Finite Supermodular Design with Interdependent Valuations

Preliminary and Incomplete

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Abstract

This paper studies supermodular mechanism design in environments with finite type spaces and interdependent valuations. In such environments, it is difficult to implement social choice functions in ex-post equilibrium, hence Bayesian Nash equilibrium becomes the appropriate equilibrium concept. The requirements for agents to play a Bayesian equilibrium are strong, so we propose mechanisms that are robust to bounded rationality and help guide agents towards an equilibrium. In quasi-linear environments that allow for informational and allocative externalities we show that any mechanism that implements a social choice function can be converted into a supermodular mechanism that implements the original social choice function's decision rule. We show that the supermodular mechanism can be chosen in a way that minimizes the size of the equilibrium set and provide two sets of sufficient conditions: for general decision rules and for decision rules that satisfy a certain requirement. This is followed by conditions for supermodular implementation with a unique equilibrium.

Keywords: Implementation, mechanisms, multiple equilibrium problem, learning, strategic complementarities, supermodular games.

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1 Introduction

For the mechanism designer that wants to create contracts, taxes, or other institutions with a certain objective in mind there is a trade off between the simplicity of the mechanism and ensuring that all possible equilibria yield a desired outcome. Why would agents immediately play the right equilibrium or how could they play an equilibrium of a game they do not fully understand? The question of equilibrium play in mechanism design combines the multiple equilibrium problem and the concern about bounded rationality of players. While there are mechanisms that solve the first issue, there are hardly any that address departures from full rationality.

Supermodular mechanism design is a tool that is equipped to handle both the multiple equilibrium problem and departures from full rationality. In these mechanisms, agents' strategies are complements, meaning that an agent wants to take a higher strategy when others do the same. In view of Milgrom and Roberts (12), supermodular mechanisms have extremal equilibria, and the interval in between gives the amplitude of the multiple equilibrium problem. Using this interval, it becomes possible to minimize the multiplicity problem, to measure it, and sometimes to solve it. This paper describes how to build supermodular mechanisms where this interval, and so the multiplicity problem, are minimized in an environment with finite type spaces and interdependent valuations.

Supermodular mechanisms are robust to boundedly rational behaviors. The interval between the extremal equilibria contains all the iteratively undominated strategy profiles, and all the limit points of adaptive and sophisticated learning dynamics (Milgrom and Roberts (12) and (13)).[1] These theoretical properties are corroborated by strong experimental evidence, showing how convergence to the equilibrium is significantly better for supermodular games (Chen and Gazzale (3), Healy (7), Chen and Plott (1) and Chen and Tang (2)). As such, supermodular mechanisms have very desirable properties in terms of convergence and learning when agents are not at equilibrium. There are many examples where supermodular mechanisms could be used to approach an objective through iterations: A principal designing supermodular contracts to approach revenue maximization, a government applying a supermodular tax system to approach the efficient public goods level, the traffic authorities setting up toll-systems (Sandholm (14) and (15)) to minimize congestion, etc.

Supermodular games are also attractive in an implementation framework, because their mixed strategy equilibria are locally unstable under monotone adaptive dynamics, such as Cournot dynamics and fictitious play (Echenique and Edlin (6)). Ruling out mixed strategy equilibria is common in implementation theory and often arbitrary; but

[1] Vives (16) reports a related result for learning à la Cournot.

it is sensible in supermodular implementation. To the contrary, many pure-strategy equilibria are stable. In a parameterized supermodular game, all equilibria that are increasing in the parameter are stable, such as the extremal equilibria (Echenique (5)).

In environments with finite type spaces and interdependent valuations it is difficult to implement social choice functions in ex-post equilibrium, hence Bayesian Nash equilibrium becomes the appropriate equilibrium concept. The requirements for agents to play a Bayesian equilibrium are strong. Thus, the good learning and convergence properties of supermodular mechanisms are especially valuable in this context, as they ensure robustness to bounded rationality and help guide agents towards an equilibrium.

A mechanism for which truthful revelation is ex post incentive compatible is robust to relaxations of the assumption that agents' information is exogenous. Agents have incentives to try to learn about the opponents' private information only if there are some profiles of opponents' types for which it is optimal to misreport one's own type, i.e. when truthtelling is not an ex post Nash equilibrium. While being a desirable property, ex post incentive compatibility is rarely a feasible one. In particular, in the case of interdependent valuations, when each agent's valuation depends not only on his private information but also on the private information of his opponents, truthful revelation is ex post incentive compatible only under very restrictive assumptions (Cremer and McLean (4)).

Most of the literature on mechanisms for which truthtelling is ex post incentive compatible focuses on the case of single dimensional private information (see McLean and Postlewaite (9) for a good summary of the literature). Jehiel et al. (8) indeed show that this assumption is necessary for truthful revelation to be an ex post equilibrium. Multidimensional private information in the context of interdependent valuations makes ex post incentive compatibility attainable only for trivial outcome functions.Therefore, in interdependent value environments, there is hope for truthtelling ex post equilibria only in the case of one-dimensional private signals, which is a substantial restriction in many settings. Since only trivial allocation rules that choose the same alternative irrespective of agents' signals (announcements) are ex post implementable in generic mechanism design settings with multidimensional signals and interdependent valuations, Bayesian implementation plays a very important role in these environments.

In quasi-linear environments we apply the theory of supermodular mechanism design to the case of finite type spaces and interdependent valuations. In this setting many social choice functions can be implemented with a supermodular mechanism without any additional assumptions on the valuation functions. The result is established by turning an existing mechanism into one that induces a supermodular game by adding a function to each agent's transfer, as proposed by Mathevet (11). We show that the supermodular mechanism can be chosen in a way that minimizes the size of the

equilibrium set and provide two sets of sufficient conditions: for general decision rules and for decision rules that satisfy a certain requirement. This is followed by conditions for supermodular implementation with a unique equilibrium.

2 Finite Supermodular Design: The Framework

Consider n agents, each endowed with quasilinear preferences over a set of alternatives. The set of players will be denoted by N. An alternative is a vector $(x,t) = (x_1,\ldots,x_n,t_1,\ldots,t_n)$, where x_i is an element of a compact set $X_i \subset \mathbb{R}$ and $t_i \in \mathbb{R}$ for all $i \in N$. In this environment, x_i is interpreted as agent i's allocation and t_i is the money transfer i receives. Each agent i has a finite type space Θ_i and information is incomplete. There is a common prior with density ϕ on Θ known to the mechanism designer. Types are assumed to be independently distributed, and ϕ has full support. Each agent i's preferences over alternatives are represented by a bounded utility function $u_i(x,t_i,\theta) = V_i(x,\theta) + t_i$, where $V_i : X_i \times \Theta_i \to \mathbb{R}$ is referred to as i's valuation. This formulation allows for allocational externalities, as V_i depends on all dimensions of the allocation x, rather than just on x_i. It also captures the case of informational externalities (interdependent valuations) since the valuation function of agent i depends not only on his own type θ_i, but also on the true types of his opponents, which are not observed by i.

A mechanism designer wishes to implement an allocation for each realization of types. This objective is represented by a decision rule $x : \Theta \mapsto (x_i(\theta))_i$. To this end, the designer sets up a transfer scheme $t_i : \Theta \to \mathbb{R}$ for each i. A mechanism is denoted by $\Gamma = (\{\Theta_i\},(x,t))$ and it describes the strategic situation into which agents are put. Agents are asked to announce a type, and from the vector of announced types, an allocation and a transfer accrue to each agent.[2] The pair $f = (x,t)$ is called a social choice function. We adopt the conventional notation where $\hat{\theta}_i$ is i's announced type, $\hat{\theta}_{-i}$ is the announced types of i's opponents, and $\hat{\theta}$ denotes the announced types of all players. The (ex-post) utility function of player i in Γ is $u_i^\Gamma(\hat{\theta},\theta) = V_i(x(\hat{\theta}),!a) + t_i(\hat{\theta})$. A pure strategy for agent i under incomplete information is a function $\hat{\theta}_i : \Theta_i \to \Theta_i$ that maps true types into announced types. Strategy $\hat{\theta}_i(\cdot)$ is called a deception. Agent i's (ex-ante) utility function in Γ is $U_i^\Gamma(\hat{\theta}_i(.),\hat{\theta}_{-i}(.)) = E_\theta[u^\Gamma(\hat{\theta}(\theta),\theta)]$.

This paper is concerned with supermodular mechanisms. To define these mechanisms, several definitions are in order. Given two partially ordered sets Y and Z, a function $g : Y \times Z \to \mathbb{R}$ such that $g : (y,z) \mapsto g(y,z)$ has increasing (decreasing) differences in (y,z) if, whenever $y \geq y'$ and $z \geq z'$, $g(y,z) - g(y',z) \geq (\leq) g(y,z') - g(y',z')$; g satisfies the single-crossing property in (y,z) if, whenever $y \geq y'$ and $z \geq z'$, $g(y,z') \geq$

[2] Most of the paper is concerned with direct mechanisms.

$g(y', z')$ implies $g(y, z) \geq g(y', z)$ and $g(y, z') > g(y', z')$ implies $g(y, z) > g(y', z)$. If g has decreasing differences in (y, z), then variables y and z are said to be substitutes. If g has increasing differences or satisfies the single-crossing property in (y, z), then y and z are said to be complements.

A game is a tuple $(N, \{S_i, u_i\})$ where N is the finite set of players; each $i \in N$ has a strategy space S_i, and a payoff function $u_i : \prod_{i \in N} S_i \to \mathbb{R}$. Generic element of S_i are denoted s_i, and $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. Subsets of the real line are endowed with the Euclidean topology.

Definition 1 *A game $\mathcal{G} = (N, \{S_i, u_i\})$ is supermodular if for all $i \in N$,*

1. *S_i is compact;*
2. *u_i is bounded, and has increasing differences in (s_i, s_{-i});*
3. *u_i is upper-semicontinuous in s_i for each s_{-i}, and continuous in s_{-i} for each s_i.*

There are three stages at which it is relevant to formulate the game induced by mechanism Γ: Ex-ante, interim and ex-post (complete information). Let $\mathcal{G}(\theta) = (N, \{\Theta_i, u_i^\Gamma(\cdot, \theta)\})$ be the game induced by mechanism Γ ex-post. Let $\mathcal{G} = (N, \{\Theta_i^{\Theta_i}, U_i^\Gamma\})$ be the ex-ante Bayesian game induced by Γ. Among these three formulations of the game, the paper considers supermodularity at the ex-post level, because this is the strongest requirement. If the ex-post game is always supermodular, then the game will be supermodular in its ex-ante and interim formulations.

Definition 2 *A social choice function $f = (x, t)$ is (truthfully) supermodular implementable if truthtelling, i.e. $\hat{\theta}_i(\theta_i) = \theta_i$ for all i, is a Bayesian equilibrium of $\mathcal{G}$ and if $\mathcal{G}(\theta)$ is supermodular for each θ.*

For all $i \in N$ let $>_i = (>_i^1, >_i^2)$ be a pair of ordering relations such that $>_i^1$ generates a complete order on Θ_i and $>_i^2$ completely orders $\Theta_{-i} = \times_{j \neq i} \Theta_j$. The product order on Θ_{-i}, obtained from the orders $>_j^1$, $j \neq i$, is denoted by $>_{-i}$. Notice that complete orders exist for all sets Θ_i, because types are finite.

Definition 3 *An order $>_i^2$ is said to be consistent with the product order $>_{-i}$ on Θ_{-i} if whenever θ'_{-i} and θ''_{-i} are ordered under the product order $>_{-i}$, they are ordered in the same way under $>_i^2$.*

Note that while $>_i^2$ is completely consistent with the product order, it also orders elements that are unordered under the product order.

Definition 4 *A profile of orders $(>^1, >^2) = (>_i^1, >_i^2)_{i \in N}$ is consistent if for all i, $>_i^2$ is consistent with the product order $>_{-i}$ on Θ_{-i}.*

3 Minimal Supermodular Implementation

3.1 A General Result

The starting point of the analysis is the class of supermodular implementable social choice functions. Corollary 1 in Mathevet (2010) states that when type spaces are finite, for any valuation functions, if the social choice function $f = (x, t)$ is implementable, then there exist transfers t^{SM} such that (x, t^{SM}) is also supermodular implementable.

Therefore, the class of supermodular implementable social choice functions is the same as the class of implementable social choice functions. Transfers are pivotal to this result: It is always possible to add complementarities through the transfers without affecting the interim expected utility in equilibrium. It is also worthwhile pointing out that if the initial social choice function satisfies ex ante or interim participation constraints, then so does (x, t^{SM}).

Definition 5 *Define the ordering relation $\succeq_{ID}$ on the space of transfer functions such that $\tilde{t} \succeq_{ID} t$ if for all $i \in N$ and for all $\theta_i'' >_i^1 \theta_i'$ and $\theta_{-i}'' >_{-i} \theta_{-i}'$, $\tilde{t}_i(\theta_i'', \theta_{-i}'') - \tilde{t}_i(\theta_i'', \theta_{-i}') - \tilde{t}_i(\theta_i', \theta_{-i}'') + \tilde{t}_i(\theta_i', \theta_{-i}') \geq t_i(\theta_i'', \theta_{-i}'') - t_i(\theta_i'', \theta_{-i}') - t_i(\theta_i', \theta_{-i}'') + t_i(\theta_i', \theta_{-i}')$.*

Note that while the relation $\succeq_{\text{ID}}$ is transitive and reflexive, it is not antisymmetric. Denote the set of $\succeq_{\text{ID}}$ equivalence classes of transfers by $\mathcal{T}$.

The next proposition is equivalent to Proposition 2 in Mathevet (2010). It provides the basis for the definition of minimal implementation. It shows that if a transfer function t'' generates more complementarities than a transfer function t', and both induce a supermodular game, then the game induced by t'' has a larger interval prediction than the interval prediction of the game induced my t'. This implies that the objective of minimizing the equilibrium set coincides with the objective of minimizing the complementarities introduced by the transfers, while maintaining the supermodularity of the game.

For any $t \in \mathcal{T}$ and supermodular implementable social choice function $f = (x, t)$, let $\bar{\theta}^t(\cdot)$ and $\underline{\theta}^t(\cdot)$ denote the extremal Bayesian equilibria of the game induced by the mechanism.

Proposition 1 *For any supermodular implementable social choice functions (x, t'') and (x, t'), if $t'' \succeq_{ID} t'$, then $[\underline{\theta}^{t'}(\cdot), \bar{\theta}^{t'}(\cdot)] \subset [\underline{\theta}^{t''}(\cdot), \bar{\theta}^{t''}(\cdot)]$.*

Since the equilibrium interval grows with the complementarities of the transfers, a social choice function $f = (x, t^*)$ will be minimally supermodular implementable if the transfers t^* generate the weakest possible complementarities while ensuring supermodular implementation. This gives the tightest equilibrium prediction around the truthful equilibrium.

3.2 Minimal Implementation under a Given Order

Supermodularity is defined on the product order $>_{-i}$ for every $i \in N$. However, in order to define minimal implementation, we have to define families of transfers within $\mathcal{T}$ that are comparable on a more refined level than supermodular implementation on $>_{-i}$. In particular, we look at families of $t \in \mathcal{T}$ that ensure supermodularity on the complete order $>_i^2$ for every $i \in N$.

Definition 6 *Let $\mathcal{F}(x, >^2)$ be the family of transfer functions t such that $t \in \mathcal{F}(x, >^2)$ if (x, t) is supermodular implementable and $(>^1, >^2)$ is consistent.*

For ease of exposition we will suppress the dependence of $\mathcal{F}$ on x and will instead write $\mathcal{F}(>^2)$ in subsequent uses of the notation.

Definition 7 *A social choice function $f = (x, t^*)$ is minimally supermodular implementable over a family $\mathcal{F}(>^2)$ if it is supermodular implementable and $t \succeq_{ID} t^*$ for all transfers $t \in \mathcal{F}(>^2)$.*

The following theorem establishes the minimal supermodular implementability result under a specific profile of complete orders.

Theorem 1 *Assume a consistent profile of orders $(>^1, >^2)$. If $f = (x, t)$ is implementable, then there exist t^* such that (x, t^*) is minimally supermodular implementable over a family $\mathcal{F}(>^2)$.*

The conclusion of the theorem is quite powerful. It says that any implementable social choice function can be minimally supermodular implemented. Notice that there are no further restrictions on the decision rule x. There are many possible transfers that can transform an implementable mechanism into a supermodular implementable mechanism for a given profile of orders. Among these, the transfers t^* are the best possible transfers in terms of minimizing the equilibrium set such that (x, t^*) is supermodular implementable.

Since there are finitely many types, there are also finitely many (consistent) profiles of orders $(>^1, >^2)$. The following corollary states that there exist transfers that give the smallest equilibrium set among the class of transfers that minimally supermodular implement a decision rule x. Therefore, those are the transfers that give the smallest equilibrium set over all transfers that supermodular implement a social choice function $f = (x, t)$.

Corollary 1 *If $f = (x, t)$ is implementable, then there exist t^{**} and an order profile $>^*$ such that (x, t^{**}) is minimally supermodular implementable under $>^*$ and gives the smallest equilibrium set among all minimally supermodular implementable (x, t^*).*

3.3 Minimal Implementation with Order Reducibility

So far no restrictions have been imposed on the decision rule x of the implementable social choice function $f = (x, t)$, which constitutes the starting point of most of the results in this paper. In the previous section we required that supermodularity be satisfied over a complete order of the opponents' type space. Finding the overall best transfers called for the consideration of many different consistent orders and many "smallest boxes", so that the t that ensures the overall smallest box could be chosen.

This section looks at which decision rules can be minimally supermodular implemented without imposing any particular complete order on the opponents' type space Θ_{-i}, except for the usual product order. In environments with finite type spaces, all implementable social choice functions whose decision rule satisfies a certain condition on the are minimally supermodular implementable. This condition, which we refer to as *"order reducibility"* is defined as follows.

Definition 8 *A decision rule $x : \Theta \to (x_i(\theta))$ is order reducible if, for each $i \in N$ there are increasing functions $r_i : \Theta_{-i} \to R_i$ such that R_i is completely ordered, $x_i(\theta) = x_i(\theta_i, r_i(\theta_{-i}))$, and if r_i'' is the successor of r_i' in R_i, then $r_i'' = r_i(\theta_i'')$ and $r_i' = r_i(\theta_i')$ where θ_{-i}'' is a successor of θ_{-i}' in Θ_{-i}.*

This condition is trivially satisfied when there are only two players. In the case of more than two players, order reducibility requires that elements of Θ_{-i} that are not comparable between each other according to the product order $>_{-i}$ get grouped with elements that are comparable to each other according to the product order or in a group of their own that is consistent with the product order. This ensures that we have a mapping between the product order $>_{-i}$ on Θ_{-i} and the complete order on R_i that is consistent with the product order. However, rather than completely ordering all the elements of Θ_{-i}, the complete order on R_i puts them into groups that are consistent with the product order, but allow for more elements to be compared.

At this point we need a definition of minimal supermodular implementability that does not rely on a complete order on Θ_{-i}.

Definition 9 *A social choice function $f = (x, t^*)$ is minimally supermodular implementable if it is supermodular implementable and $t \succeq_{ID} t^*$ for all transfers $t \in \mathcal{T}$ such that (x, t) is supermodular implementable.*

Theorem 2 *Let $f = (x, t)$ be a social choice function where x is order reducible. If f is implementable, then there exist t^* such that (x, t^*) is minimally supermodular implementable.*

4 Uniqueness

In this section we provide sufficient conditions for unique supermodular implementation, which implies that truthtelling is the only possible equilibrium. This is a natural extension of the preceding discussion on minimizing the interval prediction. In the case of a unique equilibrium, the induced game is dominance solvable and all learning dynamics converge to the equilibrium.

Before providing the results, some definitions and notational simplifications are in order. Player i's interim utility at type θ_i against θ_{-i} is denoted as

$$u_i(\hat{\theta}_i, \hat{\theta}_{-i}, \theta_i) = E_{\theta_{-i}}[V_i(x(\hat{\theta}_i, \hat{\theta}_{-i}), \theta) + t_i(\hat{\theta}_i, \hat{\theta}_{-i})].$$

Define for any $\hat{\theta}_i'' >_i^1 \hat{\theta}_i'$

$$\Delta u_i(\hat{\theta}_{-i}, \theta_i) = u_i(\hat{\theta}_i'', \hat{\theta}_{-i}, \theta_i) - u_i(\hat{\theta}_i', \hat{\theta}_{-i}, \theta_i)$$

and

$$\Delta V_i(\hat{\theta}_{-i}, \theta) = V_i(x(\hat{\theta}_i'', \hat{\theta}_{-i}, \theta) - V_i(x(\hat{\theta}_i', \hat{\theta}_{-i}, \theta).$$

Since there are finitely many types, there are numbers $K_i^j(\theta_i)$ and $\gamma_i(\theta_{-i})$ such that for all $\hat{\theta}_{-i}'' >^1 \hat{\theta}_{-i}'$

$$\Delta u_i(\hat{\theta}_{-i}'', \theta_i) - \Delta u_i(\hat{\theta}_{-i}', \theta_i) \leq d_i(\hat{\theta}_i'', \hat{\theta}_i') \sum_{j \neq i} K_i^j(\theta_i) E_{\theta_j}[d_j(\hat{\theta}_j'', \hat{\theta}_j')]$$

and for all $\theta_i'' >_i^1 \theta_i'$

$$\Delta V_i(\hat{\theta}_{-i}, \theta_i'', \theta_{-i}) - \Delta V_i(\hat{\theta}_{-i}, \theta_i', \theta_{-i}) \geq \gamma_i(\theta_{-i}) d_i(\hat{\theta}_i'', \hat{\theta}_i') d_i(\theta_i'', \theta_i').$$

Denote the truthful strategy by $\hat{\theta}_i^T(\cdot)$. Assume the prior is full support. Let

$$\varepsilon_i = \min_{\{\hat{\theta}_i(\cdot) : \hat{\theta}_i(\cdot) \neq \hat{\theta}_i^T(\cdot)\}} E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i), \theta_i)]$$

be an indicator of the gap between i's types. As we get closer to the continuous case, $\varepsilon_i \to 0$. Let

$$\Psi(\hat{\theta}_i(\cdot)) = \left\{ g_i : \Theta_i \to \Theta_i \middle| g_i(\cdot) \neq \hat{\theta}_i(\cdot) \text{ and } \hat{\theta}_i(\theta_i) >_i^1 g_i(\theta_i) \text{ whenever } \hat{\theta}_i(\theta_i) \neq g_i(\theta_i) \right\}$$

be the set of deceptions that are "smaller" than deception $\hat{\theta}_i(\cdot)$. Define the predecessor of any deception $\hat{\theta}_i(\cdot)$ as the function $\psi_{\hat{\theta}_i} \in \Psi(\hat{\theta}_i(\cdot))$ such that

$$E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i), \psi_{\hat{\theta}_i}(\theta_i))] \leq E_{\theta_i}[d_i(\hat{\theta}_i(\theta_i), g_i(\theta_i))]$$

whenever $g_i(\cdot) \in \Psi(\hat{\theta}_i(\cdot))$.

Theorem 3 *Let $f = (x, t)$ be a supermodular implementable social choice function. If for $\hat{\theta}^*(\cdot) > (or <)\ \hat{\theta}^T(\cdot)$, there exist i and θ_i such that*

$$\sum_{j \neq i} K_i^j(\theta_i) E_{\theta_j}[d_j(\theta_j^*(\theta_j), \theta_j)] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})] d_i(\psi_{\hat{\theta}_i}(\theta_i), \theta_i) < 0,$$

then $\hat{\theta}^*(\cdot)$ is not a Bayesian equilibrium.

Proposition 2 *If $(n-1)\max_{j \neq i} E_{\theta_i}[K_j^i(\theta_i)] < E_{\theta_{-i}}[\gamma_i(\theta_{-i})]$ for all player i, then there exists $\underline{\delta}$ such that for all $\delta < \underline{\delta}$ all Bayesian equilibria are at most δ-away from the truthful equilibrium.*

5 Conclusion (to be added)

6 Proofs

Proof of Proposition 1. (Following the proof to Proposition 2 in Mathevet (11)) Let (x, t'') and (x, t') be any supermodular implementable social choice functions such that $t'', t' \in \mathcal{T}$ and $t'' \succeq_{\text{ID}} t'$. Since we are in the case of supermodular implementation, the induced game $\mathcal{G}$ has a smallest and a greatest equilibrium and a truthful equilibrium in between them. Denote the truthtelling strategy by $\theta_i^T(\cdot)$, that is $\theta_i^T(\theta_i) = \theta_i$. With slight abuse of notation, let us use $\underline{\theta}_i$ and $\bar{\theta}_i$ to denote the constant strategies where agent i always announces his lowest and highest type, respectively. Define $\mathcal{G}_\ell$ to be the same game as $\mathcal{G}$ except that the strategy spaces are restricted to $[\underline{\theta}_i, \theta_i^T(\cdot)]$ for each player i. Likewise, let $\mathcal{G}_u$ be the game $\mathcal{G}$ where the strategy spaces are restricted to $[\theta_i^T(\cdot), \bar{\theta}_i]$ for every i. Since $\mathcal{G}$ is supermodular, so are the modified games $\mathcal{G}_\ell$ and $\mathcal{G}_u$, by definition. As such, $\mathcal{G}_\ell$ and $\mathcal{G}_u$ each have a smallest and a largest equilibrium. In particular, $\mathcal{G}_\ell$ has truthtelling as its largest equilibrium and the same smallest equilibrium as $\mathcal{G}$. Similarly, $\mathcal{G}_u$ has truthtelling as its smallest equilibrium and the same largest equilibrium as $\mathcal{G}$. Define the ex-ante expected utility function of player i as $U_i(\hat{\theta}(\cdot), t) = E_\theta[V_i(x(\hat{\theta}(\theta)), \theta) + t_i(\hat{\theta}(\theta))]$. Below we show that (i) $U_i(\hat{\theta}(\cdot), t)$ has decreasing differences in $(\hat{\theta}_i(\cdot), t)$ in game $\mathcal{G}_\ell$; (ii) $U_i(\hat{\theta}(\cdot), t)$ has increasing differences in $(\hat{\theta}_i(\cdot), t)$ in game $\mathcal{G}_u$. Using Theorem 6 in Milgrom and Roberts (1990) this allows us to make inference about how the extremal equilibria in each modified game vary in response to change in the transfers t as compared according to $\succeq_{\text{ID}}$. However, since one extremal equilibrium in each modified game is always truthtelling, this analysis will show how the other extremal equilibrium, i.e. the untruthful one, varies with changes in t evaluated on $\succeq_{\text{ID}}$.

Before proceeding with the proofs of (i) and (ii), note that all transfers t_i such that (x, t) is implementable have the same expected value $E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ up to a constant.[3] Therefore, if (x, t'') and (x, t') are both implementable social choice functions, it must be that $E_{\theta_{-i}}[t_i''(\hat{\theta}_i, \theta_{-i})] = E_{\theta_{-i}}[t_i'(\hat{\theta}_i, \theta_{-i})]$.

Consider the game $\mathcal{G}_\ell$. Choose any deceptions $\theta_i''(\cdot) > \theta_i'(\cdot)$, both smaller than θ_i, and any $\hat{\theta}_{-i}(\cdot)$ such that $\hat{\theta}_j(\theta_j) \leq \theta_j$ for all θ_j and $j \neq i$, so that we are in $\mathcal{G}_\ell$. First, we want to show that $U_i(\hat{\theta}(\cdot), t)$ has decreasing differences in $(\hat{\theta}_i(\cdot), t)$ in game $\mathcal{G}_\ell$. Since $t'' \succeq_{\text{ID}} t'$, we have

$$\begin{aligned} &t_i''(\theta_i''(\theta_i), \theta_{-i}) - t_i''(\theta_i''(\theta_i), \hat{\theta}_{-i}(\theta_{-i})) - t_i''(\theta_i'(\theta_i), \theta_{-i}) + t_i''(\theta_i'(\theta_i), \hat{\theta}_{-i}(\theta_{-i})) \\ &- t_i'(\theta_i''(\theta_i), \theta_{-i}) + t_i'(\theta_i''(\theta_i), \hat{\theta}_{-i}(\theta_{-i})) + t_i'(\theta_i'(\theta_i), \theta_{-i}) - t_i'(\theta_i'(\theta_i), \hat{\theta}_{-i}(\theta_{-i})) \geq 0. \quad (6.1) \end{aligned}$$

Taking expectations over θ_{-i} and using that $E_{\theta_{-i}}[t_i''(\hat{\theta}_i(\theta_i), \theta_{-i})] = E_{\theta_{-i}}[t_i'(\hat{\theta}_i(\theta_i), \theta_{-i})]$

[3] See Proposition 23.D.2 in (10).

we get

$$E_{\theta_{-i}}[t_i'(\theta_i''(\theta_i),\hat{\theta}_{-i}(\theta_{-i}))] - E_{\theta_{-i}}[t_i'(\theta_i'(\theta_i),\hat{\theta}_{-i}(\theta_{-i}))] \\ - E_{\theta_{-i}}[t_i''(\theta_i''(\theta_i),\hat{\theta}_{-i}(\theta_{-i}))] + E_{\theta_{-i}}[t_i''(\theta_i'(\theta_i),\hat{\theta}_{-i}(\theta_{-i}))] \geq 0. \quad (6.2)$$

Let us define the interim utility function of player i type θ_i against deception $\hat{\theta}_{-i}(\cdot)$ as

$$u_i(\hat{\theta}_i(\theta_i),\hat{\theta}_{-i}(\cdot),\theta_i,t) = E_{\theta_{-i}}[V_i(x(\hat{\theta}_i(\theta_i),\hat{\theta}_{-i}(\theta_{-i})),\theta) + t_i(\hat{\theta}_i(\theta_i),\hat{\theta}_{-i}(\theta_{-i}))]. \quad (6.3)$$

Adding and subtracting $E_{\theta_{-i}}[V_i(x(\theta_i'(\theta_i),\hat{\theta}_{-i}(\theta_{-i})),\theta)] + E_{\theta_{-i}}[V_i(x(\theta_i''(\theta_i),\hat{\theta}_{-i}(\theta_{-i})),\theta)]$ from the above inequality does not change anything and allows us to regroup terms and arrive at

$$u_i(\hat{\theta}_i''(\theta_i),\hat{\theta}_{-i}(\cdot),\theta_i,t'') - u_i(\hat{\theta}_i''(\theta_i),\hat{\theta}_{-i}(\cdot),\theta_i,t') \\ - u_i(\hat{\theta}_i'(\theta_i),\hat{\theta}_{-i}(\cdot),\theta_i,t'') + u_i(\hat{\theta}_i'(\theta_i),\hat{\theta}_{-i}(\cdot),\theta_i,t') \leq 0. \quad (6.4)$$

Thus, the interim utility function $u_i(\hat{\theta}_i(\theta_i),\hat{\theta}_{-i}(\cdot),\theta_i,t)$ has decreasing differences in $(\hat{\theta}_i(\cdot),t)$. Taking expectations over θ_i of the last inequality we obtain

$$U_i(\hat{\theta}_i''(\cdot),\hat{\theta}_{-i}(\cdot),t'') - U_i(\hat{\theta}_i''(\cdot),\hat{\theta}_{-i}(\cdot),t') - U_i(\hat{\theta}_i'(\cdot),\hat{\theta}_{-i}(\cdot),t'') + U_i(\hat{\theta}_i'(\cdot),\hat{\theta}_{-i}(\cdot),t') \leq 0. \quad (6.5)$$

That is, $U_i(\hat{\theta}(\cdot),t)$ has decreasing differences in $(\hat{\theta}_i(\cdot),t)$ for all $\hat{\theta}_{-i}(\cdot)$ and i. It follows from Theorem 6 in Milgrom and Roberts (1990) that the smallest equilibrium in $\mathcal{G}_\ell$ is decreasing in t. An analogous argument applies for the largest equilibrium in $\mathcal{G}_u$. To prove that we look at deceptions $\theta_i''(\cdot) > \theta_i'(\cdot) > \theta_i$, and any $\hat{\theta}_{-i}(\cdot)$ such that $\hat{\theta}_j(\theta_j) \geq \theta_j$ for all θ_j and $j \neq i$, so that we are in $\mathcal{G}_u$.As a result the sign in 6.1 and, ultimately, in 6.5 is reversed, which implies that $U_i(\hat{\theta}(\cdot),t)$ has increasing differences in $(\hat{\theta}_i(\cdot),t)$ for all $\hat{\theta}_{-i}(\cdot)$ and i. Therefore, the greatest equilibrium in $\mathcal{G}_u$ is increasing in t. Q.E.D

Proof of Theorem 1 Take a consistent profile of orders $(>^1,>^2)$. Under this profile of orders,for every $i \in N$, each element $\theta_i \in \Theta_i$ is assigned an index k according to the complete order $>_i^1$ and each element $\tau \in \Theta_{-i}$ is assigned an index q according to the complete order order $>_i^2$. Suppose that $f = (x,t)$ is implementable. Letting

$$\delta_i(\hat{\theta}_k,\hat{\tau}_q) = -\sum_{l=1}^{k-1}\sum_{z=1}^{q-1}\min_{\theta\in\Theta}[V_i(x(\hat{\theta}_{l+1},\hat{\tau}_{z+1}),\theta) - V_i(x(\hat{\theta}_l,\hat{\tau}_{z+1}),\theta) \\ - V_i(x(\hat{\theta}_{l+1},\hat{\tau}_z),\theta) + V_i(x(\hat{\theta}_l,\hat{\tau}_z),\theta)]. \quad (6.6)$$

for all $\hat{\theta}_k \in \Theta_i$ and $\hat{\tau}_q \in \times_{j\neq i}\Theta_j$, we define

$$t_i^*(\hat{\theta}_k,\hat{\tau}_q) = \delta_i(\hat{\theta}_k,\hat{\tau}_q) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_k,\theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_k,\theta_{-i})] \quad (6.7)$$

and show that (x,t^*) is minimally supermodular implementable.

Step 1. We show that t_i^* has smaller one-step supermodularity than any t_i such that (x,t) is supermodular implementable.

Let us define the one-step supermodularity of $V_i(x(\cdot),\theta)$ at any given announcement $(\hat{\theta}_k,\hat{\tau}_q)$ as

$$g_i(k,q;\theta) = V_i(x(\hat{\theta}_{k+1},\hat{\tau}_{q+1}),\theta) - V_i(x(\hat{\theta}_k,\hat{\tau}_{q+1}),\theta) - V_i(x(\hat{\theta}_{k+1},\hat{\tau}_q),\theta) + V_i(x(\hat{\theta}_k,\hat{\tau}_q),\theta). \quad (6.8)$$

For notational simplicity, we define

$$\begin{aligned} d_i(l,z) &= \min_{\theta\in\Theta}[V_i(x(\hat{\theta}_{l+1},\hat{\tau}_{z+1}),\theta) - V_i(x(\hat{\theta}_l,\hat{\tau}_{z+1}),\theta) \\ &- V_i(x(\hat{\theta}_{l+1},\hat{\tau}_z),\theta) + V_i(x(\hat{\theta}_l,\hat{\tau}_z),\theta)] \\ &= \min_{\theta\in\Theta} g_i(l,z;\theta). \end{aligned} \quad (6.9)$$

Since the one-step supermodularity of t_i^* is equivalent to the one-step supermodularity of δ_i we have

$$\begin{aligned} s_i(k,q) &= \delta_i(\hat{\theta}_{k+1},\hat{\tau}_{q+1}) - \delta_i(\hat{\theta}_k,\hat{\tau}_{q+1}) - \delta_i(\hat{\theta}_{k+1},\hat{\tau}_q) + \delta_i(\hat{\theta}_k,\hat{\tau}_q) \\ &= -\sum_{l=1}^{k}\sum_{z=1}^{q} d_i(l,z) + \sum_{l=1}^{k-1}\sum_{z=1}^{q} d_i(l,z) + \sum_{l=1}^{k}\sum_{z=1}^{q-1} d_i(l,z) - \sum_{l=1}^{k-1}\sum_{z=1}^{q-1} d_i(l,z) \\ &= -d_i(k,q) \end{aligned} \quad (6.10)$$

as the one-step supermodularity of t_i^* (and δ_i).

Therefore, the one-step supermodularity of $(V_i + t_i^*)$ is given by

$$\begin{aligned} g_i(k,q;\theta) + s_i(k,q) &= V_i(x(\hat{\theta}_{k+1},\hat{\tau}_{q+1}),\theta) - V_i(x(\hat{\theta}_k,\hat{\tau}_{q+1}),\theta) - V_i(x(\hat{\theta}_{k+1},\hat{\tau}_q),\theta) \\ &+V_i(x(\hat{\theta}_k,\hat{\tau}_q),\theta) - \min_{\theta\in\Theta}[V_i(x(\hat{\theta}_{k+1},\hat{\tau}_{q+1}),\theta) - V_i(x(\hat{\theta}_k,\hat{\tau}_{q+1}),\theta) \\ &-V_i(x(\hat{\theta}_{k+1},\hat{\tau}_q),\theta) + V_i(x(\hat{\theta}_k,\hat{\tau}_q),\theta)] \geq 0 \end{aligned} \quad (6.11)$$

for all $\hat{\theta}_k$, $\hat{\tau}_q$, θ_i, k, q, and i.

Denote the one-step supermodularity of transfer t_i as $sm_1(t_i;k,q)$, that is:

$$sm_1(t_i;k,q) = t_i(\hat{\theta}_{k+1},\hat{\tau}_{q+1}) - t_i(\hat{\theta}_k,\hat{\tau}_{q+1}) - t_i(\hat{\theta}_{k+1},\hat{\tau}_q) + t_i(\hat{\theta}_k,\hat{\tau}_q).$$

For all transfers t such that (x,t) is supermodular implementable, it must hold that $g_i(k,q;\theta) + sm_1(t_i;k,q) \geq 0$ for all $\theta\in\Theta$, which is equivalent to:

$$sm_1(t_i;k,q) \geq -\min_{\theta\in\Theta}[V_i(x(\hat{\theta}_{k+1},\hat{\tau}_{q+1}),\theta) - V_i(x(\hat{\theta}_k,\hat{\tau}_{q+1}),\theta) - V_i(x(\hat{\theta}_{k+1},\hat{\tau}_q),\theta) + V_i(x(\hat{\theta}_k,\hat{\tau}_q),\theta)] = s_i(k,q). \quad (6.12)$$

The above shows that if (x,t) is supermodular implementable then the one-step supermodularity of transfers t is necessarily greater than the one-step supermodularity of transfers t^*, which establishes Step 1.

Step 2. We show that the (multiple-step) supermodularity of any function of two variables is a sum of one-step supermodularities. Let us define the "(η,γ)-step supermodularity" of any function $t_i(\hat{\theta}_k,\hat{\tau}_q)$ of as

$$SM_{(\eta,\gamma)}(t_i;k,q) \quad = \quad t_i(\hat{\theta}_{k+\eta},\hat{\tau}_{q+\gamma}) \;-\; t_i(\hat{\theta}_k,\hat{\tau}_{q+\gamma}) \;-\; t_i(\hat{\theta}_{k+\eta},\hat{\tau}_q) \;+\; t_i(\hat{\theta}_k,\hat{\tau}_q). \quad (6.13)$$

Note that

$$t_i(\hat{\theta}_{k+\eta},\hat{\tau}_{q+\gamma}) = sm_1(t_i;k+\eta-1,q+\gamma-1) + t_i(\hat{\theta}_{k+\eta-1},\hat{\tau}_{q+\gamma}) + t_i(\hat{\theta}_{k+\eta},\hat{\tau}_{q+\gamma-1}) - t_i(\hat{\theta}_{k+\eta-1},\hat{\tau}_{q+\gamma-1}), \quad (6.14)$$

and so it follows from (6.13) that

$$SM_{(\eta,\gamma)}(t_i;k,q;) = \big[sm_1(t_i;k+\eta-1,q+\gamma-1) + t_i(\hat{\theta}_{k+\eta-1},\hat{\tau}_{q+\gamma}) + t_i(\hat{\theta}_{k+\eta},\hat{\tau}_{q+\gamma-1}) - t_i(\hat{\theta}_{k+\eta-1},\hat{\tau}_{q+\gamma-1})\big] - t_i(\hat{\theta}_k,\hat{\tau}_{q+\gamma}) - t_i(\hat{\theta}_{k+\eta},\hat{\tau}_q) + t_i(\hat{\theta}_k,\hat{\tau}_q)). \quad (6.15)$$

Note that

$$t_i(\hat{\theta}_{k+\eta-1},\hat{\tau}_{q+\gamma}) = sm_1(t_i;k+\eta-2,q+\gamma-1) + t_i(\hat{\theta}_{k+\eta-2},\hat{\tau}_{q+\gamma}) + t_i(\hat{\theta}_{k+\eta-1},\hat{\tau}_{q+\gamma-1}) - t_i(\hat{\theta}_{k+\eta-2},\hat{\tau}_{q+\gamma-1}), \quad (6.16)$$

and therefore it follows from (6.15) that

$$\begin{aligned}
&SM_{(\eta,\gamma)}(t_i;k,q) = sm_1(t_i;k+\eta-1,q+\gamma-1) \\
&+ \big[sm_1(t_i;k+\eta-2,g+\gamma-1) + t_i(\hat{\theta}_{k+\eta-2},\hat{\tau}_{q+\gamma}) + t_i(\hat{\theta}_{k+\eta-1},\hat{\tau}_{q+\gamma-1}) - t_i(\hat{\theta}_{k+\eta-2},\hat{\tau}_{q+\gamma-1})\big] \\
&+ t_i(\hat{\theta}_{k+\eta},\hat{\tau}_{q+\gamma-1}) - t_i(\hat{\theta}_{k+\eta-1},\hat{\tau}_{q+\gamma-1}) - t_i(\hat{\theta}_k,\hat{\tau}_{q+\gamma}) - t_i(\hat{\theta}_{k+\eta},\hat{\tau}_q) + t_i(\hat{\theta}_k,\hat{\tau}_q) \\
&= \sum_{n=1}^{2} sm_1(t_i;k+\eta-n,q+\gamma-1) + t_i(\hat{\theta}_{k+\eta-2},\hat{\tau}_{q+\gamma}) - t_i(\hat{\theta}_{k+\eta-2},\hat{\tau}_{q+\gamma-1}) \\
&+ t_i(\hat{\theta}_{k+\eta},\hat{\tau}_{q+\gamma-1}) - t_i(\hat{\theta}_k,\hat{\tau}_{q+\gamma}) - t_i(\hat{\theta}_{k+\eta},\hat{\tau}_q) + t_i(\hat{\theta}_k,\hat{\tau}_q). \qquad (6.17)
\end{aligned}$$

Proceeding iteratively with this process of substitution and regrouping of terms for

$n = (1, \ldots, \eta)$ we obtain

$$\begin{aligned} SM_{(\eta,\gamma)}(t_i; k, q) &= \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q+\gamma-1) + t_i(\hat{\theta}_k, \hat{\tau}_{q+\gamma}) - t_i(\hat{\theta}_k, \hat{\tau}_{q+\gamma-1}) \\ &+ t_i(\hat{\theta}_{k+\eta}, \hat{\tau}_{q+\gamma-1}) - t_i(\hat{\theta}_k, \hat{\tau}_{q+\gamma}) - t_i(\hat{\theta}_{k+\eta}, \hat{\tau}_q) + t_i(\hat{\theta}_k, \hat{\tau}_q) \\ &= \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q+\gamma-1) \\ &+ t_i(\hat{\theta}_{k+\eta}, \hat{\tau}_{q+\gamma-1}) - t_i(\hat{\theta}_k, \hat{\tau}_{q+\gamma-1}) - t_i(\hat{\theta}_{k+\eta}, \hat{\tau}_q) + t_i(\hat{\theta}_k, \hat{\tau}_q) \\ &= \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q+\gamma-1) + SM_{(\eta,\gamma-1)}(t_i; k, q). \end{aligned} \tag{6.18}$$

Iterrating on Equation (6.18) for $m = 1, \ldots, \gamma - 1$ we obtain:

$$\begin{aligned} SM_{(\eta,\gamma)}(k, q; \theta_i) &= \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q+\gamma-1) + SM_{(\eta,\gamma-1)}(t_i; k, q) \\ &= \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q+\gamma-1) + \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q+\gamma-2) + SM_{(\eta,\gamma-2)}(t_i; k, q) \\ &= \sum_{n=1}^{\eta} \sum_{m=1}^{\gamma-1} sm_1(t_i; k+\eta-n, q+\gamma-m) + SM_{(\eta,1)}(t_i; k, q). \end{aligned} \tag{6.19}$$

Now, using the fact that

$$\begin{aligned} SM_{(\eta,1)}(k, q) &= t_i(\hat{\theta}_{k+\eta}, \hat{\tau}_{q+1}) - t_i(\hat{\theta}_k, \hat{\tau}_{q+1}) - t_i(\hat{\theta}_{k+\eta}, \hat{\tau}_q) + t_i(\hat{\theta}_k, \hat{\tau}_q) \\ &= sm_1(t_i; k+\eta-1, q) + SM_{(\eta-1,1)}(t_i; k, q) \\ &= \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q) \end{aligned} \tag{6.20}$$

and plugging this into Equation (6.19) we obtain

$$\begin{aligned} SM_{(\eta,\gamma)}(t_i; k, q) &= \sum_{n=1}^{\eta} \sum_{m=1}^{\gamma-1} sm_1(t_i; k+\eta-n, q+\gamma-m) + \sum_{n=1}^{\eta} sm_1(t_i; k+\eta-n, q) \\ &= \sum_{n=1}^{\eta} \sum_{m=1}^{\gamma} sm_1(t_i; k+\eta-n, q+\gamma-m) \\ &= \sum_{l=k}^{k+\eta-1} \sum_{z=q}^{q+\gamma-1} sm_1(t_i; l, z). \end{aligned} \tag{6.21}$$

Thus, the multiple-step supermodularity of any function of two ordered variables is equal to the sum of one-step supermodularities, which establishes Step 2.

Step 3. Conclusion. Note that

$$E_{\theta_{-i}}[t_i^*(\hat{\theta}_k, \theta_{-i})] = E_{\theta_{-i}}[\delta_i(\hat{\theta}_k, \theta_{-i})] - E_{\theta_{-i}}[\delta_i(\hat{\theta}_k, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_k, \theta_{-i})] = E_{\theta_{-i}}[t_i(\hat{\theta}_k, \theta_{-i})] \tag{6.22}$$

and therefore transfers t_i and t_i^* have the same expected value given that all other agents report their types truthfully. That is, assuming truthful reporting, the expected utility of an agent is the same under t_i and t_i^*. Since (x, t) is truthfully implementable, the above discussion implies that (x, t^*) is also truthfully implementable.

Using the result eshablished in Step 2, the (η, γ)-step supermodularity of $V_i(x(\cdot), \theta)$ at any given announcement $(\hat{\theta}_k, \hat{\tau}_q)$ can now be written as:

$$\begin{aligned} G_i^{(\eta,\gamma)}(k, q; \theta) &= V_i(x(\hat{\theta}_{k+\eta}, \hat{\tau}_{q+\gamma}), \theta) - V_i(x(\hat{\theta}_k, \hat{\tau}_{q+\gamma}), \theta) \\ &\quad - V_i(x(\hat{\theta}_{k+\eta}, \hat{\tau}_q), \theta) + V_i(x(\hat{\theta}_k, \hat{\tau}_q), \theta) \\ &= \sum_{l=k}^{k+\eta-1} \sum_{z=q}^{q+\gamma-1} g_i(l, z; \theta). \end{aligned} \tag{6.23}$$

and the (η, γ)-step supermodularity of t_i^* is analogously given by

$$\begin{aligned} S_i^{(\eta,\gamma)}(k, q) &= \delta_i(\hat{\theta}_{k+\eta}, \hat{\tau}_{q+\gamma}) - \delta_i(\hat{\theta}_k, \hat{\tau}_{q+\gamma}) - \delta_i(\hat{\theta}_{k+\eta}, \hat{\tau}_q) + \delta_i(\hat{\theta}_k, \hat{\tau}_q) \\ &= -\sum_{l=1}^{k+\eta-1} \sum_{z=1}^{q+\gamma-1} d_i(l, z) + \sum_{l=1}^{k-1} \sum_{z=1}^{q+\gamma-1} d_i(l, z) + \sum_{l=1}^{k+\eta-1} \sum_{z=1}^{q-1} d_i(l, z) - \sum_{l=1}^{k-1} \sum_{z=1}^{q-1} d_i(l, z) \\ &= -\sum_{l=k}^{k+\eta-1} \sum_{z=q}^{q+\gamma-1} d_i(l, z). \end{aligned} \tag{6.24}$$

It is straightforward to check that $G_i^{(\eta,\gamma)}(k, q; \theta) + S_i^{(\eta,\gamma)}(k, q) \geq 0$ for all $\hat{\theta}_k, \hat{\tau}_q, \theta, k, q, \eta, \gamma$ and i and, therefore, t^* is supermodular implementable.

Moreover, Step 1 says that t^* has the smallest one-step supermodularity among all supermodular transfers t. Combined with Step 2, this establishes that (x, t^*) is minimally supermodular implementable under the chosen order profile $>= (>_i)_{i \in N}$. Q.E.D

Proof of Corollary 1 In the proof of Theorem 1, we constructed transfers that minimally supermodular implemented the decision rule x under some order $>= (>_i)_{i \in N}$. For each such order $>$, we can compute the distance between the largest and the smallest equilibrium in the ex-ante induced game, using some metric. Each $>_i$ is a pair of complete orders on finite sets, hence for each i there are finitely many $>_i$. Since there are finitely many agents, there exist finitely many $>$. As a result, there is an order $>$ which gives the smallest distance between extremal equilibria. Q.E.D

Proof of Theorem 2 Define z^+ to be the immediate successor of z and z^- to be the immediate predecessor of z, where z is an element of a completely ordered set. Let k be the index assigned to each $\theta_i \in \Theta_i$ according to the complete order $>_i^1$. Suppose that $f = (x, t)$ is implementable and x is order reducible. Letting

$$\delta_i(\hat{\theta}_{ik}, \hat{\theta}_{-i}) = -\sum_{l=1}^{k-1} \sum_{z=r_i(\underline{\theta}_{-i})}^{r_i^-(\hat{\theta}_{-i})} \min_{\theta \in \Theta} [V_i(x(\hat{\theta}_{l+1}, z^+), \theta) - V_i(x(\hat{\theta}_l, z^+), \theta) - V_i(x(\hat{\theta}_{l+1}, z), \theta) + V_i(x(\hat{\theta}_l, z), \theta)] \tag{6.25}$$

for all $\hat{\theta}_{ik} \in \Theta_i$ and $\hat{\theta}_{-i} \in \times_{j \neq i} \Theta_j$, we define

$$t_i^*(\hat{\theta}_{ik}, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_k, \hat{\theta}_{-i}) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_k, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_k, \theta_{-i})] \tag{6.26}$$

and show that (x, t^*) is minimally supermodular implementable.

The intuition for the proof is the following. The transfers t^* ensure minimal supermodular implementation when one-steps on Θ_i and R_i are considered. There are steps in Θ_{-i} that correspond to zero, one, or multiple steps in R_i. The zero and one step cases are minimally supermodular implemented since t^* have the smallest possible one-step supermodularity while still ensuring supermodularity. When a one-step in Θ_{-i} corresponds to multiple steps in R_i, there could be transfers $\tilde{t}$ that have a smaller multi-step supermodularity than t^*. Suppose this is the case. By the result established in Step 2 of Theorem 1, these transfers will necessarily have a smaller one-step supermodularity than t^* on at least one of the intermediate steps. Thus, they will not be ensuring that the game is supermodular and are therefore not in the class of transfers $\mathcal{T}$. A more formal proof to follow here.

Proof of Theorem 3 By way of contradiction, suppose that strategy profile $\theta^*(\cdot) > \theta^T(\cdot)$ is an equilibrium so that player i's best-response to $\theta_{-i}^*(\cdot)$ is $\theta_i^*(\cdot)$. Consider player i's interim utility at type θ_i when announcing $\hat{\theta}_i(\cdot)$ against $\theta_{-i}^*(\cdot)$:

$$u_i(\hat{\theta}_i(\theta_i), \theta_{-i}^*(\cdot), \theta_i) = E_{\theta_{-i}}[V_i(x(\hat{\theta}_i(\theta_i), \theta_{-i}^*(\theta_{-i})), \theta) + t_i(\hat{\theta}_i(\theta_i), \theta_{-i}^*(\theta_{-i}))].$$

Therefore, for all deceptions $\hat{\theta}_i(\cdot) \in [\theta_i^T(\cdot), \theta_i^*(\cdot)]$,

$$\Delta u_i(\theta_{-i}^*(\theta_{-i}), \theta_i) = u_i(\theta_i^*(\theta_i), \theta_{-i}^*(\theta_{-i}), \theta_i) - u_i(\hat{\theta}_i(\theta_i), \theta_{-i}^*(\theta_{-i}), \theta_i) \geq 0 \tag{6.27}$$

for all type θ_i. We will show that this condition is not satisfied if the inequality in the theorem holds, i.e. there must be a player for which a smaller deception is strictly better than $\theta_i^*(\cdot)$.

We know that there exist numbers $\{K_i^j(\theta_i)\}$ such that

$$E_{\theta_{-i}}[\Delta u_i(\theta_{-i}^*(\theta_{-i}), \theta_i)] \leq E_{\theta_{-i}}[\Delta u_i(\theta_{-i}, \theta_i)] + d_i(\theta_i^*(\theta_i), \hat{\theta}_i(\theta_i)) \sum_{j \neq i} K_i^j(\theta_i) E_{\theta_j}[d_j(\theta_j^*(\theta_j), \theta_j)]. \tag{6.28}$$

Since the social choice function (x,t) is implementable, the transfers $\{t_i\}$ induce truthful revelation. Therefore, it must be that for all θ_i and i the incentive compatibility constraint is satisfied, that is:

$$E_{\theta_{-i}}[V_i(x(\hat{\theta}_i(\theta_i),\theta_{-i}),\hat{\theta}_i(\theta_i),\theta_{-i})] - E_{\theta_{-i}}[V_i(x(\theta_i^*(\theta_i),\theta_{-i}),\hat{\theta}_i(\theta_i),\theta_{-i})] \geq$$
$$E_{\theta_{-i}}[t_i(\theta_i^*(\theta_i),\theta_{-i})] - E_{\theta_{-i}}[t_i(\hat{\theta}_i(\theta_i),\theta_{-i})]. \quad (6.29)$$

This implies that

$$\begin{aligned}
\Delta u_i(\theta_{-i},\theta_i) &= E_{\theta_{-i}}[V_i(x(\theta_i^*(\theta_i),\theta_{-i}),\theta_i,\theta_{-i})] - E_{\theta_{-i}}[V_i(x(\hat{\theta}_i(\theta_i),\theta_{-i}),\theta_i,\theta_{-i})] \\
&\quad + E_{\theta_{-i}}[t_i(\theta_i^*(\theta_i),\theta_{-i})] - E_{\theta_{-i}}[t_i(\hat{\theta}_i(\theta_i),\theta_{-i})] \\
&\leq E_{\theta_{-i}}[V_i(x(\theta_i^*(\theta_i),\theta_{-i}),\theta_i,\theta_{-i})] - E_{\theta_{-i}}[V_i(x(\hat{\theta}_i(\theta_i),\theta_{-i}),\theta_i,\theta_{-i})] \\
&\quad + E_{\theta_{-i}}[V_i(x(\hat{\theta}_i(\theta_i),\theta_{-i}),\hat{\theta}_i(\theta_i),\theta_{-i})] - E_{\theta_{-i}}[V_i(x(\theta_i^*(\theta_i),\theta_{-i}),\hat{\theta}_i(\theta_i),\theta_{-i})] \\
&= E_{\theta_{-i}}[\Delta V_i(\theta_{-i},\theta_i,\theta_{-i})] - E_{\theta_{-i}}[\Delta V_i(\theta_{-i},\hat{\theta}_i(\theta_i),\theta_{-i})] \\
&\leq -E_{\theta_{-i}}[\gamma_i(\theta_{-i})]d_i(\theta_i^*(\theta_i),\hat{\theta}_i(\theta_i))d_i(\hat{\theta}_i(\theta_i),\theta_i). \quad (6.30)
\end{aligned}$$

It follows from (6.28) and (6.30) that

$$\frac{[\Delta u_i(\theta_{-i}^*(\theta_{-i}),\theta_i)]}{d_i(\theta_i^*(\theta_i),\hat{\theta}_i(\theta_i))} \leq \sum_{j\neq i} K_i^j(\theta_i)E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})]d_i(\hat{\theta}_i(\theta_i),\theta_i). \quad (6.31)$$

Recall that $\hat{\theta}_i(\cdot) < \theta_i^*(\cdot)$. Therefore, the closer we are to $\theta_i^*(\theta_i)$ the further we are from θ_i, and thus if the rhs of (6.31) is to be strictly negative it has to be at $\hat{\theta}_i(\theta_i) = p(\theta_i^*(\theta_i))$. If this is the case, then (6.31) implies that $[\Delta u_i(\theta_{-i}^*(\theta_{-i}),\theta_i)] < 0$, a contradiction with (6.27). Note that there is a profile $\theta^*(\cdot)$ within a δ-neighborhood of the truthful equilibrium such that $p(\theta_i^*(\cdot)) = \theta_i^T(\cdot)$ for all i. For such profile, the theorem is not of any use and so the smallest size of the equilibrium set predicted by the theorem cannot be below δ. Q.E.D

Proof of Proposition 2 Consider any strategy profile $\theta^*(\cdot) > \theta^T(\cdot)$. Choose i such that $E_{\theta_i}[d(\theta_i^*(\theta_i),\theta_i)] \geq E_{\theta_j}[d(\theta_j^*(\theta_j),\theta_j)]$ for all j. If the inequality in Theorem 3 were violated for all θ_i, then it would be that

$$\sum_{j\neq i} E_{\theta_i}[K_i^j(\theta_i)]E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)] - E_{\theta_{-i}}[\gamma_i(\theta_{-i})]E_{\theta_i}[d_i(p_i(\theta_i^*(\theta_i)),\theta_i)] < 0. \quad (6.32)$$

Since

$$\frac{\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)]}{E_{\theta_i}[d_i(p_i(\theta_i^*(\theta_i)),\theta_i)]} \max_{j\neq i} E_{\theta_i}[K_j^i(\theta_i)] \leq (n-1)\max_{j\neq i} E_{\theta_i}[K_j^i(\theta_i)],$$

the assumption of the theorem implies

$$\sum_{j\neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j),\theta_j)] \max_{j\neq i} E_{\theta_i}[K_j^i(\theta_i)] < E_{\theta_{-i}}[\gamma_i(\theta_{-i})]E_{\theta_i}[d_i(\theta_i^*(\theta_i),\theta_i)].$$

There must be $\underline{\delta}$ small enough such that for all smaller δ, $p_i(\theta_i^*(\cdot))$ is so close to $\theta_i^*(\cdot)$ that

$$\sum_{j \neq i} E_{\theta_j}[d_j(\theta_j^*(\theta_j), \theta_j)] \max_{j \neq i} E_{\theta_i}[K_j^i(\theta_i)] < E_{\theta_{-i}}[\gamma_i(\theta_{-i})] E_{\theta_i}[d_i(p_i(\theta_i^*(\theta_i)), \theta_i)],$$

thereby violating our original assumption (6.32). Q.E.D

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