DYNAMIC STRATEGIC INFORMATION TRANSMISSION

By

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Abstract

This paper studies strategic information transmission in a dynamic environment where, each period, a privately informed expert sends a message and a decision maker takes an action. Our main result is that, in contrast to a static environment, full information revelation is possible. The gradual revelation of information and the eventual full revelation is supported by the dynamic rewards and punishments. The construction of a fully revealing equilibrium relies on two key features. The first feature is that the expert is incentivized, via appropriate actions, to join separable groups in which she initially pools with far-away types, then later reveals her type. The second feature is the use of trigger strategies. The decision maker is incentivized by the reward of further information revelation if he chooses the separation-inducing actions, and the threat of a stop in information release if he does not. Our equilibrium is non-monotonic. With monotonic partition equilibria, full revelation is impossible.

Keywords: asymmetric information; cheap talk; dynamic strategic communication; full information revelation.

1   Introduction

The seminal paper on strategic information transmission by Crawford and Sobel (1982) is a model of choice for a number of applications, ranging from economics and political science, to philosophy and biology.[1] In that paper, a biased and privately informed expert and a decision-maker interact only once. The conflict of interest results in coarse information revelation, and in some cases, in no information at all. We study strategic information transmission in a dynamic extension of the Crawford and Sobel setup. Each period the expert sends a message and a decision maker takes an action. Only the expert knows the state of the world which remains constant over time. We maintain all other features of the Crawford and Sobel (1982) environment, in particular, that the interests of the expert and the decision maker differ.

There are many environments in which information transmission is dynamic. Many sequential decisions have to take place, and the decision maker seeks the expert's advice prior to each one of them. The earlier literature on dynamic communication has focused on the role of reputation, see, for example, Sobel (1985), Morris (2000), and Ottaviani and Sorensen (2006), (2006b). Our paper focuses on the dynamics of strategic communication.

Our most surprising, and difficult to establish finding, is to show that full information revelation is possible. The construction of the fully revealing equilibrium relies on two key features. The first is the use of the separable groups: types which are far apart initially pool together and then separate. As such, there is gradual revelation of information. The second feature is that the dynamic nature of the game enables the use of "trigger strategies," in which the expert promises better advice in the future if the decision maker chooses an action beneficial for the expert now. That is, the use of trigger strategies is used to incentivize the decision maker, and to facilitate a gradual release of information that leads eventually to full revelation.

In a nutshell, in a multi-period interaction, communication can be facilitated by generating appropriate posteriors and by the use of trigger strategies, whereby the expert employs a signaling rule, that makes future information revelation credible if his advice is followed early on. We now provide more details on the roles of separable groups and trigger strategies in our construction of a fully revealing equilibrium.

We show that, over time, it is possible to divide all states into separable groups. A separable group is a set of types which are sufficiently far apart that each would reveal the truth, rather than mimic any other type in his group. Therefore, we create histories after which it is common knowledge that the decision maker puts probability one on a particular separable group, at which point the types in this group separate. The division of all types into separable groups is quite delicate, because, given that there is a continuum of types, we need to form a continuum of such

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[1]For a survey with applications across disciplines see Sobel (2008).

groups. The expert anticipates that if at some point he joins a separable group, then he will forgo his informational advantage. For the expert to join the separable group containing his true type, we have to make sure that he does not want to mimic a close-by type, who is supposed to join a different separable group. This is done via appropriate separation-inducing actions. The decision-maker is willing to choose such actions, despite their being non-myopically optimal, because the expert threatens to babble otherwise.

We emphasize another necessary condition for full information revelation. The fully revealing equilibrium cannot be a monotonic partition equilibrium. In a monotonic partition equilibrium, the expert employs a uniform signaling rule where each type sends one message with probability one, and the set of types that choose the same message is connected. We prove that if attention is restricted to monotonic partition equilibria, learning stops. Moreover, we argue that non-monotonic equilibria can be strictly Pareto superior to all dynamic monotonic equilibria. Overall, we conclude that monotonic partition equilibria constitute a special class of equilibria in our dynamic setup, whereas this is a canonical class of equilibria in the static setup of Crawford and Sobel (1982). Non-monotonic (or non-convex) equilibria also play also a prominent role in Baliga and Sjőstrőm (2011), who highlight the role of pooling extremists. Such equilibria also appear in Baliga and Sjőstrőm (2008), who, building on Baliga and Sjőstrőm (2004), show how strategic ambiguity can reduce arm proliferation.

Welfare properties of equilibria are another difference between the dynamic and the static strategic communication games. Crawford and Sobel (1982) show that, under some assumptions, both the expert and the decision maker ex-ante (before the state is realized) prefer equilibria with higher number of partitions. We provide an example that shows that it is not necessarily the case for dynamic equilibria. More partitions of the possible states of the world (i.e. larger number of equilibrium vectors of actions) do not guarantee higher expected payoff for the expert and the decision maker. The reason is as follows. In the dynamic setup the intertemporal nature of the indifference conditions that pin down the cutoffs that characterize partition equilibria allow for a much richer set of possibilities which overturn the static Pareto comparison results. A similar phenomenon occurs when the communication is noisy, as shown in an example of the working paper version of Blume, Board, and Kawamura (2007). In their example, a two-step partition Pareto dominates a three-step partition. Finally, we present an example in which dynamic monotonic partition equilibria can be strictly Pareto superior to playing the most informative one-shot equilibrium in the first period, and babbling thereafter.

Our work shows that the nature of dynamic strategic communication is quite distinct from its static counterpart. In the static case, because of the conflict of interest between the decision-maker and the expert, nearby expert types have an incentive to pool together, precluding full

information revelation. The single-crossing property also implies that in equilibrium, the action is a monotonic step function of the state. These two forces make complex signaling (even though possible) irrelevant, given that it is observationally equivalent to partitional signaling, and imply that only local incentive compatibility matters. In the dynamic setup the key difference is that today's communication sets up the stage for tomorrow's communication. Complex signaling helps in the dynamic setup, because it can lead to posteriors where possible states are sufficiently far apart that they have no incentive to mimic each other. Sustaining in equilibrium such posteriors requires complex signaling where one needs necessarily to worry about global incentive compatibility.

Related Literature

Crawford and Sobel (1982) is the seminal contribution on strategic information transmission. That paper has inspired an enormous amount of theoretical work and myriads of applications. Here we study a dynamic extension. Other papers that look at dynamic relations between one expert and one receiver differ from ours because, as already mentioned, their focus is on the sender's reputation. Other papers have a dynamic aspect because they allow for multi-round communication protocols, but with a single round of action(s). Aumann and Hart (2003) characterize geometrically the set of equilibrium payoffs when a long conversation is possible. In that paper, two players–one informed and one uniformed, play a finite simultaneous move game. The state of the word is finite, and players engage in direct (no mediator) communications (possibly infinitely long exchange of messages), before both choosing simultaneously costly actions. In contrast, in our model only the informed party sends cheap messages–the uniformed party chooses actions and the state is infinite. Krishna and Morgan (2004) add a long communication protocol to Crawford and Sobel (1982)'s game, whereas Goltsman Hoerner, Pavlov and Squintani (2009) characterize such optimal protocols. Forges and Koessler (2008a, 2008b) allow for a long protocol in a setup where messages can be certifiable. In all those papers once the communication phase is over, the decision maker chooses one action. In our paper, there are multiple rounds of communication and actions (each expert's message is followed by an action of the decision maker).

In our setup, the dynamic nature of communication enables full information revelation. In contrast, full information revelation is not possible in the dynamic setup of Andrelini, Gerardi, and Lagunoff (2008) who consider dynamic strategic communication is a dynastic game and show that if preferences are not fully aligned "full learning" equilibria do not exist. Renault, Solan, and Vieille (2011) examine dynamic sender-receiver games, in a setup where the information is not fully persistent and the state is finite. They restrict attention to Markov equilibria and characterize the set of equilibrium payoffs. In contrast, we assume fully persistent information and a state drawn from a continuum, and focus mainly on non-markovian equilibria. Ivanov (2011) allows for

a dynamic communication protocol in a setup where the expert is also initially uninformed and the decision maker controls the quality of information available to the expert.

Our model bears some similarities to the static strategic communication with multiple receivers. In those models the expert cares about a sequence of actions, but in contrast to our model, those actions are chosen by different individuals. An important difference is that in our model the receiver cares about the entire vector of actions chosen, compared to those models where each receiver cares only about his own action. Still, some of the properties of the equilibria that we obtain also appear in the models with multiple receivers. For example, our non-monotonic example presented in Section 4 resembles Example 2 of Goltzman and Pavlov (2008). It is also similar to Example 2 in Krishna and Morgan (2004).[2]

Full information revelation is possible in other variations of the Crawford and Sobel (1982) setup: When the decision maker consults two experts as in Battaglini (2002), Eso and Fong (2008) and Ambrus and Lu (2010);[3] when information is completely or partially certifiable, as in Mathis (2008); and when there are lying costs and the state is unbounded as in Kartik, Ottaviani and Squintani (2007). In the case of multiple experts, playing one against the other is the main force that supports truthful revelation. In the case of an unbounded state, lying costs become large and support the truth. In the case of certifiable information, one can exploit the fact that messages are state-contingent to induce truth telling. All these forces are very different from the forces behind our fully revealing construction.

2   The Environment

We extend the classical model of Crawford and Sobel (1982) to a dynamic setting. There are two players: an Expert - E (sender) and a Decision Maker - DM (receiver) who interact for finitely many periods $t = 1, ..., T$. The expert knows the state of the world $\theta \in [0, 1]$ which is constant over time and is distributed according to the c.d.f. F. At the beginning of each period t, the expert sends a (possibly random) message m_t which depends on the realization of the state θ to the decision maker. The decision maker then updates his beliefs about the state and chooses an action $y_t \in \mathbb{R}$ that affects both players' payoffs. There is a conflict of interest because, for all states θ, the expert's ideal choice of action differs from the decision maker's. This is captured by a scalar b, the bias parameter of the expert. The discount factor of the expert and the decision maker is $\delta_i \in [0, 1]$ for $i \in \{E, DM\}$. When the state is θ and the decision maker chooses actions $y^T = (y_1, .., y_T)$, the expert's payoff is given by

$$U^E(y^T, \theta, b) = \Sigma_{t=1}^T \delta_E^{t-1} u^E(y_t, \theta, b),$$

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[2]Equilibria can be non-monotonic also in environments where the decision maker consults two experts as in Krishna and Morgan (2001).

[3]But full information is not possible in the setup of Levy and Razin (2007).

and the decision maker's payoff is given by

$$U^{DM}(y^T, \theta) = \Sigma_{t=1}^T \delta_{DM}^{t-1} u^{DM}(y_t, \theta).$$

We assume that period payoffs $u^E(y_t, \theta)$ and $u^{DM}(y_t, \theta, b)$ satisfy the conditions imposed by Crawford and Sobel (1982), namely that (i) u^i is twice differentiable, for all θ; (ii) $\frac{\partial u^i(y, \theta)}{\partial y} = 0$ for some y for $i \in \{E, DM\}$; (iii) that $\frac{\partial^2 u^i(...)}{\partial y^2} < 0$; and (iv) a single crossing condition, $\frac{\partial^2 u^i(...)}{\partial \theta \partial y} > 0$. Conditions (ii) and (iii) guarantee a unique maximum in y.

Let $y^{DM}(\theta)$ and $y^E(\theta)$ denote, respectively, the decision maker's and the expert's most preferred action when the true state θ is known by both players . The single crossing condition ensures that $y^{DM}(\theta)$ and $y^E(\theta)$ are increasing in θ. We assume that, regardless of the state θ, a conflict of interest exists: $y^{DM}(\theta) \neq y^E(\theta)$, for all θ.

The decision maker observes his payoffs only at the end of the game. He does not observe his payoff at each period t. Otherwise, the problem would be trivial because the decision maker can learn the expert's information by simply inverting his payoff.

A *strategy profile* $\sigma = (\sigma_i)_{i=E,DM}$, specifies a strategy for each player. Let h_t denote a history that contains all the reports submitted by the expert $m^{t-1} = (m_1, ..., m_{t-1})$, and all actions chosen by the decision maker $y^{t-1} = (y_1, ..., y_{t-1})$ up to, but not to including stage t. The set of all feasible histories at t is denoted by H_t. A behavioral strategy of the expert, σ_E, consists of a sequence of signaling rules that map $[0, 1] \times H_t$ to a probability distribution over reports $\mathcal{M}$. Let $q(m | \theta, h_t)$ denote the probability that the expert reports message m at history h_t when his type is θ. A strategy for the decision maker, σ_{DM}, is a sequence of maps from H_t to actions. We use $y_t(m | h_t) \in \mathbb{R}$ to denote the action that the decision maker chooses at h_t given a report m. A *belief system*, μ, maps H_t to the set of probability distributions over $[0, 1]$. Let $F(\theta | h_t)$ denote the decision maker's beliefs about the experts's type after a history h_t, $t = 1, ..., T$. A strategy profile σ and a belief system μ is an assessment. We seek strategy profiles and belief systems that form *Perfect Bayesian Equilibria, (PBE)*.

In the paper we use the terminology as follows.

Definition 1 *An equilibrium is called babbling if for all m with $q(m | \theta, h_t) > 0$, all $\theta \in [0, 1]$, all h_t and t, we have that $y_t(m | h_t) = \hat{y}$.*

In other words, we call an equilibrium babbling, if the same action is induced with probability one for all states $\theta \in [0, 1]$ and all $t \in T$.

Definition 2 *We call a signaling rule q uniform, if $q(m | \theta, h_t)$ is uniform, with support on $[\theta_i, \theta_{i+1}]$ if $\theta \in [\theta_i, \theta_{i+1}]$.*

Definition 3 *A partition equilibrium is one in which, at each period t and history h_t, the expert employs only uniform signaling rules.*

In other words, at a partition equilibrium, the expert follows a pure strategy in which, for any message m, the set of types sending message m is connected.

Definition 4 *An equilibrium is fully revealing if there exists a $\hat{T} \leq T$ such that after all histories along the equilibrium path leading to $\hat{T}$, the expert sends for all $\theta \in [0, 1]$ a different message with probability one, inducing $y_t(\theta) = \theta$ from $t = \hat{T}$ on.*

3 Uniform Signaling

We first briefly summarize the static cheap talk game of Crawford and Sobel (1982) in which uniform signaling rules are the canonical form of communication in static games. We then study properties of uniform signaling in our dynamic setup.

3.1 The Canonical Static Communication

We start by reviewing the Crawford and Sobel (1982) findings. For the static communication and decision game they show that all equilibria are equivalent to partition equilibria. The expert employs simple uniform signaling rules. He follows a pure strategy in which, for any message m, the set of types sending message m is connected. This type of communication induces actions which are increasing step functions of the state. Hence, intervals of types pool together by sending the same message, ultimately, inducing the same action by the decision maker. This implies that communication is coarse. Even though the state θ takes a continuum of values, only finitely many different actions are induced.

The intuition behind this result can be summarized as follows. Fix an equilibrium of the one-shot game and let $y(\theta)$ denote an action induced when the state is θ. The conflict of interest between the expert and the decision maker ($y^{DM}(\theta) \neq y^E(\theta)$, for all θ) implies that there are at most finitely many actions induced. The fact that u^E is strictly concave in y, together with single-crossing and the finiteness of actions, implies that there are at most finitely many states θ where the expert is indifferent between two induced actions. Then, $y(\theta)$ can be, without loss of generality, taken to be single-valued. Because y takes at most finitely many values, it is an increasing step function. The state space is divided into a finite number of subintervals, and y takes a different value in each subinterval. Importantly, Crawford and Sobel (1982) show that, without loss of generality, equilibrium induced actions, y can be taken to arise from uniform signaling rules. This result follows from the observation that all messages inducing the same action y can be replaced by a single message. Therefore, more complex signaling rules play no role in the static setup.

3.2 Uniform Signaling: A Special Kind of Dynamic Communication

We now focus on simple partitional communication protocols (uniform signalling) and study their properties in our dynamic setup. We show two results. The first result is that with monotonic partition equilibria the decision maker never learns the truth. The second result establishes that when the bias is so large that only one action is induced at an equilibrium of the static game, the same is true also in dynamic monotonic partition equilibria. Finally, an example shows that dynamic partitions can differ and can be more informative compared to static ones.

In constructing partition equilibria, the main difference from the static case is that the expert's indifference conditions that characterize the cutoffs of the partitions are intertemporal. This makes the explicit solution of the resulting difference equation difficult even in simple environments, such as in the uniform quadratic loss case. To overcome this difficulty, we notice that at some point in the game the expert's indifference conditions reduce to the static ones and then apply the results of Crawford and Sobel (1982).

Using this approach, we first show that among the class of monotonic partition equilibria, there exists no equilibrium in which the decision maker eventually learns the state of the world for all $\theta \in [0, 1]$. That is, there exist no fully revealing equilibria.

Proposition 1 *For all T, there exist no fully revealing monotonic partition equilibria.*

This result follows almost immediately from Crawford and Sobel (1982). A short sketch of the argument is as follows. First, note that by definition, in a fully revealing monotonic partition equilibrium, there must exist a period in the game at which a continuum of possible actions is induced. Now, suppose by contradiction that there exists such an equilibrium. Then, there exists a period $\hat{T} \leq T$ in which the last subdivision occurs, with $y_t(\theta) = \theta$, for all $t \geq \hat{T}$. At $\hat{T}$ the expert's indifference condition reduces to the static one. Since $y_t(\theta) = \theta$ (respectively $y_t(\theta + \varepsilon) = \theta + \varepsilon$) from $\hat{T}$ onwards then:

$$\left(T - \hat{T}\right)\left(\delta + \delta^2 + ... + \delta^{T-\hat{T}}\right) u^E\left(y_t(\theta), \theta\right) \geq \left(T - \hat{T}\right)\left(\delta + \delta^2 + ... + \delta^{T-\hat{T}}\right) u^E\left(y_t(\theta + \varepsilon), \theta\right),$$

and similarly for state $\theta + \varepsilon$. These conditions are equivalent to the static conditions in Crawford and Sobel (1982), who proved that they imply that at most finitely many actions can be induced at an equilibrium of a static game, a contradiction to full revelation.

We now proceed to show that if all static equilibria are equivalent to babbling then all dynamic monotonic partition equilibria are equivalent to babbling.

Proposition 2 *If all static equilibria are equivalent to the babbling equilibrium, then all dynamic monotonic partition equilibria are equivalent to babbling.*

Proof. See Appendix A. ∎

It is also easy to see that the payoffs of both the expert and the decision maker of any static equilibrium can be achieved as a per period payoffs in the dynamic communication game. We can construct an "equivalent" (in terms of payoffs) dynamic equilibrium as follows: In the first period, all types follow the same strategies as in the static equilibrium. For all $t > 1$ experts of all types "babble", and the decision maker repeats the same action she took in period 1. It is trivial to verify that these strategies constitute an equilibrium.

In Appendix B, we show an example when in our dynamic setup ($T = 2$) both players can be strictly better-off compared to the best static equilibrium. In that example, $\delta_E = \delta_{DM} = 1$, the state is uniformly distributed on $[0, 1]$ and preferences satisfy

$$u^E(y, \theta) = -(y - \theta - b)^2 \text{ and } u^{DM}(y, \theta, b) = -(y - \theta)^2, \tag{1}$$

with $b = \frac{1}{12}$. At the most informative static equilibrium the state space is divided in two sub-intervals: $[0, \frac{1}{3}]$ and $[\frac{1}{3}, 1]$ inducing two actions $\frac{1}{6}$ and $\frac{4}{6}$. We show that when $T = 2$, there exists a partition equilibrium where the state space is divided ultimately in three sub-intervals: $[0, 0.25]$, $[0.25, 0.45833]$ and $[0.458\,33, 1]$.

However, in dynamic settings it is also possible to have equilibria with more partitions that are inferior to ones with less (we present an example of such an equilibrium in Appendix C). This happens because a larger ultimate number of partitions may require extensive pooling earlier on, inducing overall lower welfare. This finding is in contrast to Crawford and Sobel (1982), who show that under their Assumption M equilibria can be easily Pareto ranked: both the expert and the decision maker ex-ante (before the state is realized) prefer the equilibrium with the highest number of partitions and actions induced.[4] Our findings suggest that Pareto comparisons in dynamic cases are less straightforward, even if we restrict attention to monotonic partitional equilibria.

We proceed to study the role of complex signalling in our dynamic game.

4 An Example with Complex Signaling and Dynamic Information Revelation

In this section we present an example where the expert employs a complex signalling rule that induces an action of the decision maker that is not monotonic in the state. We show that, for a range of biases, this equilibrium is superior to all monotonic ones.

In the example below, the bias is so severe that if communication and the decision were static

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[4]The equilibrium with the largest number of partitions is the only equilibrium that satisfies the "no incentive to separate" (NITS) condition (Chen, Kartik and Sobel (2008)).

(one round), as in Crawford and Sobel (1982), all equilibria are equivalent to babbling. We show, that even in these extreme bias situations, some information is released even with two rounds. This equilibrium has the feature that the decision maker learns the state quite precisely when the news is either horrific, or terrific, but remains agnostic for intermediate levels.

Example 1 *Dynamic equilibria can be non-monotonic*

Consider the two period example where $\delta_E = \delta_{DM} = 1$, types are uniformly distributed on $[0, 1]$ and preferences are given by (1). Let the expert use the signaling rule as follows.

In period 1, the expert for states in $[0, \underline{\theta}) \cup (\bar{\theta}, 1]$ sends a message $m_{1(1)}$ with probability 1. For states in $[\underline{\theta}, \bar{\theta}]$, he sends a message $m_{1(2)}$ with probability 1.

In period 2, the expert adopts the following signaling rule. For states in $[0, \underline{\theta})$ the expert sends a message $m_{2(1)}$ with probability 1, for states in $[\underline{\theta}, \bar{\theta}]$ he sends a message $m_{2(2)}$ with probability 1, and for states in $(\bar{\theta}, 1]$ he sends a message $m_{2(3)}$ with probability 1. To sum up, we look for an equilibrium with the following signaling rule:

$$\text{types in } [0, \underline{\theta}) \text{ send message sequence } A = (m_{1(1)}, m_{2(1)}),$$
$$\text{types in } [\underline{\theta}, \bar{\theta}] \text{ send message sequence } B = (m_{1(2)}, m_{2(2)}),$$
$$\text{types in } (\bar{\theta}, 1] \text{ send message sequence } C = (m_{1(1)}, m_{2(3)}).$$

With this signaling rule, the optimal actions of the decision maker for periods 1 and 2 are, respectively, given by:

$$
\begin{aligned}
t &= 1: \ y_{1(1)} = \frac{\underline{\theta}^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)}; \ y_{1(2)} = \frac{\underline{\theta} + \bar{\theta}}{2}. \\
t &= 2: \ y_{2(1)} = \frac{\underline{\theta}}{2}; \ y_{2(2)} = \frac{\underline{\theta} + \bar{\theta}}{2}; y_{2(3)} = \frac{1 + \bar{\theta}}{2}.
\end{aligned}
$$

After any out of equilibrium message the decision maker assigns probability 1 to the state belonging to $[\underline{\theta}, \bar{\theta}]$, inducing an action equal to $y^{out} = \frac{\underline{\theta} + \bar{\theta}}{2}$. With these out-of equilibrium beliefs no type of the expert has an incentive to send an out-of-equilibrium message.

In order for this to be an equilibrium, type $\underline{\theta}$ must be indifferent between message sequence A and B:

$$
-(\frac{\underline{\theta}^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)} - \underline{\theta} - b)^2 - (\frac{\underline{\theta}}{2} - \underline{\theta} - b)^2 = -2(\frac{\underline{\theta} + \bar{\theta}}{2} - \underline{\theta} - b)^2 \tag{2}
$$

and type $\bar{\theta}$ must be indifferent between message sequence B and C :

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$$-(\frac{\underline{\theta}^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)} - \bar{\theta} - b)^2 - (\frac{1 + \bar{\theta}}{2} - \bar{\theta} - b)^2 = -2(\frac{\underline{\theta} + \bar{\theta}}{2} - \bar{\theta} - b)^2. \tag{3}$$

At $t = 2$ it must also be the case that type $\underline{\theta}$ prefers $m_{2(1)}$ to $m_{2(3)}$, and the reverse for type $\bar{\theta}$:

$$-(\frac{\underline{\theta}}{2} - \underline{\theta} - b)^2 \geq -(\frac{1 + \bar{\theta}}{2} - \underline{\theta} - b)^2 \tag{4}$$

$$-(\frac{1 + \bar{\theta}}{2} - \bar{\theta} - b)^2 \geq -(\frac{\underline{\theta}}{2} - \bar{\theta} - b)^2. \tag{5}$$

A solution of the system of equations (2) and (3) gives an equilibrium if $0 \leq \underline{\theta} < \bar{\theta} \leq 1$. We solved this system numerically and found that the highest bias that makes this non-monotonic equilibrium survive is $b = 0.256$. For $b = 0.256$, we obtain $\underline{\theta} = 0.0581$, $\bar{\theta} = 0.9823$. The corresponding optimal actions for period 1 are: $y_{1(1)} = 0.253$, $y_{1(2)} = 0.52$, and for period 2 are: $y_{2(1)} = 0.029$, $y_{2(2)} = 0.52$, $y_{2(3)} = 0.991$. The first period action is non-monotonic. It is straightforward to check that the indifference conditions (2) and (3) suffice for global incentive compatibility.[5]

An important feature of the constructed non-monotonic equilibrium is that it exists even though the only monotonic partition equilibrium in this (dynamic) game is a trivial babbling equilibrium. For bias $b = 0.256$, the only equilibrium in the static game is babbling. Proposition 2 then implies that the only monotonic equilibrium in the dynamic game is babbling. At this equilibrium the decision maker takes the action $y^B = 0.5$. The expert's and the decision maker's expected disutilities are respectively given by: -0.298 and -0.167. Both the expert and the decision maker are strictly better-off at our non-monotonic equilibrium, where the disutilities are given by -0.275 for the expert and by -0.144 for the decision maker, compared to the most informative static equilibrium.

This example highlights how complex signalling can lead to better communication in a multi-stage games. By pooling together the best and the worst states at $t = 1$, the expert is willing at $t = 2$ to reveal whether the state is very good or very bad. It also has the following implication.

Proposition 3 *There exists non-monotonic equilibria that are Pareto superior to all monotonic partition equilibria.*

Given that for preferences given by (1) and $b > 0.25$, the only equilibrium in the static game is babbling, this Proposition follows immediately from the preceding discussion.

We now construct a non-monotonic equilibrium with even higher welfare properties. The key is to allow the expert to condition his future communication of the previously chosen action of the decision maker. We call such strategies trigger strategies.

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[5] In other words, we need to check that all types in $[0, 0.058)$ prefer message sequence A, types in $[0.058, 0.982]$ prefer message sequence B and types in $(0.982, 1]$ prefer message sequence C. A way to do that is to graph $U(A, \theta) - U(B, \theta)$ and $U(B, \theta) - U(C, \theta)$ and verify that $U(A, \theta) - U(B, \theta) > 0$ for all $\theta \in [0, 0.058)$, whereas $U(A, \theta) - U(B, \theta) < 0$ for $\theta > 0.0580$, and that $U(B, \theta) - U(C, \theta) > 0$ for all $\theta < 0.982$, whereas $U(B, \theta) - U(C, \theta) < 0$ for all $\theta > 0.982$.

Now the expert uses the following signalling rule: if the state is in $[0, 0.0642] \cup [0.9919, 1]$ he tells the DM at $t = 1$ to choose 0.2615 and to choose 0.528 otherwise. The first action is above the myopic best response, which is $y_{1(1)} = \frac{\theta^2 - \bar{\theta}^2 + 1}{2(\underline{\theta} - \bar{\theta} + 1)} = 0.14008$. If the DM chooses the action recommended by the expert, then the expert at $t = 2$ reveals whether the state is $[0, 0.0642]$ or $[0.9919, 1]$; otherwise, the expert threatens to babble. It is routine to check that this is an equilibrium. Moreover, at this equilibrium, the expert and the decision maker are even better off compared to our original non-monotonic equilibrium. Their disutilities are respectively given by: -0.2674 for the expert and by -0.1409 for the decision maker. The reason for the improvement, is that relying on a threat, we relax the best response constraint of the decision maker which increases the expected payoff of expert and, as a consequence, that of the decision maker (since their ex ante expected payoffs differ by b^2).

Our main result in the section that follows is that the dynamic set-up correlates the incentives of the expert and DM in such a way that *full information revelation* is possible. We present this next.

5   Main Result: Full information revelation in dynamic cheap talk games

One of the most stark results of the static strategic communication game is that there is no equilibrium with full information revelation. Although the state can take a continuum of values, the expert sends at most finitely many signals to the decision maker. That is, substantial amount of information is not transmitted. In this section, for our dynamic strategic communication model we construct an equilibrium that is fully revealing. The construction of the equilibrium relies on two tools: the use of separable groups, and the use of trigger strategies. These tools have no leverage in single-round communications, but are powerful in dynamic communications.

5.1   Simple Case: Learning the Truth when the Expert is Myopic

We first show how to construct a fully-revealing equilibrium when the expert is myopic. The example shows how we use separable groups of far away agents who mimic each other but not the nearby types. In the case of the myopic expert, there is no need to use trigger strategies which we introduce in the example of the patient expert.

There are two essential ingredients of this example. First, the set of types that choose the same message at $t = 1$ are sufficiently far apart so they can be separated at $t = 2$. That is, each message is chosen by a separable group of types (a group of types which are apart). Second, the average of types in each group is constant. The first period action is then flat. Irrespective of the state, all

experts induce the same action. This implies that the expert does not care which group he joins (since a myopic expert cares only about the 1st-period action, which is constant across groups).

Example 2 *Fully revealing equilibrium with impatient experts* $(\delta_E = 0)$.

Consider the uniform quadratic loss case as in (1) when the bias is $b < \frac{1}{16}$ and $T = 2$. The expert is impatient $(\delta_E = 0)$. The construction works for any discount factor for the decision maker. In period 1, the expert for states $\{\frac{1}{8} - \varepsilon; \frac{3}{8} + \varepsilon, \frac{4}{8} + \varepsilon, 1 - \varepsilon\}$ sends message m_ε, for $\varepsilon \in [0, \frac{1}{8}]$. For states $\{\frac{1}{8} + \tilde{\varepsilon}, \frac{7}{8} - \tilde{\varepsilon}\}$, the expert sends a message $m_{\tilde{\varepsilon}}$ for $\tilde{\varepsilon} \in (0, \frac{1}{4})$. That is, we have two types of separable groups: indexed by ε and $\tilde{\varepsilon}$. Given this signalling rule, the best response of the decision maker at $t = 1$ is to choose:

$$
\begin{aligned}
y(m_\varepsilon) &= \frac{\frac{1}{8} - \varepsilon + \frac{3}{8} + \varepsilon + \frac{4}{8} + \varepsilon + 1 - \varepsilon}{4} = 0.5 \text{ for all } \varepsilon \in [0, \frac{1}{8}], \\
y(m_{\tilde{\varepsilon}}) &= \frac{\frac{1}{8} + \tilde{\varepsilon} + \frac{7}{8} - \tilde{\varepsilon}}{2} = 0.5 \text{ for all } \tilde{\varepsilon} \in (0, \frac{1}{4}).
\end{aligned}
$$

In period 2, the expert tells the truth, inducing an action equal to the state. After any out-of-equilibrium message at $t = 1$, the decision maker assigns equal probability to all states, leading to action $y^{out} = 0.5$. After any out-of-equilibrium message at $t = 2$, the decision maker believes that the state is equal to the smallest state that is possible, leading to an action equal to $y^{out} = 0$.

If the expert is impatient $(\delta_E = 0)$, then for any $b < \frac{1}{16}$ we argue that this is an equilibrium.

First, notice that all messages (even out-of-equilibrium ones) induce the same action at $t = 1$. Hence, at least from the $t = 1$ perspective, all types of the experts are indifferent among them. Now consider, for example, the history after m_ε was chosen at stage 1. At this history the posterior of the decision maker at $t = 2$ puts weight $\frac{1}{4}$ on the following types: $\frac{1}{8} - \varepsilon; \frac{3}{8} + \varepsilon, \frac{4}{8} + \varepsilon$ and $1 - \varepsilon$. If at stage 2 each of these types sends a different message, namely $m_{2(k)}$, where $k \in \{\frac{1}{8} - \varepsilon; \frac{3}{8} + \varepsilon, \frac{4}{8} + \varepsilon, 1 - \varepsilon\}$, then after each of these messages, the decision maker chooses

$$
y(m_{2(k)}) = k.
$$

To see that at $t = 2$ the expert has no incentive to deviate, we consider, for instance, type $\theta = \frac{3}{8} + \varepsilon$ and show that he does not have incentive to mimic type $\theta + \frac{1}{8}$:

$$
\begin{aligned}
-(\theta - \theta - b)^2 &> -(\theta + \frac{1}{8} - \theta - b)^2, \\
(\frac{1}{8} - b - b)\frac{1}{8} &> 0,
\end{aligned}
$$

which is satisfied if $b < \frac{1}{16}$. A similar argument holds for other types. Additionally, any out-of equilibrium message at $t = 2$, leads to $y^{out} = 0$ which is always worse than $y_2(\theta) = \theta$ since

$-b^2 \geq -(b + \theta)^2$ for all $\theta \in [0, 1]$. The situation after a message $m_{\tilde{\varepsilon}}$ is analogous.

This construction does not apply when the expert is patient, because it does not provide the expert with incentives to join the "right" separable group as we discuss below. We turn to the case of a patient expert next.

5.2   Full Information Revelation with a Patient Expert

The main contribution of the paper is to construct a fully revealing equilibrium when the expert is patient.

Theorem 1 *Suppose that $\delta^{expert} = \delta^{DM} = 1$ and that the preferences of the expert and of the decision maker are given by* (1). *For any bias $b \leq \frac{1}{61}$, there is an infinite set of priors F, and a horizon T^*, for which a fully revealing equilibrium exists whenever $T \geq T^*$.*

We provide a sketch of the proof and leave the details to the appendix.

There are two key constructs in the proof.

The first is the use of the separable groups as in the case of the myopic expert above. Rather than having intervals of types pool together we construct pairs of far-away types who pool together in the initial periods. The advantage is that we no longer need to worry about these paired types mimicking nearby types after they join the separable group. Their only options are to tell the truth, or to mimic each other. Of course, an important part of the proof is to ensure that these paired types indeed would prefer to join the "right" separable group. In our construction for a myopic expert, the induced action in the first period, $u(\cdot)$, was flat as the expert's incentives were static. In fact, that proof relied only on constructing separable groups without the use of trigger strategies. For the case of the patient expert, the construction of the separable groups is significantly more involved as it has to take into account the dynamic incentives. In particular, the actions that the decision maker takes in the initial stages cannot be flat. If the expert cares about the future, and, say the first-period action is flat, the expert will join the initial separable group that leads to the best future action. Hence, for a patient expert, we need to choose initial action functions which provide incentives to join the truthful separable group. If type θ knows that some type θ' will get a more favorable action in the revelation phase, then type θ's group must induce an initial action which is more favorable to type θ than that induced by (θ')'s group.

The second key part of the construction is the use of the trigger strategies. The need for the trigger strategies is closely related to the construction of the separable groups. In the revelation stage the decision maker gets rewarded as the experts reveal themselves. However, this revelation is obviously costly for the expert. To induce the expert to separate at the revelation stage the earlier actions taken by the decision maker have to provide appropriate rewards to the expert. These

expert-incentivizing action functions turn out to be incompatible with the myopically optimal actions from the DM's perspective. Therefore, to make the DM cooperate with the equilibrium, we need to use trigger strategies. The expert threatens that if the DM rejects his early advice, the expert will stop revealing information. If the DM follows expert's advice, DM will be rewarded with eventual full revelation of the state.

Before we proceed with the proof, it is useful to simplify notation and work with a scaled type space by dividing all actions and types by b. When we say that "type $\theta \in [0, \frac{1}{b}]$ recommends $u(\theta)$ in period 1, for disutility $(u(\theta) - \theta - 1)^2$", we mean that (in the unscaled type space) type θb recommends action $u(\theta)b$, for disutility $(u(\theta)b - \theta b - b)^2 = b^2 (u(\theta) - \theta - 1)^2$.

We first partition the scaled type space $[0, \frac{1}{b}]$ into four intervals, with endpoints $[0, \theta_1, \theta_2, \theta_3, \frac{1}{b}]$.

The separable groups are as follows. At time $t = 1$, each type $\theta \in [0, \theta_1]$ is paired and initially pools with a partner $g(\theta) \in [\theta_2, \theta_3]$, to recommend a sequence of actions $(u_1(\theta), u_2(\theta))$. If the DM follows both recommendations, then types $\{\theta, g(\theta)\}$ separate at time $T - 2$. Each type $\theta \in [\theta_1, \theta_2]$ is paired and initially pools with a partner $h(\theta) \in [\theta_3, \theta_4]$, to recommend a sequence of actions $(v_1(\theta), v_2(\theta))$, who then reveal the truth at time $T - \tau$ if the DM has previously followed all recommendations. If the DM does not follow any advice and chooses a different action then the one recommended by the expert, the expert never reveals the truth after the deviation by the DM.

For the purpose of the outline of the proof, take the endpoints $\theta_1, \theta_2, \theta_3$ as given, along with the partner functions $g : [0, \theta_1] \rightarrow [\theta_2, \theta_3]$; $h : [\theta_1, \theta_2] \rightarrow [\theta_3, \theta_4]$, and recommendation functions u_1, u_2, v_1, v_2. In the appendix, we derive the parameters and functions that work, and provide the full details of how to construct fully revealing equilibria.

We now describe the strategy for the expert and for the decision maker. For notational purposes it is useful to further subdivide the experts into three groups: I, II, and III.

At time $t = 1$, there are then three groups of experts. Group I consists of types $\theta^I \in [\theta_1, \theta_2]$ with their partners $h(\theta^I) \in [\theta_3, 1/b]$. Group II consists of types $\theta^{II} \in [0, \theta_1]$ with their partners $g(\theta^{II}) \in [\theta_2, \theta_3]$. Group III consists of types $\theta^{III} \in [0, \theta_1]$ and their partners $g(\theta^{III}) \in [\theta_2, \theta_3]$ who are not in Group II. In other words, we divided the types in interval $[0, \theta_1]$ and their counterparts $g(\cdot) \in [\theta_2, \theta_3]$ into two groups: Group II and Group III: Group II consists of all types whose first-period messages coincide with those of a Group I pair, and Group III consists of all remaining types in $[0, \theta_1] \cup [\theta_2, \theta_3]$.[6]

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[6] As explained in the appendix (Section 12.2.1 and the two paragraphs leading up to it), it is impossible to construct a fully revealing equilibrium if all information sets at $t = 0$ contain only two types: the DM would, at some such information sets, necessarily have an incentive to deviate. We get around this problem by making sure that each recommendation sent by a Group I pair (where the DM's IC constraint may be violated) is also sent by a Group II pair (where the DM's IC constraint is satisfied); Group III simply consists of all remaining agents.

From time $t = 1$ up to time[7] $t = 2\alpha_0$ the experts make recommendations and the decision maker takes actions as follows. Types in Group I recommend v_1: $v_1\left(\theta^I\right) = v_1\left(h\left(\theta^I\right)\right)$. Types in Group II recommend the action u_1: $u_1\left(\theta^{II}\right) = u_1\left(g\left(\theta^{II}\right)\right)$. Importantly, the recommendations for the types in Group I and Group II coincide: for every θ^I in Group I, there exists θ^{II} in Group II with $v_1\left(\theta^I\right) = u_1\left(\theta^{II}\right)$. This is the reason why we for the ease of exposition have a subdivison into Group II and III. Upon observing these recommendations for this period, the DM believes that it came from a group of types $\left\{\theta^I, h\left(\theta^I\right), \theta^{II}, g\left(\theta^{II}\right)\right\}$. Types in Group III (as Group II) also have a recommendation function u_1 and send a recommendation: $u_1\left(\theta^{III}\right) = u_1\left(g\left(\theta^{III}\right)\right)$ ($u_1\left(\theta^{III}\right) = u_1\left(g\left(\theta^{III}\right)\right) \neq v_1\left(\theta^I\right) = v_1\left(h\left(\theta^I\right)\right) = u_1\left(\theta^{II}\right) = u_1\left(g\left(\theta^{II}\right)\right)$. The decision maker then can separate Group III from Groups I and II. That is, upon getting a recommendation $u_1\left(\theta^{III}\right) = u_1\left(g\left(\theta^{III}\right)\right)$, the decision maker believes that he is facing a group of types $\left\{\theta^{III}, g\left(\theta^{III}\right)\right\}$. We also need to specify what happens for the off-path recommendation – the DM treats it as a mistake and believes he faces a group $\left\{\theta^{III} = 0; g\left(\theta^{III} = 0\right)\right\}$ (we refer to the appendix for the exact details).

At $t = 2\alpha_0$, the decision maker separates three groups I, II, III. Group I recommends the action v_2. Groups II and III continue with the recommendation u_1. The beliefs of the DM then are (given the respective actions) that he is facing $\left\{\theta^I, h\left(\theta^I\right)\right\}$, or $\left\{\theta^{II}, g\left(\theta^{II}\right)\right\}$ or $\left\{\theta^{III}, g\left(\theta^{III}\right)\right\}$. These recommendations continue up to time $t = 2\alpha_a$ (in Lemma D7.1, we will specify $2\alpha_a \geq 2\alpha_0$).

At $t = 2\alpha_a \geq 2\alpha_0$, Group II and III switch to the recommendation u_2. Group I continues recommending action v_2. There is no additional revelation of information here. The switch to action u_2 is needed (we will expand on this below) to ensure that the correct separable groups were chosen at time $t = 1$.

At time $T - \tau$, types in Group I get separated into θ^I and $h\left(\theta^I\right)$. That is, from time $t = T - \tau$ to time T these types, respectively, recommend θ^I and $h\left(\theta^I\right)$. Note that the function $h\left(\cdot\right)$ is chosen to have a property $\left|\theta - h\left(\theta\right)\right| \geq 2$ to ensure separation in this revelation stage.[8] In the appendix, we show that τ depends on the horizon T and approaches T as the bias b gets close to zero.

At time $T - 2$, types in Group II and III get separated. That is, from time $t = T - 2$ to time T the types, respectively, recommend θ^{II} and $g\left(\theta^{II}\right)$, or θ^{III} and $g\left(\theta^{III}\right)$. Note that from time $2\alpha_0$ Groups II and Group III were already separated (but the types within each respective group still were pooled). At time $T - 2$ the final separation of the paired types occurs. Note that the function $g\left(\cdot\right)$ is chosen to have a property $\left|\theta - g\left(\theta\right)\right| \geq 2$ to ensure separation in this revelation stage.

The strategies of the expert switch to babbling if at any time the decision maker does not follow any prescriptions by the expert (the details are in the appendix).

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[7] We have time denoted by $2\alpha_0$ and, subsequently, by $2\alpha_a$ to be consistent with the construction in the appendix.
[8] We have 2 on the right hand side because we operate in the type space rescaled by b.

To summarize – there are three key phases. The initial phase is at time $t = 1$ is when the types are paired and choose their separable groups. The second stage is information transmission in which the experts give recommendation (u_1, u_2) and (v_1, v_2). In these phases, the DM refrains from taking the myopically optimal option and sticks to the prescribed recommendations in order to be rewarded at the revelation phase. One can think of these two stages loosely as providing the correct incentives for the expert to join the separable groups. The third stage is revelation stage in which the paired types separate.

Finally, we briefly comment on the construction of the functions (u_1, u_2) and (v_1, v_2) which are given parametrically in the appendix. They are designed such that the following incentive constraints are satisfied.[9] The first set of constraints can be thought of as local incentive compatibility constraints–that is, those applying within each type θ's interval $[\theta_i, \theta_{i+1}]$. These (dynamic) incentive compatibility constraints ensure that, say, the agent $\theta \in [0, \theta_1]$ prefers to induce actions $u_1(\theta)$, $u_2(\theta)$, and then reveal his type θ, rather than e.g. following the sequence $\left(u_1(\theta'), u_2(\theta'), \theta'\right)$ prescribed for some other type θ' in the same interval $[0, \theta_1]$. That is, the ones applying *within* the interval $[0, \theta_1]$ (and, respectively, for the paired types $g(\cdot)$ in $[\theta_2, \theta_3]$) or within the interval $[\theta_1, \theta_2]$ (and, respectively, for the paired types $h(\cdot)$ in $[\theta_3, 1/b]$). These (dynamic) incentive compatibility constraints ensure that, say, the agent $\theta \in [0, \theta_1]$ prefers to follow actions u_1, u_2 and then to reveal its type (analogously, this should be true for this expert's counterpart and for the experts in other intervals and their counterparts. The second and third set of constraints and the conditions imposed on the recommendation functions can be thought of as (global) incentive compatibility constraints, ensuring that types in each respective Group I, Group II, and Group III do not want to mimic a type in another group. That is, the expert does not want to deviate *across* the intervals. As explained in the appendix, this boils down to two conditions: our second constraint requires that each endpoint type be indifferent between his two possible equilibrium sequences (for example, type θ_1 must be indifferent between recommending $u_1(\theta_1), u_2(\theta_1)$, then revealing the truth at time $T-2$, and recommending $v_1(\theta_1), v_2(\theta_1)$, then revealing the truth at time $T - \tau$). Our third constraint requires that the average action be increasing in θ, for all θ : this must hold both within each interval (relates to a second-order convexity condition), and at the discontinuity points $\theta_1, \theta_2, \theta_3$ (relates to the requirement that, while θ_i is indifferent between the sequences prescribed at the right endpoint of $[\theta_{i-1}, \theta_i]$ and the left endpoint of $[\theta_i, \theta_{i+1}]$, all types below θ_i prefer the first sequence, all types above θ_i prefer the second).[10]

---

[9] Of course, the endpoints $(\theta_1, \theta_2, \theta_3)$, the pairing functions g and h, and the functions (u_1, u_2) and (v_1, v_2) have to be constructed simultaneously as we do in the appendix.

[10] The reason we need two actions prior to revelation of the truth $(u_1, u_2$ or $v_1, v_2)$ is that with just one function (say u or v), the requirement that each type send the *same* initial recommendation as at least one other "partner", turns out to be incompatible with the local and global IC constraints (in particular, the requirement that type θ find the initial action of type $\theta + \varepsilon$ less attractive than his own, yet the average action is increasing whenever discontinuous). Two actions give us more flexibility, since, for example, to make type $\theta + \varepsilon$'s sequence unattractive

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The natural question arises: why do we need more than one reward phases? The reason is as follows. In order to ultimately achieve full-information revelation the expert types must voluntarily join separable groups, revealing some information. Suppose that we only had one reward phase. If the decision maker uses the information released up to that point to choose his action, then the expert would not find it optimal to join the separable group. In other words, rewards sufficient to incentivize the expert require the decision maker to choose an action that differs from the myopic best response. In order, then for the deviation to the myopic best response not to be attractive, it must be the case that the DM remains sufficiently agnostic about the state of the world. This is achieved by having two separable pairs of types pool together during phase 1 and then breaking into to two pairs in phase 2. At the beginning of the second phase, the reward to the expert is smaller because the division to separable pairs has already occurred. Put it differently, the reason why one need to have two actions u_1 and u_2 is to ensure that there is indifference at the endpoints, and that the average action is increasing. Having only one action (say, u_1) will only ensure that condition the indifference at the endpoints is satisfied.

6   Concluding Remarks

This paper shows that dynamic strategic communication differs from its static counterpart. Our most striking result is that fully revealing equilibria exist. We also presented a series of examples that show how the canonical results of the static strategic communication games may not hold in the dynamic settings.

The main novel ingredient of our model is that there are many (finite) rounds of communication and after each round an action is chosen. The dynamic considerations of the expert allow us to group together types that are far apart, forming "separable" groups which is the key ingredient behind the construction of fully revealing equilibria and of non-monotonic equilibria. The dynamic setup of our model also allows us to use trigger strategies.

Our analysis highlights the nature of incentive compatibility in dynamic incentive settings. One important characteristic is that dynamic incentive constraints can bind for a disconnected set of types. Moreover, the fact that posteriors are endogenous, implies that earlier rounds in the game can be used to induce posteriors that eventually allow the decision maker to learn the truth. At those nodes in the game the information rents of the expert disappear.

The forces that we identify can be present in other dynamic environments where there is asymmetric information and limited commitment as to what will happen later in the game. Think, for example, a dynamic contracting environment where there is limited commitment about the future,

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to type θ we can either move it further away from type θ's bliss point, *or* increase the variance by pulling the first two recommendations further apart.

or, more generally, a dynamic mechanism design problem, or some other dynamic game where one imposes the requirement of sequential rationality. In those models as well, past behavior setups the stage for future behavior because posteriors are endogenous. And, in contrast to the vast majority of the recent literature on dynamic mechanisms design,[11] one needs to worry about global incentive compatibility despite the fact that stage payoffs satisfy the single-crossing property.

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[11]In recent years motivated by the large number and the importance of applications there has been substantial work on dynamic mechanism design. See, for example, the survey of Bergemann and Said (2010), and the references therein, or Athey and Segal (2007) who characterize a dynamic version of the VCG mechanism that is efficient and balances the budget.

A Proof of Proposition 2

When we restrict attention to monotonic partition equilibria there will be some point in the game where the last subdivision of an interval into smaller partitions occurs. This can either happen at the last period, or at some period before last. Assume without loss, that one interval is partitioned into two inducing actions $y_1\left(m_1\right)$ and $y_2\left(m_2\right)$. If it happens in the last period T of the game, then the indifference condition of the expert type - call that type $\hat{\theta}$- that is indifferent between y_1 and y_2 is the static one. But then, if a subinterval of the original state space can divided into two, it cannot be the case that all static equilibria are equivalent babbling. This follows by Corollary 1 of Crawford and Sobel (1982)).

Now, suppose that the last subdivision happens at a period before the last. Say at period $\hat{T}$. Given that no further subdivision occurs after $\hat{T}$, when the expert sends a message that induces y_1 (resp. y_2) at $\hat{T}$ the same action is induced for the remaining number of periods $T - \hat{T}$. Then the indifference condition for type $\hat{\theta}$ expert is

$$\left(T - \hat{T}\right) u^E\left(y_1, \hat{\theta}\right) \geq \left(T - \hat{T}\right) u^E\left(y_2, \hat{\theta}\right),$$

which is equivalent to static. But then, as before, this subdivision at $\hat{T}$ is not possible given that all static equilibria are equivalent babbling.

Observe that all the arguments in this proof go through even if we allow for trigger strategies. This is because at the point where the last subdivision occurs, it is impossible to incentivize the decision maker to do something worse than his myopic best response. To see this, note at that point, either the game ends - when $\hat{T} = T$, so there is no time to reward the decision maker for choosing a suboptimal action at T or $\hat{T} < T$. Suppose that $\hat{T} < T$, and assume without loss, that one interval is partitioned into two inducing actions $\tilde{y}_1\left(m_1\right)$ and $\tilde{y}_2\left(m_2\right)$ and suppose that $\tilde{y}_1$ generates a payoff for the DM different from the myopic best response y_1, then in order for the DM to follow $\tilde{y}_1$ it must hold

$$\left(T - \hat{T}\right) E_\theta\left[u^{DM}\left(\tilde{y}_1, \theta\right) | m_1\right] > \left(T - \hat{T}\right) E_\theta\left[u^{DM}\left(y_1, \theta\right) | m_1\right],$$

which contradicts the fact that y_1 is the myopic best response given m_1. Hence, the DM cannot choose actions that give him a payoff different from the payoff at the myopic best response. This together with the strict concavity of u^{DM} imply that $y_1 = \tilde{y}_1$.

B Monotonic partition equilibria with more partitions

Suppose that $\delta_E = \delta_{DM} = 1$, types are uniformly distributed on $[0, 1]$ and preferences satisfy (1). Suppose that bias $b = \frac{1}{12}$. Using standard arguments, one can establish that game has only two equilibria,[12] a babbling equilibrium and an equilibrium with two partitions, where the space is divided in two sub-intervals: $[0, \frac{1}{3}]$ and $[\frac{1}{3}, 1]$ inducing two actions $\frac{1}{6}$ and $\frac{4}{6}$. Now we show that when

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[12] The largest number of subintervals that the type space can be divided into is the largest integer that satisfies

$$-2bp^2 + 2bp + 1 > 0, \tag{6}$$

whose solution is

$$\left\langle -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2}{b}} \right\rangle, \tag{7}$$

and where $\langle x \rangle$ denotes the smallest integer greater than or equal to x.

$T = 2$, there exists a monotonic partition equilibrium where in the end the state space is divided in three sub-intervals.

We look for an equilibrium with the following signaling rule:

$$\text{types in } [0, \theta_1] \text{ send message sequence } A = (m_{1(1)}, m_{2(1)}),$$
$$\text{types in } [\theta_1, \theta_2] \text{ send message sequence } B = (m_{1(2)}, m_{2(2)}),$$
$$\text{types in } [\theta_2, 1] \text{ send message sequence } C = (m_{1(2)}, m_{2(3)}).$$

With this signaling rule in the first period the interval $[0, 1]$ is partitioned into $[0, \theta_1]$ and $[\theta_1, 1]$. At stage 2, $[\theta_1, 1]$ is partitioned further into $[\theta_1, \theta_2]$ and $[\theta_2, 1]$. We can express θ_2 in terms of θ_1 as follows:

$$\theta_2 = \left(\frac{1 - \theta_1}{2} - \frac{4}{12}\right) + \frac{2}{12} + \theta_1 = \frac{1}{2}\theta_1 + \frac{1}{3}. \tag{8}$$

The second period actions induced are $y_{2(1)} = \frac{\theta_1}{2}$, $y_{2(2)} = \frac{3}{4}\theta_1 + \frac{1}{6}$ and $y_{2(3)} = \frac{1}{4}\theta_1 + \frac{2}{3}$, whereas the first period actions induced are $y_{1(1)} = \frac{\theta_1}{2}$ and $y_{1(2)} = \frac{1 + \theta_1}{2}$.

After any out-of- equilibrium message the decision maker assigns probability one to the state belonging in $[0, \theta_1]$ inducing $y^{out} = \frac{\theta_1}{2}$. With these out-of equilibrium beliefs it is immediate to see that no type has an incentive to send an out-of-equilibrium message.

At an equilibrium θ_1 must satisfy the following indifference condition:

$$-\left(\frac{1 + \theta_1}{2} - \theta_1 - \frac{1}{12}\right)^2 - \left(\frac{\frac{1}{2}\theta_1 + \frac{1}{3} + \theta_1}{2} - \theta_1 - \frac{1}{12}\right)^2 = -2\left(\frac{\theta_1}{2} - \theta_1 - \frac{1}{12}\right)^2,$$

which with the help of (8) gives us 3 partitions with cutoffs $\theta_1 = 0.25$ and in turn $\theta_2 = 0.458\,33$. Using this value of θ_1 we get that: $y_{1(1)} = y_{2(1)} = 0.125$, $y_{1(2)} = 0.625$, $y_{2(2)} = 0.35417$ and $y_{2(3)} = 0.72917$.

In constructing this strategy profile we have imposed that type θ_2 is indifferent between inducing action $y_{2(2)}$ and $y_{2(3)}$ at $t = 2$ and that type θ_1 is indifferent between inducing action sequence $y_{1(1)}$ and $y_{2(1)}$ and $y_{1(2)}$ and $y_{2(2)}$. Now we want to verify that these conditions are sufficient for global incentive compatibility. At $t = 2$ the game is isomorphic to the static one, where the fact that θ_2 is indifferent between $y_{2(2)}$ and $y_{2(3)}$ implies that all types above θ_2 prefer $y_{2(3)}$ and all types below θ_2 prefer $y_{2(2)}$. To verify that types below θ_1 prefer message sequence A inducing $y_{1(1)}$ and $y_{2(1)}$, and types above θ_1 prefer message sequence B inducing $y_{1(2)}$ and $y_{2(2)}$, we plot the difference $U(A, \theta) - U(B, \theta)$ and show that it is positive for all $\theta < \theta_1$ and negative for $\theta > \theta_1$:
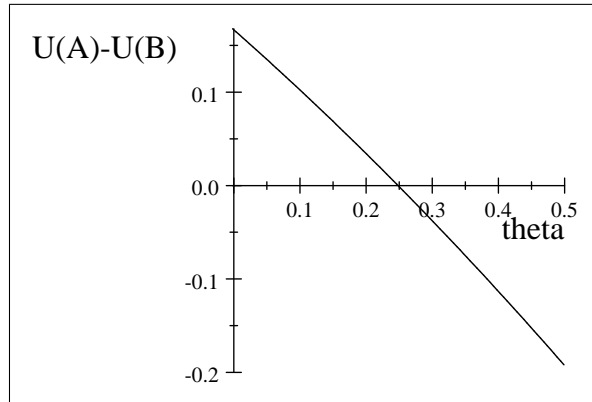
At the dynamic equilibrium we constructed here the expert's welfare is -0.065 and the decision maker's welfare is given by -0.0517, which are both higher than the welfare at the equilibrium where the most informative static equilibrium is played in the first period, and babbling in the second, which are respectively -0.069 and -0.055.

C Pareto comparisons of dynamic cheap talk equilibria

The following examples demonstrate that equilibria with more partitions can be Pareto inferior to the equilibria with fewer partitions

Take $\delta_E = \delta_{DM} = 1$ and $b = 0.08$, and consider the most informative static partition equilibrium where the number of partitions is $p = 3$. At this equilibrium the state space is divided into $[0, 0.013]$, $[0.013, 0.347]$ and $[0.347, 1]$. The corresponding optimal actions of the decision maker are given by

$$y_1 = 0.0067 \quad y_2 = 0.18 \quad y_3 = 0.673,$$

from which we can calculate the ex-ante expected utility levels for the expert -0.032 and for the decision maker -0.0263. Then, at the equilibrium of the dynamic game where the most informative static equilibrium is played at $t = 1$ and babbling thereafter, the total expected utility is -0.065 for the expert, and -0.053 for the decision maker.

We now construct a dynamic equilibrium where the type space is subdivided into more subintervals, but both players' ex-ante expected payoffs are lower. We look for an equilibrium with the following signaling rule:

> types in $[0, \theta_1]$ send message sequence $(m_{1(1)}, m_{2(1)})$
>
> types in $[\theta_1, \theta_2]$ send message sequence $(m_{1(2)}, m_{2(2)})$
>
> types in $[\theta_2, \theta_3]$ send message sequence $(m_{1(2)}, m_{2(3)})$
>
> types in $[\theta_3, 1]$ send message sequence $(m_{1(3)}, m_{2(4)})$.

So types are partitioned into four intervals in stage 2, but in stage 1, the types in $[\theta_1, \theta_2]$ and $[\theta_2, \theta_3]$ pool together to send the same message $m_{1(2)}$. Since the signaling rule does not depend on the decision maker's action at stage 1, the decision maker will choose the following myopically optimal actions:

$$y_{1(1)} = y_{2(1)} = \frac{\theta_1}{2},$$
$$y_{1(2)} = \frac{\theta_1 + \theta_3}{2}, \ \ y_{2(2)} = \frac{\theta_1 + \theta_2}{2}, \ \ y_{2(3)} = \frac{\theta_2 + \theta_3}{2},$$
$$y_{1(3)} = y_{2(4)} = \frac{1 + \theta_3}{2}.$$

After any out-of-equilibrium message the decision maker assigns probability one to the state belonging in $[0, \theta_1]$ inducing $y^{out} = \frac{\theta_1}{2}$. With these out-of-equilibrium beliefs it is immediate to see that no type has an incentive to deviate.

In equilibrium, type θ_1 is indifferent between action sequences $\{y_{1(1)}, y_{2(1)}\}$ and $\{y_{1(2)}, y_{2(2)}\}$, type θ_2 is indifferent between 2nd-period actions $y_{2(2)}$ and $y_{2(3)}$, and type θ_3 is indifferent between action sequences $\{y_{1(2)}, y_{2(3)}\}$ and $\{y_{1(3)}, y_{2(4)}\}$. Therefore, equilibrium cutoffs are the solution to

the following system of equations:[13]

$$2\left(\frac{\theta_1}{2} - \theta_1 - b\right)^2 - \left(\frac{\theta_1 + \theta_3}{2} - b - \theta_1\right)^2 - \left(\frac{\theta_1 + \theta_2}{2} - b - \theta_1\right)^2 = 0,$$

$$\left(\frac{\theta_1 + \theta_2}{2} - b - \theta_2\right)^2 - \left(\frac{\theta_2 + \theta_3}{2} - b - \theta_2\right)^2 = 0,$$

$$2\left(\frac{1 + \theta_3}{2} - b - \theta_3\right)^2 - \left(\frac{\theta_1 + \theta_3}{2} - b - \theta_3\right)^2 - \left(\frac{\theta_2 + \theta_3}{2} - b - \theta_3\right)^2 = 0.$$

At $b = 0.08$, the only solution that gives numbers in $[0, 1]$ is $\theta_1 = 0.0056, \theta_2 = 0.015, \theta_3 = 0.345$, and the actions induced for $t = 1$ and for $t = 2$ are respectively given by $y_{1(1)} = y_{2(1)} = 0.00278$, $y_{1(2)} = 0.175$, $y_{2(2)} = 0.0105$, $y_{2(3)} = 0.18$ and $y_{1(3)} = y_{2(4)} = 0.673$. This implies the following total ex-ante expected utility for the expert -0.066, which is lower than $2(-0.033) = -0.0656$. The utility for the decision-maker is -0.053 which is lower than $2(-0.026) = 0.052$.

This example illustrates that although the interval is divided into more subintervals here, both players strictly worse off compared to the one where the most informative static equilibrium is played in the first period and babbling thereafter. The feature that less partitions lead to higher ex-ante welfare for both players also appears in example 1 of Blume, Board and Kawamura (2007).

D Proof of Theorem 1

We will prove by construction that a fully revealing equilibrium exists. We first choose the endpoints $\theta_1, \theta_2, \theta_3$ described in the proof outline: for any bias $b < \frac{1}{61}$, define a_γ by

$$(a_\gamma - 2 + 2e^{-a_\gamma})e^2 - a_\gamma = \frac{1}{b}$$

and then set

$$\theta_3 = \frac{1}{b} + a_\gamma, \ \theta_2 = \theta_3 - 2, \ \theta_1 = \theta_2 - \theta_3 e^{-2} \tag{9}$$

It will be convenient to describe types parametrically, via functions $x : [-2, 0] \to [0, \theta_1]$, $g : [-2, 0] \to [\theta_2, \theta_3]$, $z : [a_\gamma, 0] \to [\theta_1, \theta_2]$, and $h : [a_\gamma, 0] \to [\theta_3, \frac{1}{b}]$. Then, let $u_1(a, \alpha_a)$, $u_2(a, \alpha_a)$ denote the first, second recommendations of types $x(a), g(a)$ (for all $a \in [-2, 0]$), and let $v_1(a, \alpha_0), v_2(a, \alpha_0)$ denote the first, second recommendations of types $(z(a), h(a))$ (for all $a \in [a_\gamma, 0]$). With this notation, Groups I,II, III described in the text are as follows:

$$\begin{aligned}
\text{Group } I &= \{z(a), h(a) \mid a \in [a_\gamma, 0]\} \\
\text{Group } II &= \{x(a), g(a) \mid a \in [-2, 0], \text{ and } \exists \ a' \in [a_\gamma, 0] \text{ with } v_1(a', \alpha_0) = u_1(a, \alpha_a)\} \\
\text{Group } III &= \{x(a), g(a) \mid a \in [-2, 0], \text{ and } x(a), g(a) \notin \text{Group II}\}
\end{aligned}$$

In our proposed equilibrium construction, each Group I pair $\{z(a), h(a)\}$ recommends $v_1(a, \alpha_0)$ for $2\alpha_0$ periods, then $v_2(a, \alpha_0)$ for $T - \tau - 2\alpha_0$ periods, then reveals the truth at time $T - \tau$; each Group II pair $\{x(a), g(a)\}$ recommends $u_1(a, \alpha_a)$ for $2\alpha_a$ periods, then $u_2(a, \alpha_a)$ for $2(1 - \alpha_a)$ periods, then separates and reveals the truth at time $T - 2$; and moreover, the recommendation $u_1(a, \alpha_a)$ coincides with the recommendation $v_1(a', \alpha_0)$ of some Group I pair $\{z(a'), h(a')\}$. Group III is identical to Group II, except that their recommendations *do not* coincide with those of any Group I pair.

We also specify the following off-path strategy for the expert: if the DM ever deviates, by rejecting a recommendation that the expert made, then (i) if the expert himself has *not* previously

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[13]It is trivial to check exactly as we did in previous examples that these indifference conditions suffice for global incentive compatibility.

deviated: send no further recommendations (equivalently, repeat the current recommendation in all subsequent periods). And (ii) if the expert *has* observably deviated in the past, behave as if the deviation did not occur. (For example, if he sends the initial recommendation $u_1(0, \alpha_0)$ prescribed for types $\{x(0), g(0)\}$, but then follows this with anything other than recommendation $u_2(0, \alpha_0)$ at time $2\alpha_0$, subsequently behave as if the deviation never occurred and he indeed sent $u_2(0, \alpha_0)$ at time $2\alpha_0$).

D.1 Optimality for the Expert

We prove that the expert wishes to follow the prescribed recommendation strategy via three propositions. Proposition D1 specifies strategies and beliefs for the DM such that the expert has no incentive to send an out-of-equilibrium recommendation sequence, so we can restrict to making sure that he does not wish to mimic any other type. Proposition D2 shows that in the prescribed revelation phase, the expert indeed finds it optimal to reveal the truth, provided that there have been no previous deviations. It remains only to show that the expert has no incentive to deviate prior to the prescribed revelation phase - by mimicking the initial recommendations of some other type - which we show in Proposition D3.

D.1.1 Expert Optimality: Off-Path Behavior

We specify the following strategy and beliefs for the DM:
DM Strategy and Beliefs: If there are no detectable deviations by the expert (i.e., he sends the equilibrium recommendation sequence for some type $\theta \in [0, \frac{1}{b}]$), then follow all recommendations, using Bayes' rule to assign beliefs at each information set. Following deviations: (i) If the expert observably deviates at time 0 (sending an off-path initial recommendation), subsequently adopt the strategy/beliefs that would follow if the expert had instead sent the recommendation $u_1(0, \alpha_0)$ prescribed for types $\{x(0), g(0)\}$; (ii) If the expert observably deviates on his 2nd recommendation (i.e., if an initial recommendation $u_1(a, \alpha_a)$ (or $v_1(a, \alpha_0)$) is followed by something *other* than $u_2(a, \alpha_a)$ (or $v_2(a, \alpha_0)$), ignore it as an error, and subsequently adopt the strategy/beliefs that would follow had the deviation not occurred; (iii) If the expert deviates observably in the revelation phase, ignore it as an error, assigning probability 1 to the lowest type in the current information set, and accordingly choosing this as the myopically optimal action; (iv) And finally, if the DM himself deviates, rejecting some recommendation by the expert, then he subsequently maintains the current (at time of deviation) beliefs, anticipating that the expert will subsequently repeat the current (at time of deviation) recommendation, and ignoring any other recommendations as errors.
Proposition D1: Under the above strategy and beliefs prescribed for the DM, the expert has no incentive to choose an off-path recommendation sequence.
Proof of Proposition D1: Follows trivially from the specified strategy and beliefs for the DM: (i) a deviation at time zero is equivalent to mimicking type $x(0)$ (who recommends $u_1(0, \alpha_0)$ at time $t = 0$); (ii) a deviation on the 2nd recommendation has no effect, since the DM ignores it; (iii) a deviation in the revelation phase, if there have been no previous recommendations, is equivalent to mimicking the strategy of the lowest type in the DM's current (pre-revelation) information set; and (iv) if the DM *has* previously deviated, then (by point (iv) of the above strategy-belief specification) he will chose whichever action was myopically optimal at the time of deviation, regardless of the expert's message; therefore, babbling is optimal for the expert, since his message has no effect on the DM's action. ∎

D.1.2 Expert Optimality: Truth Revelation Phase

Proposition D2: In the prescribed revelation phase, (i) if there have been no previous deviations by the DM, then the expert finds it optimal to reveal the truth; (ii) if the DM has ever deviated, then the expert finds it optimal to babble (eg. by remaining silent).
Proof of Proposition D2: Part (ii) follows immediately from Proposition D1 (iv). For part (i): our specification of the expert strategy is such that at time $2\alpha_0$, the DM's information set contains at most two types: either a pair $\{x(a), g(a)\}$ (in which case he plans to choose $g(a)$ if the expert

recommends $g(a)$, $x(a)$ otherwise), or a pair $\{z(a), h(a)\}$ (in which case he plans to choose $h(a)$ if the expert recommends it, $z(a)$ otherwise). So, it suffices to show that each type would rather tell the truth than mimic his partner, which requires (in our rescaled state space) simply that all paired types be at least 2 units apart. We have,

$$\min_{a \in [-2,0]} |g(a) - x(a)| = \theta_2 - \theta_1$$

$$\min_{a \in [a_\gamma, 0]} |h(a) - z(a)| = \theta_3 - \theta_2$$

By the endpoints specified at the beginning of this proof, $\theta_2 - \theta_1 = (a_\gamma - 2 + 2e^{-a_\gamma}) \geq 8$ (for $b < \frac{1}{61}$), and $\theta_3 - \theta_2 = 2$. ∎

D.1.3 Expert Optimality: Initial Recommendations

Propositions D1,D2 imply that once the expert has sent the initial recommendation (u_1 or v_1) prescribed for some type θ, it is optimal to follow also the continuation recommendations prescribed for that type. So, the only time when it could possibly be profitable to deviate is at time $t = 0$: we need to make sure that each type θ prefers to send the proposed equilibrium sequence of recommendations, rather than the sequence prescribed for any other type θ'.[14] We now choose parametrizations of functions x, g, z, h, along with action function u_1, u_2, v_1, v_2, which guarantee that the expert indeed finds it optimal to send the prescribed initial recommendation:

Proposition D3: Let the action functions and type parametrizations be as follows:

$$x(a) = \theta_3 + a - \theta_3 e^a, \; g(a) = \theta_3 + a, \; z(a) = \frac{1}{b} + a - 2e^{a - a_\gamma}, \; h(a) = \frac{1}{b} + a \tag{10}$$

$$u_1(a, \alpha_a) = \theta_3 + K - \frac{T-2}{2}a - \sqrt{\frac{1 - \alpha_a}{\alpha_a}}\sqrt{T-2}\sqrt{C_u + a\left(K - \frac{T}{4}a\right)} \tag{11}$$

$$u_2(a, \alpha_a) = \theta_3 + K - \frac{T-2}{2}a + \sqrt{\frac{\alpha_a}{1 - \alpha_a}}\sqrt{T-2}\sqrt{C_u + a\left(K - \frac{T}{4}a\right)} \tag{12}$$

$$v_1(a, \alpha_0) = \theta_3 + \frac{2K - \tau(a - a_\gamma)}{T - \tau} - \frac{\sqrt{\frac{\tau(T - \tau - 2\alpha_0)}{\alpha_0}}\sqrt{\frac{(T-\tau)(T-2)}{\tau}C_u + \left(\frac{T-\tau-2}{\tau}\right)K^2 + 2K(a - a_\gamma) - \frac{T}{2}(a - a_\gamma)^2}}{T - \tau} \tag{13}$$

$$v_2(a, \alpha_0) = \theta_3 + \frac{2K - \tau(a - a_\gamma)}{T - \tau} + \frac{\sqrt{\frac{4\tau\alpha_0}{T - \tau - 2\alpha_0}}\sqrt{\frac{(T-\tau)(T-2)}{\tau}C_u + \left(\frac{T-\tau-2}{\tau}\right)K^2 + 2K(a - a_\gamma) - \frac{T}{2}(a - a_\gamma)^2}}{T - \tau} \tag{14}$$

for constants C_u, K, and for now taking T, α_0, α_a as given (T is the horizon, and α_a, α_0 relate to the duration of recommendations u_1, v_1 as described in the strategies above). Also set τ (length of the revelation phase for types in Group I) according to

$$\frac{\tau}{T - 2} = \beta \equiv \frac{(\theta_2 - \theta_1)(\theta_2 - \theta_1 - 2)}{(\theta_4 - \theta_1)(\theta_4 - \theta_1 - 2)} \tag{15}$$

Then, for all types $\theta, \theta' \in [0, \frac{1}{b}]$, expert type θ prefers his equilibrium recommendation sequence to that sent by type θ', and in particular has no incentive to deviate at time $t = 0$ from the prescribed strategy.
Proof of Proposition D3:

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[14] This is what the text refers to as "providing incentives to join the right separable group". We need to make sure, for example, that type $\theta = 0$ prefers to induce the action sequence $(u_1(0), u_2(0), 0)$, rather than e.g. the sequence that type $\theta' \neq 0$ is supposed to send; by Propositions D1,D2, the choice to follow a different recommendation sequence can only be made at time $t = 0$.

Let $D_u(\theta'|\theta)$ denote the disutility to type θ from following the recommendation sequence pre-scribed for a type $\theta' \in [0, \theta_1] \cup [\theta_2, \theta_3]$, and let $D_v(\theta'|\theta)$ denote the disutility to type θ from following the strategy prescribed for a type $\theta' \in [\theta_1, \theta_2] \cup [\theta_3, \frac{1}{b}]$. The proof proceeds through two main Lem-mas. Lemma D3.1 proves that the expert strategy is "locally" incentive compatible: for each interval $[\theta_i, \theta_{i+1}]$, $i \in \{0, 1, 2, 3\}$, no expert type $\theta \in [\theta_i, \theta_{i+1}]$ wishes to mimic any other type $\theta' \in [\theta_i, \theta_{i+1}]$ from the same interval. Lemma D3.2 proves that the expert strategy is also "globally" incentive compatible: no expert type wishes to mimic any type θ' from any other interval. The proofs will use calculations obtained below in Lemmas D3.3 and D3.4.

Lemma D3.1 (Local IC): For each interval $[\theta_i, \theta_{i+1}]$, with $i = 0, 1, 2, 3$, and any pair of types $\theta, \theta' \in [\theta_i, \theta_{i+1}]$, the disutility to type θ from mimicking type θ' is (weakly) *increasing* in $|\theta' - \theta|$, thus minimized when $|\theta' - \theta| = 0$. Therefore, for each $\theta \in [\theta_i, \theta_{i+1}]$, truth-telling is (weakly) better than mimicking any other type in the interval.

Proof of Lemma D3.1:

Differentiating disutility expressions (20), (21), (22), and (23) (obtained below in Lemma D3.3) gives

$$
\begin{aligned}
\frac{dD_u(x(a)|\theta)}{dx(a)} &= \frac{dD_u(g(a)|\theta)}{dx(a)} + \frac{2(T-2)(x(a)-\theta-1)x'(a) - 2(T-2)(g(a)-\theta-1)g'(a)}{x'(a)} \\
&= 0 + 2(T-2)\left(\theta_3 + a - \theta_3 e^a - \theta - 1 - \frac{\theta_3 + a - \theta - 1}{1 - \theta_3 e^a}\right) \quad (\text{by } (10)) \\
&= 2(T-2)\left(\frac{\theta_3 e^a}{\theta_3 e^a - 1}\right)(x(a)-\theta) \quad (16)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dD_v(z(a)|\theta)}{dz(a)} &= \frac{dD_v(h(a)|\theta)/da}{z'(a)} + \frac{2\tau(z(a)-\theta-1)z'(a) - 2\tau(h(a)-\theta-1)h'(a)}{z'(a)} \\
&= 2\tau\left(\frac{2e^{a-a_\gamma}}{2e^{a-a_\gamma}-1}\right)(z(a)-\theta) \quad (17)
\end{aligned}
$$

$$
\frac{dD_u(g(a)|\theta)}{dg(a)} = 0 \quad (18)
$$

$$
\frac{dD_v(h(a)|\theta)}{dh(a)} = 0 \quad (19)
$$

Consider first a type $\theta \in [0, \theta_1]$. By (16), noting that $\frac{\theta_3 e^a}{\theta_3 e^a - 1} > 0$ (since $\theta_3 e^a \geq \theta_3 e^{-2} = \theta_2 - \theta_1 \geq 8$, by Proposition D2), we see that $\frac{dD_u(x(a)|\theta)}{dx(a)}$ has the same sign as $(x(a) - \theta)$. So if $x(a) - \theta > 0$, then $D_u(x(a)|\theta)$ is *increasing* in $x(a)$, thus increasing in $(x(a) - \theta)$; while if $x(a) - \theta < 0$, then $D_u(x(a)|\theta)$ is increasing in $(-x(a))$, thus increasing in $\theta - x(a)$. Combined, these establish that $D_u(x(a)|\theta)$ is strictly increasing in $|x(a) - \theta|$, as desired.

Next consider a type $\theta \in [\theta_1, \theta_2]$. By (17), noting that $\left(\frac{2e^{a-a_\gamma}}{2e^{a-a_\gamma}-1}\right) > 0$ (since $a \in [a_\gamma, 0]$ implies $2e^{a-a_\gamma} \geq 2$), we see that $\frac{dD_v(z(a)|\theta)}{dz(a)}$ has the same sign as $z(a) - \theta$, and is thus positive (disutility increasing in $z(a) - \theta$) if $z(a) > \theta$, and negative (disutility increasing in $\theta - z(a)$) if $z(a) < \theta$. Combined, these establish that $D_u(z(a)|\theta)$ is strictly increasing in $|z(a) - \theta|$, as desired.

By (18) and (19), the disutility to type θ from mimicking a type $g(a) \in [\theta_2, \theta_3]$ or $h(a) \in [\theta_3, \frac{1}{b}]$ is independent of the particular type $g(a), h(a)$ chosen. Thus, $D_u(g(a)|\theta)$ is weakly increasing (in fact constant) in $|g(a) - \theta|$, and $D_u(h(a)|\theta)$ is weakly increasing (constant) in $|h(a) - \theta|$, completing the proof. ■

Lemma D3.2: For every interval $[\theta_i, \theta_{i+1}]$ $(i = 0, 1, 2, 3)$, and every $\theta \in [\theta_i, \theta_{i+1}]$, following the prescribed (truthful) recommendation sequence is better than mimicking any type θ' drawn from any *other* interval $[\theta_j, \theta_{j+1}]$ with $j \neq i$.

Proof of Lemma D3.2:

Consider first a type $\theta \in [0, \theta_1]$. By Lemma D3.1, truth-telling is better than mimicking any

other type $\theta' \in [0, \theta_1]$, in particular type $\theta_1 = x(-2)$. By Lemma D3.4 (i) (below), type $\theta \in [0, \theta_1]$ prefers type $x(-2)$'s sequence to type $z(0)$'s sequence; and by Lemma D3.1, it is better to mimic type $z(0) = \theta_1$, than any other type $z(a) \in (\theta_1, \theta_2]$ (since $z(a) > \theta$ implies that $D_v(z(a)|\theta)$ is increasing in $z(a) - \theta$); together, these establish that mimicking a type $\theta' \in [\theta_1, \theta_2]$ is not optimal. By Lemma D3.4 (ii), type $\theta \leq \theta_2$ prefers $z(a_\gamma)$'s sequence (right endpoint of $[\theta_1, \theta_2]$) to $g(-2)$'s sequence (left endpoint of $[\theta_2, \theta_3]$); and by Lemma D3.1, disutility to type θ from mimicking type $g(a) \in [\theta_2, \theta_3]$ is independent of a; together, this implies that type θ also does not want to mimic any type $g(a) \in [\theta_2, \theta_3]$. And finally, by Lemma D3.4 (iii), type $\theta \leq \theta_2$ prefers the sequence prescribed for type $g(0)$ (right endpoint of $[\theta_2, \theta_3]$) to that prescribed for type $h(a_\gamma)$ (left endpoint of $[\theta_3, \frac{1}{b}]$), which (by Lemma D3.1) yields the same utility as mimicking any other type $h(a) \in [\theta_3, \frac{1}{b}]$, thus it is not optimal to mimic any type $h(a) \in [\theta_3, \frac{1}{b}]$. This establishes that type $\theta \in [0, \theta_1]$ does not wish to mimic any type θ' from any other interval.

Next consider type $\theta \in [\theta_1, \theta_2]$. By Lemma D3.1, truth-telling is better than mimicking any other type $z(a) \in [\theta_1, \theta_2]$, in particular type $z(0) = \theta_1$; by Lemma D3.4 (i), type $\theta \geq \theta_1$ prefers the sequence prescribed for type $z(0)$, to that prescribed for type $x(-2)$; and by Lemma D3.1, it is better to mimic $x(-2)$ (right endpoint of $[0, \theta_1]$) than any other type $x(a) \in [0, \theta_1]$, since $D_u(x(a)|\theta)$ is increasing in $|\theta - x(a)|$ and we have here $\theta > x(a)$; together, this implies that type θ does not wish to mimic any type $\theta' \in [0, \theta_1]$. The proof that he doesn't wish to mimic any type $g(a) \in [\theta_2, \theta_3]$ or $h(a) \in [\theta_3, \frac{1}{b}]$ is identical to the one given in the previous paragraph.

Now consider a type $\theta \in [\theta_2, \theta_3]$. As explained in the previous two paragraphs, following the truthful recommendation sequence yields the same utility as mimicking any other type $g(a) \in [\theta_2, \theta_3]$ or $h(a) \in [\theta_3, \theta_4]$, so we just need to make sure that it is not optimal to mimic types $\theta' \in [0, \theta_1] \cup [\theta_1, \theta_2]$. By Lemma D3.4 (ii), type $\theta \geq \theta_2$ prefers type $g(-2)$'s sequence (left endpoint of $[\theta_2, \theta_3]$) to type $z(a_\gamma)$'s sequence (right endpoint of $[\theta_1, \theta_2]$); by Lemma D3.1, type $z(a_\gamma)$'s sequence is more attractive to type $\theta \geq \theta_2$ than the one prescribed for any other (further-away) type $z(a) \in [\theta_1, \theta_2]$; combined, this establishes that mimicking a type $z(a) \in [\theta_1, \theta_2]$ is not optimal. By Lemma D3.4 (i), it is better to mimic type $z(0)$'s sequence than $x(-2)$'s sequence, which in turn is better (by Lemma D3.1) than any other type $x(a)$'s sequence. Thus, it is not optimal to mimic any type $x(a) \in [0, \theta_1]$, completing the proof for types $\theta \in [\theta_2, \theta_3]$.

The argument that types $\theta \in [\theta_3, \frac{1}{b}]$ don't wish to mimic types from other intervals is identical to the proof in the previous paragraph (for types $\theta \in [\theta_2, \theta_3]$).

This completes the proof of Lemma D3.2. ∎

D.1.4 Expert Optimality: Preliminary Calculations

Lemma D3.3: Given the type parametrizations and action functions given in Proposition D3, disutility expressions $D_u(\theta'|\theta)$, $D_v(\theta'|\theta)$ are given by

$$D_u(x(a)|\theta) = D_u(g(a)|\theta) + (T-2)(x(a) - \theta - 1)^2 - (T-2)(g(a) - \theta - 1)^2 \qquad (20)$$

$$D_u(g(a)|\theta) = T(\theta_3 - \theta - 1)^2 + 4K(\theta_3 - \theta - 1) + 2K^2 + 2(T-2)C_u \qquad (21)$$

$$D_v(z(a)|\theta) = D_v(h(a)|\theta) - 2\tau(h(a) - z(a))\left(\frac{h(a) + z(a)}{2} - \theta - 1\right) \qquad (22)$$

$$D_v(h(a)|\theta) = 2K^2 + 2(T-2)C_u + 4(\theta_3 - \theta - 1)K + T(\theta - \theta_3 + 1)^2 \qquad (23)$$

Proof of Lemma D3.3:

The disutility $D_u(g(a)|\theta)$ to expert type θ from following the strategy prescribed for type $g(a) \in$

$[\theta_2, \theta_3]$, using $(11), (12),$ is

$$2\alpha_a \left(u_1(a) - \theta - 1\right)^2 + 2(1 - \alpha_a)\left(u_2(a) - \theta - 1\right)^2 + (T-2)\left(g(a) - \theta - 1\right)^2 \tag{24}$$

$$= \quad 2(1 - \alpha_a)\left(\theta_3 + K - \frac{T-2}{2}a - \theta - 1 + \sqrt{\frac{\alpha_a}{1-\alpha_a}}\sqrt{T-2}\sqrt{C_u + a\left(K - \frac{T}{4}a\right)}\right)$$

$$+2\alpha_a \left(\theta_3 + K - \frac{T-2}{2}a - \theta - 1 - \sqrt{\frac{1-\alpha_a}{\alpha_a}}\sqrt{T-2}\sqrt{C_u + a\left(K - \frac{T}{4}a\right)}\right)^2 + (T-2)\left(x(a) - \theta - 1\right)^2$$

Expanding gives [15]

$$D_u(g(a)|\theta) = 2\left(\theta_3 - \theta - 1 + K - \frac{T-2}{2}a\right)^2 + 2(T-2)\left(C_u + a\left(K - \frac{T}{4}a\right)\right) + (T-2)\left(\theta_3 + a - \theta - 1\right)^2$$

If we now expand this expression, the coefficients on a^2, a reduce to zero (this is due to our choice $g(a) = \theta_3 + a$), leaving

$$D_u(g(a)|\theta) = 2\left(\theta_3 - \theta - 1 + K\right)^2 + 2(T-2)C_u + (T-2)\left(\theta_3 - \theta - 1\right)^2$$

which rearranges to expression (21).

The disutility to type θ from following the strategy prescribed for type $x(a) \in [0, \theta_1]$, $D_u(x(a)|\theta)$, is given by (24), just replacing $g(a)$ with $x(a)$: this gives the desired expression (20).

The disutility to type θ from following the strategy prescribed for type $h(a) \in [\theta_3, \theta_4]$ is

$$D_v(h(a)|\theta) = 2\alpha_0 \left(v_1(a) - \theta - 1\right)^2 + (T - \tau - 2\alpha_0)\left(v_2(a) - \theta - 1\right)^2 + \tau\left(h(a) - \theta - 1\right)^2$$

Again, the coefficients on the square root terms in v_1, v_2 were chosen to make both disutility and average action independent of α_0: substituting $(13), (14)$ into the above expression and expanding, we get

$$D_v(h(a)|\theta) \quad = \quad (T-\tau)\left(\theta_3 + \frac{2K + \tau a_\gamma}{T - \tau} - \frac{\tau}{T - \tau}a - \theta - 1\right)^2 + \tau\left(h(a) - \theta - 1\right)^2$$

$$+ 2\tau \left(\frac{(T-2)}{\tau}C_u + \frac{\left(\frac{T-\tau-2}{\tau}\right)K^2 + 2K\left(a - a_\gamma\right) - \frac{T}{2}\left(a - a_\gamma\right)^2}{T - \tau}\right)$$

Substituting in $h(a) = \theta_4 + a$, using $\theta_3 = \theta_4 + a_\gamma$, and expanding, we find (this is due to our choice $h'(a) = 1$) that the coefficients on both a^2, a reduce to zero, so that our expression simplifies further to (23). Finally, using the fact that the strategies for types $h(a), z(a)$ differ only in the revelation phase, so

$$D_v(h(a)|\theta) - D_v(z(a)|\theta) = \tau\left(h(a) - \theta - 1\right)^2 - \tau\left(z(a) - \theta - 1\right)^2$$

we obtain (22). This completes the proof. ∎

Lemma D3.4: (utility at the endpoints)

Under the expressions given in Proposition D3, we have that (i) at $\theta_1 = x(-2) = z(0)$: type θ (weakly) prefers type $x(-2)$'s recommendation sequence to $z(0)$'s sequence iff $\theta \in [0, \theta_1]$; (ii) at $\theta_2 = z(a_\gamma) = g(-2)$, type θ prefers $z(a_\gamma)$'s sequence to $g(-2)$'s sequence iff $\theta \in [0, \theta_2]$; and (iii) at $\theta_3 = g(0) = h(a_\gamma)$, all types are indifferent between the sequences sent by types $g(0), h(a_\gamma)$.[16]

---

[15] Note that the coefficients on the square roots were chosen to make this independent of α_a, as mentioned in Appendix E - Derivations.

[16] For example, consider part (i). In our construction, type θ_1 is both the right endpoint $x(-2)$ of the interval

Proof of Lemma D3.4:

At $\theta_1 = x(-2) = z(0)$, we have (using the expressions in Lemma D3.3 and simplifying) that $D_v(z(0)|\theta) - D_u(x(-2)|\theta)$ equals

$$(T-2)(\theta_2 - \theta_1)(\theta_1 + \theta_2 - 2\theta - 2) - \tau(\theta_4 - \theta_1)(\theta_4 + \theta_1 - 2\theta - 2)$$

Using $\tau(\theta_4 - \theta_1) = (T-2)\frac{(\theta_2 - \theta_1)(\theta_2 - \theta_1 - 2)}{(\theta_4 - \theta_1 - 2)}$ (by (15)), this simplifies to

$$D_v(z(0)|\theta) - D_u(x(-2)|\theta) = 2(T-2)(\theta_2 - \theta_1)(\theta_4 - \theta_2)\left(\frac{\theta_1 - \theta}{\theta_4 - \theta_1 - 2}\right) \tag{25}$$

This is negative, meaning that type θ prefers $z(0)$'s strategy to $x(-2)$'s strategy, iff $\theta > \theta_1$, thus establishing part (i).

At $\theta_2 = g(-2) = z(a_\gamma)$, we have (by (22) and (21))

$$
\begin{aligned}
D_u(g(-2)|\theta) - D_v(z(a_\gamma)|\theta) &= \tau(\theta_3 - \theta_2)(\theta_3 + \theta_2 - 2\theta - 2) \\
&= 4\tau(\theta_2 - \theta) \quad \text{(using } \theta_3 - \theta_2 = 2) \tag{26}
\end{aligned}
$$

This is negative, meaning that type θ prefers $g(-2)$'s strategy to $z(a_\gamma)$'s strategy, iff $\theta > \theta_2$, proving part (ii).

At θ_3, we have (by (21) and (23)),

$$D_u(g(0)|\theta) - D_v(h(a_\gamma)|\theta) = 0 \tag{27}$$

so that all types are indifferent between the strategies prescribed for type $g(0) = \theta_3$, $h(a_\gamma) = \theta_3$, as desired to complete the proof. ∎

D.2 Optimality for the DM

Let the expert strategy be as specified in the previous subsection, using the action functions and parametrizations from Proposition D3, with $\tau = \beta(T-2)$ as in (15). Recall that we had the following free parameters: constants K, C_u, the horizon T, a number $\alpha_0 \in [0,1]$, and numbers $\alpha_a \in [0,1] \ \forall a \in [-2,0]$. We wish to show that the specified strategies constitute a fully revealing PBE: since we established expert optimality in the previous section, and since the beliefs and off-path strategies specified for the DM (see Proposition D1) trivially satisfy all PBE requirements, all that remains is to prove that the DM's on-path strategy is optimal.

So, to prove Theorem 1, we just need to prove that we can choose K, C_u, α_0, the α_a's, and a horizon T[17] such that whenever $b < \frac{1}{61}$, there exist (many) priors μ s.t. if the DM holds the corresponding Bayesian beliefs at each information set, then he finds it optimal to follow all expert recommendations (assuming no observable deviations).

Recall the following features of the expert strategy: at time $t = 2\alpha_0$, Group I and II pairs separate, with Group I then revealing the truth at time $T - \tau$; pairs from Groups II and III reveal the truth at time $T - 2$. It is immediately clear that during the "revelation phase", when the expert's recommendation is equal (with probability 1) to the true state, the DM indeed finds it optimal to follow the recommendation. In between time $2\alpha_0$ (when the DM learns whether the state lies in Group I, II, or III) and the group's corresponding revelation time (τ or $T-2$), no new information is revealed, but rejecting any (on-path) recommendation by the expert will cause the expert to subsequently babble rather than reveal the truth. So, the best possible deviation is

---

$[0, \theta_1]$, and the left endpoint $z(0)$ of the interval $[\theta_1, \theta_2]$: part (i) says that type θ_1 is indifferent between the two sequences prescribed for his type, and that everyone below θ_1 prefers the strategy of type $x(-2)$, everyone above θ_1 prefers the strategy of type $z(0)$.

[17]We prove here that there exists some T^* that works; we can trivially extend the construction to hold for all $T \geq T^*$, as in the theorem, simply by beginning with a babbling phase.

to choose the myopically optimal action in all subsequent periods. If such a deviation is optimal at some time t, then it is optimal also prior to time t (when information is the same, but the "reward phase" (following revelation of the truth) is further away). So, to prove that the DM does not wish to deviate, it is sufficient to prove that deviating is not optimal at the *earliest* time at which new information is revealed. We summarize this in the following observation:

Observation D4: If the DM cannot gain by deviating at time $t \in \{0, 2\alpha_0\}$, then the prescribed strategy is optimal.

So, we just need to prove that we can choose our free parameters such that (for a range of priors and $b < \frac{1}{61}$) the DM does not wish to deviate at time $t = 0$ or $t = 2\alpha_0$. We do so via three main propositions. All three rely on calculations from Lemma D7.1 below, which shows that for the constants K, C_u in $(28), (29)$ and $T - \tau \leq 4$, we can choose α_0 and the α_a's (with $\alpha_a \geq \alpha_0 \; \forall a$) such that the range of $v_1(\cdot, \alpha_0)$ is contained in the range of $u_1(\cdot)$. So every recommendation sent by a Group I pair is also sent by a Group II pair, for at least as long. Taking these values as given, Proposition D5 will prove that the DM has no incentive to deviate at time $t = 2\alpha_0$ if he gets a message $v_2(a, \alpha_0)$ from a Group I pair, for a range of posteriors. Proposition D6 proves that with these constants, there is a range of probability measures such that whenever $b < \frac{1}{61}$, it is strictly optimal for the DM to follow the prescribed recommendation at time $t = 0$ if the DM believes that it came from any pair $\{x(a), g(a)\}$, which implies also that he does not wish to deviate at time $t = 2\alpha_0$ if he realizes he is facing a Group II pair (again, we prove this for a range of posteriors). It remains only to prove that there are infinitely many priors μ generating the Bayesian posteriors in Propositions D5, D6, which we show in Proposition D7. This will complete the proof that we have a fully revealing PBE.

D.2.1 Optimality for the DM: Deviations at time $t = 2\alpha_0$

Proposition D5: Suppose that the DM receives recommendation $v_2(a)$ at time $t = 2\alpha_0$, and assigns probabilities $q_a, 1 - q_a$ to the two types $z(a), h(a)$ in his information set. Also set

$$
C_u = \frac{1 - \alpha_0}{\alpha_0} \frac{K^2}{T - 2} \tag{28}
$$

$$
K = \frac{\alpha_0 \tau a_\gamma \left(1 + \sqrt{\frac{(T - 2\alpha_0)(T - \tau)}{2\tau \alpha_0}}\right)}{(T - \tau - 2\alpha_0)} \tag{29}
$$

There exists a range of values for $\frac{q_a}{1 - q_a}$ such that (i) whenever $\beta a_\gamma^2 < 8$, we can choose a horizon T such that the DM's strategy is strictly optimal in the limit as $\alpha_0 \to 1$; and (ii) whenever $\beta a_\gamma^2 > 8$, we can choose a horizon T such that the DM's strategy is strictly optimal in the limit as $\alpha_0 \to 0$.[18]

Proof of Proposition D5:

If the DM follows recommendation $v_2(a)$ (expecting to choose this action until time $T - \tau$, then learn the truth), his expected disutility is

$$
(T - \tau - 2\alpha_0) \left(q_a \left(v_2(a) - z(a)\right)^2 + (1 - q_a) \left(v_2(a) - h(a)\right)^2\right) + \tau(0)
$$

The best possible deviation is to instead choose myopically optimal action $q_a z(a) + (1 - q_a)h(a)$ in all remaining $T - 2\alpha_0$ periods, for disutility

$$
(T - 2\alpha_a) \left(q_a \left(q_a z(a) + (1 - q_a)h(a) - z(a)\right)^2 + (1 - q_a) \left(q_a z(a) + (1 - q_a)h(a) - h(a)\right)^2\right)
$$
$$
= (T - 2\alpha)q_a(1 - q_a) \left(h(a) - z(a)\right)^2
$$

---

[18] Lemma D7.1 will show that we need only an upper bound on α_0 for biases satisfying the condition $\beta a_\gamma^2 > 8$ in (ii), and a lower bound on α_0 for biases satisfying $\beta a_\gamma^2 < 8$; with β specified by (15), $\beta a_\gamma^2 > 8 \Leftrightarrow a_\gamma < -3.18$.

So to prove the Proposition, we need to show that there are q_a's s.t. the following inequality holds:

$$
\begin{aligned}
0 \;\geq\; & \left(q_a \left(v_2(a) - z(a)\right)^2 + (1 - q_a)\left(v_2(a) - h(a)\right)^2 \right) - \frac{(T - 2\alpha_0)}{(T - \tau - 2\alpha_0)} q_a (1 - q_a)\left(h(a) - z(a)\right)^2 \\
\;=\; & q_a \left(2\left(\frac{v_2(a) - h(a)}{h(a) - z(a)} \right) + 1 \right) + \left(\frac{v_2(a) - h(a)}{h(a) - z(a)} \right)^2 - \frac{T - 2\alpha_0}{T - \tau - 2\alpha_0} q_a (1 - q_a)
\end{aligned}
\tag{30}
$$

This is easiest to satisfy (RHS is minimized) at

$$
q_a = q_a^* \equiv \frac{1}{2} - \frac{\left(\frac{v_2(a) - h(a)}{h(a) - z(a)} \right) + \frac{1}{2}}{\frac{T - 2\alpha_0}{T - \tau - 2\alpha_0}}
\tag{31}
$$

where (30) becomes

$$
\frac{\phi^2 - 1}{4\phi^2} \left(2\left(\frac{v_2(a) - h(a)}{h(a) - z(a)} \right) + \phi + 1 \right) \left(2\left(\frac{v_2(a) - h(a)}{h(a) - z(a)} \right) + 1 - \phi \right) \;\leq\; 0
\tag{32}
$$

$$
\text{with } \phi^2 \;=\; \frac{T - 2\alpha_0}{T - \tau - 2\alpha_0}
\tag{33}
$$

Thus, a sufficient condition for the DM's IC constraint (30) to hold is

$$
-\phi - 1 \leq 2\left(\frac{v_2(a) - h(a)}{h(a) - z(a)} \right) \leq \phi - 1
\tag{34}
$$

which implies both that (32) holds (so the DM does not want to deviate), and that q_a^* (from (31)) is indeed a probability, since we have

$$
2\left(\frac{v_2(a) - h(a)}{h(a) - z(a)} \right) + 1 + \phi^2 \;>\; 2\left(\frac{v_2(a) - h(a)}{h(a) - z(a)} \right) + 1 + \phi \geq 0 \Rightarrow q_a^* < 1
$$

$$
2\left(\frac{v_2(a) - h(a)}{h(a) - z(a)} \right) + 1 - \phi^2 \;<\; 2\left(\frac{v_2(a) - h(a)}{h(a) - z(a)} \right) + 1 - \phi \leq 0 \Rightarrow q_a^* = 0
$$

Substituting $C_u = \frac{1 - \alpha_0}{\alpha_0} \frac{K^2}{T - 2}$ (from (28)) into (14), we obtain that $v_2(a, \alpha_0) - h(a)$ equals

$$
\theta_3 + \frac{2K - \tau(a - a_\gamma)}{T - \tau} + \frac{\sqrt{\frac{4\tau\alpha_0}{T - \tau - 2\alpha_0}} \sqrt{\frac{(T - \tau)(T - 2)}{\tau} \frac{1 - \alpha_0}{\alpha_0} \frac{K^2}{T - 2} + \left(\frac{T - \tau - 2}{\tau} \right) K^2 + 2K(a - a_\gamma) - \frac{T}{2}(a - a_\gamma)^2}}{T - \tau} - h(a)
$$

Simplifying, and using $h(a) - z(a) = 2^{a - a_\gamma}$ and $\theta_3 - \theta_4 = a_\gamma$ (from (10) and (9)), this implies that

$$
\begin{aligned}
2\frac{v_2(a, \alpha_0) - h(a)}{h(a) - z(a)} \;&=\; \frac{2K - T(a - a_\gamma) + \sqrt{\frac{2\alpha_0}{T - \tau - 2\alpha_0}} \sqrt{2\frac{T - \tau - 2\alpha_0}{\alpha_0} K^2 + 2\tau\left(2K(a - a_\gamma) - \frac{T}{2}(a - a_\gamma)^2 \right)}}{(T - \tau)e^{a - a_\gamma}} \\
&=\; \frac{k - ty + \sqrt{k^2 + (\phi^2 - t)(2ky - ty^2)}}{e^y}
\end{aligned}
$$

where $k \equiv \frac{2K}{T - \tau}$, $t \equiv \frac{T}{T - \tau}$, and $y \equiv a - a_\gamma$ (with ϕ defined in (33)). So by (34), we wish to show that

for all $y \in [0, -a_\gamma]$,

$$\frac{k - ty + \sqrt{k^2 + (\phi^2 - t)(2ky - ty^2)}}{e^y} \in [-\phi - 1, \phi - 1] \tag{35}$$

By construction, the value of K specified in (29) sets the square root portion of v_1, v_2 equal to zero at $a = 0 \Leftrightarrow y = -a_\gamma$ (see Lemma D7.1), so we have

$$k = a_\gamma \left(\phi^2 - t + \phi \sqrt{\phi^2 - t} \right)$$

We first prove (ii). Suppose $\beta a_\gamma^2 > 8$, in which case Lemma D7.1 requires $\frac{T - \tau}{2} \leq 2$, and $\alpha_0 \leq \overline{\alpha_0}$ (for some $\overline{\alpha_0} \in (0, 1)$). Consider what happens as $\alpha_0 \to 0$ (in which case we trivially satisfy $\alpha_0 \leq \overline{\alpha_0}$): then $\phi^2 - t = \frac{T - 2\alpha_0}{T - \tau - 2\alpha_0} - \frac{T}{T - \tau} \to 0$, implying also that $k \to 0$, so we have

$$\min_{y \in [0, -a_\gamma]} \left(k - ty + \sqrt{k^2 + (\phi^2 - t)(2ky - ty^2)} + (1 + \phi)e^y \right) \quad \to \quad \min_{y \in [0, -a_\gamma]} \left(-\phi^2 y + (\phi + 1)e^y \right)$$

$$\max_{y \in [0, -a_\gamma]} \left(k - ty + \sqrt{k^2 + (\phi^2 - t)(2ky - ty^2)} + (1 - \phi)e^y \right) \quad \to \quad \max_{a \in [0, -a_\gamma]} \left(-\phi^2 y + (1 - \phi)e^y \right)$$

In the second line, the limit is clearly decreasing in y (noting that $\phi \to \sqrt{\frac{T}{T - \tau}} > 1$), thus maximized at $y = 0$: here, the value is $1 - \phi$, the desired upper bound in (35). In the first line, note that if we set ϕ s.t.

$$(\phi + 1)e > \phi^2 \Leftrightarrow 1 > \ln \frac{\phi^2}{\phi + 1} \Leftrightarrow \phi < 3.4959$$

then the limiting expression $(-\phi^2 y + (1 + \phi)e^y)$, which is minimized at $y = \ln \frac{\phi^2}{\phi + 1} < 1$, has a minimum value of

$$-\phi^2 \ln \frac{\phi^2}{\phi + 1} + \phi^2 = \phi^2 \left(1 - \ln \frac{\phi^2}{\phi + 1} \right)$$

which is positive, thus trivially satisfying the desired lower bound in (35). So, any $\phi < 3.4959$ satisfies (strictly) the DM's ex-post constraint in the limit as $\alpha_0 \to 0$, so by continuity the constraint holds also for all α_0 sufficiently close to zero. To complete the proof of (ii), we just need to show that we can choose a horizon T, satisfying the requirement $\frac{T - \tau}{2} \equiv \Delta \leq 2$ (from Proposition D6 and Lemma D7.1), which corresponds to α_0 near zero and $\phi < 3.4959$. We have, recalling that (15) (specifically $\tau = \beta(T - 2)$) implies $T = \frac{2\Delta - 2\beta}{1 - \beta}$, so $t = \frac{T}{T - \tau} = \frac{\Delta - \beta}{\Delta(1 - \beta)}$, that

$$\alpha_0 \to 0 \Rightarrow \phi^2 \to \frac{\Delta - \beta}{\Delta(1 - \beta)} \; (= t)$$

So we need

$$\frac{\Delta - \beta}{\Delta(1 - \beta)} < (3.4959)^2 \Leftrightarrow \left(1 - (3.4959)^2(1 - \beta) \right) \Delta < \beta$$

For $\beta < .9182$, the LHS of this inequality is negative, so the constraint $\phi < 3.4959$ does not bind. For $\beta > .9182$, the constraint (34) requires $\Delta \leq \frac{\beta}{1 - (3.4959)^2(1 - \beta)}$; together with the constraint $\Delta \leq 2$ (from Proposition D6, Lemma D7.1), we obtain that for $\beta a_\gamma^2 > 8$, the DM's strategy is optimal for

any horizon T satisfying

$$T < \begin{cases} \frac{2(1-\beta)}{1-\beta} & \text{if } \beta \leq .9182 \\ \min\left\{\frac{2(1-\beta)}{1-\beta}, \frac{2(3.4959)^2}{1-(3.4959)^2(1-\beta)}\right\} & \text{if } \beta > .9182 \end{cases} \tag{36}$$

This completes the proof of (ii): for $\beta a_\gamma^2 > 8$, if we set T according to (36) and α_0 near zero, then there exist posteriors (the q_a's) such that the DM's gain to deviating at time $2\alpha_0$ is strictly negative, at all information sets.

Finally we prove (i). Suppose $\beta a_\gamma^2 < 8$, in which case Lemma D7.1 uses $\frac{T-\tau}{2} \equiv \Delta = 2$ and requires $\alpha_0 \geq \underline{\alpha_0}$, for some $\underline{\alpha_0} \in [0, 1]$. So, set $\Delta = 2$, and consider what happens as $\alpha_0 \to 1$: then

$$t \to \frac{T}{4} = \frac{2-\beta}{2(1-\beta)}, \quad \phi^2 \to \frac{T-2}{T-\tau-2} = \frac{1}{1-\beta}, \quad k \to \frac{\beta a_\gamma}{2(1-\beta)}\left(1 + \sqrt{\frac{2}{\beta}}\right)$$

Define $f(y) \equiv k - ty + \sqrt{k^2 + (\phi^2 - t)(2ky - ty^2)}$; we want to prove that

$$-(1+\phi) \leq \frac{f(y)}{e^y} \leq \phi - 1 \text{ for all } y \in [0, -a_\gamma] \tag{37}$$

To this end, note that

$$f'(y) = -t + \frac{(\phi^2 - t)(k - ty)}{\sqrt{k^2 + (\phi^2 - t)(2ky - ty^2)}} < 0 \text{ (by } \phi^2 - t > 0, y > 0, \text{ and } k < 0)$$

$$f''(y) = \frac{-(\phi^2 - t)k^2\phi^2}{\left(k^2 + (\phi^2 - t)(2ky - ty^2)\right)^{\frac{3}{2}}} < 0$$

This implies that $f(y)$ reaches a maximum (over $y \in [0, -a_\gamma]$) at $y = 0$, and lies above the straight line connecting the points $(0, f(0))$ and $(-a_\gamma, f(-a_\gamma))$: since we have $f(-a_\gamma) = k + ta_\gamma$ (the square root term is zero here by construction), and $f(0) = k + \sqrt{k^2} = 0$, this line $\widetilde{f}$ is given by

$$\widetilde{f}(y) - \widetilde{f}(0) = \frac{\widetilde{f}(-a_\gamma) - \widetilde{f}(0)}{-a_\gamma}(y - 0) \Rightarrow \widetilde{f}(y) = \left(\frac{-k}{a_\gamma} - t\right)y$$

Substituting in $k = \frac{\beta a_\gamma}{2(1-\beta)}\left(1 + \sqrt{\frac{2}{\beta}}\right)$, this becomes $\widetilde{f}(y) = -\left(\frac{1+\sqrt{\frac{\beta}{2}}}{1-\beta}\right)y$. Then, the upper bound in (37) follows from the fact that

$$\max_{y \in [0, -a_\gamma]} \frac{f(y)}{e^y} \leq \frac{\max_{y \in [0, -a_\gamma]} f(0)}{\min_{y \in [0, -a_\gamma]} e^y} = 0 \leq \phi - 1$$

while the lower bound in (37) follows from

$$\min_{y \in [0, -a_\gamma]} \frac{f(y)}{e^y} \geq \min_{y \in [0, -a_\gamma]} \frac{\widetilde{f}(y)}{e^y} = \min_{y \in [0, -a_\gamma]} -\left(\frac{1+\sqrt{\frac{\beta}{2}}}{1-\beta}\right)\frac{y}{e^y} = \frac{-\left(1+\sqrt{\frac{\beta}{2}}\right)}{(1-\beta)e}$$

So, using $\phi = \frac{1}{\sqrt{1-\beta}}$, to show that $\min_{y \in [0, -a_\gamma]} \frac{f(y)}{e^y} + (1+\phi) \geq 0$, it suffices to show that

$$\frac{-\left(1 + \sqrt{\frac{\beta}{2}}\right)}{(1-\beta)e} + 1 + \sqrt{\frac{1}{1-\beta}} \geq 0 \Leftrightarrow e \geq \frac{\left(1 + \sqrt{\frac{\beta}{2}}\right)}{1 - \beta + \sqrt{1-\beta}}$$

Noting that the RHS is clearly increasing in β, it may be easily verified numerically that the above expression holds for all $\beta \leq 0.82035$, which is implied by the range under consideration ($a_\gamma \geq -3.2 \Rightarrow \beta < 0.8$). This completes the proof for $a_\gamma \geq -3.2 \Leftrightarrow \beta a_\gamma^2 < 8$, showing that if we set $T = \frac{4-2\beta}{1-\beta}$, then all DM IC constraints are slack in the limit as $\alpha_0 \to 1$ (and so, by continuity, hold for all α_0 sufficiently high, which satisfy the bounds in Lemma D7.1). This completes the proof. ∎

D.2.2 Optimality for the DM: Deviations at time $t = 0$

Proposition D6: For all $b < \frac{1}{61} \Leftrightarrow a_\gamma \leq -1.7726$, and for all $a \in [-2, 0]$, there exists a range of values p_a such that if the DM assigns probabilities $p_a, 1 - p_a$ to types $x(a), g(a)$, then his gain to deviating at information set $\{x(a), g(a)\}$ is strictly negative at any time $t \geq 0$.

Proof of Proposition D6: As explained in Observation D4, it suffices to prove that the gain to deviating is negative at time $t = 0$. Substituting $x(a) = \theta_3 + a - \theta_3 e^a$, $g(a) = \theta_3 + a$, and $C_u = 0$ into (55), we obtain that the DM's gain to deviating at information set $\{x(a), g(a)\}$, if he assigns probability p to type $x(a)$, is

$$
\begin{aligned}
&2p \left(K - \frac{T}{2}a + \theta_3 e^a \right)^2 + 2(1-p) \left(K - \frac{T}{2}a \right)^2 + 2(T-2)a \left(K - \frac{T}{4}a \right) - Tp(1-p) \left(\theta_3 e^a \right)^2 \\
&= 2K^2 + 4K \left(p\theta_3 e^a - a \right) + T \left(p\theta_3 e^a - a \right)^2 - (T-2)p(\theta_3 e^a)^2 \quad (38)
\end{aligned}
$$

The derivative of this expression w.r.t. a is

$$
\begin{aligned}
&2(p\theta_3 e^a - 1) \left[2K - Ta + Tp\theta_3 e^a - \frac{(T-2)p \left(\theta_3 e^a \right)^2}{(p\theta_3 e^a - 1)} \right] \\
&\leq 2(p\theta_3 e^a - 1) \left[2K - Ta - ((1-p)T - 2)\theta_3 e^a \right] \quad (39)
\end{aligned}
$$

So, it suffices to choose a value p s.t.

$$2K + 2T - ((1-p)T - 2)\theta_3 e^{-2} \leq 0 \quad (40)$$

$$2K^2 + 4K \left(p\theta_3 e^{-2} + 2 \right) + T \left(p\theta_3 e^{-2} + 2 \right)^2 - (T-2)p(\theta_3 e^{-2})^2 \leq 0 \quad (41)$$

and then set $p(a) = p$ for all $a \in [-2, 0]$: the first inequality guarantees that the expression in (39) is negative (so the DM's ex ante gain to deviating is decreasing in a, thus the gain is largest - so the IC constraint is most difficult to satisfy - at $a = -2$); the second inequality says that this maximal gain, obtained by evaluating (38) at $a = -2$, is negative.

Suppose first that $a_\gamma \in [-2, -1.7726]$, and consider setting $K = -4, p = \frac{1}{4}$: in this case, (39), and (38) at $a = -2$, become (respectively)

$$-8 + 2T - \left(\frac{3}{4}T - 2 \right)x \leq 0 \Leftrightarrow T \geq \frac{8(x-4)}{3x-8}$$

$$T \left(\frac{1}{4}x + 2 \right)^2 - 4x - \frac{1}{4}x^2 (T-2) \leq 0 \Leftrightarrow \frac{8x(x-8)}{3x-8} \leq (x-8)T$$

where $x \equiv \theta_3 e^{-2} = a_\gamma - 2 + 2e^{-a_\gamma}$; so, since $x \geq 8 \Leftrightarrow \frac{8x}{3x-8} \leq 4$, a sufficient condition for both inequalities to hold is $T \geq 4$ and $x \geq 8$.

Next, suppose $a_\gamma \leq -2$. It is sufficient to show that both expressions $(39), (38)$ hold at $p = \frac{1}{2}$, in which case (38) becomes

$$\theta_3 e^{-2} \geq 4 \left(\frac{K + T + \sqrt{\left(\frac{T-2}{2}\right)\left((K+4)^2 + 4(T-4)\right)}}{T-4} \right) \tag{42}$$

This implies also that $\theta_3 e^{-2} \geq \frac{4(K+T)}{T-4}$, so that (39) holds $p = \frac{1}{2}$. So, we just need to show that (42) holds, which implies that the DM's gain to deviating at $\{x(a), g(a)\}$ is largest at $a = -2$, and satisfied at $a = -2$ (hence satisfied also for all $a \in (-2, 0]$). For this, we will use the notation from (the proof of) Proposition D5:

$$\phi = \sqrt{\frac{T - 2\alpha_0}{T - \tau - 2\alpha_0}}, t = \frac{T}{T - \tau}, \Delta = \frac{T - \tau}{2}, k \equiv \frac{K}{\Delta} = a_\gamma \left(\phi^2 - t + \phi\sqrt{\phi^2 - t} \right)$$

Suppose first that $\beta a_\gamma^2 > 8$: for this range, we proved Proposition D5 using $\Delta < 2$ and $\alpha_0 \to 0$ (which implies also that $\phi^2 - t \to 0$, hence $K \to 0$), in which case (42) becomes

$$a_\gamma - 2 + 2e^{-a_\gamma} \geq 4 \left(\frac{T + \sqrt{2T(T-2)}}{T-4} \right) = 4 \left(\frac{T + \sqrt{2T(T-2)}}{T-4} \right)$$

Rearranging to solve for T, we need $T \geq \frac{4x^2}{x^2 - 8x - 16}$, with $x \equiv (a_\gamma - 2 + 2e^{-a_\gamma})$; since

$$\beta a_\gamma^2 > 8 \Rightarrow a_\gamma < -\sqrt{8} \Rightarrow x > 29 \Rightarrow \frac{4x^2}{x^2 - 8x - 16} < 6$$

it follows that for large α_0 and p near $\frac{1}{2}$, ex ante incentive compatibility for the DM holds, at every information set of the form $\{x(a), g(a)\}$, for any $T \geq 6$.
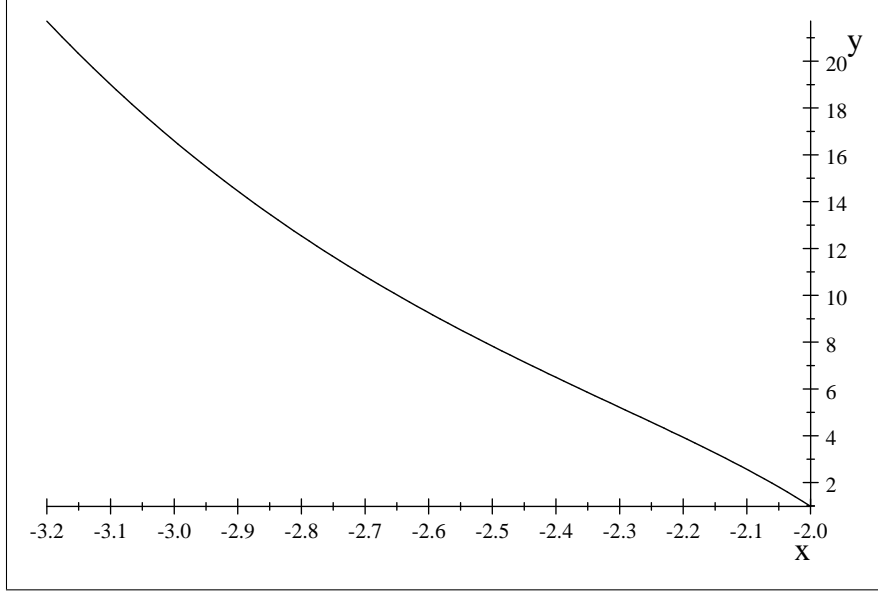
Suppose next that $\beta a_\gamma^2 < 8$ (and $a_\gamma \leq -2$) : for this range, we proved Lemma DM2 using $\alpha_0 \to 1$ and $\Delta = 2$, in which case

$$T = \frac{4 - 2\beta}{1 - \beta}, \phi^2 = \frac{1}{1 - \beta}, t = \frac{2 - \beta}{2(1 - \beta)}, \text{ and } K = \frac{\beta a_\gamma}{1 - \beta}\left(1 + \sqrt{\frac{2}{\beta}}\right)$$

Substituting into (42), we need

$$a_\gamma - 2 + 2e^{-a_\gamma} - \left(\frac{2}{\beta}\right)\left(\beta a_\gamma\left(1 + \sqrt{\frac{2}{\beta}}\right) + 4 - 2\beta + \sqrt{\left((1-\beta)\left(\frac{\beta a_\gamma}{1-\beta}\left(1 + \sqrt{\frac{2}{\beta}}\right) + 4\right)^2 + 8\beta\right)}\right) \geq 0$$

Here is a graph of the above expression for $a_\gamma \in [-3.2, -2]$ (using $\beta = \frac{(a_\gamma - 2 + 2e^{-a_\gamma})(a_\gamma - 4 + 2e^{-a_\gamma})}{2e^{-a_\gamma}(2e^{-a_\gamma} - 2)}$, and noting that $\beta a_\gamma^2 < 8 \Rightarrow a_\gamma \geq -3.2$) :

positive, as desired.

This completes the proof of Proposition D6, establishing that for every $a \in [-2, 0]$ and every $a_\gamma \leq -2$, there exists a value $\widehat{p} \in (0, 1)$ s.t., using the parameters that satisfy Proposition D5 and Lemma D7.1, , the DM's gain to deviating at information set $\{x(a), g(a)\}$ is strictly negative if he assigns probability $\widehat{p}$ to type $x(a)$. ∎

D.2.3 Optimality for the DM: Completing the Proof

As explained at the beginning of this section, it remains only to prove that there are many priors μ generating posteriors which satisfy the conditions in Proposiions D5, D6, which we prove here:

Proposition D7: For any continuous functions $p : [-2, 0] \to [0, 1]$, $q : [a_\gamma, 0] \to [0, 1]$, and $r : [a_\gamma, 0] \to [0, 1]$, there exists a probability measure μ such that, in our construction, a Bayesian DM will hold the following beliefs: (i) $\Pr(x(a)|\{x(a), g(a)\}) = p(a)$; (ii) $\Pr(z(a)|\{z(a), h(a)\}) = q(a)$; (iii) $\Pr(\{z(a), h(a)\} \mid \{z(a), h(a), x(\widehat{a}), g(\widehat{a})\} = r(a)$.

Proof of Proposition D7: Bayesian beliefs satisfy

$$
\frac{\Pr(x(a)|\{x(a), g(a)\})}{\Pr(g(a)|\{x(a), g(a)\})} = \frac{\mu(x(a))}{\mu(g(a))} \cdot \left| \frac{x'(a)}{g'(a)} \right| = \frac{\mu(x(a))}{\mu(g(a))}(\theta_3 e^a - 1)
$$

$$
\frac{\Pr(z(a)|\{z(a), h(a)\})}{\Pr(h(a)|\{z(a), h(a)\})} = \frac{\mu(z(a))}{\mu(h(a))} \cdot \left| \frac{z'(a)}{h'(a)} \right| = \frac{\mu(z(a))}{\mu(h(a))}\left(2e^{a-a_\gamma} - 1\right)
$$

$$
\frac{\Pr(\{z(a), h(a)\})}{\Pr(\{x(\widehat{a}), g(\widehat{a})\})} = \frac{\mu(z(a))}{\mu(\widehat{x}(a))} \frac{p(\widehat{a})}{q(a)} \frac{2e^{a-a_\gamma} - 1}{\theta_3 e^{\widehat{a}(a)} - 1} \cdot \frac{1}{|\widehat{a}'(a)|}
$$

where $\widehat{a}(a) = u_1^{-1}(v_1(a'))$ (as explained in the section Derivations below). We want the first expression to equal $\frac{p(a)}{1-p(a)}$, the second expression to equal $\frac{q(a)}{1-q(a)}$, and the third expression to equal $\frac{r(a)}{1-r(a)}$. It is straightforward to construct such a measure μ : for example, for each $a \in [-2, 0]$, set

$$
\mu(x(a)) = \frac{1}{M}
$$

with M a constant to be determined (this assigns a probability to each type in $[0, \theta_1]$). Then, assign

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priors to the types $g(a) \in [\theta_2, \theta_3]$ using this and the first expression above, i.e. set

$$\mu(g(a)) = \mu(x(a))(\theta_3 e^a - 1)\left(\frac{1-p(a)}{p(a)}\right) = \frac{(\theta_3 e^a - 1)}{M}\frac{1-p(a)}{p(a)}$$

Next, for each $a \in [a_\gamma, 0]$, set $\mu(z(a))$ to satisfy the 3rd bound above, i.e.

$$\mu(z(a) = \frac{\mu(x(\widehat{a}(a)))\frac{p(\widehat{a})}{q(a)}\frac{2e^{a-a_\gamma}-1}{\theta_3 e^{\widehat{a}(a)}-1}\cdot|\widehat{a}'(a)|}{\frac{r(a)}{1-r(a)}} = \frac{\frac{q(a)}{p(\widehat{a})}\frac{\theta_3 e^{\widehat{a}(a)}-1}{2e^{a-a_\gamma}-1}\cdot|\widehat{a}'(a)|}{M}\frac{r(a)}{1-r(a)}$$

(This assigns a prior to each type $z(a) \in [\theta_1, \theta_2]$). And similarly, now use this and the second expression above to assign a prior to each type $h(a) \in [\theta_3, \theta_4]$. Finally, choose M so that the total measure of the type space integrates to 1 (This is possible since $(11), (13)$ imply that u_1, v_1 and their derivatives w.r.t. a are finite and non-zero except perhaps at a single point a, from which it follows that $|\widehat{a}'(a)|$ is finite (again, except perhaps at a single point a)). Therefore, all our measures $\mu(x(a)), \mu(g(a)), \mu(z(a))$ are finite numbers divided by M; so, integrating over the state space results in a finite number divided by M. Choose M so that this number equals 1. ∎

D.2.4 Optimality for the DM: Preliminary Calculations

Lemma D7.1: Define

$$[\underline{\alpha_0}, \overline{\alpha_0}] \equiv \left[\frac{(T-\tau)+\beta a_\gamma\sqrt{\frac{T-\tau}{2\beta}}}{2+(-\beta a_\gamma)\left(\frac{2(T-2)}{T}+\frac{T-4}{T}\sqrt{\frac{T-\tau}{2\beta}}\right)}, 1-(T-\tau-2)\left(\sqrt{\frac{1}{2\beta(T-\tau)}}-\frac{1}{-\beta a_\gamma}\right)\right]$$

Then (i) $T - \tau \leq 4$ implies that $[\underline{\alpha_0}, \overline{\alpha_0}] \cap [0,1] \neq \emptyset$, with $\underline{\alpha_0} < 0$ whenever $\beta a_\gamma^2 > 8$, and $\overline{\alpha_0} > 1$ whenever $\beta a_\gamma^2 < 8$; (ii) $\alpha_0 \in [\underline{\alpha_0}, \overline{\alpha_0}]$ implies that u_t, v_t are both real-valued, and (iii) $\alpha_0 \in [\underline{\alpha_0}, \overline{\alpha_0}]$ implies that for every $a \in [a_\gamma, 0]$, there exists a pair $(\widehat{a}(a), \alpha(a))$, with $\alpha(a) \geq \alpha_0$ and $\widehat{a}(a) \in [-2, 0]$, such that

$$u_1(\widehat{a}(a), \alpha(a)) = v_1(a, \alpha_0)$$

Proof of Lemma D7.1: We first prove (i). Note that

$$T - \tau < \frac{\beta a_\gamma^2}{2} \Rightarrow (T-\tau) < \sqrt{\frac{\beta a_\gamma^2(T-\tau)}{2}} = -\beta a_\gamma\sqrt{\frac{T-\tau}{2\beta}} \Rightarrow \underline{\alpha_0} < 0$$

$$T - \tau > \frac{\beta a_\gamma^2}{2} \Rightarrow \sqrt{\frac{1}{2\beta(T-\tau)}} < \sqrt{\frac{1}{\beta^2 a_\gamma^2}} \Rightarrow \overline{\alpha_0} > 1$$

So, $\beta a_\gamma^2 > 8$ implies that we need only verify that $\overline{\alpha_0} > 0$, while if $\beta a_\gamma^2 < 8$, we need $\underline{\alpha_0} < 1$. In the first case,

$$\beta > \frac{1}{2} \text{ and } (T-\tau) \in [1,4] \Rightarrow (T-\tau-2)^2 \leq (T-\tau) < 2\beta(T-\tau)$$

$$\Rightarrow T-\tau-2 < \frac{1}{\sqrt{\frac{1}{2\beta(T-\tau)}}} < \frac{1}{\sqrt{\frac{1}{2\beta(T-\tau)}}+\frac{1}{\beta a_\gamma}}$$

which rearranges to give $\overline{\alpha_0} > 0$. In the second case, rearranging the inequality $\underline{\alpha_0} < 1$ gives

$$\frac{T}{2} < \frac{(T-2)}{T-\tau-2}(-\beta a_\gamma)\left(1 + \sqrt{\frac{T-\tau}{2\beta}}\right) = \left(\frac{-\beta a_\gamma}{1-\beta}\right)\left(1 + \sqrt{\frac{T-\tau}{2\beta}}\right)$$

Setting $\Delta \equiv \frac{T-\tau}{2} = \frac{(1-\beta)T+2\beta}{2} \Leftrightarrow \frac{T}{2} = \frac{\Delta - \beta}{1-\beta}$, this rearranges to the inequality $a_\gamma < 1 - \sqrt{\frac{\Delta}{\beta}}$, which is implied by $a_\gamma \leq -1.7726 \Rightarrow \beta \geq .4172$ and $\Delta \leq 2$. This establishes (i).

Now for (ii) and (iii): with C_u as specified by (28), the equations in $(11), (12)$ become

$$u_1(a, \alpha_a) - \theta_3 = K - \frac{T-2}{2}a - \sqrt{\frac{1-\alpha_a}{\alpha_a}\frac{1-\alpha_0}{\alpha_0}K^2 + \frac{1-\alpha_a}{\alpha_a}(T-2)a\left(K - \frac{T}{4}a\right)} \tag{43}$$

$$v_1(a, \alpha_0) - \theta_3 = \frac{2K - \tau(a - a_\gamma) - \sqrt{\frac{T-\tau-2\alpha_0}{2\alpha_0}}\sqrt{2\frac{T-\tau-2\alpha_0}{\alpha_0}K^2 + 2\tau\left(2K(a-a_\gamma) - \frac{T}{2}(a-a_\gamma)^2\right)}}{T-\tau} \tag{44}$$

We first prove (ii). To show that $v_1(\cdot)$ is real-valued, note that the term in square roots is decreasing in a (derivative w.r.t. a has the same sign as $2K - T(a - a_\gamma)$, which is negative by $K < 0$ and $a \geq a_\gamma$), so we just need to show that minimum value, which is at $a = 0$, is non-negative: that is, we need

$$0 \leq \frac{T-\tau-2\alpha_0}{\tau\alpha_0}K^2 - 2Ka_\gamma - \frac{T}{2}a_\gamma^2$$

The value of K specified in (29) is precisely the negative root of this equation, so the above inequality holds by construction. For $u_1(\cdot)$ to be real-valued, we need

$$0 \leq \frac{1-\alpha_0}{\alpha_0}K^2 + (T-2)\min_{a \in [-2,0]} a\left(K - \frac{T}{4}a\right)$$

The 2nd term is either minimized at $a = 0$ (in which case the expression clearly holds), or at $a = -2$, in which case it holds (for $K < 0$) iff

$$K \leq \frac{\alpha_0(T-2)}{1-\alpha_0}\left(1 - \sqrt{\frac{T-2\alpha_0}{\alpha_0(T-2)}}\right)$$

For this to be satisfied by the K-value in our construction, we need

$$\frac{\alpha_0\tau(-a_\gamma)\left(1 + \sqrt{\frac{(T-2\alpha_0)(T-\tau)}{2\tau\alpha_0}}\right)}{(T-\tau-2\alpha_0)} \geq \frac{\alpha_0(T-2)}{1-\alpha_0}\left(\sqrt{\frac{T-2\alpha_0}{\alpha_0(T-2)}} - 1\right)$$

$$\Leftrightarrow \frac{(-\beta a_\gamma)(1-\alpha_0)}{(T-\tau-2\alpha_0)} \geq \frac{T(1-\alpha_0)}{\left(\sqrt{\alpha_0(T-2)(T-2\alpha_0)}\left(1 + \sqrt{\frac{T-\tau}{2\beta}}\right) + (T-2\alpha_0)\sqrt{\frac{T-\tau}{2\beta}} + \alpha_0(T-2)\right)}$$

(2nd line obtained from 1st by (i) replacing τ with $\beta(T-2)$ in the LHS numerator; (ii) multiplying both sides by $\frac{1-\alpha_0}{\alpha_0(T-2)\left(1 + \sqrt{\frac{(T-2\alpha_0)(T-\tau)}{2\tau\alpha_0}}\right)}$; and finally (iii) on the resulting RHS, multiplying both numerator and denominator by $\sqrt{\alpha_0(T-2)}\left(\sqrt{T-2\alpha_0} + \sqrt{\alpha_0(T-2)}\right)$). Note that

$\sqrt{\alpha_0(T-2)(T-2\alpha_0)} \geq \alpha_0(T-2)$, so the RHS is at most

$$\frac{T(1-\alpha_0)}{\left(\alpha_0(T-2)\left(1+\sqrt{\frac{T-\tau}{2\beta}}\right)+(T-2\alpha_0)\sqrt{\frac{T-\tau}{2\beta}}+\alpha_0(T-2)\right)} = \frac{T(1-\alpha_0)}{\left((T-2\alpha_0+\alpha_0(T-2))\sqrt{\frac{T-\tau}{2\beta}}+2\alpha_0(T-2)\right)}$$

Therefore, to prove the desired inequality, it is sufficient to show that the following holds:

$$\frac{(-\beta a_\gamma)}{(T-\tau-2\alpha_0)} \geq \frac{T}{\left((T-2\alpha_0+\alpha_0(T-2))\sqrt{\frac{T-\tau}{2\beta}}+2\alpha_0(T-2)\right)}$$

$$\Leftrightarrow \quad \alpha_0 \geq \frac{(T-\tau)+\beta a_\gamma\sqrt{\frac{T-\tau}{2\beta}}}{2+(-\beta a_\gamma)\left(\frac{2(T-2)}{T}+\frac{T-4}{T}\sqrt{\frac{T-\tau}{2\beta}}\right)}$$

The RHS is precisely the value $\underline{\alpha_0}$ defined in the Lemma, so $\alpha_0 \geq \underline{\alpha_0}$ is sufficient to guarantee that $u_t(\cdot)$ is real-valued. This completes the proof of (ii).

Finally, to prove (iii): by (43) and (44) we have $u_1(0,\alpha_0) = K - \frac{1-\alpha_0}{\alpha_0}\sqrt{K^2} = K\left(1+\frac{1-\alpha_0}{\alpha_0}\right) = \frac{K}{\alpha_0}$, and $u_1(-2,1) = K+T-2$. So, if we set $\alpha(0) = \alpha_0$ and $\alpha(-2) = 1$, then $\left[\frac{K}{\alpha_0}, K+T-2\right] \subseteq [u_1^{\min}, u_1^{\max}]$. On the other hand,

$$\frac{T-\tau}{\tau}\frac{dv_1(a,\alpha_0)}{da} = -1 + \frac{\left(\frac{T}{2}(a-a_\gamma)-K\right)}{\sqrt{K^2+\frac{\tau\alpha_0}{T-\tau-2\alpha_0}\left(2K(a-a_\gamma)-\frac{T}{2}(a-a_\gamma)^2\right)}}$$

$$\frac{T-\tau}{2}\frac{d^2v_1(a,\alpha_0)}{da^2} = \frac{\frac{T}{2}\sqrt{K^2+\frac{\alpha_0}{T-\tau-2\alpha_0}\tau\left(2K(a-a_\gamma)-\frac{T}{2}(a-a_\gamma)^2\right)}^{-2}+\frac{\tau\alpha_0}{T-\tau-2\alpha_0}\left(\frac{T}{2}(a-a_\gamma)-K\right)^2}{\sqrt{K^2+\frac{\tau\alpha_0}{T-\tau-2\alpha_0}\left(2K(a-a_\gamma)-\frac{T}{2}(a-a_\gamma)^2\right)}^{-3}}$$

Clearly the second expression is positive, so we have that

$$a \geq a_\gamma \Rightarrow \frac{T-\tau}{\tau}\frac{dv_1(a,\alpha_0)}{da} \geq \frac{T-\tau}{\tau}\frac{dv_1(a,\alpha_0)}{da}\bigg|_{a=a_\gamma} = -1+\frac{(-K)}{\sqrt{K^2}} = 0$$

That is, $v_1(\cdot)$ is strictly increasing on $[a_\gamma, 0]$, with

$$v_1^{\min}-\theta_3 = v_1(a_\gamma,\alpha_0)-\theta_3 = \frac{2K-\sqrt{\frac{T-\tau-2\alpha_0}{2\alpha_0}}\sqrt{2\frac{T-\tau-2\alpha_0}{\alpha_0}K^2}}{T-\tau} = \frac{2K\left(1+\frac{T-\tau-2\alpha_0}{2\alpha_0}\right)}{T-\tau} = \frac{K}{\alpha_0}$$

$$v_1^{\max}-\theta_3 = v_1(0,\alpha_0)-\theta_3 = \frac{2K+\tau a_\gamma-\sqrt{\frac{T-\tau-2\alpha_0}{2\alpha_0}}\sqrt{2\tau\left(\frac{T-\tau-2\alpha_0}{\tau\alpha_0}K^2-2Ka_\gamma-\frac{T}{2}a_\gamma^2\right)}}{T-\tau}$$
$$= \frac{2K+\tau a_\gamma}{T-\tau}$$

So we immediately have $v_1^{\min} = u_1(0,\alpha_0)$. To show that $v_1^{\max} \leq u_1(-2,1)$, we need to show that

$$K+T-2 \geq \frac{2K+\tau a_\gamma}{T-\tau} \Leftrightarrow \frac{(T-\tau-2)}{(T-2)}K > \frac{\tau a_\gamma}{T-2}-(T-\tau)$$

Substituting in our value $K = \dfrac{a_\gamma \left(\tau \alpha_0 + \sqrt{\frac{(T-2\alpha_0)(\tau \alpha_0)(T-\tau)}{2}} \right)}{(T-\tau-2\alpha_0)}$, using $\tau = \beta(T-2)$, and dividing by a_γ (which flips the inequality), this becomes

$$\frac{a_\gamma \left(\tau \alpha_0 + \sqrt{\frac{(T-2\alpha_0)(\tau \alpha_0)(T-\tau)}{2}} \right)}{(T-\tau-2\alpha_0)} > \frac{\beta a_\gamma - (T-\tau)}{1-\beta} \Leftrightarrow \frac{(T-2)\left(\beta \alpha_0 + \sqrt{\frac{(T-2\alpha_0)(\beta \alpha_0)(T-\tau)}{2(T-2)}} \right)}{(T-\tau-2\alpha_0)} < \frac{\beta + \frac{(T-\tau)}{(-a_\gamma)}}{1-\beta}$$

For this, it is sufficient to prove the following (which replaces the 2nd numerator term on the LHS with its maximum value (at $\alpha_0 = 1$)):

$$\frac{(T-2)\left(\beta \alpha_0 + \sqrt{\frac{\beta(T-\tau)}{2}} \right)}{(T-\tau-2\alpha_0)} < \frac{\beta + \frac{(T-\tau)}{-a_\gamma}}{1-\beta}$$

Multiplying both sides by $(1-\beta)(T-\tau-2\alpha_0)$, this rearranges as

$$(T-\tau-2)\sqrt{\frac{\beta(T-\tau)}{2}} - (T-\tau-2\alpha_0)\frac{(T-\tau)}{-a_\gamma} < \beta(T-\tau)(1-\alpha_0)$$

Replacing the LHS with the upper bound at $\alpha_0 = 1$, and dividing through by $\beta(T-\tau)$, we obtain the following sufficient condition:

$$\alpha_0 \leq 1 - (T-\tau-2)\left(\sqrt{\frac{1}{2\beta(T-\tau)}} - \frac{1}{-\beta a_\gamma} \right)$$

The RHS is precisely the value $\overline{\alpha_0}$ defined in the Lemma: that is, $\alpha_0 \leq \overline{\alpha_0}$ implies that $[v_1^{\min}, v_1^{\max}] \subseteq [u_1(0, \alpha_0), u_1(-2, 1)]$, which, by continuity, implies that for every $a \in [a_\gamma, 0] \Leftrightarrow v_1(a) \in [v_1^{\min}, v_1^{\max}]$, there exists $\widehat{a}(a) \in [-2, 0]$ and $\alpha_a \geq \alpha_0$ s.t. $v_1(a, \alpha_0) = u_1(\widehat{a}(a), \alpha_a)$. This proves (iii). ∎

E   Appendix E: Derivations

In this section, we explain how the functions and parameters in our fully revealing construction were chosen.

E.1   For the Expert:

Suppose we wanted an equilibrium in which each type $\theta \in [0, \theta_1]$ pools with a type $g(\theta) \in [\theta_2, \theta_3]$, to recommend an action $u_1(\theta)$ in period 1, $u_2(\theta)$ in period 2, and then reveal the truth at time $T - 2$. The disutility to type θ from following the strategies prescribed for types $\theta', g(\theta')$ is then

$$D_u(\theta'|\theta) = \left(u_1(\theta') - \theta - 1 \right)^2 + \left(u_2(\theta') - \theta - 1 \right)^2 + (T-2)\left(\theta' - \theta - 1 \right)^2 \tag{45}$$
$$D_u(g(\theta')|\theta) = \left(u_1(\theta') - g(\theta) - 1 \right)^2 + \left(u_2(\theta') - g(\theta) - 1 \right)^2 + (T-2)\left(g(\theta') - g(\theta) - 1 \right)^2$$

In order for this to be an equilibrium, it must be that $D_u(\theta'|\theta)$ reaches a minimum over $[0, \theta_1]$ at $\theta' = \theta$ (so that type θ earns a lower disutility by telling the truth than by mimicking any other type θ' in the interval $[0, \theta_1]$), and that $D_u(g(\theta')|g(\theta))$ reaches a minimum at $g(\theta') = g(\theta)$. We can do this by simply choosing functions that satisfy the corresponding first- and second-order conditions:

beginning with the F.O.C.'s, we need

$$0 \;=\; \frac{1}{2}\frac{d}{d\theta'}D_u(\theta'|\theta)\bigg|_{\theta'=\theta} = u_1'(\theta)\left(u_1(\theta)-\theta-1)\right) + u_2'(\theta)\left(u_2(\theta)-\theta-1\right) - (T-2) \tag{46}$$

$$0 \;=\; \frac{1}{2}\frac{d}{d\theta'}D_u(\theta'|\theta)\bigg|_{\theta'=\theta} = u_1'(\theta)\left(u_1(\theta)-g(\theta)-1\right) + u_2'(\theta)(u_2(\theta)-\theta-1) - (T-2)g'(\theta) \tag{47}$$

Subtracting the 2nd expression from the 1st, we get

$$\left(u_1'(\theta) + u_2'(\theta)\right)(g(\theta) - \theta) = (T-2)(1 - g'(\theta))$$

If we define $a(\theta) \equiv \ln\frac{g(\theta)-\theta}{g(\theta_1)-\theta_1}$, so that $a'(\theta) = \frac{g'(\theta)-1}{g(\theta)-\theta}$, this becomes

$$u_1'(\theta) + u_2'(\theta) = -(T-2)a'(\theta) \Rightarrow u_1(\theta) + u_2(\theta) = k_u - (T-2)a(\theta), \; k_u \text{ a constant} \tag{48}$$

Now: the disutility from telling the truth is

$$D_u(\theta|\theta) \;\equiv\; D_u(\theta) = (u_1(\theta)-\theta-1)^2 + (u_2(\theta)-\theta-1)^2 + (T-2) \tag{49}$$

$$\Rightarrow \frac{D_u'(\theta)}{2} = (u_1'(\theta)-1)(u_1(\theta)-\theta-1) + u_2'(\theta)(u_2(\theta)-\theta-1)$$

Substituting (46) into this expression, we get

$$\frac{D_u'(\theta)}{2} \;=\; T - 2 - (u_1(\theta) + u_2(\theta)) + 2(\theta+1)$$

$$=\; T + 2\theta - k_u + (T-2)a(\theta) \;\text{ (by (48))}$$

Integrating w.r.t. θ, we get

$$D_u(\theta) = D_u(0) + 2\theta\,(T + \theta - k_u) - 2(T-2)\int_0^\theta (a(\theta'))d\theta'$$

Setting $u_1(0) \equiv u_0$ and using expression (48) to obtain $u_2(0) = k_u - (T-2)a(0) - u_1(0) = k_u - u_0$, this becomes

$$D_u(\theta) \;=\; \underbrace{(u_0-1)^2 + (k_u - u_0 - 1)^2 + (T-2)}_{D_u(0)} + 2\theta(T + \theta - k_u) + 2(T-2)\int_0^\theta a(\theta')d\theta'$$

$$=\; 2\left(\frac{k_u}{2} - u_0\right)^2 + 2\left(\frac{k_u}{2} - \theta - 1\right)^2 + 2\theta\,(T-2) + 2(T-2)\int_0^\theta a(\theta')d\theta' + (T-2)$$

It will be convenient to change variables: rather than describing g as a function from $[0,\theta_1] \to [\theta_2,\theta_3]$, and using $a(\theta) \equiv \ln\frac{g(\theta)-\theta}{g(0)-0}$, we "flip" variables, describing each type $\theta \in [0,\theta_1]$ as a parametric function $x(a)$ of the variable a, and each type in $[\theta_2,\theta_3]$ as a parametric function $g(a)$ of the variable a, where a takes on all values between 0 and $a_1 = \ln\frac{g(\theta_1)-\theta_1}{g(0)-0}$, and $g(a), x(a)$ hold the relationship $(g(a) - x(a)) = (g(0) - x(0))\,e^a$. With this, rewriting $\int_0^\theta a(\theta')d\theta'$ as $\int_0^a sx'(s)ds$, and

noting that $\theta = \int_0^a x'(s)ds$, our above disutility expression for type $\theta = x(a)$ becomes

$$D_u(x(a)) = 2\left(\frac{k_u}{2} - x(a) - 1\right)^2 + 2(T-2)\int_0^a (s+1)x'(s)ds + (T-2) + 2(T-2)C_u \quad (50)$$

where $C_u \equiv \frac{\left(\frac{k_u}{2} - u_0\right)^2}{T-2}$ may be any non-negative constant. Setting this equal to type $x(a)'s$ truth-telling disutility (evaluate (49) at $\theta = x(a)$), using $u_2(a) = k - (T-2)a - u_1(a)$ (from (48)), and solving for $u_1(a), u_2(a)$, we obtain

$$u_1(a) = \frac{k_u}{2} - \frac{T-2}{2}a - \sqrt{T-2}\sqrt{C_u + \frac{k_u}{2}a - a(x(a)+1) - \frac{T-2}{4}a^2 + \int_0^a (s+1)x'(s)ds} \quad (51)$$

with $u_2(a) = k_u - (T-2)a - u_1(a)$. Evaluating this at $x(a) = \theta_3 + a - \theta_3 e^a$ and $\frac{k_u}{2} = K + \theta_3$ gives precisely our expression $u_1(a, \alpha_a)$ in (11) evaluated at $\alpha_a = \frac{1}{2}$; the expressions in (11), (12) were "rescaled" (via the coefficients on the square roots) such that both disutility and average actions are independent of α_a.

Now, for our S.O.C.'s: differentiating (45) w.r.t. θ' gives

$$\frac{1}{2}\frac{dD_u(\theta'|\theta)}{d\theta'} = \underbrace{u_1'(\theta')\left(u_1(\theta') - \theta' - 1\right) + u_2'(\theta')\left(u_2(\theta') - \theta' - 1\right) - (T-2)}_{=0 \text{ by } (46)}$$

$$+ (\theta' - \theta)\underbrace{\left(u_1'(\theta') + u_2'(\theta') + T - 2\right)}_{=\frac{d}{d\theta'}(u_1(\theta')+u_2(\theta')+(T-2)\theta')}$$

This implies that a sufficient condition for truth-telling to indeed yield a *minimum* on disutility is that the average action induced by each type θ, $u_1(\theta) + u_2(\theta) + (T-2)\theta$, be increasing: in this case, $\frac{dD_u(\theta'|\theta)}{d\theta'}$ is positive for any $\theta' > \theta$ (as type θ contemplates mimicking types θ' further above him, disutility increases, making him worse off), and negative for $\theta' < \theta$ (as he moves further below the truth, disutility increases, also making him worse off), but zero at $\theta' = \theta$: thus, telling the truth is better than mimicking any other type in the interval.

To sum up, this has shown that given arbitrary interval endpoints $\theta_1, \theta_2, \theta_3$, functions $x : [a_1, 0] \to [0, \theta_1]$ and $g : [a_1, 0] \to [\theta_2, \theta_3]$, and with $a_1 \equiv \ln\frac{g(a_1)-x(a_1)}{g(0)-x(0)} = \ln\frac{\theta_2-\theta_1}{\theta_3-0}$ and $g(a) - x(a) = \theta_3 e^a$, if we want an equilibrium in which types $x(a), g(a)$ recommend $u_1(a)$ for one period, then $u_2(a)$ for one period, then separate and reveal the truth, then truth-telling satisfies the F.O.C. for disutility minimization iff u_1, u_2 are as specified by (51) and $u_2(a) = k_u - (T-2)a - u_1(a)$. If we additionally impose the requirement that average action be increasing in type, then we satisfy also the S.O.C.'s; this requires that each of $x'(a), g'(a)$ is either negative or ≥ 1. Analogously, for arbitrary functions $z : [a_\gamma, 0] \to [\theta_1, \theta_2]$, $h : [a_\gamma, 0] \to [\theta_3, \theta_4]$, with $a_\gamma = \ln\frac{h(a_\gamma)-z(a_\gamma)}{h(0)-z(0)} = \frac{\theta_3-\theta_2}{\theta_4-\theta_1}$ and $h(a) - z(a) = (\theta_4 - \theta_1)e^a$, if we want an equilibrium in which types $z(a), h(a)$ recommend $v_1(a)$ for $\frac{T-\tau}{2}$ periods, then $v_2(a)$ for $\frac{T-\tau}{2}$ periods, then separate and reveal the truth, the F.O.C.'s for truth-telling to minimize disutility yield the following equations:

$$v_1(a) = \frac{k_v}{2} - \frac{\tau}{T-\tau}a - \sqrt{\frac{2\tau}{T-\tau}}\sqrt{C_v + \frac{k_v}{2}a - a(z(a)+1) - \frac{\tau}{2(T-\tau)}a^2 + \int_0^a (s+1)z'(s)ds} \quad (52)$$

with $v_2(a) = k_v - \frac{2\tau}{T-\tau}a - v_1(a)$, k_v and C_v constants. And the S.O.C.'s, guaranteeing that truth-telling indeed yields a disutility *minimum* over the interval, reduce to the requirement that each of $z'(a), h'(a)$ is either negative or ≥ 1. The proof that no expert type wishes to deviate after the

initial recommendation follows almost trivially from the prescribed strategies.

It remains to show that no expert type wishes to mimic the initial recommendation of any type from any *other* interval. This reduces to the additional requirements that at each endpoint $\theta_i \in \{\theta_1, \theta_2, \theta_3\}$, the average action is non-decreasing at θ_i (if discontinuous), and type θ_i is indifferent between the two sequences that he can induce. Our construction chooses the specific parameterizations $g(a) = \theta_3 + a$ and $h(a) = \theta_4 + a$, with $x(a) = g(a) - \theta_3 e^a$, $z(a) = h(a) - (\theta_4 - \theta_1)e^a$. Then we have $x'(a) = 1 - \theta_3 e^a \leq 1 - \theta_3 e^{-2}$, $z'(a) = 1 - 2e^{a-a_\gamma} \leq -1$, and $g'(a) = h'(a) = 1$, which clearly satisfy the S.O.C.'s (provided that $\theta_3 e^{-2} \geq 2$; we in fact will restrict to $\theta_3 e^{-2} \geq 8$). With this, the expressions in (51), (52) become (with $K \equiv \frac{k_u}{2} - \theta_3$)

$$u_1(a) = K + \theta_3 - \frac{T-2}{2}a - \sqrt{T-2}\sqrt{C_u + Ka - \frac{T}{4}a^2} \tag{53}$$

$$v_1(a) = \frac{k_v}{2} - \frac{\tau}{T-\tau}a - \sqrt{\frac{2\tau}{T-\tau}}\sqrt{C_v + \left(\frac{k_v}{2} - \theta_4\right)a - \frac{T}{2(T-\tau)}a^2} \tag{54}$$

We chose $a_1 = -2$ and $g'(a) = 1$ because this is in fact the *only* way that the indifference constraint at θ_2 can hold simultaneously with both the indifference constraint at θ_3, and the increasing-average-action requirement at θ_2. We chose $h'(a) = 1$ just for simplicity. With it, the remaining increasing-average-action constraints do not bind, and the indifference conditions reduce to the following requirements on the relationships between k_v, C_v, τ (parameters from the v_t-functions) and k_u, C_u, T (parameters from the u_t-functions):

$$k_v = \theta_3 + \frac{2K + \tau a_\gamma}{T - \tau}$$

$$C_v = \frac{(T-2)C_u}{\tau} + \frac{\frac{T-\tau-2}{\tau}K^2 - 2Ka_\gamma - \frac{T}{2}a_\gamma^2}{(T-\tau)}$$

$$\frac{\tau}{T-2} = \beta \equiv \frac{(\theta_2 - \theta_1)(\theta_2 - \theta_1 - 2)}{(\theta_4 - \theta_1)(\theta_4 - \theta_1 - 2)}$$

With this, the expressions in (53), (54) simplify exactly to the expressions for $u_1(a, \alpha_a)$ in (11) and $v_1(a, \alpha_0)$ in (13), at $\alpha_a = \frac{1}{2}$, $\alpha_0 = \frac{T-\tau}{4}$; in Proposition D3, we then rescaled (11), (13) for other values of α_a, α_0 in such a way that incentives are not affected.

E.2   For the DM:

Suppose the DM receives the recommendation $u_1(a, \alpha_a)$ in period 1. If he assigns probabilities $(p_a, 1 - p_a)$ to types $x(a), g(a)$, then his disutility from following all recommendations is

$$p_a\left(2\alpha_a\left(u_1(a, \alpha_a) - x(a)\right)^2 + 2(1 - \alpha_a)\left(u_2(a, \alpha_a) - x(a)\right)^2\right)$$

$$+(1 - p_a)\left(2\alpha_a\left(u_1(a, \alpha_a) - g(a)\right)^2 + 2(1 - \alpha_a)\left(u_2(a, \alpha_a) - g(a)\right)^2\right)$$

Substituting in the expression for $u_1(a, \alpha_a), u_2(a, \alpha_a)$ from (11), (12), and expanding, this becomes

$$2p_a\left(K + \theta_3 - \frac{T-2}{2}a - x(a)\right)^2 + 2(1 - p_a)\left(K + \theta_3 - \frac{T-2}{2}a - g(a)\right)^2 + 2(T-2)C_u + 2(T-2)a\left(K - \frac{T}{4}a\right)$$

The best possible deviation is to choose the myopically optimal action $p_a x(a) + (1 - p_a)g(a)$ in all T periods, resulting in disutility

$$Tp_a \left(p_a x(a) + (1 - p_a)g(a) - x(a)\right)^2 + T(1 - p_a)\left(p_a x(a) + 1 - p_a)g(a) - g(a)\right)^2$$
$$= Tp_a(1 - p_a)\left(g(a) - x(a)\right)^2$$

Therefore, incentive compatibility of our strategies for the DM refers that the following expression (the gain to deviating at $\{x(a), g(a)\}$) be weakly negative for all $a \in [-2, 0]$:

$$2p_a \left(K + \theta_3 - \frac{T-2}{2}a - x(a)\right)^2 + 2(1 - p_a)\left(K + \theta_3 - \frac{T-2}{2}a - g(a)\right)^2 \tag{55}$$

$$+2(T-2)C_u + 2(T-2)a\left(K - \frac{T}{4}a\right) - Tp_a(1 - p_a)\left(g(a) - x(a)\right)^2$$

Substituting in $g(a) = \theta_3 + a$, $x(a) = \theta_3 + a - \theta_3 e^a$ and solving for K, we obtain that the DM's gain to deviating at information set $\{x(a), g(a)\}$ is negative if and only if

$$K \in \left[a - p_a\theta_3 e^a - \sqrt{T-2}\Delta(a), \ a - p_a\theta_3 e^a + \sqrt{T-2}\Delta(a)\right] \tag{56}$$

$$\text{where } \Delta(a) \equiv \sqrt{\frac{1}{2}p_a(1 - p_a)(\theta_3 e^a)^2 + p_a(\theta_3 a e^a) - \frac{a^2}{2} - C_u}$$

For there to exist a value of K which satisfies this expression, we need $\Delta(a)$ to be real-valued, i.e. the term in square roots must be positive; at $a = -2$, this holds iff

$$p_a \in \left[\frac{\theta_3 e^{-2} - 4}{2(\theta_3 e^{-2})} - \frac{1}{2}\sqrt{\frac{\theta_3 e^{-2} - 8}{\theta_3 e^{-2}}}, \ \frac{\theta_3 e^{-2} - 4}{2(\theta_3 e^{-2})} + \frac{1}{2}\sqrt{\frac{\theta_3 e^{-2} - 8}{\theta_3 e^{-2}}}\right]$$

which in turn is *possible* (for some belief system) to satisfy only if $\theta_3 \geq 8e^2$. For our construction, with $g' = h' = 1$, this corresponds to

$$a_\gamma = \ln \frac{\theta_3 - \theta_2}{\theta_4 - \theta_1} = \ln \frac{2}{-a_\gamma + 2 + \theta_3 e^{-2}} \leq -1.7726 \Leftrightarrow b \leq \frac{1}{60.885}$$

That is, it is possible to satisfy the DM IC constraints in a straightforward manner only if the bias satisfies $b \leq \frac{1}{60.885}$, which is why our construction specifies $b < \frac{1}{61}$.

Similarly, if the DM receives the recommendation $v_1(a)$ in period 1 and assigns probabilities $q_a, 1 - q_a$ to types $z(a), h(a)$, his maximum gain to deviating is

$$q_a \left(2\alpha_0 (v_1(a, \alpha_0) - z(a))^2 + (T - \tau - 2\alpha_0)(v_2(a, \alpha_0) - z(a))^2\right)$$

$$+(1 - q_a)\left(2\alpha_0 (v_1(a, \alpha_0) - h(a))^2 + (T - \tau - 2\alpha_0)(v_2(a, \alpha_0) - h(a))^2\right) - Tq_a(1 - q_a)(h(a) - z(a))^2$$

Recalling that the expressions in $(13), (14)$ were scaled to make the above expression independent of α_0, we can without loss of generality set $\alpha_0 = 1$, in which case v_1 is given by (54), and $v_2 =$

$k_v - \frac{2\tau}{T-\tau}a - v_1$; substituting into the above expression for the DM's gain to deviating, we obtain

$$(T - \tau)\left(q_a\left(\frac{k_v}{2} - \frac{\tau}{T-\tau}a - z\right)^2 + (1 - q_a)\left(\frac{k_v}{2} - \frac{\tau}{T-\tau}a - h\right)^2\right) + 2\tau C_v$$

$$+2\tau a\left(\frac{k_v}{2} - \theta_4 - \frac{T}{2(T-\tau)}a\right) - Tq_a(1 - q_a)(h - z)^2 \tag{57}$$

Setting $h(a) = \theta_4 + a$, $z(a) = \theta_4 + a - (\theta_4 - \theta_1)e^a = \theta_4 + a - 2e^{a-a_\gamma}$ and solving for $\frac{k_v}{2}$, we obtain that the DM's gain to deviating at $\{z(a), h(a)\}$ is negative if and only if

$$\frac{k_v}{2} \in \left[\theta_4 + a - q_a\left(2e^{a-a_\gamma}\right) - \sqrt{\frac{2\tau}{T-\tau}}\widetilde{\Delta}(a), \ \theta_4 + a - q_a\left(2e^{a-a_\gamma}\right) + \sqrt{\frac{2\tau}{T-\tau}}\widetilde{\Delta}(a)\right]$$

$$\text{where } \widetilde{\Delta}(a) \equiv \sqrt{\frac{1}{2}q_a(1 - q_a)(2e^{a-a_\gamma})^2 + q_a\left(2ae^{a-a_\gamma}\right) - \frac{a^2}{2} - C_v}$$

This constraint by itself is problematic. To understand the difficulty, note that at $a = a_\gamma$, there exists a value of k_v satisfying the above expression only if $\widetilde{\Delta}(a_\gamma)$ is real-valued, requiring

$$2q_a(1 - q_a + a_\gamma) - \frac{a_\gamma^2}{2} - C_v \geq 0$$

We showed in the previous paragraph that the IC constraints at information sets of the form $\{x(a), g(a)\}$ can only hold if $a_\gamma \leq -1.7726$, in this case, the first term in the above inequality is negative (since we need $q_a \geq 0$ and $1 + a_\gamma < 0$), the second term is clearly negative, and the third must be negative (i.e. we need $C_v \geq 0$) in order for the functions v_1, v_2 to be real-valued at $a = 0$. Therefore, if the DM finds it optimal to follow all recommendations sent by pairs $\{x(a), g(a)\}$, then he necessarily will have an incentive to deviate if his information set contains only types $\{z(a_\gamma), h(a_\gamma)\} = \{\theta_2, \theta_3\}$.

To solve this problem, we will "bunch" pairs - scaling our action functions such that whenever the DM would have an incentive to deviate after a recommendation $v_1(a)$ sent by a pair $\{z(a), h(a)\}$, he believes that the recommendation $v_1(a)$ is *also* sent (for the same length of time) by a pair $\{x(a'), g(a')\}$, and such that the expected benefit to following the recommendation (likelihood that it was sent by the pair $(x(a'), g(a'))$, times the gain to following the recommendation conditional on this pair), exceeds the cost to following the recommendation (which is the likelihood that it was sent by pair $(z(a), h(a))$, times the benefit to deviating conditional on this pair).

E.2.1   DM: Bunching and Beliefs

Our incentive constraints for the DM were specified in terms of arbitrary probabilities p_a, q_a, which in turn depend both on his prior (the measure μ over the state space $[0, 1]$), and on the precise details of our construction. We assume that the DM is Bayesian: if he believes that the expert's first-period messages are given by a function $M : [0, 1] \to \mathbb{R}$, with the property that

$$M(x) = M(p(x))$$

for all x in some interval $[\underline{x}, \overline{x}] \subseteq [0, 1]$, and $p : [\underline{x}, \overline{x}] \to [0, 1]\backslash[\underline{x}, \overline{x}]$ some continuous differentiable function (i.e., types x and $p(x)$ are "partners" who follow the same messaging strategy), then, after receiving the message $m = M(x) = M(p(x))$, the DM's beliefs satisfy

$$\frac{\Pr(x|m)}{\Pr(p(x)|m)} = \frac{\mu(x)}{\mu(p(x))}\left|\frac{1}{p'(x)}\right| \tag{58}$$

This says that the likelihood of type x relative to $p(x)$ is equal to the unconditional likelihood ratio (determined by the prior μ), times a term which depends on the shape of the p-function, in particular due to its influence on the size of the interval of p-types compared to the size of their partner interval, $[\underline{x}, \overline{x}]$. To understand this formula, consider an example in which μ is uniform and $p(\cdot)$ is linear, say $p(x) = \alpha + \beta x$. In this case, the interval $[p(\underline{x}), p(\overline{x})]$ is β times as large as the interval $[\underline{x}, \overline{x}]$, so intuitively, it is as if the message sent by type x is sent by β "copies" of type $p(x)$: therefore, the DM's beliefs assign β times as much weight to type $p(x)$ as to type x, which is precisely what our formula says. Beliefs are assigned analogously after period 1.

For our particular construction, our belief formula in (58) becomes:

- after a message (or message sequence) sent by types $\{x(a), g(a)\}$, $a \in [-2, 0]$:

$$\frac{p_a}{1 - p_a} \equiv \frac{\Pr(x(a))}{\Pr(g(a))} = \frac{\mu(x(a))}{\mu(g(a))} \cdot \left| \frac{x'(a)}{g'(a)} \right| = \frac{\mu(x(a))}{\mu(g(a))} (\theta_3 e^a - 1)$$

- after a message (or message sequence) sent by types $\{z(a)b, h(a)b\}$, $a \in [a_\gamma, 0]$:

$$\frac{q_a}{1 - q_a} \equiv \frac{\Pr(z(a))}{\Pr(h(a))} = \frac{\mu(z(a))}{\mu(h(a))} \cdot \left| \frac{z'(a)}{h'(a)} \right| = \frac{\mu(z(a))}{\mu(h(a))} (2e^{a - a_\gamma} - 1)$$

- and, after a message sent by types $\{z(a), h(a), x(\widehat{a}(a)), g(\widehat{a}(a))\}$, with $u_1(\widehat{a}(a)) = v_1(a)$:

$$\frac{\Pr(x(\widehat{a}(a)))}{\Pr(z(a))} = \frac{\mu(x(\widehat{a}))}{\mu(z(a))} \cdot \left| \frac{x'(\widehat{a})}{z'(a)} \right| \cdot |\widehat{a}'(a)|$$

so, denoting $\widehat{I}(\widehat{a}) = \{x(\widehat{a}), g(\widehat{a})\}$ and $I(a) = \{z(a), h(a)\}$ (with $a \in [a_\gamma, 0]$ and $\widehat{a}(a)$ defined by $u_1(\widehat{a}(a), \alpha_a) = v_1(a, \alpha_0)$), and $I^*(a) = \widehat{I}(\widehat{a}) \cup I(a)$, the DM's beliefs at the pooled (4-type) information set $I^*(a)$ satisfy

$$\begin{aligned}
\frac{P^*}{1 - P^*} &\equiv \frac{\Pr(\widehat{I}(\widehat{a}))}{\Pr(I(a))} = \frac{\Pr(x(\widehat{a})) \cdot \left(1 + \frac{\Pr(g(\widehat{a}))}{\Pr(x(\widehat{a}))}\right)}{\Pr(z(a)) \cdot \left(1 + \frac{\Pr(h(a))}{\Pr(z(a))}\right)} \\
&= \frac{\mu(x(\widehat{a}))}{\mu(z(a))} \cdot \left(\frac{\theta_3 e^{\widehat{a}(a)} - 1}{2e^{a - a_\gamma} - 1}\right) |\widehat{a}'(a)| \cdot \left(\frac{q_a}{p_a}\right)
\end{aligned}$$

(with p_a, q_a as defined in the first two bullet points; the final term in the first line numerator is $1 + \frac{1 - p_a}{p_a} = \frac{1}{p_a}$, and similarly final denominator term is $1 + \frac{1 - q_a}{q_a} \frac{1}{q_a}$).

By "bunching", we mean that we will create the 4-type information sets described in the third bullet point above. More precisely, we choose the parameters α_0, α_a to make sure that the range of v_1 is contained in the range of u_1, so that any initial recommendation sent by a pair $\{x(a), g(a)\}$ is also sent by a pair $\{z(a), h(a)\}$ (see Lemma D7.1).

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