relations as well, such as triangles, tetrahedron, etc). The dimension of a simplicial complex is the maximum dimension of any of its simplices. As the simplest application of the model we have chosen Moore neighborhoods with interactions with nearest and next-nearest neighbors (square lattices where each player has eight neighbors). Moore Lattices have many triangles and thus the concept of group connectivity is easily applicable.

The paper is also related to the literature on contagion. Tristan Boyer (2010) identifies structural properties of iterated neighborhoods which are key to contagion by efficient behavior. In general Prisoners' Dilemmas with deterministic imitation efficient behavior spread if and only if it has a chance of being imitated, which requires that it generates high payoffs. This can only happen if players are, to a certain extent, segregated and remains so as contagion unfolds. Therefore for arbitrary regular networks efficient contagion depends strongly on the extent to which play mostly happens within (rather than across) the iterated neighborhoods of some initial seed group. Morris (2000) formulates very general results on contagion in arbitrary, infinite order networks and with best reply dynamics. His approach uses the concept of cohesion, i.e., the extent to which groups of individuals have interactions within the group rather than outside the group. Our results on minimal cooperation structures from which cooperation can spread have a similar flavor. Taking agents to be in a square lattice reduce the number of defectors surrounding cooperators, but not enough to avoid the invasion from the former ones. Therefore group connectivity allows cooperators to huddle together in concentrated groups, although their members are still exposed to defectors.

The concept and modeling of group connectivity could be applied to some local interaction models and help explain some features not yet clarified. For instance, the survival of very small groups in hostile environments, as long as the members of

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such groups have a strong inner cohesion. Specifically, the existence of "lumpy" cities where small ethnic/or sub-culture groups can survive in homogeneous environments generating persistent "islands of diversity".

The interaction of these structures in a global system generates a complex global behavior. We would like to analyze how the existence of group connections helps increase cooperative behavior. The heterogeneity and correlation introduced by the simplex structure produces a highly inhomogeneous system which is not compatible with the standard techniques of the mean-field approach and/or pair-approximation, more suitable for analyzing homogeneous systems. Given that, we have run some numerical simulations which help develop some results and intuition regarding the global cooperative behavior.

The paper is organized as follows. Section 2 proposes a modeling of group connectivity. Section 3 cares about the Spatial Prisoners' Dilemma game in a Moore neighborhood and its extension to the simplicial complex game. A measure of link intensity is also presented. Section 4 analyzes the minimal structures of cooperation survival, and section 5 offers the results of some numerical simulations about the global behavior of the system. Concluding remarks close the paper.

2 Concepts

Imagine a table with four people having a conversation at a restaurant; this event is understood as a social network with four nodes. If every one can hear everyone else, then in graph theory this network is represented by a complete graph on four vertices. Now imagine a situation of four people playing the game "phone" so that each person may only whisper in another person's ear. This scenario is again modeled by a complete graph on four vertices. But the situations are extremely

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different! One-dimensional graph theory does not capture the distinction between a single 4-person conversation and six 2-person conversation.

Define *group connectivity* as a situation in which interactions among individuals are of a higher-dimensional character, i.e., the interaction of n individuals is not simply the aggregation of the interaction of the $n(n-1)/2$ pairs.

1. High-dimensional connectivity: From networks to simplicial complexes

The concept of high-dimensional connectivity is introduced to analyze group connectivity among agents in a network. The key idea is to consider that groups of agents can be linked beyond pair-wise interactions. In standard networks, a link could be understood as having dimension one. In this model, for instance, a link of dimension two would represent a connection among three nodes and such connection has deeper implications that those of an aggregation of the three one-to-one- connections between them.

To study the above situations, one must turn from (1-dimensional) graph to a higher-dimensional model. The two most popular such models are hypergraphs, simplicial sets and/or simplicial complexes. A hypergraph is like a graph, except that edges can connect more than two nodes. One draws a hypergraph by drawing a set of dots and enclosing certain subsets of them within circles. Therefore, hypergraps are not closed under taking subsets[1].

Simplicial sets, on the other hand, are visualized as multi-dimensional polygonal

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[1]In many applications, there may be groups that interact in a way that subgroups do not. In a conference call any person can speak to the whole group but cannot speak to a given subgroup. For example, consider the academic reseach network, where every researcher is a vertex, and every coauthorship is a (hyper) edge. If three researchers are the coauthors on a paper, that does not imply that some subset of them is also the set of coauthors of a paper. Coauthorship is a relation that is not closed under taking subsets.

shapes made up of nodes, edges, triangles, and higher-dimensional triangles like tetrahedra. An n-dimensional triangle, called an n-simplex, can be thought of as the "polyhedral hull" of $n+1$ vertices. Thus, a 1-dimensional triangle, or 1-simplex, is the polyhedral hull of two vertices: it is simply an edge. Likewise, a 2-simplex is the hull of three vertices and is hence a triangle; a 3-simplex is a tetrahedron; and a 0-simplex is just a vertex. In general, a simplicial set is the union of many of such vertices, edges, triangles, etc.

To make this a little more clear, let us emphasize that any graph is a 1-dimensional simplicial set. Now suppose that we want to construct a 2-simplicial set. Begin with a graph G, and choose a set of three edges (a,b), (b,c) and (a,c) which form a triangle inside G:
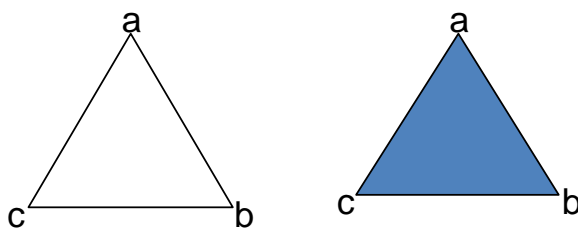

Figure 1

Given this triangle, one may attach in a 2-simplex to G, filling in the triangle abc. Thus, a 2-dimensional simplicial set looks like a graph except that some triangles are filled. To create a 3-dimensional simplicial set, begin with a 2-dimensional simplicial set, choose some grouping of four triangles that form an empty tetrahedron, and filled them. The appropriate simplicial set model for our "table of four" is the tetrahedron. This solid shape represents the idea that the relationship (the conversation) is taking place between four entities in a *shared space.* The relationship is "closed under taking subsets". For the situation in which the four people can only whisper to each other, we instead use the simplicial set consisting only of the six edges of the tetrahedron.

This graph is just the complete graph on four vertices.

A Simplicial Complex is a simplicial set in which not two simplices have the same set of vertices. Abstract Simplicial Complex can be described as a higher dimensional version of some network[2]. When we connect nodes with edges we have a simplicial complex. Moving to a higher dimension, a triangle is known as a 2-simplex. Note that a triangle has three faces which are segments, i.e., 1-simplices. In turns, every 1-simplex has two faces which are 0-simplices. We can therefore think that our 2-simplices must be faces of something else we should call a 3-simplex. This is exactly the case, as 3-simplices are defined as tetrahedrons,i.e., pyramids with three faces, each of which is a triangle. The "face" of a simplex is the natural generalization of the face of a polyhedron and then given a simplex we can consider its faces and then the faces of its faces, and so on. This lead us to the notion of the m-face of a simplex. Notice that the face of an n-simplex is always an $n+1$-simplex. We can now define a simplicial complex as the union of several simplices, possibly of different dimensions, such that if the intersection of two simplices is not empty, then the intersection itself is an m-face for both simplices.

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[2]For instance, one might want to consider two research papers with the same set of authorsas two different simplices linking the same vertices. For this, one must use simplicial sets, not simplicial complexes.
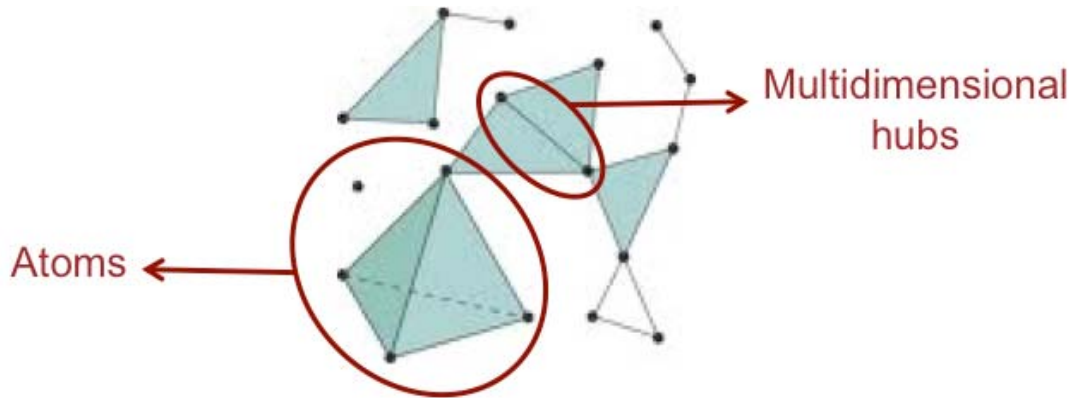
Figure 2

High-dimensional connectivity is modelled by associating an abstract simplicial complex to a network[3] in such a way that when a subset of $n + 1$ nodes has a high-dimensional connection, then the n-dimensional simplex formed by these $n + 1$ nodes belongs to the simplicial complex.

We study network high-dimensional connectivity by associating a stylized form of such simplicial complexes to a spatial evolutionary Prisoner's Dilemma (PD).

3 The Evolutionary Prisoner's Dilemma game

3.1 The spatial Prisoner's Dilemma game

Players are located on a two-dimensional square lattice of $N \times N$ nodes, interacting in a Moore neighborhood, i.e., each of them will interact with its eights nearest

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[3]We wish to point out that the idea of using simplices to describe and model social interactions is not totally new, as it was developed by R. Atkin (1972, 1974) in his theory of Q-analisis. However, Atkins uses simplicial complexes to represent relations in a cartesian product of two sets (binary relations). In our viewpoint, simplicial complexes indicate actual social entities, whereas relations, either binary and n-ary alike, that comprise them are, literally, their faces. Just as human body is more than a mere aggegate of its organs, social groups are more than their constituents.
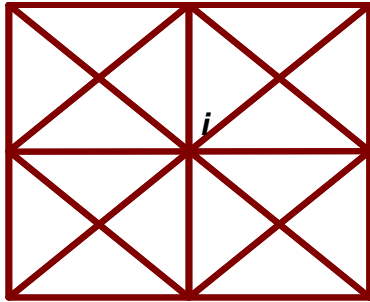
neighbors.

Fig. 3. Moore Neighborhood

Interaction takes place through a Prisoners's Dilemma (PD), defined by the following payoff matrix (payoffs for the row player are given):

$$
\begin{array}{c c}
 & \begin{array}{c c} C & D \end{array} \\
\begin{array}{c} C \\ D \end{array} & \left(\begin{array}{c c} 1 & T \\ S & 0 \end{array} \right)
\end{array}
$$

The parameters are the temptation payoff $1 \leq T \leq 2$, and the sucker's payoff $S \leq 0$, this last one representing the risk in cooperating. For a first exploration of parameters, we will restrict ourselves to the riskless case $S = 0$, to make cooperation easier (the so called the *weak prisoners's dilemma*). We note however, that changing $S \in [-1, 1]$ and $T \in [0, 2]$ allows us to explore the whole variety of 2×2 symmetric games.

Each agent plays simultaneously the PD with each of her eight immediate neighbors and her strategy, to Cooperate C or to Defect D, is the same in all these games. After a round of the game, each player i collects her payoffs by adding up the payoffs obtained from her individual interactions with all their neighbors, represented by $\Pi_i = \sum_{j=1}^{8} \pi_{ij}$. Subsequently, they update their strategy using the *proportional update rule* (which, when played in a complete network converges to the *replicator dynamics*). This rule posits that every player, say i, looks at its neighborhood and

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chooses one of its neighbors, say j, with equal probability and copies her strategy with some probability $P(s_{i,t} \to s_{j,t+1})$. Let $P(ij)$ denote the probability that player i observes the strategy of player j, then $P(ij) = \frac{1}{8}$ as we have a lattice with eight neighbors per node, and

$$P(s_{i,t} \to s_{j,t+1}) = \frac{1}{8}[\theta(\Pi_j - \Pi_i) \times (\Pi_j - \Pi_i)]$$

where $\theta(\Pi_j - \Pi_i)$ is the Heaviside function

$$\theta(\Pi_j - \Pi_i) = \begin{cases} 0 & \text{if } \Pi_j - \Pi_i \leq 0 \\ 1 & \text{if } \Pi_j - \Pi_i > 0 \end{cases}$$

After all players have checked on possible updates of their strategies, all payoffs are reset to zero, implying that we are modelling one-shot interactions among memoryless agents, and the process is repeated. Eventually, after a number of interactions, all agents reach a stationary value, and the level of cooperation, measured as the percentage of cooperative actions taken at a given time, fluctuates around some constant value. For the case of the PD, unless $T - 1$ is small ($T \leq 1.2$) and S is very small, the evolution converges to full defection, i.e., all the agents defect all the time.

Clustering explains much of the replicator dynamics in Moore Lattices. Let x be the the global density of cooperators in a population without structure, and let $\widehat{x}$ the local density in a network with structured population. Notice that x is a global variable, whereas $\widehat{x}$ is defined for each player. As a result, the effect of populations structure can be understood as the replacement of the global density x by the player-dependent local densities, in the dynamics that drives the evolution of the population, up to a time scale factor. Obviously, all local densities do not evolve equally, but some of them do feature an increase caused by the correlation that arise from the spatial structure. The local densities fluctuate over the population

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in the initial random conditions, with cooperators more or less connected to other cooperators. Those with small $\widehat{x}$ eventually disappear, while those with large $\widehat{x}$ convert, with high probability, their defective neighbors to cooperators. This is the point where the large clustering players its crucial role: newly converted cooperators will be connected not only to the cooperators whose strategy they have just adopted, but also to some of her neighbors (because of the network clustering), which are, with high probability, cooperators as well (because of the high $\widehat{x}$ of the initial cooperator). Hence the new cooperator will also have a large local density of cooperators. In the stationary state every neighborhood verifies that $\widehat{x} = x_e$, the population will freeze at that configuration, because in that case all players would obtain the same payoff.

3.2   The simplicial complex game:

3.3   Triads: A measure of link intensity

Associate to the above spatial PD game a 2-dimensional simplicial complex whose simplices (triangles) represent the groups of three neighbor nodes with 2-dimensional connection, denoted as *triads (*groups of three connections*)*. Suppose that player i belongs to a triad.

Two agents are '*closer*', the higher the number of triads they share in common. Denote by Δ_{ij} the *intensity* of the link between agent i and j and define:

$$\Delta_{ij} = \Delta^{\alpha}$$

where $\Delta > 1$ and $\alpha \in [0, 4]$ is the number of triads i and j jointly pertain to.
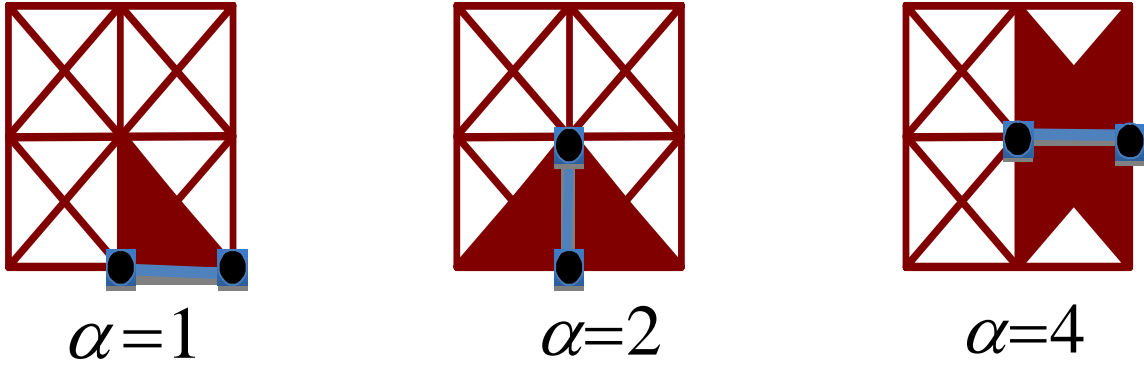
Figure 4

The model introduces link heterogeneity with some specific structure. If agents i and j have link intensity 2 (they belong to 2 triads), it means that there exist two other agents m and n to whom i and j are respectively connected with link intensity 1. This argument is extended to any other link intensity 3 and 4.

Group connectivity may have several consequences for the players' behavior. Firstly (*information seeking*), the probability that player i observes an agent inside the simplex may be higher. Consider a player, say i, denote by ij the link between player i and player j and recall that $P(ij)$ denotes the probability that player i observes the strategy of player j.

$$P(ij) = \frac{1}{Z}\Delta_{ij}$$

where $Z = \left(\sum_{j=1}^{8} \Delta_{ij}\right)^{-1}$ is the normalization constant and $\Delta_{ij} > 1$ is the parameter defining the higher order dimensionality. By above,

$$\Delta_{ij} = \begin{cases} 1 & \text{if } ij \text{ is a regular link, i.e., they do not belong to any triad} \\ \Delta & \text{if } ij \text{ have link intensity 1 (are an edge of a triad)} \\ \Delta^2 & \text{if } ij \text{ have link intensity 2 (are the common edge of two triads)} \\ \Delta^3 & \text{if } ij \text{ have link intensity 3 (are the common edge of three triads)} \\ \Delta^4 & \text{if } ij \text{ have link intensity 4 (are the common edge of four triads)} \end{cases}$$

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Note that a link can belong to at most four triads with our choice of network (the generalization of the idea to more complex networks is obvious).

Secondly (*imitation*), the probability that i imitates the behavior of an agent inside the triad she belongs to is higher. This translates to player i weighting agent j's payoffs by her link intensity with him. Therefore, the updating of strategy will be:

$$P(s_{i,t} \rightarrow s_{j,t+1}) = \frac{1}{Z}\Delta_{ij}[\theta(\Delta_{ij}\Pi_j - \Pi_i) \times (\Delta_{ij}\Pi_j - \Pi_i)]$$

Notice that a player i in a triad can imitate a neighbor j belonging to the same triad even with $\Pi_j < \Pi_i$.

Finally (*payoff externality*), the payoffs of the PD's played within the triad have a higher weight than those coming from neighbors outside the group, i.e.,

$$\Pi_i = \sum_{j=1}^{8}\Delta_{ij}\pi_{ij}$$

The following table summarizes the main differences from the standard model:

	Standard spatial PD	Model with triads
Information: Prob. to observe agent j's behavior, $P(ij)$	$\frac{1}{8}$	$\dfrac{\Delta_{ij}}{\left(\sum\limits_{j=1}^{8}\Delta_{ij}\right)^{-1}}$
Imitation: Prob. to imitate agent j's strategy, $P(s_{i,t}\rightarrow s_{j,t+1})$	$\frac{1}{8}[\theta(\Pi_j-\Pi_i) \times (\Pi_j-\Pi_i)]$	$\dfrac{\Delta_{ij}}{\left(\sum\limits_{j=1}^{8}\Delta_{ij}\right)^{-1}} \times$ $[\theta(\Delta_{ij}\Pi_j-\Pi_i) \times (\Delta_{ij}\Pi_j-\Pi_i)]$
Payoff externality: Π_i	$\sum\limits_{j=1}^{8}\pi_{ij}$	$\sum\limits_{j=1}^{8}\Delta_{ij}\pi_{ij}$

Table 1

We have then a family of different replicator dynamics, which are defined by the distinct group connectivity effects being taken into account. Recall that in the stationary state of the replicator dynamics in homogeneous Moore lattices every neighborhood verified that $\widehat{x} = x_e$, the population freezes at that configuration, because in that case all players will obtain the same payoff. With degree heterogeneous players, the population never reaches the regular configuration, but the local densities $\widehat{x}$ fluctuate around x_e, and the players' strategies oscillates accordingly. Therefore degree heterogeneity and network clustering give rise to different dynamic processes.

4   The minimal structures of cooperation survival.

We want to investigate the *minimal atoms* of cooperation survival or *cooperation kernel* of the spatial PD game in Moore neighborhoods when all the agents adjust their strategies at the same time and there are triadic connections. It is clear that in a Moore neighborhood with simultaneous strategy updating, a single cooperator surrounded by defector cannot survive. In fact for pair-wise connections this is certainly true for $T < 1$. For $T > 1$, cooperator can only survive (and grow) if they form clusters. Nowak and May (1993) show that the minimal cooperation atoms are square shaped clusters, with a minimum of $4C$, provided that T is small enough[4].

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[4]In fact the results of Nowak and May (1993) are for deterministic imitation. Specifically, they show that for $T < 3/2$, all square shaped clusters, with a minimum of $4C$ can grow. However, for $T \in (2, 3)$, a 2×2 cluster will dissapear, but a 3×3 C-cluster (or any larger square) will persist -without gaining or loosing. If $T > 3$ all cooperators will disappear. These analysis suggests the existence of three classes of parameter regions: (i) of $T < 5/3$, only C-clusters keep growing; ii) if $T > 8/5$, only D-clusters can keep growing and iii) if $T \in (8/5, 5/3)$ then both C and D clusters

To better compare our results with those of Nowak and May (1993), we leave aside the information seeking and imitation effects and only consider the payoff externalities of a triad. At the end of each period, an agent may either retain her strategy or choose a strategy played by one of her eight neighbors, depending on their payoffs. These payoffs, in turn, depend of the strategies of the next Moore neighborhoods. To illustrate our results let us consider a 5×5 torus with a triad of cooperators in a sea of defectors as figure 5.a below shows.
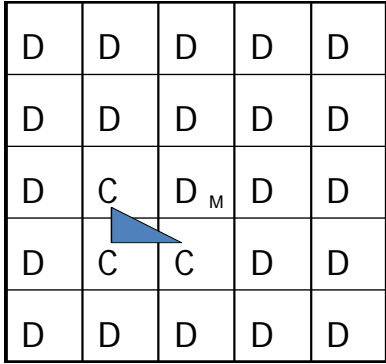
D	D	D	D	D
D	D	D	D	D
D	C	D $_M$	D	D
D	C	C	D	D
D	D	D	D	D

Fig. 5.a. A triad of C's

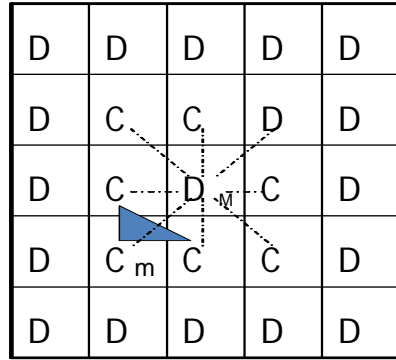| D	D	D	D	D
D	C	C	D	D
D	C	D $_M$	C	D
D	C $_m$	C	C	D
D	D	D	D	D

Fig.5.b. Propagation of C's

Let Π_{D_M} be the payoff of the defector player with the highest payoffs (he plays the PD game against three cooperators), therefore his initial payoffs are $\Pi_{D_M} = 3T$. Obviously, with only pair-wise connections, cooperation will not survive. Therefore, the triadic connection is necessary to maintain cooperation and since $\Pi_C = \sum_{j=1}^{8} \Delta_{ij}\pi_{ij} = 2\Delta_{ij} = 2\Delta$, then, letting as long as $\Delta > \frac{3}{2}T$, the cluster of cooperation will survive but need not grow. What about propagation of strategy C? As the strategy updating is stochastic and simultaneous, we consider the sufficient condition for propagation. Suppose that cooperation has been imitated by some players

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can keep growing.

and consider again player D_M. If all his neighbors having a connection with the triangle of C's change their strategies to C, then he will get at most $\Pi_{D_M} = 7T$, while player C_m, will only get $\Pi_{C_m} = 2\Delta$ again (see figure 5.b. above). Therefore, if $\Delta > \frac{7}{2}T$, cooperation will propagate and contagion around the cooperative kernel will occur. Notice that once cooperation propagates beyond the triad, the conditions for a further spread of strategy C depend on T. For instance, consider figure 6.a below, the payoffs of D_M are now $\Pi_{D_M} = 5T$ and those of C_m, $\Pi_{C_m} = 7$, therefore for $T < \frac{7}{5}$, cooperation will be maintained in the left lower 3×3 cooperation torus. Also notice that the highest payoff for defector on the lattice upper and right borders are $\Pi_D = 6T$ and thus some C's of the second upper row and of the last row may change to D. In general, we will find an area of influence around the cooperation kernel. It starts from a 3×3 cooperator cluster and continues with a fluctuation area (this area is easily seen at simulations), where players at the lattice borders may alternate their choices between C and D. The defectors will be along the borders of the lattice. Notice however that figure 6.b. is for illustrative purposes since for small lattices the only absorbing states are C or D, and the above result will drive the system to full cooperation in lattices of small size.

Fig.6.a. Local contagion

Fig.6.b. Influence area

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Now consider two triads with a common face (see figure 7.a). Here there are two types of cooperators: those belonging only to a triangle with payoffs of $\Pi_{C_B} = 2\Delta$, and those sharing the common face of the two triangles with payoffs $\Pi_{C_A} = 2\Delta + \Delta^2$. Obviously, without triadic link intensity, cooperation will disappear, since $\Pi_{C_B} = 2 < 3T = \Pi_{D_M}$. Clearly, since $\Pi_{D_M} = 3T$ and $\Pi_{C_B} = 2\Delta$, then for $\Delta > \frac{3}{2}T$, the kernel of cooperation will not disappear. Now suppose as in figure 6.b.that some players connected with the two triads have imitated their strategy, then $\Pi_{D_M} = 7T$ and $\Pi_{C_B} = 2\Delta + 2$ and therefore for $\Delta > \frac{7}{2}T - 1$, cooperation[5] will propagate. Notice that this bound is a sufficient condition, for instance consider D_m in figure 7.b and suppose that the left hand neighbors and the right hand neighbors have turn to cooperators, then the bound for cooperation to spread is $\Delta > \frac{5}{2}T - 1$. Also notice that the condition on Δ, when we consider any D up or down of any C_A and compare payoffs is that $\Delta > \sqrt{5T - 1} - 1$.
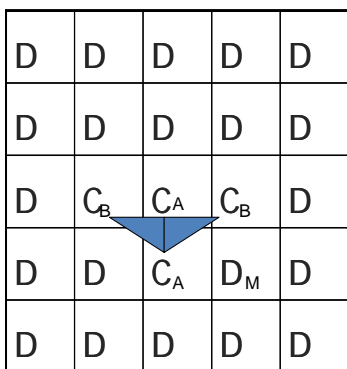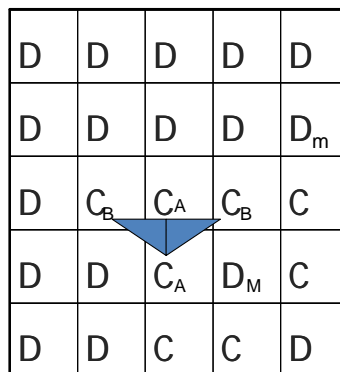

Fig.7.a. Two triads of C's

Fig.7.b. Some contagion

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[5] The above condition on Δ would be slighty different in a 4×4 torus. In this later case the sufficient condition is that $\Delta > 4T - \frac{5}{2}$.

Suppose now that $\sqrt{5T-1} - 1 < \Delta < \frac{7}{2}T - 1$, so that players C_B have turn to D. This situation is displayed at figure 8.a. First, notice that if $\Delta > 2$, cooperation between the pair of players in the common face of the two triangles will survive. Furthermore, it is easily shown that for $\Delta > T + \sqrt{T(T+3) - 1} > 2$ (since $T > 1$), the payoffs of C_{A_1} are higher than those of D_M (notice that for $T \geq 1.2$, then $T + \sqrt{T(T+3) - 1} < \frac{7}{2}T - 1$), and therefore cooperation between the pair of players in the common face of the two triangles not only will survive but it may even spread.
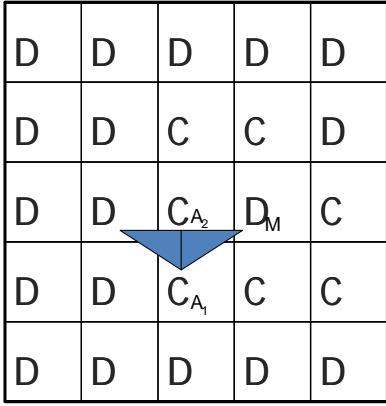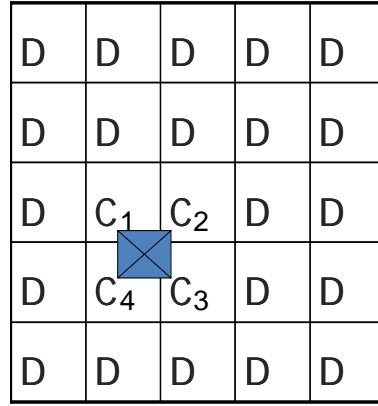
D	D	D	D	D
D	D	C	C	D
D	D	C_{A_2}	D_M	C
D	D	C_{A_1}	C	C
D	D	D	D	D

Fig.8.a

D	D	D	D	D
D	D	D	D	D
D	C_1	C_2	D	D
D	C_4	C_3	D	D
D	D	D	D	D

Fig.8.b. A 2×2 C-cluster

Finally, consider the envelope-shaped cooperation kernel: a 2×2 cluster of co-operators (see figure 8.b. above). At this cooperation kernel, each C is connected to three other C's. Only C_1 and C_4 are in a common faced of the two triangles[6], and therefore $\Pi_{C_1} = 2\Delta + \Delta^2 = \Pi_{C_4}$, and $\Pi_{C_2} = 3\Delta = \Pi_{C_3}$. Each defector only faces two cooperators and then as long[7] as $\Delta > \max\{\frac{2}{3}T, \sqrt{2T+1} - 1\}$, the cluster will survive and grow. Suppose that cooperation spreads from the cooperation cluster and consider both the payoff of the defector to the right of C_2, who gets at most

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[6] Notice that the simplicial complex is of dimension three, while the lattice only has dimension two.

[7] This depends on the value of T: for $T = 3/2$, the two arguments are equal.

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$\Pi_D = 5T$ and those of the defector to the left of C_1 ($\Pi_D = 5T$) and compare them with $\Pi_{C_3} = 3\Delta + 1$ and $\Pi_{C_4} = 2\Delta + \Delta^2 + 1$, respectively. Then, whenever[8] $\Delta > \frac{5}{3}T - \frac{1}{3}$, local contagion will occur.

The above analysis is summarized in the following Proposition.

Proposition 1 : *The minimal cooperation structures that ensure the survival and propagation of cooperation in a torus of 5×5 or higher numbers of players with at least a triadic connection are: 1) A triad of cooperators. The cooperative triad will survive whenever $\Delta > \frac{3}{2}T$, and for $\Delta > \frac{7}{2}T$ cooperation will propagate. 2) A cluster of two triads with a common face whenever $\Delta > \frac{7}{2}T - 1$. 3) Two isolate cooperators belonging to a common face of two triads will survive and propagate cooperation, whenever $\frac{7}{2}T - 1 > \Delta > T + \sqrt{T(T+3)} - 1 > 2$. 4) A 2×2 cluster of cooperators, as long as $\Delta > \frac{5}{3}T - \frac{1}{3}$. Contagion from these structures occurs at the influence area of the specific cooperation atom, it is local and fluctuates in the borders of the lattice.*

These results are in contrast with those of classical segregation models (Schelling, 1972), because very small groups can survive in hostile environments, as long as the members of such groups have a strong inner cohesion. This could explain, for instance, the existence of "lumpy" cities where small ethnic/or sub-culture groups can survive and/or the persistence of "islands of diversity" in homogeneous environments.

5 Global Behavior

Once we have isolated the minimal structures of cooperation survival we face the issue of the dynamic of a Moore Lattice with several triads non necessarily of coop-

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[8]Notice that for a 4×4 torus, the conditions are: $\Delta > \sqrt{8T} - 1$ (for $T \leq 3$).

eration. Because the state in which all agents are defectors is absorbing, the system may drive cooperators to extinction. It will important to know whether there exist absorbing states containing cooperators.

As it is known, the direct calculations of an evolutionary binary game on a degree homogeneous network with N nodes, gives rise to a huge Markov chain of 2^N states. For this reasons several approximation have been proposed to tackle this problem. The most commonly known is the pair-approximation methodology which provide analytical insight into the relationship between lattice structure and population dynamics. However, the pair-approximation that yields reasonably good results for random networks it is not adequate to study regular lattices, specially if they have network clustering. This is still worse if the lattice is inhomogeneous as in our case. In this case, simulations are excellent for developing intuition regarding spatial processes.

We have conducted some simulation exploring the asymptotic behavior of the spatial Prisoners's Dilemma in a Moore neighborhood with triads, taking two values of the temptation parameter $T = 1.1, 1.2$. The duration of every run is 2000 iterations (understood as 2000 sweeps over every node of the lattice, sequentially) and the results are averaged over 100 realizations for each set of parameters. The simulations calculate the frequency of cooperators as a function of parameter Δ, for random initial conditions with a 50% of cooperators and a 10% of triads[9]. These triads may have only three cooperators, only three defectors or both. In fact, these initial conditions imply that the probability of a triad of only cooperators is quite small -about 1/8- and that of bigger cooperation structures is even smaller. The

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[9]Simulations have also been conducted for different number of triads in the network. The comparison shows that different numbers of privileged triangles do not change significantly the results.

results for the model without triads are those for $\Delta = 1$.

Recall that we have a family of different replicator dynamics, which are defined by the distinct group connectivity effects being taken into account. Let us denote them by *RP(SEEK,IMIT,PF)* where *SEEK,IMIT,PF* is 0 if the effect is not active and 1 if it is active.. Because we wish to analyze the impact of each of the effects of the group connectivity in isolation and the impact of all of them together, the simulation have focused on:

RP(1,0,0) Preferential information seeking with both standard imitation and payoffs

RP(0,1,0) Preferential imitation with both standard imitation and payoffs

RP(0,0,1) Preferential payoffs with both standard information and imitation

RP(1,1,1) Preferential information, imitation and payoffs.

Main Results:

For isolated effects the simulations show that preferential information seeking $(RP(1,0,0))$, i.e., the probability that players in triads observe the other agents inside the simplex with a higher probability, does not make differences in the long run as compared with the one without triads. Figure 9 shows that starting from a probability of 50% of cooperators, their frequency is around the 70% for values of $T = 1.1$ and of 40% for $T = 1.2$, for $\Delta = 1$ (the baseline case). This frequency does not change with preferential information seeking. The key idea here is that even with small probability neighbors outside the triads are going to be observed so that in the long run the impact of preferential information seeking is not important.
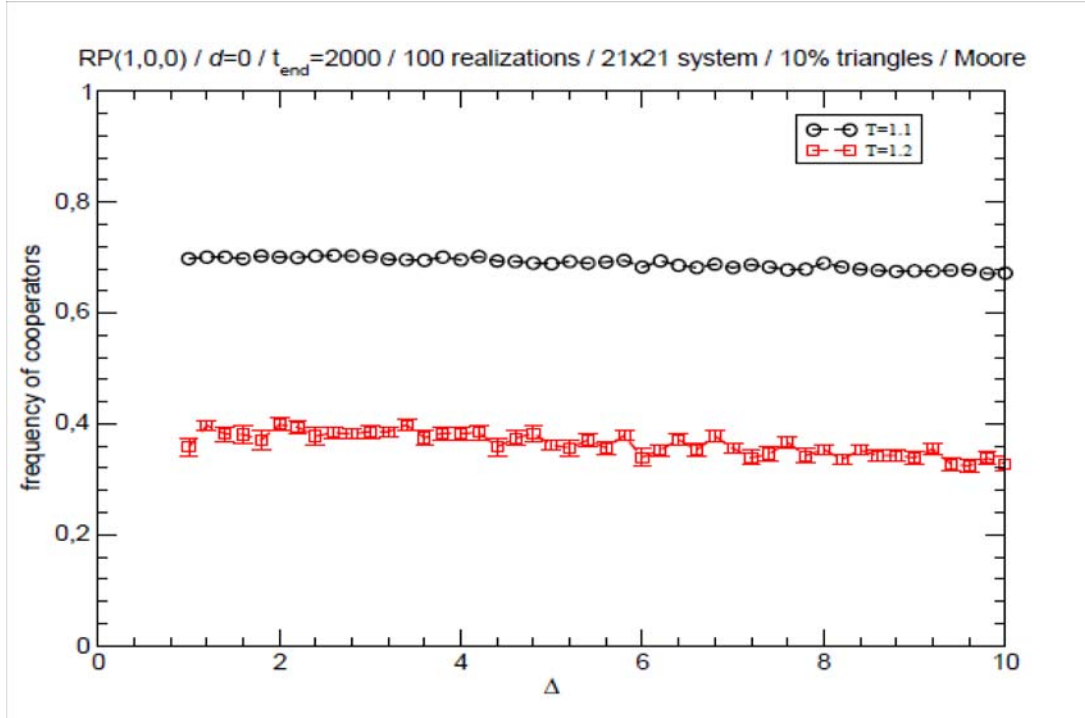
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Figure 9. Information seeking. RP(1,0,0)

Only imitation of the neighbors in the triad $(RP(0,1,0))$ need not increase cooperation by itself, unless the lattice has at least several triads of cooperators. As it was already mention, the system has a 10% of triads but the probability of a triad with three cooperators is of 1/8, this meaning that the more abundant triads are those mixing cooperators and defectors. These result matches that of Eshel, Samuelson and Shaked (1998), who need a proportion of 60% of Altruist players to drive the system to cooperation. Furthermore, players inside the triad loose the correlation generated by clusters of cooperators in the complete lattice and then the frequency of cooperators goes down as compared to the baseline ($\Delta = 1$). In figure 10, for $T = 1.1$, the frequency of cooperators exhibits an initial decay of the cooperation level which goes up a little bit. However, for larger values of T the frequency level never goes up again and remains constant and lower than without

preferential imitation. This result stresses the influence of the global correlation generated by clusters of cooperators in the dynamics of the system.
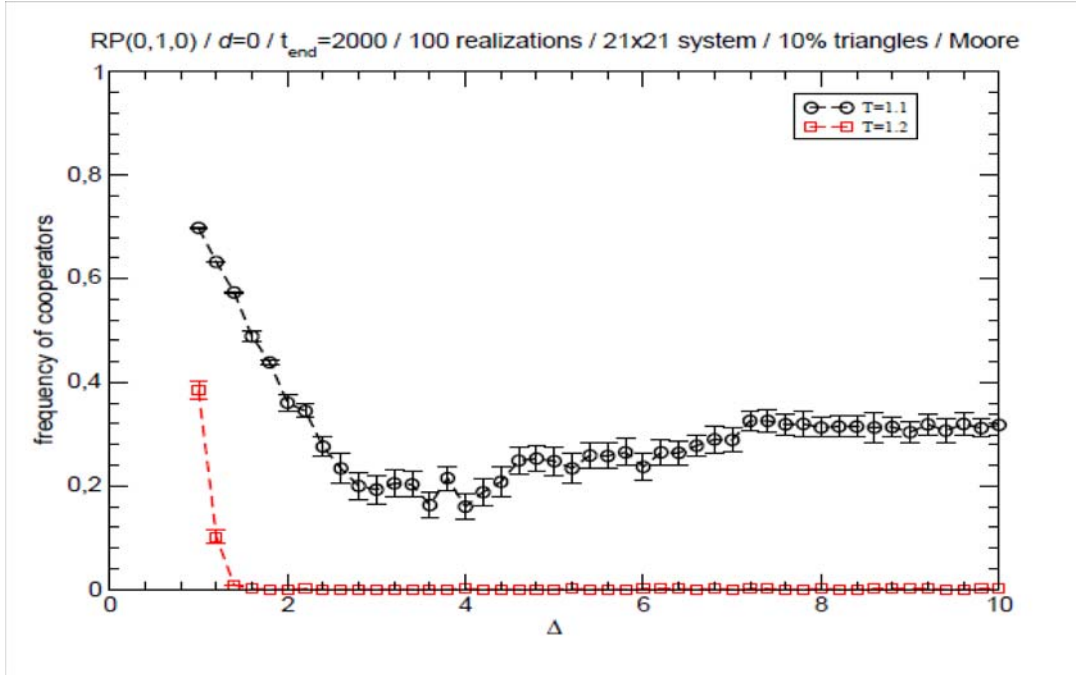

Figure 10. Imitation. RP(0,1,0)

However, payoff externalities (*RP(0,0,1)*) are vital to enhance cooperation. Figure 11 exhibits a maximum of cooperation around $\Delta = 3$, where the frequency of cooperators is maximal (around the 98%) and then it decreases a little bit. Interestingly enough, this value of Δ is very close to the critical value specified in Proposition 1 for the spread of cooperation from the minimal structures of cooperation, even though now the triads may contain both cooperators and defectors. The reason of the posterior decay in the frequency of cooperators is that with so many of them, the 2% of defectors gain a lot and some cooperator will change strategies. As Δ increases this tendency is reversed and the global frequency of cooperator stabilizes around the 90%, which is higher than without the payoff externality (70%).
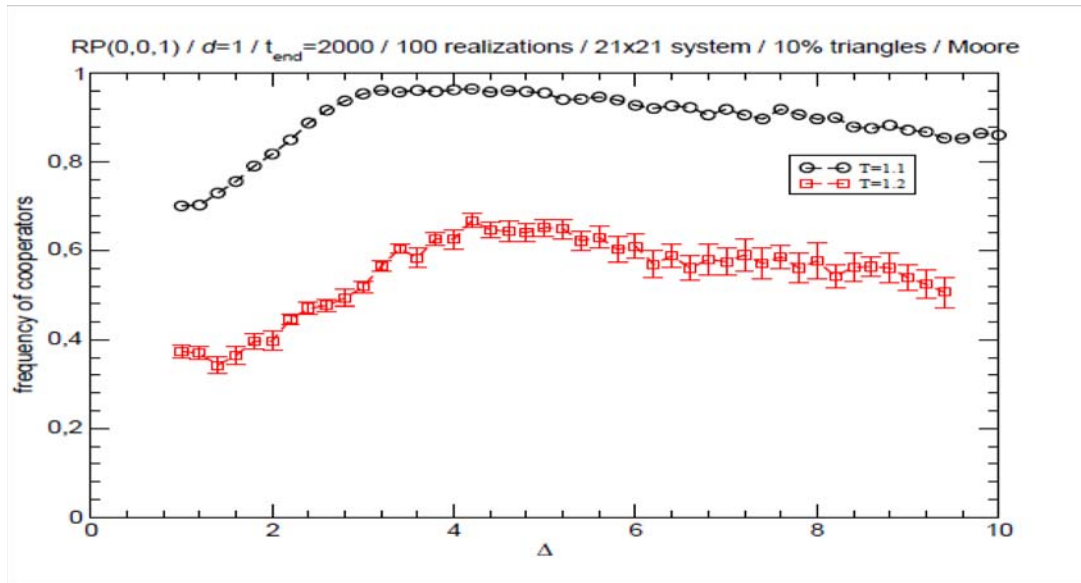
25

Figure 11. Payoff externality. RP(0,0,1)

When we combine the three effects together $(RP(1,1,1))$, we observe in figure 12 that, again, when $\Delta \geq 2.4$ cooperation is fully achieved[10], for $T = 1.1$, and when $\Delta \geq 4.2$, for $T = 1.2$. The figure also shows that for low values of Δ ($\Delta \leq 1.6$) the imitation effect dominates the dynamics and accordingly cooperation decays. This is easily explained by proposition 1, since any triad of cooperators needs at least a $\Delta = 1.65$, for $T = 1.1$, and a $\Delta = 1.8$, for $T = 1.2$, to survive. As Δ increases the payoff externality starts offsetting the imitation effect and for Δ big enough the system converge to full cooperation.

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[10] Notice again that the triads may contain both cooperators and defectors.
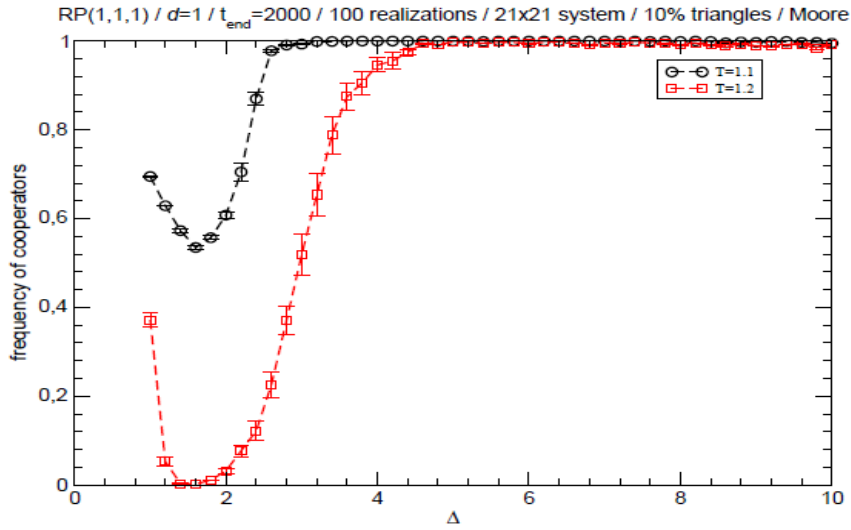
26

Figure 12. All the effects together. RP(1,1,1)

6   Conclusions

This papers has offered a modeling of "group connectivity" by proposing a generalization of the concept of a graph. The new approach captures not only binary relations between agents in a network but also high-order relations among subsets of them.

This approach enables to characterize the minimal structures of cooperation survival in a Spatial Prisoners' Dilemma in a Moore network and may help explain the existence of persistent "islands of cooperation" in hostile environments, as long as the members of such groups have a strong inner cohesion. Thus, group connectivity may explain the coexistence of different and small ethnic groups in the midst of neighborhoods, i.e., lumpiness and spatial concentration phenomena.

The term "population viscosity" was coined by Hamilton (1964). A population is viscous if individual do not move far away from their places of birth. The lim-

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ited dispersal facilitates the evolution of cooperation by increasing the degree of relatedness among interacting individuals. We show here, in contrast, that local interaction within a spatial array and group connectivity, by themselves, enable cooperative behavior to persist indefinitely. As already known, cooperators fare poorly when exposed to many defectors, while defectors fare well when exposed to cooperators. Therefore, cooperators will survive if they form clusters and hence defectors will be always in the lattice boundaries. The replicator-like dynamics, the local nature of the interactions and group connectivity are all important to the spread of cooperation. Taking agents to be in a square lattice reduce the number of defectors surrounding cooperators, but not enough to avoid the invasion from the former ones. Therefore group connectivity allows cooperators to huddle together in concentrated groups, although their members are still exposed to defectors.

However, much work remains to be done. Leaving aside the concrete applications on spatial concentration phenomena in urban economics and/or the economics of agglomerations, the analysis of multidimensional connectivity in general networks beyond Moore neighborhoods will be useful to understand more complex geometries of cooperation such as multidimensional hubs and structural holes.

Finally, in our model the preferential connection are exogenous and fixed. A natural extension of the model would need to consider the formation and coevolution of these structures under the network dynamics.

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