CONTINUOUS TIME CONTESTS

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This paper introduces a contest model in continuous time in which each player decides when to stop a privately observed Brownian motion with drift and incurs costs depending on his stopping time. The player who stops his process at the highest value wins a prize.
Under mild assumptions on the cost function, we prove existence and uniqueness of the Nash equilibrium outcome, even if players have to choose bounded time stopping strategies. We derive a closed form of the equilibrium strategy and distribution. If the noise parameter goes to zero, the equilibrium converges to, and thus selects the symmetric equilibrium of an all-pay contest. For positive noise levels, results differ from those of all-pay contests—for instance, participants make positive profits. Moreover, for two players and constant costs, the profits of each participant increase for higher costs of research or lower productivity of each player. Hence, participants prefer a contest design, which impedes research progress.

KEYWORDS: Contests, All-pay Contests, Discontinuous Games.

1. INTRODUCTION

Two types of models are predominant in the literature on contests, races, and tournaments. In one of these, there is no feedback about the performance measure or standings throughout the competition at all, while the other one considers full feedback about the performance of each player at all points in time. The former category includes all-pay contests with complete information (Hillman and Samet, 1987; Siegel, 2009, 2010), Tullock contests (Tullock, 1980), silent timing games (Karlin, 1953; Park and Smith, 2008), and models with additive noise in the spirit of Lazear and Rosen (1981). The latter category contains wars of attrition (Maynard Smith, 1974; Bulow and Klemperer, 1999), races (Aoki, 1991; Hörner, 2004; Anderson and Cabral, 2007), and contest models with full observability such as Harris and Vickers (1987) and Moscarini and Smith (2007).

In this paper, we want to analyze an intermediate case in which there is partial feedback about the performance measure. More precisely, a player learns about his own stochastic research progress over time, but he does not observe the progress of the other players or their effort decisions. A good example for this

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setting is an R&D contest. Each participant is well-informed about his own progress, but often uninformed about the progress of his competitors.

Formally, our model is an n-player contest in which each player decides when to stop a privately observed Brownian motion (X_t) with drift μ and diffusion coefficient σ. As long as a player exerts effort, i.e., does not stop the process, he incurs flow costs of $c(X_t)$. The player who stops his process at the highest value wins a prize.

Under mild assumptions on the cost function—it has to be continuous and bounded away from zero—we show that the game has a unique Nash equilibrium outcome. This outcome is feasible in stopping strategies which stop almost surely before a fixed time $T < \infty$. Hence, provided the contest length is above a threshold, the equilibrium is independent of the contest length. In equilibrium, each player makes positive expected profits. For two players and constant costs, these profits increase if the productivity (drift) of both players decreases or the costs increase. Hence, participants prefer a contest design, which impedes progress.

The formal analysis proceeds as follows. Proposition 1 and Theorem 1 establish existence and uniqueness of the equilibrium distribution. The existence proof first characterizes the equilibrium distribution $F(x)$ of values at the stopping time $X_\tau = x$ uniquely up to its endpoints. We then use a Skorokhod embedding approach to show that there exists a stopping strategy, which induces this distribution. This technique from probability theory (e.g., Skorokhod, 1961, 1965; for a survey, see Obłój, 2004) was first introduced to game theory in Seel and Strack (2009).

Moreover, we verify a condition from a recent paper in probability theory (Ankirchner and Strack, 2011) to show that there exists a bounded time stopping strategy—a strategy that stops almost surely before a fixed time $T < \infty$—which induces the equilibrium distribution. As most real-world contests have a fixed deadline, this result fortifies the predictions of the model. It is also one of main technical contributions of the paper, since this technique is also applicable to other models without observability.

We then analyze the characteristics of the equilibrium distribution. As uncertainty vanishes, it converges to the symmetric equilibrium of an all-pay contest—see Siegel (2009, 2010)—by Theorem 2. In the special case of constant costs, the equilibrium converges to the symmetric equilibrium of an all-pay auction. On the one hand, the model offers a microfoundation for the use of all-pay auctions to scrutinize environments in which uncertainty is not a crucial ingredient; on the other hand, it gives an equilibrium selection result between the all-pay auction equilibria in Baye, Kovenock, and de Vries (1996). Moreover, this result serves as a benchmark to discuss how our predictions differ from all-pay models if $\sigma > 0$.

For any $\sigma > 0$, Proposition 2 shows that all players make positive expected profits in equilibrium. Intuitively, agents use their private information about their progress, which arrives continuously over time, to generate rents. The intuition is similar to an all-pay contest, in which players have incomplete information about the valuation of their rivals—see, e.g., Hillman and Riley (1989), Amann and Leininger (1996), or Moldovanu and Sela (2001).

Finally, we analyze the special case of two players and constant costs. We derive a closed-form solution for the profits of each player, which depends only on the fraction $y = \frac{2\mu^2}{c\sigma^2}$ (Proposition 4). In particular, profits increase as costs c increase, infinitesimal variance σ^2 increases, or productivity μ decreases (Theorem 3). Hence, contestants prefer to have mutually worse technologies.

There are many possible applications for the model. For instance, most R&D competitions would suit the modeling framework, since firms are not informed about the research of their rivals; for concrete examples of such competitions, see Taylor (1995).

1.1. *Related Literature*

In a companion paper (Seel and Strack, 2009), we analyze a model in which players do not have any costs of research, but have a (usually negative) drift and face a bankruptcy constraint. The driving forces of both models differ substantially. In particular, in the present paper, contestants trade-off higher costs versus a higher winning probability, whereas in Seel and Strack (2009) the trade-off is between winning probability and risk. Also, the applications of Seel and Strack (2009) are related to finance, while the present paper is in spirit of the contest literature.

The paper entails a direct extension of the literature on *silent timing games*—see, e.g., Karlin (1953). This literature scrutinizes, among others, our setting for the case without uncertainty. Intuitively, adding uncertainty allows us to have a model with partial learning throughout the contest.

With a similar motivation, Taylor (1995) also analyzes a model in which players only learn about their own stochastic research success. In his T-period model, however, the highest draw in a single period determines this success. The resulting equilibrium stopping rule is a threshold strategy, which stops whenever a player has a draw above a deterministic, time-independent value.

We proceed as follows. Section 2 sets up the model. In Section 3, we prove that an equilibrium exists and is unique. Section 4 discusses the relation to all-pay contests and derives the main comparative statics results. Section 5 concludes. Most proofs are relegated to the appendix.

2. THE MODEL

There are $n < \infty$ agents indexed by $i \in \{1, 2, \ldots, n\} = N$ who face a stopping problem in continuous time. At each point in time $t \in \mathbb{R}_+$, agent i privately observes the realization of a stochastic process $(X_t^i)_{t \in \mathbb{R}_+}$ with

$$X_t^i = x_0 + \mu t + \sigma B_t^i \,.$$

The constant x_0 denotes the starting value of all processes; without loss of generality, we assume $x_0 = 0$. The drift $\mu \in \mathbb{R}_+$ is the common expected change of each process X_t^i per time, i.e., $\mathbb{E}(X_{t+\Delta}^i - X_t^i) = \mu\Delta$. The noise term is an n-dimensional Brownian motion (B_t) scaled by $\sigma \in \mathbb{R}_+$.

2.1. *Strategies*

A pure strategy of player i is a stopping time τ^i. This stopping time depends only on the realization of his process X_t^i, as the player only observes his own process.[1] Mathematically, the agents' stopping decision until time t has to be $\mathcal{F}_t^i$-measurable, where $\mathcal{F}_t^i = \sigma(\{X_s^i : s < t\})$ is the sigma algebra induced by the possible observations of the process X_s^i before time t. We require stopping times to be bounded by a real number $T < \infty$ such that $\tau^i < T$ almost surely.

To incorporate mixed strategies, we allow for randomized stopping times—progressively $\mathcal{F}_t^i$-measurable functions $\tau^i(\cdot)$ such that, for every $r^i \in [0,1]$, the value $\tau^i(r^i)$ is a stopping time. Intuitively, agents draw a random number r^i from the uniform distribution on $[0,1]$ before the game and play a stopping strategy $\tau^i(r^i)$.[2]

2.2. *Payoffs*

The player who stops his process at the highest value wins a prize $p > 0$. Ties are broken randomly. Each player incurs a flow cost $c : \mathbb{R} \to \mathbb{R}_{++}$, which is independent of the time t, until he stops. The payoff π^i is thus

$$\pi^i = \frac{p}{k} \mathbf{1}_{\{X_{\tau^i}^i = \max_{j \in N} X_{\tau^j}^j\}} - \int_0^{\tau^i} c^i(X_t^i) dt \ ,$$

where $k = |\{i \in N : X_{\tau^i}^i = \max_{j \in N} X_{\tau^j}^j\}|$ is the number of agents who stop with the highest value. All agents maximize their expected profit $\mathbb{E}(\pi^i)$. We henceforth normalize p to 1, since agents only care about the trade-off between winning probability and cost-prize ratio. The cost function satisfies the following mild assumption:

Assumption 1 *For every $x \in \mathbb{R}$, the cost function $c : \mathbb{R} \to \mathbb{R}_{++}$ is continuous and bounded away from zero on $[x, \infty)$.*

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[1]The equilibrium of the model would be the same if the stopping decision was reversible and stopped processes were constant.

[2]The unique equilibrium outcome of this paper can be obtained in pure strategies, but we include mixing to make the results more general.

2.3. *A Brief Discussion of the Technology*

The Brownian Motion specification entails the possibility that research success might decrease over time. There are several possible interpretations of this feature. In an R&D setting, for instance, the value of the innovation (say prototype of a fighter jet) may depend on the market prize of the components. Hence, one might interpret the decrease in innovation value for one player as an increase in the prize of a component of that players prototype. Another interpretation is that a part of the innovation gets destroyed (for instance a car component). Similarly, a worker might be hired by another firm or simply forget something.

3. EQUILIBRIUM CONSTRUCTION

In this section, we first establish some necessary conditions on the distribution functions in equilibrium. In a second step, we prove existence and uniqueness of the Nash equilibrium outcome, and calculate the equilibrium distributions depending on the cost function.

Every strategy of agent i induces a (potentially non-smooth) cumulative distribution function (cdf) $F^i : \mathbb{R} \to [0,1]$ of his stopped process $F^i(x) = \mathbb{P}(X^i_{\tau^i} \leq x)$. We denote the winning probability of player i if he stops at $X^i_{\tau^i} = x$, given the distributions of the other players, by

$$u^i(x) = \mathbb{P}(\max_{j \neq i} X^j_{\tau^j} \leq x) = \prod_{j \neq i} F^j(x)\,.$$

Denote the endpoints of the support of the cdf of player i by $\overline{x}^i = \sup\{x : F^i(x) < 1\}$ and $\underline{x}^i = \inf\{x : F^i(x) > 0\}$. Let $\underline{x} = \max_{i \in N} \underline{x}^i$ and $\overline{x} = \max_{i \in N} \overline{x}^i$. In the next step, we establish a series of auxiliary results that are crucial to prove uniqueness of the equilibrium distribution.

Lemma 1 *At least two players stop with positive probability on every interval $I = (a,b) \subset [\underline{x}, \overline{x}]$.*

Lemma 2 *No player places a mass point in the interior of the state space, i.e., for all i, for all $x > \underline{x}$: $\mathbb{P}(X^i_{\tau^i} = x) = 0$. At least one player has no mass at the left endpoint, i.e., $F^i(\underline{x}) = 0$, for at least player i.*

We omit the proof of Lemma 2, since it is just a specialization of the standard logic in static game theory with a continuous state space; see, e.g., Burdett and Judd (1983). Intuitively, in equilibrium, no player can place a mass point in the interior of the state space, since no other player would then stop slightly below the mass point. This contradicts Lemma 1.

Lemma 3 *All players have the same right endpoint, $\overline{x}^i = \overline{x}$, for all i.*

LEMMA 4 *All players have the same expected profit in equilibrium. Moreover, each player loses for sure at $\underline{x}$, i.e., $u^i(\underline{x}) = 0$, for all i.*

LEMMA 5 *All players have the same equilibrium distribution function $F^i = F$, for all i.*

As players have symmetric distributions, we henceforth drop the superscript i. The previous lemmata imply that each player is indifferent between any stopping strategy on his support. By Itô's lemma, it follows from the indifference inside the support that, for every point $x \in (\underline{x}, \overline{x})$, the function $u(\cdot)$ must satisfy the second order ordinary differential equation (ODE)

$$c(x) = \mu u'(x) + \frac{\sigma^2}{2} u''(x)\,. \tag{1}$$

As (1) is a second order ODE, we need two boundary conditions to determine $u(\cdot)$ uniquely. One boundary condition is $u(\underline{x}) = 0$. We determine the other one in the following lemma:

LEMMA 6 *In equilibrium, $u'(\underline{x}) = 0$.*

The idea of the proof in the appendix is simple. If the derivative was negative, $u'(\underline{x}) < 0$, there would a profitable deviation at $\underline{x}$, which stops in the neighborhood of $\underline{x}$ rather than at the point itself.

Imposing the two boundary conditions, the solution to equation (1) is unique. To calculate it, we define $\phi(x) = \exp(\frac{-2\mu x}{\sigma^2})$ as a solution of the homogeneous equation $0 = \mu u'(x) + \frac{\sigma^2}{2} u''(x)$. To solve the inhomogeneous equation (1), we apply the variation of the constants formula. We then use the two boundary conditions to calculate the unique solution candidate. Finally, we rearrange with Fubini's Theorem to get

$$u(x) \quad = \quad \begin{cases} 0 & \text{for } x < \underline{x} \\ \frac{1}{\mu} \int_{\underline{x}}^{x} c(z)(1 - \phi(x - z)) \mathrm{d}z & \text{for } x \in [\underline{x}, \overline{x}] \\ 1 & \text{for } \overline{x} < x\,. \end{cases}$$

By symmetry of the equilibrium strategy, the function $F : \mathbb{R} \to [0, 1]$ satisfies $F(x) = \sqrt[n-1]{u(x)}$. Consequently, the unique candidate for an equilibrium distribution F is

$$F(x) = \begin{cases} 0 & \text{for all } x < \underline{x} \\ \sqrt[n-1]{\frac{1}{\mu} \int_{\underline{x}}^{x} c(z)(1 - \phi(x - z)) \mathrm{d}z} & \text{for all } x \in [\underline{x}, \overline{x}] \\ 1 & \text{for all } \overline{x} < x\,. \end{cases}$$

In the next step, we verify that F is a cumulative distribution function, i.e., that F is nondecreasing and that $\lim_{x \to \infty} F(x) = 1$.

LEMMA 7 *F is a cumulative distribution function.*

PROOF: By construction of F, $F(\underline{x}) = 0$. Clearly, F is increasing on $(\underline{x}, \overline{x})$, as the derivative with respect to x,

$$F'(x) = \frac{F(x)^{2-n}}{(n-1)}(\frac{2}{\sigma^2}\int_{\underline{x}}^{x} c(z)\phi(x-z)\mathrm{d}z)\,,$$

is greater than zero for all $x > \underline{x}$. It remains to show that there exists an $x > \underline{x}$ such that $F(x) = 1$.

$$\begin{aligned} F(x)^{n-1} &= \frac{1}{\mu}\int_{\underline{x}}^{x} c(z)(1-\phi(x-z))\mathrm{d}z \\ &\geq \frac{1}{\mu}\inf_{y\in[\underline{x},\infty)} c(y)\left(x-\underline{x}-\frac{\sigma^2}{2\mu}(1-\phi(x-\underline{x}))\right) \\ &\geq \frac{1}{\mu}\inf_{y\in[\underline{x},\infty)} c(y)(x-\underline{x}-\frac{\sigma^2}{2\mu}) \end{aligned}$$

Assumption 1 implies that the cost function $c(\cdot)$ is bounded away from zero. Consequently, $\inf_{y\in[\underline{x},\infty)} c(y)$ is strictly greater than zero. Continuity of F implies that there exists a point $\overline{x} > \underline{x}$ such that $F(\overline{x}) = 1$. *Q.E.D.*

The next lemma derives a necessary condition for a distribution F to be the outcome of a strategy τ.

LEMMA 8 *If $\tau \leq T < \infty$ is a bounded stopping time that induces the continuous distribution $F(\cdot)$, i.e., $F(z) = \mathbb{P}(X_\tau \leq z)$, then $1 = \int_{\underline{x}}^{\overline{x}} \phi(x)F'(x)\mathrm{d}x$.*

PROOF: Observe that $(\phi(X_t))_{t\in\mathbb{R}_+}$ is a martingale. Hence, by Doob's optional stopping theorem, for any bounded stopping time τ,

$$1 = \phi(X_0) = \mathbb{E}[\phi(X_\tau)] = \int_{\underline{x}}^{\overline{x}} \phi(x)F'(x)\mathrm{d}x\,.$$

Q.E.D.

We use the necessary condition from Lemma 8 to prove that the equilibrium distribution is unique.

PROPOSITION 1 *There exists a unique pair $(\underline{x}, \overline{x}) \in \mathbb{R}^2$ such that the distribution*

$$F(x) = \begin{cases} 0 & \text{for all } x \leq \underline{x} \\ \sqrt[n-1]{\frac{1}{\mu}\int_{\underline{x}}^{x} c(z)(1-\phi(x-z))dz} & \text{for all } x \in (\underline{x}, \overline{x}) \\ 1 & \text{for all } x \geq \overline{x} \end{cases}$$

is the unique candidate for an equilibrium distribution.

PROOF: As F is continuous, the right endpoint $\overline{x}$ satisfies $1 = \int_{\underline{x}}^{\overline{x}} F'(x;\underline{x},\overline{x})\mathrm{d}x$. Since $F'(x;\underline{x},\overline{x})$ is independent of $\overline{x}$, we drop the dependency in our notation. By the implicit function theorem,

$$\frac{\partial \overline{x}}{\partial \underline{x}} = -\frac{-\overbrace{F'(\underline{x};\underline{x})}^{=0} + \int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x;\underline{x})\mathrm{d}x}{F'(\overline{x};\underline{x})} = -\frac{\int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x;\underline{x})\mathrm{d}x}{F'(\overline{x};\underline{x})}. \tag{2}$$

Lemma 8 states that any feasible distribution satisfies $1 = \int_{\underline{x}}^{\overline{x}} F'(x;\underline{x})\phi(x)\mathrm{d}x$. Applying the implicit function theorem to this equation gives us

$$\begin{aligned} \frac{\partial \overline{x}}{\partial \underline{x}} &= -\frac{-\overbrace{F'(\underline{x};\underline{x})}^{=0} + \int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x;\underline{x})\phi(x)\mathrm{d}x}{F'(\overline{x};\underline{x})\phi(\overline{x})} \\ &= -\frac{\int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x;\underline{x}) \overbrace{\phi(x-\overline{x})}^{<1} \mathrm{d}x}{F'(\overline{x};\underline{x})} \\ &< -\frac{\int_{\underline{x}}^{\overline{x}} \frac{\partial}{\partial \underline{x}} F'(x;\underline{x})\mathrm{d}x}{F'(\overline{x};\underline{x})}. \end{aligned} \tag{3}$$

The last step follows, because $\frac{\partial}{\partial \underline{x}} F'(x;\underline{x}) \geq 0$. Hence, conditions 2 and 3 cross exactly once. Thus, in equilibrium, the left and right endpoint are unique. *Q.E.D.*

Henceforth, we write $F(\cdot)$ to refer to the unique equilibrium distribution.

LEMMA 9 *Every strategy that induces the unique distribution F from Proposition 1 is an equilibrium strategy.*

PROOF: Define $\Psi(\cdot)$ as the unique solution to (1) with the boundary conditions $\Psi(\underline{x}) = 0$ and $\Psi'(\underline{x}) = 0$. By construction, the process $\Psi(X_t^i) - \int_0^t c(X_s^i)\mathrm{d}s$ is a martingale and $\Psi(x) = u(x)$ for all $x \in [\underline{x},\overline{x}]$. As $\Psi'(x) < 0$ for $x < \underline{x}$ and $\Psi'(x) > 0$ for $x > \overline{x}$, $\Psi(x) > u(x)$ for all $x \notin [\underline{x},\overline{x}]$. For every stopping time S, we use Itô's Lemma to calculate the expected value

$$\begin{aligned} \mathbb{E}[u(X_S) - \int_0^S c(X_t)\mathrm{d}t] &\leq \mathbb{E}[\Psi(X_S) - \int_0^S c(X_t)\mathrm{d}t] \\ &= \Psi(X_0) = u(X_0) = \mathbb{E}(u(X_\tau)). \end{aligned}$$

The last equality results from the indifference of every agent to stop immediately with the expected payoff $u(X_0)$ or to play the equilibrium strategy with the expected payoff $\mathbb{E}(u(X_\tau))$. *Q.E.D.*

So far, we have verified that a bounded stopping time $\tau \leq T < \infty$ is an equilibrium strategy if and only if it induces the distribution $F(\cdot)$, i.e., $F(z) = \mathbb{P}(X_\tau \leq z)$. To show that the game has a Nash equilibrium, it remains to establish the existence a bounded stopping time inducing $F(\cdot)$. The problem of finding a stopping time τ such that a Brownian motion stopped at τ has a given centered probability distribution F, i.e., $F \sim B_\tau$, is known in the probability literature as the Skorokhod embedding problem (SEP). Since its initial formulation in Skorokhod (1961, 1965), many solutions have been derived; for a survey article, see Obłój (2004). Ankirchner and Strack (2011) find conditions guaranteeing the existence of stopping times τ that are *bounded* by some real number $T < \infty$, and embed a given distribution in Brownian motion, possibly with drift.[3] They define $g(x) = F^{-1}(\Phi(x))$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(\frac{z^2}{2}) \mathrm{d}z$ is the density function of the normal distribution.

Lemma 10 (Ankirchner and Strack, 2011, Theorem 2) *Suppose that g is Lipschitz-continuous with Lipschitz constant $\sqrt{T}$. Then F can be embedded in $X_t = \mu t + B_t$.*

This auxiliary result enables us to prove the main result of this section:

Theorem 1 *The game has a Nash equilibrium.*

To prove existence of a Nash equilibrium, it remains to show the existence of a bounded stopping time τ that induces F. The proof in the appendix verifies Lipschitz continuity of the function g, which makes Lemma 10 applicable. Thus, a Nash equilibrium in bounded time stopping strategies exists, and, by Proposition 1, the equilibrium distribution F is unique.

4. EQUILIBRIUM ANALYSIS

4.1. *Convergence to the All-pay Contest*

This subsection considers the relationship between our model and the literature on all-pay contests. In a first step, we show that for vanishing noise the left endpoint of the equilibrium distribution converges to the starting point.

Lemma 11 *If the noise vanishes $\sigma \to 0$, the left endpoint $\underline{x}$ of the equilibrium distribution converges to zero, i.e., $\lim_{\sigma \to 0} \underline{x} = 0$.*

Proof: For any bounded stopping time, for any $\sigma > 0$, feasibility implies that $\underline{x} \leq 0$. By contradiction, assume there exists a constant ϵ such that $\underline{x} \leq \epsilon < 0$

[3] Ankirchner and Strack (2011) use a construction of the stopping time introduced for Brownian motion without drift in Bass (1983) and for the case with drift in Ankirchner, Heyne, and Imkeller (2008).

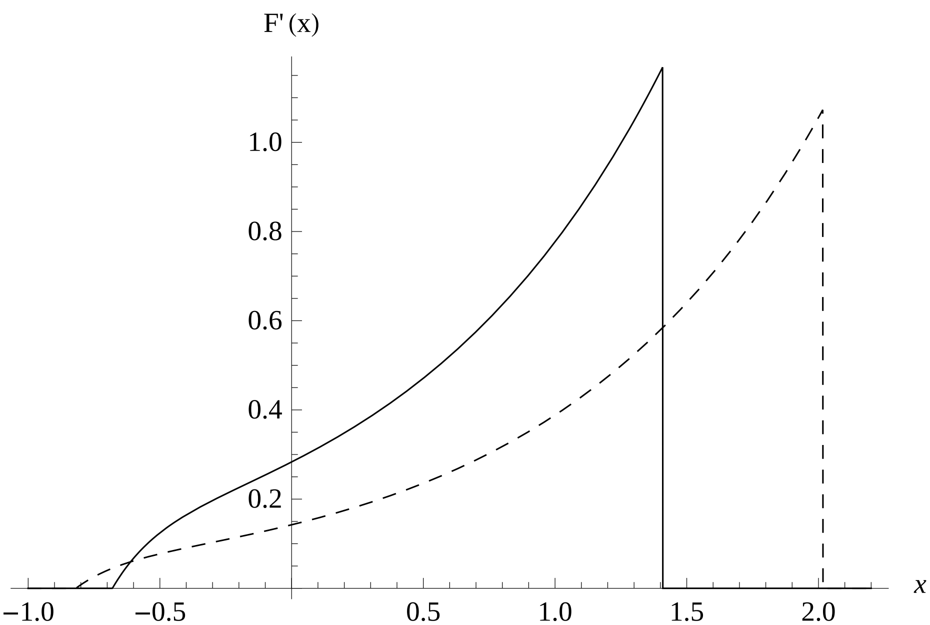

FIGURE 1.— The density function $F'(\cdot)$ for the parameters $n = 2$, $\mu = 3$, $\sigma = 1$ and the cost-functions $c(x) = \exp(x)$ solid line and $c(x) = \frac{1}{2}\exp(x)$ dashed line.

for all $\sigma > 0$. Then F' is bounded away from zero by

$$\begin{aligned} F'(x) &= \frac{F(x)^{2-n}}{n-1}\frac{2}{\sigma^2}(\int_{\underline{x}}^{x} c(z)\phi(x-z)\mathrm{d}z) \\ &\geq \frac{1}{n-1}\frac{2}{\sigma^2}\int_{\epsilon}^{x} c(z)\phi(x-z)\mathrm{d}z \\ &= \frac{1}{\mu(n-1)}\left(\inf_{y\in[\epsilon,\infty)} c(y)\right)(1-\phi(x-\epsilon)). \end{aligned}$$

For every point $x < 0$, $\lim_{\sigma\to 0}\phi(x) = \infty$. Thus, $\lim_{\sigma\to 0}\int_{\underline{x}}^{0} F'(x)\phi(x)\mathrm{d}x > 1$, which contradicts feasibility, because $\int_{\underline{x}}^{0} F'(x)\phi(x)\mathrm{d}x \leq \int_{\underline{x}}^{\overline{x}} F'(x)\phi(x)\mathrm{d}x = 1$. *Q.E.D.*

Taking the limit $\sigma \to 0$, the equilibrium distribution converges to

$$\lim_{\sigma\to 0} F(x) = \sqrt[n-1]{\frac{1}{\mu}\int_0^x c(z)\mathrm{d}z}.$$

This condition is well-known in the literature on static all-pay contests, which yields us the following theorem.

THEOREM 2 *For vanishing noise, the equilibrium distribution converges to the symmetric equilibrium distribution of an all-pay contest. In the case of constant costs, it converges to the symmetric equilibrium distribution of an all-pay auction.*

Thus, our model supports the use of all-pay auctions to analyze contests in which the variance is negligible. Figure 4.1 illustrates the similarity to the all-pay auction equilibrium if variance σ and costs $c(\cdot)$ are small in comparison to the drift μ.

Moreover, the symmetric all-pay auction has multiple equilibria—for a full characterization see Baye, Kovenock, and de Vries (1996). This paper offers a selection criterion in favor of the symmetric equilibrium, in which no participant places a mass point at zero. Intuitively, all other equilibria of the symmetric all-pay auction include mass points at zero for some players, which is not possible in our model for any positive σ by Lemma 2.

4.2. *Comparative Statics and Rent Dispersion*

Proposition 2 has linked all-pay contests with complete information to our model for the case of vanishing noise. In the following, we scrutinize how the predictions differ if there is a positive noise. In a symmetric all-pay contests with complete information, agents make zero profits in equilibrium. This does not hold true in our model for any positive level of variance σ:

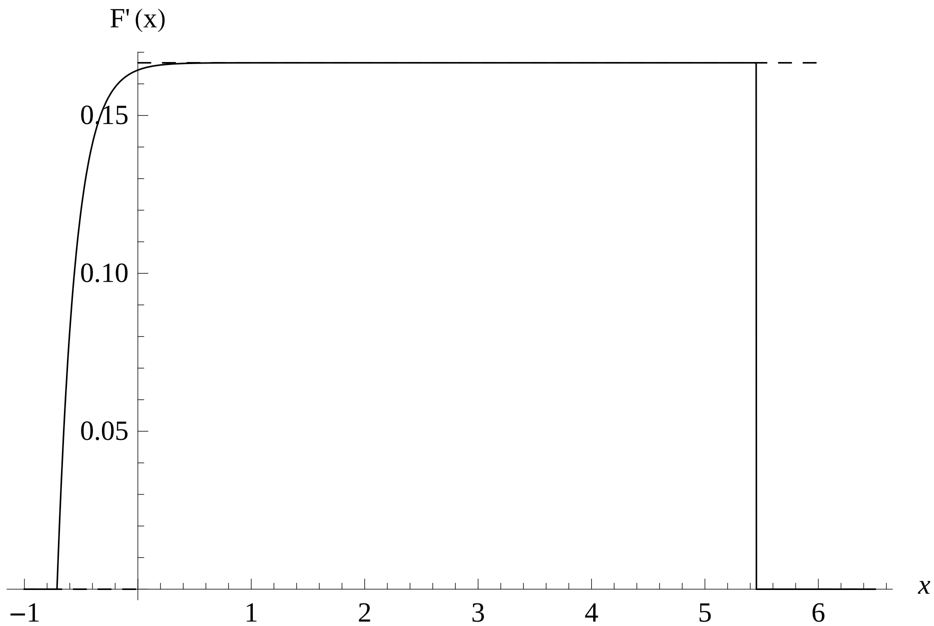

FIGURE 2.— This picture shows the density function $F'(\cdot)$ with support $[-0.71, 5.45]$ for the parameters $n = 2$, $\mu = 3$, $\sigma = 1$ and the cost-functions $c(x) = \frac{1}{2}$ (solid line) and for the same parameters the equilibrium density of the all-pay auction with support $[0, 6]$ (dashed line).

PROPOSITION 2 *In equilibrium, all agents make strictly positive expected profits.*

PROOF: In equilibrium, agents are indifferent between stopping immediately and the equilibrium strategy. Their expected profit is thus $u(0)$, which is strictly positive as $\underline{x} < 0$. *Q.E.D.*

Intuitively, private information about their research progress enables the agents to generate informational rents. A similar result is known in the literature on all-pay contests with incomplete information, see, e.g., Hillman and Riley (1989), Amann and Leininger (1996), and Moldovanu and Sela (2001). In their models, participants take a draw from a distribution prior to the contest, which determines their valuation. The valuation is private information. In contrast to this, private information about one's progress arrives continuously over time in our model.

From now on, we restrict attention to the special case of constant costs $c(x) = c$ for tractability. We define the support length as $\Delta = \overline{x} - \underline{x}$.

LEMMA 12 *If the number of players n increases, then the support length Δ remains constant and both endpoints increase.*

PROOF: If $c(x) = c$, $F(\overline{x}) - F(\underline{x})$ clearly depends only on Δ. Hence, for $F(\overline{x}) - F(\underline{x}) = 1$, Δ has to be constant. As F gets more concave if n increases, by feasibility $\underline{x} \nearrow$ and $\overline{x} \nearrow$. (Both functions intersect twice on $(\underline{x}, \overline{x})$.) *Q.E.D.*

PROPOSITION 3 *If the number of agents n increases, the expected profit of each agent decreases.*

PROOF: The function $u(x)$ depends only on $x - \underline{x}$. As n increases $\underline{x}$ increases by Lemma 12. Thus, the expected value of stopping immediately, $u(0)$, which is an optimal strategy in both cases, decreases as n increases. *Q.E.D.*

We now restrict attention to the two-player case $n = 2$. The next proposition derives a closed-form solution for the profits of each player.

PROPOSITION 4 *The equilibrium profit of each player depends only on the ratio $y = \frac{2\mu^2}{c\sigma^2}$. It is given by*

$$\frac{(1 - h(y))^2}{2y^2} - \frac{2\ln(1 - h(y)) - \ln(2y) - 1}{y},$$

with $h(y) = \exp(-y - 1 - W_0(\exp(-1 - y)))$, where W_0 is the principal branch of the Lambert W-function.

Given the previous proposition, it is simple to establish the main comparative statics result of this paper.

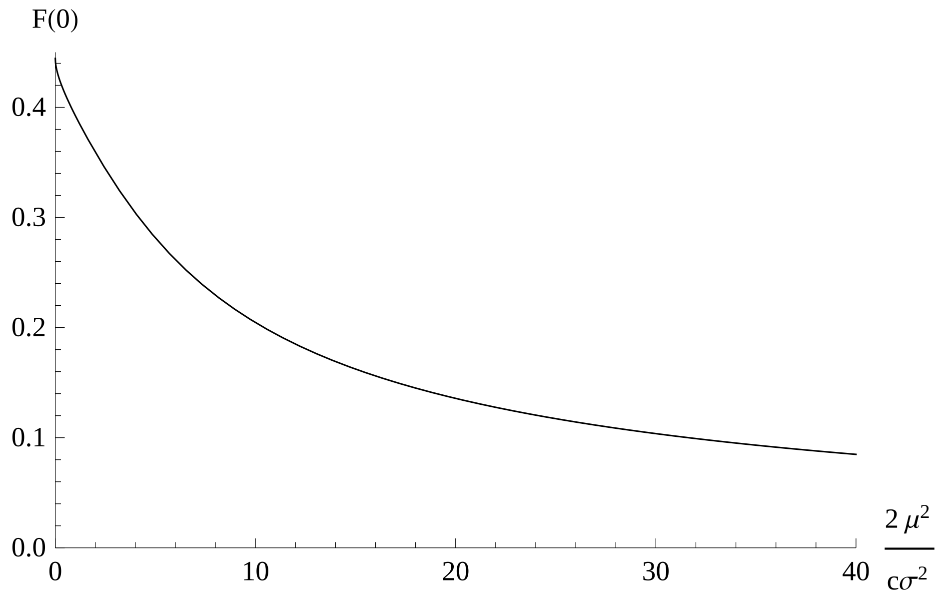

FIGURE 3.— This picture shows the equilibrium profit $F(0)$ of the agents on the y-Axes for $n = 2$ constant cost-functions $c(x) = c \in \mathbb{R}_+$ with $y = \frac{2\mu^2}{c\sigma^2}$ on the x-Axes.

THEOREM 3 *The equilibrium profit increases if costs c increase, variance σ^2 increases, or drift μ decreases.*

We can decompose the term $\frac{2\mu^2}{c\sigma^2}$, which determines the equilibrium profit of the players into two parts

$$\frac{2\mu^2}{c\sigma^2} = \underbrace{\frac{\mu}{c}}_{\text{Productivity}} \times \underbrace{\frac{2\mu}{\sigma^2}}_{\text{Signal to noise ratio}} .$$

The first term is the productivity $\frac{\mu}{c}$ of the agents. As firms get more productive, the competition gets more fierce and each firm makes less profits. The second term $\frac{2\mu}{\sigma^2}$ is the signal to noise ratio which measures the informativeness of $X^i_{\tau^i}$. Intuitively, if the signal to noise ratio decreases, the outcome $X^i_{\tau^i}$ becomes less correlated with agent i's effort choice τ^i. In turn, this reduces his incentives to exert effort and thereby the cost of his expected stopping time. As his winning probability in equilibrium remains constant, his profits are decreasing in the signal to noise ratio. Summarizing, participants prefer to have mutually worse—more costly, more random, or less productive—technologies.

Proposition 5 *The equilibrium profit of each agent is bounded from above by 4/9.*

Proof: The agents profit is decreasing in $y = \frac{2\mu^2}{c\sigma^2} \geq 0$ by Theorem 3. Hence, profits are bounded from above by $\lim_{y\to 0} u(0)$. By l'Hôpital's rule,

$$\lim_{y\to 0} \frac{(1-h(y))^2}{2y^2} - \frac{2\ln(1-h(y)) - \ln(2y) - 1}{y} = \frac{4}{9}\,.$$

Q.E.D.

Even for a perfectly uninformative signal, agents cannot extract the full prize of one from the principal without exerting any effort. Their expected equilibrium effort $\mathbb{E}(\tau^i)$ is bounded from below by

$$\begin{aligned} & \frac{4}{9} \geq \mathbb{E}(F(X^i_{\tau^i}) - c\tau^i) = \frac{1}{2} - c\mathbb{E}(\tau^i) \\ \Leftrightarrow \quad & \mathbb{E}(\tau^i) \geq \frac{1}{18c}\,. \end{aligned}$$

5. CONCLUSION AND DISCUSSION

In this paper, we have analyzed a new model of contests in continuous time in which each player learns only about his own research progress. Under mild assumptions on the cost function, the Nash equilibrium outcome in this model exists and is unique. If the research progress contains little uncertainty, the equilibrium is close to the symmetric equilibrium of a static all-pay auction. If the research outcome is uncertain, players prefer mutually higher costs of research, worse technologies and higher uncertainty. Intuitively, these factors make competition less fierce, since for a worse technology, the randomness has a higher influence on the equilibrium outcome. Hence, there is an additional way of collusion between the firms, which the principal should account for when designing the contest.

From a technical perspective, we have introduced a method to construct equilibria in continuous time games that are independent of the time horizon. Furthermore, we have introduced a constructive method to calculate a minimal time horizon that ensures the existence of such equilibria. These methodological contributions may prove fruitful in other models, and add to the general understanding of continuous time models.

6. APPENDIX

Proof of Lemma 1: As players have to use bounded time stopping strategies, each player i stops with positive probability on every subinterval of $[\underline{x}^i, \overline{x}^i]$. Hence, it suffices to show that at least two players have $\overline{x}$ as their right endpoint. Assume, by contradiction, only player i has $\overline{x}$ as his right endpoint. Denote $\overline{x}^{-i} = \max_{j\neq i} \overline{x}^j$. Then, for any $\epsilon > 0$, at $\overline{x}^{-i} + \epsilon$, player i strictly prefers to stop, which yields him the maximal possible winning probability of 1 without any additional costs. This contradicts the optimality of a strategy, which stops at $\overline{x}^i > \overline{x}^{-i} + \epsilon$. *Q.E.D.*

REMARK 1 *We write* $\tau^i_{(a,b)}(x)$ *shorthand for* $\inf\{t : X^i_t \notin (a,b) | X^i_s = x\}$ *in the next two proofs. Clearly,* $\tau^i_{(a,b)}(x)$ *is not a bounded time strategy, but we use it to bound the payoffs. Moreover, for sufficiently large time horizon* T*, the payoff from stopping at* $\min\{\tau^i_{(a,b)}(x), T\}$ *is arbitrarily close to that of* $\tau^i_{(a,b)}(x)$*.*

PROOF OF LEMMA 3: Assume $\overline{x}^j > \overline{x}^i$. For at least two players j, j', the payoff from $\tau^j_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)$ is weakly higher than from stopping at $X^j_t = \overline{x}^i$ by Lemma 1. By Lemma 2, at least one of these players—denote it j—wins with probability zero at $\underline{x}^j$. Note that $u^i(\overline{x}^i) = \prod_{h\neq i} F^h(\overline{x}^i) < \prod_{h \neq j} F^h(\overline{x}^i) = u^j(\overline{x}^i)$, because $F^i(\overline{x}^i) = 1 > F^j(\overline{x}^i)$.

Optimality of $\tau^j_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)$ implies

$$u^j(\overline{x}^i) \leq \mathbb{P}(X^j_{\tau^j} = \overline{x}^j | \tau^j_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)) u^j(\overline{x}^j) - \mathbb{E}(c(\tau^j_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)))\,.$$

On the other hand,

$$u^i(\overline{x}^i) < u^j(\overline{x}^i) \leq \mathbb{P}(X^i_{\tau^i} = \overline{x}^j | \tau^i_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)) u^i(\overline{x}^j) - \mathbb{E}(c(\tau^i_{(\underline{x}^j,\overline{x}^j)}(\overline{x}^i)))\,.$$

Hence, at $X^i_t = \overline{x}^i$, player i can profitably deviate by stopping at $\min\{\tau^i_{(a,b)}(x), T\}$ if he chooses a sufficiently large time horizon T. This contradicts the equilibrium assumption. *Q.E.D.*

PROOF OF LEMMA 4: To prove the first statement, we distinguish two cases.
(i) If at least two players have $F^i(\underline{x}) = 0$, then $u^i(\underline{x}) = 0\ \forall i$. Assume there exists a player j who makes less profit than a player i, where $\pi^i \leq \mathbb{P}(X^i_{\tau^i} = \overline{x} | \tau^i_{(\underline{x}^i,\overline{x})}(0)) - \mathbb{E}(c(\tau^i_{(\underline{x}^i,\overline{x})}(0)))$. If player j deviates to the strategy $\min\{\tau^j_{(\underline{x}^j,\overline{x})}(0), T\}$, player j gets a profit arbitrarily close to π^i; this contradicts optimality of player j's strategy.
(ii) If only one player has $F^i(\underline{x}) = 0$, then $u^i(\underline{x}) > 0$. We now consider the case in which this player i makes a weakly higher payoff than the remaining players, who make the same payoff each—otherwise the argument in the first part of the proof leads to a contradiction.
For any interval $I \in [\underline{x}, \overline{x}]$ in which player i stops with positive probability, by Lemma 1, there exists another player j who also stops in the interval. In particular, for $x \in I$, for any $\epsilon > 0$, we get

$$\mathbb{P}(X^i_{\tau^i} = \overline{x} | \tau^i_{(\underline{x},\overline{x})}(x)) + \mathbb{P}(X^i_{\tau^i} = \underline{x} | \tau^i_{(\underline{x},\overline{x})}(x)) u^i(\underline{x}) - \mathbb{E}(c(\tau^i_{(\underline{x},\overline{x})}(x))) < u^i(x) + \epsilon$$

and $\mathbb{P}(X^j_{\tau^j} = \overline{x} | \tau^j_{(\underline{x},\overline{x})}(x)) - \mathbb{E}(c(\tau^j_{(\underline{x},\overline{x})}(x))) \geq u^j(x)\ \forall j \neq i$.

For $\epsilon \to 0$, the two equations imply that $u^i(x) > u^j(x)$, for all $j \neq i$, for all x in the support of player i. Hence, $F^i(x) \leq F^j(x)\ \forall j$, for all x on the support of player i, and, by monotonicity of F^j, on $[\underline{x}, \overline{x}]$. Thus, the distribution of player i stochastically dominates that of all other players. This contradicts feasibility, since all players start at the same value and stopping times have to be bounded. The second statement of the lemma follows immediately from the proof of (ii). *Q.E.D.*

Proof of Lemma 5: Recall that all players have the same profit, and $u^i(\underline{x}) = 0\ \forall i$. Hence, if players i and j stop in each interval $I' \subset I = (a, b)$, then $u^i(x) = u^j(x)$ for all $x \in (a, b)$, since at 0, it is optimal for both to play until X_t reaches x or one endpoint. By the same argument, for any player h who does not stop on I and any player k who stops on I, we get $u^h(x) = \prod_{l \neq h} F^l(x) \leq \prod_{l \neq k} F^l(x) = u^k(x)$. This implies $F^h(x) \geq F^k(x)$.
Now take the supremum of all points $x \leq \overline{x}$ at which there exists an ϵ such that a player k does not stop in $(x - \epsilon, x)$. Clearly, at x, $F^i(x) = F^j(x)$ for all i, j. Take the highest point $\tilde{x}$ in $(\underline{x}, x - \epsilon)$ at which player k stops. Thus, $F^k(\tilde{x}) > F^i(\tilde{x})$. This contradicts $F^k(\tilde{x}) \leq F^i(\tilde{x})$ from the first part of the proof. *Q.E.D.*

Proof of Lemma 6: By definition, $u(x) = 0$, for all $x \leq \underline{x}$. Hence, the left derivative $\partial_- u(\underline{x})$ is zero. It remains to prove that the right derivative $\partial_+ u(\underline{x})$ is also zero. For a given $u : \mathbb{R} \to \mathbb{R}_+$, let $\Psi : \mathbb{R} \to \mathbb{R}$ be the unique function that satisfies the second order ordinary differential equation $c(x) = \mu \Psi'(x) + \frac{\sigma^2}{2} \Psi''(x)$ with the boundary conditions $\Psi(\underline{x}) = \partial_+ u(\underline{x})$ and $\Psi'(\underline{x}) = \partial_+ u(\underline{x})$. As $\Psi'(\underline{x}) > 0$, there exists a point $\hat{x} < \underline{x}$ such that $\Psi(\hat{x}) < 0 = u(\hat{x})$. Consider the strategy S that stops when either the point $\hat{x}$ or $\overline{x}$ is reached or at 1,

$$S = \min\{1, \inf\{t \in \mathbb{R}_+ : X_t^i \notin [\hat{x}, \overline{x}]\}\}.$$

As $u(\hat{x}) > \Psi(\hat{x})$, it follows that $\mathbb{E}(u(X_S)) > \mathbb{E}(\Psi(X_S))$. Thus,

$$\mathbb{E}(u(X_S) - \int_0^S c(X_t^i) \mathrm{d}t) > \mathbb{E}(\Psi(X_S) - \int_0^S c(X_t^i) \mathrm{d}t) .$$

Note that, by Itô's lemma, the process $\Psi(X_t^i) - \int_0^t c(X_s^i) \mathrm{d}s$ is a martingale. By Doob's optional sampling theorem, agent i is indifferent between the equilibrium strategy τ and the bounded time strategy S, i.e.,

$$\begin{aligned} \mathbb{E}(\Psi(X_S) - \int_0^S c(X_t^i) \mathrm{d}t) &= \mathbb{E}(\Psi(X_\tau) - \int_0^\tau c(X_t^i) \mathrm{d}t) \\ &= \mathbb{E}(u(X_\tau) - \int_0^\tau c(X_t^i) \mathrm{d}t) . \end{aligned}$$

The last step follows because $u(x)$ and $\Psi(x)$ coincide for all $x \in (\underline{x}, \overline{x})$. Consequently, the strategy S is a profitable deviation, which contradicts the equilibrium assumption. *Q.E.D.*

Proof of Theorem 1: The function Φ is Lipschitz continuous with constant $\frac{1}{\sqrt{2\pi}}$. Consequently, it suffices to prove the Lipschitz continuity of F^{-1} to get the Lipschitz continuity of $F^{-1} \circ \Phi$. The density $f(\cdot)$ is

$$\begin{aligned} f(x) &= \frac{F(x)^{-n+2}}{n-1} \frac{2}{\sigma^2} \int_{\underline{x}}^{x} c(z)\phi(x-z)\mathrm{d}z \\ &= \frac{F(x)^{-n+2}}{n-1} \frac{2}{\sigma^2} \left(\int_{\underline{x}}^{x} c(z)\mathrm{d}z - \mu F(x)^{n-1} \right) \end{aligned}$$

As $f(x) > 0$ for all $x > \underline{x}$, it suffices to show Lipschitz continuity of F^{-1} at 0. We substitute $x = F^{-1}(y)$ to get

$$(f \circ F^{-1})(y) \geq \frac{1}{n-1} \frac{2}{\sigma^2} \left(y^{2-n} \underbrace{(\min_{z \in [\underline{x}, \overline{x}]} c(z))}_{= \underline{c}} (F^{-1}(y) - F^{-1}(0)) - \mu y \right).$$

Rearranging with respect to $F^{-1}(y) - F^{-1}(0)$ gives

$$\begin{aligned} F^{-1}(y) - F^{-1}(0) &\leq \left(\frac{(n-1)\sigma^2}{2} (f \circ F^{-1})(y) + \mu y \right) \frac{y^{n-2}}{\underline{c}} \\ &\leq \left(\frac{(n-1)\sigma^2}{2} f(\overline{x}) + \mu \right) \frac{y^{n-2}}{\underline{c}}. \end{aligned}$$

This proves the Lipschitz continuity of $F^{-1}(\cdot)$ for $n > 2$. Note that for two agents $n = 2$ the function $F^{-1}(\cdot)$ is not Lipschitz continuous as $f(\underline{x}) = 0$. However, we show in the following paragraph that $F^{-1} \circ \Phi$ is Lipschitz continuous for $n = 2$.

$$\begin{aligned} F(x) &= \int_{\underline{x}}^{x} \frac{c(z)}{\mu} (1 - \phi(x-z)\mathrm{d}z \\ &\leq \underbrace{\left(\sup_{z \in [\underline{x}, \overline{x}]} \frac{c(z)}{\mu} \right)}_{= \overline{c}} \left(x - \underline{x} - \frac{\sigma^2}{2\mu} (1 - \phi(x - \underline{x})) \right) \end{aligned}$$

A second order Taylor expansion around $\underline{x}$ shows that, for an open ball around $\underline{x}$ and $\underline{x} < x$, we have the following upper bound

$$x - \underline{x} - \frac{\sigma^2}{2\mu}(1 - \phi(x - \underline{x})) \leq \frac{2\mu}{\sigma^2}(1 - \phi(x - \underline{x}))^2.$$

For an open ball around $\underline{x}$, we get an upper bound on $F(x) \leq \frac{2\overline{c}}{\sigma^2}(1 - \phi(x - \underline{x}))^2$ and hence the following estimate

$$1 - \phi(x - \underline{x}) \geq \sqrt{\frac{\sigma^2}{2\overline{c}} F(x)}.$$

We use this estimate to obtain a lower bound on $f(\cdot)$ depending only on $F(\cdot)$

$$\begin{aligned} f(x) &= \frac{2}{\sigma^2}\left(\int_{\underline{x}}^{x} c(z)\phi(x-z)\mathrm{d}z\right) \geq \frac{2\underline{c}}{\sigma^2}\left(\frac{\sigma^2}{2\mu}(1-\phi(x-\underline{x}))\right) \\ &\geq \frac{\underline{c}}{\mu}\sqrt{\frac{\sigma^2}{2\overline{c}}F(x)}\,. \end{aligned}$$

Consequently, there exists an $\epsilon > \underline{x}$ such that, for all $x \in [\underline{x}, \epsilon)$, we have an upper bound on $\frac{(\phi\circ\Phi^{-1}\circ F)(x)}{f(x)}$. Taking the limit $x \to \underline{x}$ yields

$$\lim_{x\to\underline{x}} \frac{(\phi\circ\Phi^{-1}\circ F)(x)}{f(x)} \leq \lim_{x\to\underline{x}} \frac{(\phi\circ\Phi^{-1}\circ F)(x)}{\frac{\underline{c}}{\mu}\sqrt{\frac{\sigma^2}{2\overline{c}}F(x)}} \leq \sqrt{\frac{2\overline{c}\mu^2}{\underline{c}^2\sigma^2}}\lim_{y\to 0}\frac{(\phi\circ\Phi^{-1})(y)}{\sqrt{y}} = 0\,.$$

Q.E.D.

Proof of Proposition 4: Rearranging the density condition $1 = F(\overline{x}) = \frac{c}{\mu}[\Delta - \frac{\sigma^2}{2\mu}(1-\phi(\Delta))]$ yields

$$\exp(\frac{-2\mu\Delta}{\sigma^2}) = -\frac{2\mu}{\sigma^2}[\Delta - (\frac{\mu}{c} + \frac{\sigma^2}{2\mu})].$$

The solution to the transcendental algebraic equation $e^{-a\Delta} = b(\Delta - d)$ is $\Delta = d + \frac{1}{a}W_0(\frac{ae^{-ad}}{b})$, where $W_0 : [-\frac{1}{e}, \infty) \to \mathbb{R}_+$ is the principal branch of the Lambert W-function. This branch is implicitly defined on $[-\frac{1}{e}, \infty)$ as the unique solution of $x = W(x)\exp(W(x))$, $W \geq -1$. Hence,

$$\Delta = \frac{\mu}{c} + \frac{\sigma^2}{2\mu}[1 + W_0(-\exp(-1 - \frac{2\mu^2}{c\sigma^2}))]\,,$$

and

$$\begin{aligned} \phi(\Delta) &= \exp(\frac{-2\mu^2}{c\sigma^2} - 1 - W_0(-\exp(-1 - \frac{2\mu^2}{c\sigma^2}))) \\ &= \exp(-1 - y - W_0(-\exp(-1-y))) \\ &= h(y)\,. \end{aligned}$$

Note that $\phi(\Delta)$ only depends on $y = \frac{2\mu^2}{c\sigma^2}$. Moreover, $h(y) = \phi(\Delta), h : \mathbb{R}_+ \to [0,1]$ is strictly decreasing in y as $W_0(\cdot)$ and $\exp(\cdot)$ are strictly increasing functions. For constant costs, the feasibility condition from Lemma 8 reduces to

$$\begin{aligned} 1 &= \int_{\underline{x}}^{\overline{x}} F'(x)\phi(x)\mathrm{d}x \\ &= \frac{c\sigma^2}{2\mu^2}[\frac{1}{2}\phi(\underline{x}) + \frac{1}{2}\phi(2\overline{x} - \underline{x}) - \phi(\overline{x})]\,. \end{aligned}$$

Dividing by $\phi(\underline{x})$ gives

$$\begin{aligned}
\phi(-\underline{x}) &= \frac{c\sigma^2}{2\mu^2}[\frac{1}{2}+\frac{1}{2}\phi(\Delta)^2-\phi(\Delta)] \\
&= \frac{1}{y}[\frac{1}{2}+\frac{1}{2}h(y)^2-h(y)] \\
&= \frac{1}{2y}(1-h(y))^2 \\
&= g(y)
\end{aligned}$$

Note that $g:\mathbb{R}_+\to[0,1]$ is strictly decreasing in y. We calculate $\underline{x}$ as

$$\underline{x} = -\phi^{-1}(\phi(-\underline{x})) = -\frac{\sigma^2}{2\mu}\ln(\frac{2\mu^2}{c\sigma^2[\frac{1}{2}+\frac{1}{2}\phi(\Delta)^2-\phi(\Delta)]}).$$

Plugging in $F(0)$ yields

$$\begin{aligned}
F(0) &= \frac{c}{\mu}[-\underline{x}-\frac{\sigma^2}{2\mu}(1-\phi(-\underline{x}))] \\
&= \frac{c\sigma^2}{2\mu^2}[\ln(\frac{\frac{2\mu^2}{c\sigma^2}}{\frac{1}{2}+\frac{1}{2}\phi(\Delta)^2-\phi(\Delta)})+\frac{\frac{1}{2}+\frac{1}{2}\phi(\Delta)^2-\phi(\Delta)}{\frac{2\mu^2}{c\sigma^2}}-1] \\
&= \frac{1}{y}[\ln(\frac{y}{\frac{1}{2}+\frac{1}{2}h(y)^2-h(y)})+\frac{\frac{1}{2}+\frac{1}{2}h(y)^2-h(y)}{y}-1] \\
&= \frac{1}{y}[g(y)-\ln(g(y))-1]
\end{aligned}$$

Hence, the value of $F(0)$ depends only on the value of the fraction $y=\frac{2\mu^2}{c\sigma^2}$ in the above way, which completes the proof. *Q.E.D.*

PROOF OF THEOREM 3: By Proposition 4, it suffices to show that the profit $F(0)$ is increasing in y. First observe that $x-\ln(x)$ is increasing in x and hence $g(y)-\ln(g(y))-1$ is decreasing in y as $g(y)$ is decreasing in y. Because $\frac{1}{y}$ is decreasing in y the product $\frac{1}{y}[g(y)-\ln(g(y))-1]$ is also decreasing in y. *Q.E.D.*

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