On the Relationship between Robustness to Incomplete Information and Noise-Independent Selection in Global Games

DAISUKE OYAMA

Faculty of Economics, University of Tokyo
Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
oyama@e.u-tokyo.ac.jp

AND

SATORU TAKAHASHI

Department of Economics, Princeton University
Princeton, NJ 08544, USA
satorut@princeton.edu

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Abstract

This note demonstrates that symmetric 3×3 supermodular games may fail to have any equilibrium robust to incomplete information. Since the global game solution in these games is known to be independent of the noise structure, our result implies that a noise-independent selection in global games may not be a robust equilibrium. *Journal of Economic Literature* Classification Numbers: C72, D82.

KEYWORDS: equilibrium selection; supermodular game; incomplete information; robustness; contagion; global game.

1 Introduction

Suppose that an analyst plans to model some strategic situation with a complete information game **g** and has a Nash equilibrium a^* of **g** in hand as his prediction of the situation. While he believes that the complete information game **g** correctly describes the situation with high probability, he is also aware that there is some uncertainty about the payoffs, so that the players may play some incomplete information game close to **g**. Is his prediction a^* still valid even in the presence of a small amount of incomplete information? Kajii and Morris (1997, KM henceforth) formalize this robustness question as follows: Nash equilibrium a^* of complete information game **g** is *robust to incomplete information* if *every* incomplete information game in which the payoffs are given by **g** with high probability has a Bayesian Nash equilibrium such that a^* is played with high probability. This notion allows for a quite rich structure of correlated types in incomplete information perturbations, making the robustness test quite stringent. Indeed, even strict Nash equilibria may fail to be robust[1] and there are games that have no robust equilibrium,[2] whereas KM and subsequent studies have obtained several sufficient conditions for robustness.[3]

In this note, we demonstrate that there is a non-empty open set of symmetric 3×3 supermodular games that have no robust equilibrium. For each game in this set, we construct a sequence of dominance-solvable incomplete information perturbations in which one action profile is played everywhere and another sequence of dominance-solvable perturbations in which another action profile is played everywhere.[4] This has an important implication regarding the relationship between robust equilibrium and noise-independent selection in global games.

Global games, first studied by Carlsson and van Damme (1993) for 2×2 games, offer a natural way of introducing incomplete information perturbations that gives rise to equilibrium uniqueness through a "contagion" effect, where correlation in beliefs is generated by noisy signals of the true state which determines the payoffs. For general supermodular games, Frankel et al. (2003, FMP henceforth) show, with a setting with one-dimensional signals, that as the signal noise vanishes, the game has a unique equilibrium

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[1] See the earlier 2×2 example by Rubinstein (1989).

[2] KM construct a $3 \times 3 \times 3$ (non-supermodular) game with a unique (strict) Nash equilibrium that has no robust equilibrium. Morris (1999) demonstrates non-existence of robust equilibrium in a symmetric 4×4 supermodular game.

[3] KM show that a p-dominant equilibrium with p sufficiently small is robust, while Ui (2001) shows that in potential games, a potential maximizer is robust. Morris and Ui (2005) introduce a generalized notion of potential that unifies and generalizes the p-dominance and the potential maximization conditions and show that a generalized potential maximizer is robust. See Oyama and Tercieux (2009) for further developments.

[4] The conditions that define this set of games have been found by Honda (2010) to show that these games have no monotone potential maximizer.

that survives iterative dominance, while the surviving equilibrium may depend on the noise distribution.[5] While the global game approach only considers a particular class of perturbations as opposed to the robustness to all elaborations,[6] in classes of games considered in the literature so far the equilibrium that is played in global games independently of the noise structure has turned out to be also a robust equilibrium.[7] This might lead one to conjecture that the global game perturbations would represent a critical class of perturbations that determines whether or not an equilibrium is robust to incomplete information.

Our result in the this note falsifies this conjecture, that noise-independent selection implies robustness, at least for global games with one-dimensional signals as FMP consider. For, Basteck and Daniëls (2010) show in the setting of FMP that generic symmetric 3×3 supermodular games have a noise-independent selection in (one-dimensional) global games,[8] and the present note shows that some of these games have no robust equilibrium. Hence, the set of incomplete information perturbations that KM's concept of robustness allows is strictly larger than the set that global games with one-dimensional signals generate. For our 3×3 games and for the equilibrium other than the noise-independent selection in one-dimensional global games, the constructed incomplete information perturbation in which this equilibrium is the unique rationalizable outcome cannot be generated by a one-dimensional global game and may in effect be considered as a "two-dimensional" perturbation (although we do not work directly with a noisy signal formulation).

2 Preliminaries

2.1 Complete Information Games

We focus on two-player games. The set of players is denoted by $\mathcal{I} = \{1, 2\}$, and for $i \in \mathcal{I}$ we write $-i$ for player $j \neq i$. Each player $i \in \mathcal{I}$ has a linearly ordered, finite set of actions $A_i = \{0, 1, \ldots, n_i\}$. These action sets are fixed throughout the analysis. A complete information game is thus represented by a profile of payoff functions $\mathbf{g} = (g_1, g_2)$, where $g_i \colon A = \prod_{i \in \mathcal{I}} A_i \to \mathbb{R}$, $i \in \mathcal{I}$. Let $\Delta(S)$ denote the set of probability distributions over a set S. We

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[5] FMP provide a symmetric 4×4 example in which different equilibria survive under different noise distributions.

[6] In fact, Oury and Tercieux (2007) and Basteck et al. (2010) show that in supermodular games, a robust equilibrium is a noise-independent selection in global games.

[7] For example, the sufficient condition for noise-independent selection provided by FMP in terms of a generalized notion of potential is also sufficient for robustness (Morris and Ui (2005)).

[8] See also FMP (Section 5) for a heuristic argument with symmetric noise distributions.

denote by $br_i(\pi_i)$ the set of player i's pure best responses to $\pi_i \in \Delta(A_{-i})$:

$$br_i(\pi_i) = \arg\max_{a_i \in A_i} g_i(a_i, \pi_i),$$

where $g_i(a_i, \pi_i) = \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) g_i(a_i, a_{-i})$.

Complete information game $\mathbf{g}$ is *supermodular* if for each $i \in \mathcal{I}$,

$$g_i(a'_i, a_{-i}) - g_i(a_i, a_{-i}) \leq g_i(a'_i, a'_{-i}) - g_i(a_i, a'_{-i})$$

whenever $a_i < a'_i$ and $a_{-i} < a'_{-i}$. It is well known that the best response correspondence of a supermodular game is nondecreasing in the stochastic dominance order. For $\pi_i, \pi'_i \in \Delta(A_{-i})$, we write $\pi_i \precsim \pi'_i$ (and $\pi'_i \succsim \pi_i$) if π'_i stochastically dominates π_i, i.e., if $\sum_{a'_{-i} \geq a_{-i}} \pi_i(a'_{-i}) \leq \sum_{a'_{-i} \geq a_{-i}} \pi'_i(a'_{-i})$ for all $a_{-i} \in A_{-i}$. If $\mathbf{g}$ is supermodular, then for each $i \in \mathcal{I}$,

$$\min br_i(\pi_i) \leq \min br_i(\pi'_i)$$
$$\max br_i(\pi_i) \leq \max br_i(\pi'_i)$$

whenever $\pi_i \precsim \pi'_i$.

2.2 ε-Elaborations and Robust Equilibria

Given the game $\mathbf{g}$, we consider the following class of incomplete information games. Each player $i \in \mathcal{I}$ has a countable set of types, denoted by T_i, and we write $T = \prod_{i \in \mathcal{I}} T_i$. The (common) prior probability distribution on T is given by P. We assume that P satisfies that $\sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) > 0$ for all $i \in \mathcal{I}$ and $t_i \in T_i$. Under this assumption, the conditional probability of t_{-i} given t_i, $P(t_{-i}|t_i)$, is well defined by $P(t_{-i}|t_i) = P(t_i, t_{-i}) / \sum_{t'_{-i} \in T_{-i}} P(t_i, t'_{-i})$. The payoff function for player $i \in \mathcal{I}$ is a bounded function $u_i \colon A \times T \to \mathbb{R}$. Denote $\mathbf{u} = (u_i)_{i \in \mathcal{I}}$. The tuple $(T, P, \mathbf{u})$ defines an incomplete information game.

A (behavioral) strategy for player i is a function $\sigma_i \colon T_i \to \Delta(A_i)$. Denote by Σ_i the set of strategies for player i, and write $\Sigma = \prod_{i \in \mathcal{I}} \Sigma_i$. For a strategy $\sigma_i \in \Sigma_i$, we denote by $\sigma_i(a_i|t_i)$ the probability that $a_i \in A_i$ is chosen at $t_i \in T_i$. For $\sigma \in \Sigma$, we write $\sigma_P \in \Delta(A)$ for the probability distribution over A generated by σ, i.e., $\sigma_P(a) = \sum_{t \in T} P(t) \prod_{i \in \mathcal{I}} \sigma_i(a_i|t_i)$ for $a \in A$.

The expected payoff to player i of type $t_i \in T_i$ playing $a_i \in A_i$ against the opponent's strategy $\sigma_{-i} \in \Sigma_{-i}$ is given by

$$U_i(a_i, \sigma_{-i}|t_i) = \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i)\, u_i\big((a_i, \sigma_{-i}(t_{-i})), (t_i, t_{-i})\big),$$

where $u_i((a_i, \sigma_{-i}(t_{-i})), t) = \sum_{a_{-i} \in A_{-i}} \sigma_{-i}(a_{-i}|t_{-i}) u_i((a_i, a_{-i}), t)$. Let $BR_i(\sigma_{-i}|t_i)$ denote the set of pure best responses of player i of type $t_i \in T_i$ against $\sigma_{-i} \in \Sigma_{-i}$:

$$BR_i(\sigma_{-i}|t_i) = \arg\max_{a_i \in A_i} U_i(a_i, \sigma_{-i}|t_i).$$

A strategy profile $\sigma \in \Sigma$ is a *Bayesian Nash equilibrium* of $(T, P, \mathbf{u})$ if for all $i \in \mathcal{I}$, all $a_i \in A_i$, and all $t_i \in T_i$,

$$\sigma_i(a_i|t_i) > 0 \Rightarrow a_i \in BR_i(\sigma_{-i}|t_i).$$

Given $\mathbf{g}$, let $T_i^{g_i}$ be the set of types t_i such that payoffs of player i of type t_i are given by g_i and he knows his payoffs:

$$\begin{aligned} T_i^{g_i} = \{t_i \in T_i \mid & \, u_i(a, (t_i, t_{-i})) = g_i(a) \\ & \text{for all } a \in A \text{ and all } t_{-i} \in T_{-i} \text{ with } P(t_i, t_{-i}) > 0\}. \end{aligned}$$

Denote $T^{\mathbf{g}} = \prod_{i \in \mathcal{I}} T_i^{g_i}$.

Definition 1. Let $\varepsilon \in [0, 1]$. An incomplete information game $(T, P, \mathbf{u})$ is an *ε-elaboration* of $\mathbf{g}$ if $P(T^{\mathbf{g}}) = 1 - \varepsilon$.

Following KM, we say that an action distribution $\mu \in \Delta(A)$ is robust if, for small $\varepsilon > 0$, every ε-elaboration of $\mathbf{g}$ has a Bayesian Nash equilibrium σ such that the action distribution it generates, σ_P, is close to μ.

Definition 2. Action distribution $\mu \in \Delta(A)$ is *robust to incomplete information* in $\mathbf{g}$ if for every $\delta > 0$, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \leq \bar{\varepsilon}$, any ε-elaboration $(T, P, \mathbf{u})$ of $\mathbf{g}$ has a Bayesian Nash equilibrium σ such that $\max_{a \in A} |\mu(a) - \sigma_P(a)| \leq \delta$.

If $\mu \in \Delta(A)$ is robust in $\mathbf{g}$, then it must be a correlated equilibrium of $\mathbf{g}$ (KM, Corollary 3.5). We say that an action profile $a \in A$ is robust in $\mathbf{g}$ if the degenerate action distribution on a (i.e., $\mu \in \Delta(A)$ such that $\mu(a) = 1$) is robust in $\mathbf{g}$.

Given $\sigma_{-i} \in \Sigma_{-i}$, let $\pi_i(\sigma_{-i}|t_i) \in \Delta(A_{-i})$ be the belief of player i of type t_i over the opponent's actions, i.e.,

$$\pi_i(\sigma_{-i}|t_i)(a_{-i}) = \sum_{t_{-i} \in T_{-i}} P(t_{-i}|t_i)\sigma_{-i}(a_{-i}|t_{-i})$$

for $a_{-i} \in A_{-i}$. Observe that

$$BR_i(\sigma_{-i}|t_i) = br_i(\pi_i(\sigma_{-i}|t_i))$$

for all $t_i \in T_i^{g_i}$.

Several sufficient conditions for robustness to incomplete information have been obtained. In particular, Morris and Ui (2005) introduce generalized notions of potential and show, among others, that a *monotone potential maximizer* (*MP-maximizer*), a special form of their generalized potential maximizer concept, is robust in supermodular games (and in games that admit a monotone potential function that is supermodular). Their result unifies and generalizes the previous results by KM in terms of p-dominance and by Ui (2001) in terms of potential maximization. On the other hand, Morris (1999) presents an example of a symmetric 4×4 supermodular game that has no robust equilibrium.

3 Result

We now restrict to 3×3 games, so that $A_1 = A_2 = \{0, 1, 2\}$, and assume that $\mathbf{g}$ is supermodular. The game $\mathbf{g}$ is symmetric if $g_1(h, k) = g_2(k, h)$ for all $h, k \in \{0, 1, 2\}$. We prove the following:

Proposition 1. *There is a non-empty open set of symmetric* 3×3 *supermodular games that have no robust equilibrium.*

The proof proceeds as follows. In Lemma 1, we present a condition under which there is a sequence of ε-elaborations with a unique Bayesian Nash equilibrium where action 2 is played everywhere. It implies that if the game satisfies this condition, no action distribution other than (the degenerate distribution on) $(2, 2)$ is robust. In Lemma 2, we then present a condition under which there is a sequence of ε-elaborations with a unique Bayesian Nash equilibrium where action 0 is played everywhere. Thus, if the game satisfies this condition, no action distribution other than $(0, 0)$ is robust. Proposition 1 follows from the fact that there is a non-empty open set of symmetric 3×3 supermodular games that satisfy the conditions in Lemmata 1 and 2 simultaneously.[9]

In fact, these conditions have been found (and shown to be satisfied by some games) by Honda (2010) as a sufficient condition for a 3×3 game to have no MP-maximizer. He shows by direct computation that these conditions imply non-existence of an MP-maximizer. Since, as shown by Morris and Ui (2005), an MP-maximizer is robust, our non-existence result of robust equilibrium gives an indirect, alternative proof of the non-existence of MP-maximizer.

4 Proof

For $p \in (0, 1/2)$, let

$$\pi^a = \left(\tfrac{1}{2}, p, \tfrac{1}{2} - p\right), \quad \pi^b = \left(\tfrac{1}{2} - p, p, \tfrac{1}{2}\right), \quad \pi^c = \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right).$$

The condition for contagion of action 2 is stated in terms of best responses to these beliefs.

Lemma 1. *If there exists* $p \in (0, 1/2)$ *such that*

$$\min br_i(\pi^a) \geq 1, \quad \min br_i(\pi^b) = 2, \quad \min br_i(\pi^c) \geq 1, \tag{1}$$

then for all $\varepsilon > 0$*, there exists an* ε*-elaboration where the strategy profile* σ^* *such that* $\sigma_i^*(2|t_i) = 1$ *for all* $t_i \in T_i$ *is the unique Bayesian Nash equilibrium.*

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[9] In these games, action profiles $(0, 0)$ and $(2, 2)$ are the only pure Nash equilibria. Oyama and Takahashi (2009) show that symmetric 3×3 supermodular *coordination* games, where the three symmetric action profiles are all Nash equilibria, generically have an MP-maximizer and hence a robust equilibrium by Morris and Ui (2005).

Proof. Let $p \in (0, 1/2)$ be such that condition (1) is satisfied. We construct a sequence of elaborations $(T, P^\varepsilon, \mathbf{u})_{\varepsilon>0}$, where $P^\varepsilon(T^{\mathbf{g}}) \to 0$ as $\varepsilon \to 0$, as follows. Let $T_i = \mathbb{Z}_+$ for each $i \in \mathcal{I}$. Define $P^\varepsilon \in \Delta(T)$ by

$$\begin{aligned}
&P^\varepsilon(1,0) = P^\varepsilon(0,1) = \frac{1}{2}\varepsilon, \\
&P^\varepsilon(\tau+1,\tau) = P^\varepsilon(\tau,\tau+1) = p\varepsilon(1-\varepsilon)^\tau, \ \tau \geq 1, \\
&P^\varepsilon(\tau+2,\tau) = P^\varepsilon(\tau,\tau+2) = \left(\frac{1}{2} - p\right)\varepsilon(1-\varepsilon)^{\tau+1}, \ \tau \geq 0,
\end{aligned}$$

and $P^\varepsilon(t_1, t_2) = 0$ otherwise; see Table 1. Define $u_i \colon A \times T \to \mathbb{R}$ for each $i \in \mathcal{I}$ by

$$u_i(a,t) = \begin{cases} g_i(a) & \text{if } t_i \neq 0, \\ 1 & \text{if } t_i = 0 \text{ and } a_i = 2, \\ 0 & \text{if } t_i = 0 \text{ and } a_i \neq 2. \end{cases}$$

That is, type 0 is a "crazy type" for which action 2 is a dominant action, and $T_i^{g_i} = \mathbb{Z}_+ \setminus \{0\}$. (The constructed elaboration is an $\varepsilon\{1 + (1-2p)(1-\varepsilon)\}$-elaboration.)

We want to show that $(T, P^\varepsilon, \mathbf{u})$ has a unique Bayesian Nash equilibrium, which plays action 2 everywhere. Consider any Bayesian Nash equilibrium σ^* of $(T, P^\varepsilon, \mathbf{u})$. We show by induction that

$$\sigma_i^*(2|\tau-1) = 1 \text{ and } \sigma_i^*(0|\tau) = 0, \ i = 1, 2, \tag{$*_\tau$}$$

for all $\tau \geq 1$. We note that by the assumption (1) and the supermodularity of g_i, for any $t_i \in T_i^{g_i}$,

$$\begin{aligned}
\min BR_i(\sigma_{-i}^*|t_i) &\geq 1 \ \text{ if } \pi_i(\sigma_{-i}^*|t_i) \succsim \pi^a = \left(\tfrac{1}{2}, p, \tfrac{1}{2} - p\right), && (2) \\
\min BR_i(\sigma_{-i}^*|t_i) &= 2 \ \text{ if } \pi_i(\sigma_{-i}^*|t_i) \succsim \pi^b = \left(\tfrac{1}{2} - p, p, \tfrac{1}{2}\right), && (3) \\
\min BR_i(\sigma_{-i}^*|t_i) &\geq 1 \ \text{ if } \pi_i(\sigma_{-i}^*|t_i) \succsim \pi^c = \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right). && (4)
\end{aligned}$$

We first show $(*_1)$. Indeed, $\sigma_i^*(2|0) = 1$ by construction, and therefore, for type $t_i = 1$, $\pi_i(\sigma_{-i}^*|1) \succsim \pi^c$, so that $\sigma_i^*(0|1) = 0$ by (4).

Assume $(*_\tau)$. Then, for type $t_i = \tau + 1$, $\pi_i(\sigma_{-i}^*|\tau+1) \succsim \pi^a$, so that $\sigma_i^*(0|\tau+1) = 0$ by (2). Therefore, for type $t_i = \tau$, we have $\pi_i(\sigma_{-i}^*|\tau) \succsim \pi^b$ (which applies also to $\tau = 1$), so that $\sigma_i^*(2|\tau) = 1$ by (3). Thus, $(*_{\tau+1})$ holds. ▮

For $q, r \in (0,1)$, $r \leq q$, let

$$\pi^d = \left(\tfrac{1+q}{2}, 0, \tfrac{1-q}{2}\right), \quad \pi^e = \left(\tfrac{1-r}{2}, 0, \tfrac{1+r}{2}\right), \quad \pi^f = \left(0, \tfrac{q+r}{2q}, \tfrac{q-r}{2q}\right).$$

The condition for contagion of action 0 is stated in terms of best responses to these beliefs.

Lemma 2. *If there exist* $q, r \in (0,1)$ *with* $0 < r \leq q < 1$ *such that*

$$\max br_i(\pi^d) = 0, \quad \max br_i(\pi^e) \leq 1, \quad \max br_i(\pi^f) = 0, \tag{5}$$

then for all $\varepsilon > 0$*, there exists an* ε*-elaboration where the strategy profile* σ^* *such that* $\sigma_i^*(0|t_i) = 1$ *for all* $t_i \in T_i$ *is the unique Bayesian Nash equilibrium.*

Proof. Let $q, r \in (0,1)$, $r \leq q$, be such that condition (5) is satisfied. We construct a sequence of elaborations $(T, P^\varepsilon, \mathbf{u})_{\varepsilon > 0}$, where $P^\varepsilon(T^{\mathbf{g}}) \to 0$ as $\varepsilon \to 0$, as follows. Let $T_i = \{\alpha, \beta\} \times \mathbb{Z}_+$ for each $i \in \mathcal{I}$. Define $P^\varepsilon \in \Delta(T)$ by

$$\begin{aligned}
P^\varepsilon((\alpha, \tau+1), (\alpha, \tau)) = P^\varepsilon((\alpha, \tau), (\alpha, \tau+1)) &= \frac{1-q}{2(1+q)}\varepsilon(1-\varepsilon)^\tau, \\
P^\varepsilon((\alpha, \tau), (\beta, \tau)) = P^\varepsilon((\beta, \tau), (\alpha, \tau)) &= \frac{q+r}{2(1+q)}\varepsilon(1-\varepsilon)^\tau, \\
P^\varepsilon((\alpha, \tau+1), (\beta, \tau)) = P^\varepsilon((\beta, \tau), (\alpha, \tau+1)) &= \frac{q-r}{2(1+q)}\varepsilon(1-\varepsilon)^\tau,
\end{aligned}$$

and $P^\varepsilon(t_1, t_2) = 0$ otherwise; see Table 2. Define $u_i \colon A \times T \to \mathbb{R}$ for each $i \in \mathcal{I}$ by

$$u_i(a, t) = \begin{cases} g_i(a) & \text{if } t_i \neq (\alpha, 0), \\ 1 & \text{if } t_i = (\alpha, 0) \text{ and } a_i = 0, \\ 0 & \text{if } t_i = (\alpha, 0) \text{ and } a_i \neq 0. \end{cases}$$

That is, type $(\alpha, 0)$ is a "crazy type"for which action 0 is a dominant action, and $T_i^{g_i} = (\{\alpha, \beta\} \times \mathbb{Z}_+) \setminus \{(\alpha, 0)\}$. (The constructed elaboration is an $\varepsilon(1+r)/(1+q)$-elaboration.)

We want to show that $(T, P^\varepsilon, \mathbf{u})$ has a unique Bayesian Nash equilibrium, which plays action 0 everywhere. Consider any Bayesian Nash equilibrium σ^* of $(T, P^\varepsilon, \mathbf{u})$. We show by induction that

$$\sigma_i^*(0|(\alpha, \tau)) = \sigma_i^*(0|(\beta, \tau)) = 1, \ \ i = 1, 2, \tag{$\star_\tau$}$$

for all $\tau \geq 0$. We note that by the assumption (5) and the supermodularity of g_i, for any $t_i \in T_i^{g_i}$,

$$\begin{aligned}
\max BR_i(\sigma_{-i}^*|t_i) = 0 \ \text{ if } \pi_i(\sigma_{-i}^*|t_i) \precsim \pi^d = \left(\tfrac{1+q}{2}, 0, \tfrac{1-q}{2}\right), && (6)\\
\max BR_i(\sigma_{-i}^*|t_i) \leq 1 \ \text{ if } \pi_i(\sigma_{-i}^*|t_i) \precsim \pi^e = \left(\tfrac{1-r}{2}, 0, \tfrac{1+r}{2}\right), && (7)\\
\max BR_i(\sigma_{-i}^*|t_i) = 0 \ \text{ if } \pi_i(\sigma_{-i}^*|t_i) \precsim \pi^f = \left(0, \tfrac{q+r}{2q}, \tfrac{q-r}{2q}\right). && (8)
\end{aligned}$$

We first show $(\star_0)$. Indeed, $\sigma_i^*(0|(\alpha, 0)) = 1$ by construction, and therefore, for type $t_i = (\beta, 0)$, $\pi_i(\sigma_{-i}^*|(\beta, 0)) \precsim \pi^f$, so that $\sigma_i^*(0|(\beta, 0)) = 1$ by (8).

Assume $(\star_\tau)$. Then, for type $t_i = (\alpha, \tau+1)$, $\pi_i(\sigma_{-i}^*|(\alpha, \tau+1)) \precsim \pi^e$, so that $\sigma_i^*(2|(\alpha, \tau+1)) = 0$ by (7). Then, for type $t_i = (\beta, \tau+1)$, we have $\pi_i(\sigma_{-i}^*|(\beta, \tau+1)) \precsim \pi^f$, so that $\sigma_i^*(0|(\beta, \tau+1)) = 1$ by (8). Therefore, going back to type $t_i = (\alpha, \tau+1)$, we now have $\pi_i(\sigma_{-i}^*|(\alpha, \tau+1)) \precsim \pi^d$, so that $\sigma_i^*(0|(\alpha, \tau+1)) = 1$ by (6). Thus, $(\star_{\tau+1})$ holds. ∎

We present two examples that satisfy the hypotheses of Lemmata 1 and 2 simultaneously. Example 1 is taken from Honda (2010). Example 2 presents a game involving some economic context, the so-called “Bilingual Game” studied by Galesloot and Goyal (1997), Goyal and Janssen (1997), and Oyama and Takahashi (2010), among others. Clearly, the conditions will continue to be satisfied with small perturbations of the payoffs.

Example 1 (Honda (2010))**.** Let the game $\mathbf{g}$ be given by

	0	1	2
0	$13, 13$	$3, 5$	$0, 0$
1	$5, 3$	$0, 0$	$13, 2$
2	$0, 0$	$2, 13$	$16, 16$

where (0, 0) and (2, 2) are the only pure Nash equilibria. One can verify that conditions 1 and 5 in Lemmata 1 and 2 are satisfied (with equalities) for $p \in (1/7, 5/32)$ and for q and r such that $q > 5/21$, $r < 1/4$, and $(15/17)q < r \leq q$, respectively (Honda (2010, Example 1)). This game thus have no robust equilibrium.

Example 2 (Bilingual Game)**.** Two players are to choose between two computer programing languages, or two types of technologies in general, A and B. Assume that A is more efficient while B is less risky: if both players choose A, then they each receive a payoff 11, while both choose B, then they receive 10; if they choose different options, then the A-player receives 0, while the B-player receives 3. Thus, (A, A) Pareto-dominates (B, B), while (B, B) pairwise risk-dominates (A, A). In this 2×2 coordination game, the risk-dominant, and Pareto-dominated, equilibrium (B, B) is robust to incomplete information.

Now suppose that a “bilingual option”, or a compatible technology, AB is available with some cost $e > 0$. An AB-player adopts A against an A-player to receive a (gross) payoff 11 and adopts B against a B-player to receive 10. If both players choose AB, then they use the efficient option A and receive 11. This situation is described by

	0	1	2
0	$11, 11$	$11, 11-e$	$0, 3$
1	$11-e, 11$	$11-e, 11-e$	$10-e, 10$
2	$3, 0$	$10, 10-e$	$10, 10$

where the actions A, AB, and B are denoted 0, 1, and 2, respectively, and with the order $0 < 1 < 2$ the game is supermodular. The profiles (A, A) and (B, B) are the only pure Nash equilibria of this game.

For this game, it is conceivable that if the cost e is large so that AB is too costly, then the game is strategically similar to the original 2×2 game, and thus the pairwise risk-dominant equilibrium (B, B) will be robust, while if e is small enough, then B will tend to be abandoned in the presence of the even less risky option AB, and thus the efficient option A will be robust. In fact, by Oyama and Takahashi (2010), it turns out that if $e > 40/19$, then (B, B) is an MP-maximizer and hence a robust equilibrium, while if $e < 5/3$, then (A, A) is an MP-maximizer and hence a robust equilibrium; in the middle case when $5/3 < e < 40/19$, the conditions in Lemmata 1 and 2 are simultaneously satisfied and therefore the game has no robust equilibrium.

5 Discussion

Let us discuss the relation to global-game noise-independent selection. Global games, first introduced by Carlsson and van Damme (1993) for binary games, represent an important class of incomplete information games in which equilibrium uniqueness arises through contagion effects along higher order beliefs, and which are simple and tractable enough to be used in various economic applications.[10] General supermodular games (with many players and many actions) are studied by FMP in the following setting. A state of the world θ is drawn from the one-dimensional real line and determines the payoffs of the players, and each player observes a noisy signal about θ. It is assumed that the payoff differences are monotone in opponents' actions (supermodularity) and in the state θ (state monotonicity), and the players have a dominant action when θ is sufficiently small or large (dominance regions). In this setting, FMP show that as the signal noise vanishes, the game has a unique equilibrium that survives iterative dominance, while the selected equilibrium may depend on the noise distribution. FMP provide a symmetric 4×4 example in which difference equilibria survive under difference noise distributions. They also provide sufficient conditions for the selection to be noise-independent. In particular, they give a heuristic argument that generic symmetric 3×3 supermodular games have a noise-independent selection, which is formally proved by Basteck and Daniëls (2010).

The crucial difference between the global game and the robustness approaches (besides the technical difference whether the type space is discrete or continuous) is that the former considers a certain critical class of payoff perturbations, while the latter allows for all perturbations. In particular, FMP employ a *one-dimensional* type space. In fact, as proved by Oury and

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[10] See the survey by Morris and Shin (2003).

Tercieux (2007) and Basteck et al. (2010), if a^* is a robust equilibrium, then it is a noise-independent selection in global games.[11]

Our result shows that the converse of this result does not hold: we demonstrated non-existence of robust equilibrium in symmetric 3×3 supermodular games, a class of games that admit noise independence in global games, thus implying that a global-game noise-independent selection may not be a robust equilibrium. That is, the global game perturbation is in general not the only possible perturbation that yields a unique equilibrium outcome.

Corollary 2. *A global-game noise-independent selection may not be a robust equilibrium.*

More precisely, the result by Basteck and Daniëls (2010) in fact shows that if the game $\mathbf{g}$ satisfies the condition (1) in Lemma 1, then $(2, 2)$ is the global game selection of $\mathbf{g}$. The incomplete information elaboration we constructed in the proof of Lemma 1 can thus be seen as a type space representation of a global game with some noise structure with a one-dimensional state space as in FMP. On the other hand, their noise-independence result implies that the elaboration we constructed in the proof of Lemma 2 for contagion of $(0, 0)$ cannot be generated by a one-dimensional global game perturbation and may be seen as a "two-dimensional" perturbation. This fact underlines the gap between robustness to incomplete information and noise independence in (one-dimensional) global games. The gap could be filled by allowing global games to have more complex noise structures, but for more general games than 3×3 games, we conjecture that the degree of required complexity would increase as the numbers of players and actions increase and would reduce the tractability and applicability of resulting global games.

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[11] See Morris and Shin (2003, Section 4.5) for a heuristic argument for this claim.

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$t_1 \backslash t_2$	0	1	2	3	4	5	$\cdots$
0		$\frac{1}{2}\varepsilon$	$\left(\frac{1}{2}-p\right)\varepsilon(1-\varepsilon)$				
1	$\frac{1}{2}\varepsilon$		$p\varepsilon(1-\varepsilon)$	$\left(\frac{1}{2}-p\right)\varepsilon(1-\varepsilon)^2$			
2	$\left(\frac{1}{2}-p\right)\varepsilon(1-\varepsilon)$	$p\varepsilon(1-\varepsilon)$		$p\varepsilon(1-\varepsilon)^2$	$\left(\frac{1}{2}-p\right)\varepsilon(1-\varepsilon)^3$		
3		$\left(\frac{1}{2}-p\right)\varepsilon(1-\varepsilon)^2$	$p\varepsilon(1-\varepsilon)^2$		$p\varepsilon(1-\varepsilon)^3$	$\left(\frac{1}{2}-p\right)\varepsilon(1-\varepsilon)^4$	
$\vdots$			$\ddots$	$\ddots$		$\ddots$	$\ddots$

Table 1: Contagion of action 2

$t_1\backslash t_2$	$(\alpha,0)$	$(\alpha,1)$	$(\alpha,2)$	$(\alpha,3)$	$(\alpha,4)$	$\cdots$
$(\alpha,0)$		$Q\varepsilon$				
$(\alpha,1)$	$Q\varepsilon$		$Q\varepsilon(1-\varepsilon)$			
$(\alpha,2)$		$Q\varepsilon(1-\varepsilon)$		$Q\varepsilon(1-\varepsilon)^2$		
$(\alpha,3)$			$Q\varepsilon(1-\varepsilon)^2$		$Q\varepsilon(1-\varepsilon)^3$	
$\vdots$				$\ddots$		$\ddots$

$t_1\backslash t_2$	$(\beta,0)$	$(\beta,1)$	$(\beta,2)$	$(\beta,3)$	$\cdots$
$(\alpha,0)$	$R_0\varepsilon$				
$(\alpha,1)$	$R_1\varepsilon$	$R_0\varepsilon(1-\varepsilon)$			
$(\alpha,2)$		$R_1\varepsilon(1-\varepsilon)$	$R_0\varepsilon(1-\varepsilon)^2$		
$(\alpha,3)$			$R_1\varepsilon(1-\varepsilon)^2$	$R_0\varepsilon(1-\varepsilon)^3$	
$\vdots$				$\ddots$	$\ddots$

$t_1\backslash t_2$	$(\alpha,0)$	$(\alpha,1)$	$(\alpha,2)$	$(\alpha,3)$	$(\alpha,4)$	$\cdots$
$(\beta,0)$	$R_0\varepsilon$	$R_1\varepsilon$				
$(\beta,1)$		$R_0\varepsilon(1-\varepsilon)$	$R_1\varepsilon(1-\varepsilon)$			
$(\beta,2)$			$R_0\varepsilon(1-\varepsilon)^2$	$R_1\varepsilon(1-\varepsilon)^2$		
$(\beta,3)$				$R_0\varepsilon(1-\varepsilon)^3$	$R_1\varepsilon(1-\varepsilon)^3$	
$\vdots$					$\ddots$	$\ddots$

Table 2: Contagion of action 0, where $Q = \frac{1-q}{2(1+q)}$, $R_0 = \frac{q+r}{2(1+q)}$, $R_1 = \frac{q-r}{2(1+q)}$