Whose Opinion Counts?
Political Processes and the Implementation Problem*

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Abstract

We augment the mechanism used in Nash implementation with a political process that collects the opinions of a subset of individuals with a fixed probability distribution. The outcome is a function of only the collected opinions. We show that the necessary – and sometimes sufficient – condition for implementation by a specific political process can be either weaker or stronger than Maskin monotonicity. We study three such processes: oligarchy, oligarchic democracy and random sampling. Oligarchy collects only the opinions of the oligarchs (a strict subset of the individuals). We present a Nash implementable social choice rule (SCR) that cannot be implemented by *any* oligarchy. Oligarchic democracy "almost always" collects the opinions of the oligarchs but sometimes, there is a referendum (i.e., everyone's opinions are collected). We show that in economic environments, every Nash implementable SCR can be implemented by oligarchic democracy in which *any* three individuals act as oligarchs. In random sampling, a sample of opinions are collected randomly. We show that in economic environments, every Nash implementable SCR can be implemented by randomly sampling opinions of 4 individuals. We also provide necessary and sufficient conditions for implementation when the planner has the flexibility to choose any political process.

Keywords: Nash Implementation; Political Process; p–Implementation; Direct Democracy; Oligarchy; Oligarchic Democracy; Random Sampling

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1 Introduction

Consider a community of n individuals having a social choice rule (SCR) that specifies the socially desirable alternatives conditional on the state of the economy (preferences, technology etc.). The community would like to design a mechanism such that individuals' incentives in the mechanism are "aligned" with its SCR, i.e., given the state, any Nash equilibrium of the mechanism must lead to a socially desirable alternative and every socially desirable alternative must be obtained under some Nash equilibrium of the mechanism. This is the implementation problem under complete information.[1] The standard mechanism used in this literature has two components: a message space for each individual and a single outcome function that maps everyone's messages to an alternative. Thus, the planner collects everyone's messages and then decides on the outcome. This is analogous to direct democracy since everyone's opinions are considered while making the decision. However, direct democracy is seldom used at national, state or even local levels. Indeed, when n is large, it will be too costly and time consuming to collect everyone's opinions while making social decisions. By the time a data collector gathers everyone's opinions, the state of the economy might change, rendering the whole exercise meaningless.

In this paper, we augment the standard mechanism with a political process that selects a senate (i.e., a subset of the individuals) with an exogenous probability distribution.[2] Thus, a mechanism now has three components: a message space for each individual, a political process, and a set of outcome functions, one for each possible senate, that map messages of the senators to alternatives. This mechanism defines a strategic-form game in which all individuals simultaneously announce their messages. The political process then selects the senate and transmits the messages of only the selected senators to the planner; the messages of other individuals are ignored.[3] Finally, the planner implements an alternative using the transmitted messages and the outcome function of the corresponding senate. We are interested in the relationship between a political process and the SCRs that a community can "implement" if it were to adopt a mechanism which uses that political process.

In designing the mechanism, the planner could face two alternative scenarios. First, the planner could be restricted to use a specific political process. In this case, we say that an SCR is implementable by the political process p or p–implementable if there exists a mechanism that uses p as the political process and the set of Nash equilibrium outcomes

[1] See Maskin and Sjöström (2002) and Serrano (2004) for surveys of implementation theory.

[2] Alternatively, one can consider political processes in which the senate is selected endogenously, e.g., through elections. This is an interesting area of research that we hope will be pursued in future.

[3] It is possible to alter the mechanism so that an individual announces a message if and only if she is selected as the senator (see Remark 2.10). There is no change in our results under this participation requirement.

of the mechanism coincides with the SCR in each state. Although lotteries over the set of alternatives can be generated in the mechanism (since the selection of the senate under p could be stochastic), only off-the-equilibrium outcomes can be lotteries. Nash equilibrium outcomes are deterministic with the same socially desirable alternative being implemented for all possible senates. Second, the planner could have the flexibility to use any political process. In this case, we say that an SCR is implementable by a political process if there exists some political process p such that the SCR is p–implementable. We study both these scenarios in this paper.

First, consider the scenario when the planner must use a specific political process. In his seminal contribution, Maskin (1999) studied Nash implementation which is equivalent to implementation by the process of direct democracy. In direct democracy, the probability of referendum (i.e., selecting the set of all individuals as the senate) is 1 and hence, messages of all individuals determine the outcome. Maskin (1999) showed that any Nash implementable SCR must be Maskin monotonoic. He further proved that if $n \geq 3$, then Maskin monotonicity and no-veto power are sufficient conditions for an SCR to be Nash implementable.[4] It follows that in economic environments (i.e., when at most $n-2$ individuals have a common most-preferred alternative) with at least 3 individuals, Maskin monotonicity is sufficient for Nash implementation. We use these results for Nash implementation as benchmarks. In particular, for each political process p, we are interested in knowing whether every SCR that is Nash implementable is also p–implementable. If the answer is yes, then replacing direct democracy with such a political process that considers the messages of a smaller number of individuals can reduce the costs and time involved in collecting everyone's opinions.[5]

We prove that a necessary condition for an SCR to be p–implementable is that it is p–monotonic. Maskin monotonicity is equivalent to p–monotonicity when p is direct democracy. More generally, if p is such that every individual has a positive chance of being selected as a senator, then p–monotonicity is weaker than Maskin monotonicity. This is because the planner can now implement lotteries over alternatives on off-the-equilibrium paths whenever p is stochastic, which provides greater flexibility than in Nash implementation where she is restricted to implement only alternatives.[6] On the other hand, if p is such that some

[4]For necessary and sufficient conditions for Nash implementation see Moore and Repullo (1990), Sjöström (1991), Danilov (1992) and Yamato (1992).

[5]We do not explicitly model such costs of collecting messages. One possibility is to assume that there is a fixed cost of collecting each message equal to c. In Section 8, we briefly discuss the implication of our results under this assumption. We must also point out that we do not consider the issue of communication and processing burdens imposed on, respectively, the individuals and planner due to large size of the message space (see, for e.g., Saijo (1988) and McKelvey (1989), who tackle this issue in the context of Nash implementation).

[6]Benoît and Ok (2008, Remark 3) provide an example to show that Maskin monotonicity is not necessary for Nash implementation if the planner can implement lotteries and use her knowledge about individuals' preferences over lotteries.

individuals have no chance of being selected as senators, then p–monotonicity is stronger than Maskin monotonicity.

Restricting attention to those p that always select at least three senators and have a positive chance of referendum, we prove that in economic environments, p–monotonicity is also sufficient for p–implementation. Since p–monotonicity is weaker than Maskin monotonicity for such p, we obtain the result that in economic environments, any Maskin monotonic SCR can be implemented by any political process that always selects at least three senators and has a positive – no matter how small – chance of referendum.

We then study three specific political processes: oligarchy, oligarchic democracy and random sampling. In oligarchy, the senate comprises of a particular strict subset of the individuals (the oligarchs) with probability 1. Thus, only the messages of the oligarchs determine the implemented alternative. Oligarchic democracy is a perturbation of the oligarchy in which the senate "almost always" comprises of the oligarchs but there is a small positive chance of referendum. That is, in an oligarchic democracy, the messages of only the oligarchs determine the implemented alternative with probability "arbitrarily" close to 1 while with the rest of the probability, the messages of all individuals determine the implemented alternative. Finally, in random sampling, a fixed number of individuals, which is strictly less than n, are selected at random to form the senate and only the messages of these randomly selected individuals determine the implemented alternative.

We show that any SCR that is implementable by oligarchy is also Nash implementable by making the outcome function under the direct democracy unresponsive to the messages of the commoners (individuals who are not in the oligarchy). However, the converse is not true: we present an SCR that is Nash implementable but not implementable by *any* oligarchy. The reason is that p–monotonicity when p is oligarchy is stronger than Maskin monotonicity since messages of the commoners are ignored in the oligarchy.

The perturbed process of oligarchic democracy, on the other hand, generates dramatically opposite conclusions. Our sufficiency result for political processes that always select at least three senators and have a positive chance of referendum gives us the following corollary: in economic environments, any Nash implementable – and sometimes even non-Maskin monotonic – SCR is implementable by oligarchic democracy that has at least three oligarchs. The identities of the oligarchs do not matter for this result; any three individuals can be designated as the oligarchs. This success of oligarchic democracy, however, does not carry over to noneconomic environments. We give an example of a noneconomic environment and an SCR that is Nash implementable but not implementable by oligarchic democracy with any subset of the individuals designated as the oligarchs. The reason for this failure of oligarchic democracy is the lack of "sufficient" diversity in the most-preferred alternatives of the oligarchs. As we show, any SCR satisfying Maskin monotonicity and no-veto power can be implemented by oligarchic democracy that has at least three oligarchs who never have a common most-preferred alternative.

In random sampling, every individual has a positive chance of being a senator. Hence, p–monotonicity when p is random sampling, or equivalently, the necessary condition for implementation by random sampling is weaker than Maskin monotonicity. However, there is no chance of referendum under random sampling. Still, we show that in those economic environments in which each individual has at most a single most-preferred alternative, the necessary p–monotonicity condition for implementation by random sampling is also sufficient if the sample size is at least 4. In arbitrary economic – and some noneconomic – environments, we show that any Maskin monotonic SCR is implementable by random sampling if the sample size is at least 4. Thus, in economic environments, instead of collecting everyone's messages, the planner can collect messages of only 4 randomly selected individuals in order to implement any Nash implementable SCR. Four is the minimal sample size that guarantees this result. We present an SCR that is Nash implementable in an economic environment but not implementable by random sampling when the sample size is smaller than 4.

Second, consider the scenario when the planner has the flexibility to use any political process. Our results for the first scenario immediately imply that an SCR is implementable by a political process only if it is p–monotonic for some political process p. The set of SCRs that are p–monotonic for some p of course includes the set of all Maskin monotonic SCRs. However, there are SCRs that are not p–monotonic for any p and hence, not implementable by a political process. Under a stronger assumption on preferences over lotteries, we show that in economic environments, any SCR that is p–monotonic for some p that always selects at least three senators is implementable by a political process.

Under mild domain restrictions, Bochet (2007) and Benoît and Ok (2008) show that Maskin monotonicity is both necessary and sufficient for implementation using lottery mechanisms. In a lottery mechanism, off-the-equilibrium outcomes can be lotteries over alternatives but equilibrium outcomes are deterministic. Maskin monotonicity remains necessary because the planner has limited information about individuals' preferences over lotteries; she only knows that these are "monotonic". Hence, she must design a mechanism that works for all possible "monotonic" preferences over lotteries that agree on the ordering of alternatives. We instead assume that the planner knows individuals' preferences over lotteries and uses this information in designing the mechanism. Therefore, for some political processes, we obtain weaker necessary conditions than Maskin monotonicity. Another difference between theirs and our setup is that in lottery mechanisms, arbitrary lotteries can be used off the equilibrium whereas the "structure" of off-the-equilibrium lotteries in our model is defined by the specific stochastic selection of senates.[7]

The rest of the paper is organized as follows. We outline the model and provide main definitions in Section 2. In Section 3, we study the scenario when the planner is restricted use a specific political process. In Sections 4, 5 and 6, we present the results for, respectively, implementation by oligarchy, oligarchic democracy and random sampling. In Section 7, we study the scenario when the planner has the flexibility to use any political process. We provide a brief conclusion in Section 8. Longer proofs are collected in Appendix.

2 Preliminaries

There is a finite set of players $N = \{1, \ldots, n\}$ with $n \geq 3$. The set of social alternatives is A, which can be infinite but not singleton. A *lottery* l is a probability distribution with a *finite* support in A. For any lottery l, let $l(a)$ denote the probability assigned by l to $a \in A$ and $A(l)$ denote the support of l. We write a for both the alternative $a \in A$ and the degenerate lottery that puts probability 1 on a. Let ΔA denotes the set of lotteries.

Let Θ be set of states with at least two elements. A typical state is denoted by θ. We assume that the players have complete information about the realized state.

Each player has a state dependent preference ordering (i.e., complete and transitive relation) $\succeq_i^\theta$ over ΔA.[8] Let $\succ_i^\theta$ and $\sim_i^\theta$ denote, respectively, the strict and indifference relations derived from $\succeq_i^\theta$.

For any i and θ, let $A_i^*(\theta) = \{a \in A : a \succeq_i^\theta a', \forall a' \in A\}$ be the set of most-preferred alternatives for i in state θ. Since A could be infinite, without additional assumptions, $A_i^*(\theta)$ could be empty.

For any i, θ, finite $A' \subseteq A$, and for all natural numbers $k \geq 1$, recursively define:

$$P_1(i, \theta, A') = \{a \in A' : a \succeq_i^\theta a', \forall a' \in A'\}$$
$$P_{k+1}(i, \theta, A') = \{a \in A' \setminus P_k(i, \theta, A') : a \succeq_i^\theta a', \forall a' \in A' \setminus P_k(i, \theta, A')\}.$$

Thus, $P_1(i, \theta, A')$ is the set of most-preferred alternatives in A' for player i in state θ, $P_2(i, \theta, A')$ is the set of second preferred alternatives in A' for player i in state θ etc.

[7] Nevertheless, in their proofs, Bochet (2007) and Benoît and Ok (2008) use simple lotteries with at most two alternatives in their respective supports. This is similar to the lottery generated under oligarchic democracy (because the senate is either the oligarchs or all individuals). In contrast to Bochet (2007) and Benoît and Ok (2008), however, commoners cannot induce a lottery in oligarchic democracy since their messages are considered only in one senate.

[8] We take these state-dependent preference orderings over ΔA as primitives of the model instead of the state-dependent preference orderings over the set of probability measures on A (set of probability measures on A is in general a superset of ΔA). This is because every outcome in the mechanisms that we study lies in ΔA.

Unless stated otherwise, we maintain the following assumption on preferences:

Assumption 2.1. *Preferences over Lotteries are Monotone*: For all $i \in N$, $\theta \in \Theta$, and $l, l' \in \Delta A$, we have

$$\sum_{k=1}^{K} \sum_{a \in P_k(i,\theta,A(l) \cup A(l'))} l(a) \geq \sum_{k=1}^{K} \sum_{a \in P_k(i,\theta,A(l) \cup A(l'))} l'(a), \forall K \geq 1 \implies l \succeq_i^\theta l'$$

and whenever at least one inequality is strict, then $l \succ_i^\theta l'$.

This assumption means that shifting probability weights from alternatives lower in the preference ordering to alternatives higher in the preference ordering generates a preferred lottery. It is weaker than imposing the independence axiom (Definition 7.4) on players' preferences.

2.1 Environment

The *environment* is $\mathcal{E} = \left\langle N, A, \left((\succeq_i^\theta)_{i \in N}\right)_{\theta \in \Theta} \right\rangle$. We consider various classes of environments, the most important being economic environment as defined by Bergemann and Morris (2008).

Definition 2.2. An environment $\mathcal{E}$ is *economic in state* θ if for any $a \in A$, there exist $i \neq j$ and alternatives a_i and a_j such that

$$a_i \succ_i^\theta a \text{ and } a_j \succ_j^\theta a.$$

An environment $\mathcal{E}$ is *economic* if it is economic in every state $\theta \in \Theta$.

Observe that $\mathcal{E}$ is an economic environment if and only if in any state θ, an alternative is most-preferred by at most $n-2$ players, i.e., $\bigcap_{i \in S} A_i^*(\theta) = \emptyset$ for all $S \subseteq N$ such that $|S| \geq n-1$.

Benoît and Ok (2008) define the following class of environments:

Definition 2.3. The environment $\mathcal{E}$ satisfies *top-coincidence condition* if for any $\theta \in \Theta$ and $S \subset N$ such that $|S| = n-1$, $\bigcap_{i \in S} A_i^*(\theta)$ is at most a singleton.

Thus, an environment that satisfies top-coincidence condition is such that in any state, any subset of $n-1$ players have at most one unanimously most-preferred alternative. Clearly, the class of environments identified by top-coincidence condition is weaker than the class of economic environments.

We define two other classes of environments.

Definition 2.4. The environment $\mathcal{E}$ satisfies *unique-top condition* if for all $i \in N$ and $\theta \in \Theta$,

$$A_i^*(\theta) \neq \emptyset \implies A_i^*(\theta) \text{ is singleton.}$$

Thus, unique-top condition requires that whenever a player has a most-preferred alternative in some state, then that alternative is unique. An example of such an environment is when all players have strict preferences. Another example is when all players have single-peaked preferences over A. Clearly, any environment that satisfies unique-top condition must also satisfy top-coincidence condition. However, there is no logical relation between the class of economic environments and the class of environments satisfying the unique-top condition, i.e., there can exist both economic environments that do not satisfy the unique-top condition and noneconomic environments that satisfy the unique top condition.

Definition 2.5. Let $S \subseteq N$. The environment $\mathcal{E}$ satisfies *diversity of top alternatives for* S (DTA-S) if for all $\theta \in \Theta$, $\bigcap_{i \in S} A_i^*(\theta) = \emptyset$, i.e., there does not exist an alternative that is unanimously most-preferred by every player in S.

Given $S \subseteq N$, there can exist both economic and noneconomic environments that satisfy DTA-S. Moreover, if $|S| \geq n-1$, then every economic environment satisfies DTA-S. However, if $|S| \leq n-2$, then there can exist economic environments that do not satisfy DTA-S.

2.2 Social Choice Rules

Social goals are embodied in a *social choice rule* (SCR), which is a nonempty-valued correspondence $F : \Theta \twoheadrightarrow A$.

The following two properties of SCRs are prominent in the literature on Nash implementation.

Definition 2.6. F is *Maskin monotonic* if whenever an alternative $a \in F(\theta)$ and $a \notin F(\theta')$ for some θ and θ', there exist player $i \in N$ and $a' \in A$ such that

$$a \succeq_i^{\theta} a' \text{ and } a' \succ_i^{\theta'} a.$$

Definition 2.7. F satisfies *no-veto power* if for any $\theta \in \Theta$ and $S \subseteq N$ such that $|S| \geq n-1$, we have

$$\bigcap_{i \in S} A_i^*(\theta) \subseteq F(\theta).$$

We will later introduce other notions of monotonicity and no-veto power among a subset of the players.

2.3 Mechanism

A social planner, who does not know the realized state, designs a mechanism in order to "implement" an SCR. The standard definition of a mechanism has a message space for each player and an outcome function that maps the messages of *all* players into the set of alternatives. We augment this definition with a political process that selects a *senate*, i.e., a subset of the players. Ultimately, the outcome is a function of only the messages of the selected senators.

Formally, let $\mathcal{N} = 2^N \setminus \emptyset$ be the set of all possible senates. By *referendum* we mean the event in which N is selected as the senate.

Definition 2.8. A *political process* p is a lottery on $\mathcal{N}$, i.e., $p = (p(S))_{S\in\mathcal{N}}$ such that

$$p(S) \geq 0, \forall S \in \mathcal{N} \text{ and } \sum_{S\in\mathcal{N}} p(S) = 1.$$

Let $\mathcal{P}$ be the set of all political processes on $\mathcal{N}$.

A *mechanism* is a triplet $\Gamma = ((M_i)_{i\in N}, p, (g^S)_{S\in\mathcal{N}})$, where

- M_i is the set of opinions or messages that player i can announce.
- $p \in \mathcal{P}$ is the political process for selecting a senate. If $S \subseteq N$ is the selected senate, then the messages of the players in S, $(m_i)_{i\in S}$, are transmitted to the planner and the messages of all players $j \notin S$ are ignored.
- $g^S : \prod_{i\in S} M_i \to A$ is the outcome function conditional on the selection of the senate $S \subseteq N$. Note that the outcome function g^S is deterministic.

Let m denote a typical element of $\prod_{i\in N} M_i$. For any $m \in \prod_{i\in N} M_i$ and $S \subseteq N$, let m^S be the projection of m into $\prod_{i\in S} M_i$ (note that if $S = N$, then $m^N = m$). That is, given any profile m of messages of all players, m^S is the profile of messages of only the players in S.

For any mechanism $\Gamma = ((M_i)_{i\in N}, p, (g^S)_{S\in\mathcal{N}})$ and message profile m, let $l[\Gamma, m]$ denote the lottery that assigns to each $a \in \{g^S(m^S) : S \in \mathcal{N}\}$, the probability $\sum_{S:g^S(m^S)=a} p(S)$.

For any realization θ, a mechanism $\Gamma = ((M_i)_{i\in N}, p, (g^S)_{S\in\mathcal{N}})$ defines a strategic-form game $\langle\Gamma, \theta\rangle$. In this game, each player $i \in N$ announces a message $m_i \in M_i$. Let m is be generated message profile. Then a senate $S \in \mathcal{N}$ is selected according to the political process p. The messages of the selected senators m^S are transmitted to the planner who then implements the alternative $g^S(m^S)$.[9] Thus, the outcome of message profile m is lottery

[9]Thus, players cannot change their messages after their selection as senators. This constraint is important for our results. If after their selection, the senators are informed about each others identities and allowed to change their messages, then the necessary condition for implementation by any political process will be stronger than the necessary condition for implementation by oligarchy. Also see Remark 2.10.

$l[\Gamma, m]$. The players' preferences are given by the profile $(\succeq_i^\theta)_{i\in N}$. Let $NE(\Gamma,\theta)$ denote the set of Nash equilibria of $\langle\Gamma,\theta\rangle$.[10]

Definition 2.9. F is *p–implementable* if there exists a mechanism $\Gamma = ((M_i)_{i\in N}, p, (g^S)_{S\in\mathcal{N}})$ such that

$$\Big\{ l[\Gamma, m] : m \in NE(\Gamma,\theta) \Big\} = F(\theta), \forall \theta \in \Theta.$$

F is *implementable by a political process* if it is p–implementable for some $p \in \mathcal{P}$.

Thus, if mechanism $\Gamma = ((M_i)_{i\in N}, p, (g^S)_{S\in\mathcal{N}})$ p–implements F, then (i) for any $m \in NE(\Gamma,\theta)$, we must have $l[\Gamma, m] \in F(\theta)$, i.e., $l[\Gamma, m]$ is deterministic with $g^S(m^S) = a \in F(\theta)$ for all S with $p(S) > 0$ and (ii) for any $a \in F(\theta)$, there exists $m \in NE(\Gamma,\theta)$ such that $l[\Gamma, m] = a$. Hence, although lottery p is used in the selection of the senators, p–implementation requires *exact* implementation of the SCR.

Remark 2.10. In the strategic-form game defined by a mechanism, every player announces a message even though a player's message is ignored when she is not selected in the senate. A more reasonable assumption is that a player announces a message if and only if she knows that she is a senator. This alternative assumption can be easily incorporated into the model. Suppose mechanism $\Gamma = ((M_i)_{i\in N}, p, (g^S)_{S\in\mathcal{N}})$ p–implements F. Consider the following extensive-form game: first, a senate $S \in \mathcal{N}$ is chosen using the political process p but the selected senators in S are not informed about each other's identities, i.e., each senator only knows that she is selected. Then the senators in S announce their messages $(m_i)_{i\in S} \in \prod_{i\in S} M_i$ and alternative $g^S((m_i)_{i\in S})$ is implemented. In this extensive-form game, every player moves at exactly one information set and thus, it has the same strategic form as $\langle\Gamma,\theta\rangle$ in each state θ. Hence, the extensive-form game also implements F in Nash equilibrium.[11] This alternative game form resembles an "opinion poll" (e.g. a telephone survey) since participants in such polls usually do not know each others' identities.

Our definition of a political process is broad enough to include several procedures like direct democracy, oligarchy, oligarchic democracy and random sampling that are used in practice. We define these political processes next.

[10] Like most literature on implementation, we restrict ourselves to pure strategies.

[11] Under the stronger assumption that players' preferences over lotteries satisfy the independence axiom, we can argue that in each state θ, the set of Perfect Bayesian equilibria of the extensive-form game coincides with the set of Nash equilibria of $\langle\Gamma,\theta\rangle$. When a player is asked to play, she finds out that she is in the senate but she does not know who else is in the senate. Due to the independence axiom, the player's incentives in the extensive-form game are the same as in $\langle\Gamma,\theta\rangle$.

2.4 Direct Democracy

We define *direct democracy*, denoted by p_D, as the political process in which the probability of referendum is 1, i.e., $p_D \in \mathcal{P}$ such that $p_D(N) = 1$. Hence, in direct democracy, every player's message is transmitted to the planner. A *direct-democratic mechanism*, $\Gamma_D = ((M_i)_{i \in N}, p_D, (g^S)_{S \in \mathcal{N}})$, is a mechanism that uses p_D to select the senate.[12] Note that for any $m \in \prod_{i \in N} M_i$, we have $l[\Gamma_D, m] = g^N(m)$.

The literature has exclusively focused on direct-democratic mechanisms. The corresponding notion of implementation is called Nash implementation.

Definition 2.11. F is *Nash implementable* if it is p_D–implementable, i.e., there exists a direct-democratic mechanism $\Gamma_D = ((M_i)_{i \in N}, p_D, (g^S)_{S \in \mathcal{N}})$ such that

$$\left\{ g^N(m) : m \in NE(\Gamma_D, \theta) \right\} = F(\theta), \forall \theta \in \Theta.$$

Maskin (1999) proves that Maskin monotonicity is necessary for Nash implementation.

Theorem 2.12 (**Maskin (1999)**). *If F is Nash implementable, then F is Maskin monotonic.*

In general environments with at least three players, Maskin (1999) shows that Maskin monotonicity and no-veto power are sufficient for Nash implementation.

Theorem 2.13 (**Maskin (1999)**). *If $n \geq 3$, then any Maskin monotonic SCR that satisfies no-veto power is Nash implementable.*

Since no-veto power is vacuously satisfied in economic environments, we have the following corollary:

Corollary 2.14. *F is Nash implementable in an economic environment with $n \geq 3$ if and only if it is Maskin monotonic.*

2.5 Oligarchy

Let S be any proper subset of N. We define S *oligarchy*, denoted by p_S, as the political process that selects S as the senate with probability 1, i.e., $p_S \in \mathcal{P}$ such that $p_S(S) = 1$. Hence, in S oligarchy, the messages of all players in S – the *oligarchs* – are transmitted to

[12] Since messages of all players are transmitted to the planner, we only need to specify the outcome function g^N as none of the other outcome functions are ever used in a direct democracy. However, we continue to list these functions in the definition of direct-democratic mechanisms just to make it consistent with the general definition of mechanisms given earlier. Similar remarks apply to other political processes.

the planner whereas the messages of all players in $N \setminus S$ – the *commoners* – are ignored. An *S-oligarchic mechanism*, $\Gamma_S = ((M_i)_{i\in N}, p_S, (g^{S'})_{S'\in\mathcal{N}})$, is a mechanism that uses p_S to select the senate. Note that for any $m \in \prod_{i\in N} M_i$, we have $l[\Gamma_S, m] = g^S(m^S)$.

Definition 2.15. Let $S \subset N$. F is *implementable by S oligarchy* if it is p_S–implementable, i.e., there exists an S-oligarchic mechanism $\Gamma_S = ((M_i)_{i\in N}, p_S, (g^{S'})_{S'\in\mathcal{N}})$ such that

$$\left\{g^S(m^S) : m \in NE(\Gamma_S, \theta)\right\} = F(\theta), \forall\theta \in \Theta.$$

2.6 Oligarchic Democracy

Let $S \subset N$. By *S-oligarchic democracy*, we mean the political process in which either the senate is S or there is a referendum but the probability of referendum is arbitrarily close to 0. To capture the idea of "arbitrarily close to 0", we define a sequence of political processes in which the probability of referendum is converging to 0.

Formally, for any $S \subset N$ and $\epsilon \in (0,1)$, let *(S,ϵ)-oligarchic democracy*, denoted by $p_{(S,\epsilon)}$, be the political process that selects S as the senate with probability $1-\epsilon$ and N as the senate with probability ϵ. Hence, in (S,ϵ)-oligarchic democracy, either the messages of only the players in S – the oligarchs – are transmitted to the planner with probability $1-\epsilon$ or the messages of all players in N are transmitted to the planner with probability ϵ. An *(S,ϵ)-oligarchic-democratic mechanism*, $\Gamma_{(S,\epsilon)} = ((M_i)_{i\in N}, p_{(S,\epsilon)}, (g^{S'})_{S'\in\mathcal{N}})$, is a mechanism that uses $p_{(S,\epsilon)}$ to select the senate. Note that for any $m \in \prod_{i\in N} M_i$, lottery $l[\Gamma_{(S,\epsilon)}, m]$ assigns probability $1-\epsilon$ to alternative $g^S(m^S)$ and probability ϵ to alternative $g^N(m)$.

Definition 2.16. Let $S \subset N$. F is *implementable by S-oligarchic democracy* if there exists a sequence of (S,ϵ_k)-oligarchic democracies $\left(p_{(S,\epsilon_k)}\right)_{k=1}^{\infty}$ such that

(i) $\lim_{k\to\infty} \epsilon_k = 0$ and

(ii) for all k, F is $p_{(S,\epsilon_k)}$–implementable, i.e., there exists an (S,ϵ_k)-oligarchic-democratic mechanism $\Gamma_{(S,\epsilon_k)} = ((M_i)_{i\in N}, p_{(S,\epsilon_k)}, (g^{S'})_{S'\in\mathcal{N}})$ such that

$$\left\{l[\Gamma_{(S,\epsilon_k)}, m] : m \in NE(\Gamma_{(S,\epsilon_k)}, \theta)\right\} = F(\theta), \forall\theta \in \Theta.$$

Condition (i) in the above definition formalizes the idea that the probability of referendum is "arbitrarily close to 0". Condition (ii) requires *exact* implementation by every political process $p_{(S,\epsilon_k)}$. In particular, if $m \in NE(\Gamma_{(S,\epsilon_k)}, \theta)$, then $l[\Gamma_{(S,\epsilon_k)}, m]$ is deterministic with $g^S(m^S) = g^N(m) \in F(\theta)$.

2.7 Random Sampling

The final political process that we consider is $\bar{n}$-random sampling. In this process, denoted by $p_{\bar{n}}$, a senate of fixed positive size $\bar{n} < n$ is randomly selected from the set of all players. Let $\mathcal{N}_{\bar{n}}$ denote the set of all senates $S \in \mathcal{N}$ such that $|S| = \bar{n}$. Thus, in random sampling, the senate is equally likely to be any $S \in \mathcal{N}_{\bar{n}}$ and the messages of such a randomly chosen sample of players are transmitted to the planner. The *$\bar{n}$-sampling mechanism*, $\Gamma_{\bar{n}} = ((M_i)_{i\in N}, p_{\bar{n}}, (g^{S'})_{S'\in\mathcal{N}})$, is a mechanism that uses $p_{\bar{n}}$ to select the senate. Note that for any $m \in \prod_{i\in N} M_i$, lottery $l[\Gamma_{\bar{n}}, m]$ is such that for all $S \in \mathcal{N}_{\bar{n}}$, it assigns probability $1/|\mathcal{N}_{\bar{n}}|$ to the alternative $g^S(m^S)$.

Definition 2.17. F is *implementable by $\bar{n}$-random sampling* if it is $p_{\bar{n}}$–implementable, i.e., there exists an $\bar{n}$-sampling mechanism $\Gamma_{\bar{n}} = ((M_i)_{i\in N}, p_{\bar{n}}, (g^S)_{S\in\mathcal{N}})$ such that

$$\left\{l[\Gamma_{\bar{n}}, m] : m \in NE(\Gamma_{\bar{n}}, \theta)\right\} = F(\theta), \forall\theta \in \Theta.$$

Thus, implementation by $\bar{n}$-random sampling requires *exact* implementation of the SCR even though a lottery is used in the sampling of the messages.

3 p–Implementation

In this section, we consider the situation when the planner is restricted to use a given political process p. Thus, the relevant notion of implementation is p–implementation. We now provide the necessary and sufficient conditions for p–implementation.

3.1 Necessary Condition

For any $p \in \mathcal{P}$ and function $h : \mathcal{N} \to A$, define the *lottery induced by* $[p, h]$, denoted by $l[p, h]$, as the lottery that assigns to each $a \in \{h(S) : S \in \mathcal{N}\}$, the probability $\sum_{S\in\mathcal{N}:\, h(S)=a} p(S)$. Also, let $\mathcal{N}(i) = \{S \in \mathcal{N} : i \in S\}$ be the set of those senates in which player i is a senator.

Definition 3.1. F is *p–monotonic* for a given $p \in \mathcal{P}$ if whenever $a \in F(\theta)$ and $a \notin F(\theta)$ for some a, θ and θ', then there exist player $i \in N$ and function $h_i : \mathcal{N} \to A$ such that

$$h_i(S) = a, \forall S \in \mathcal{N} \setminus \mathcal{N}(i) \text{ with } p(S) > 0, \quad a \succeq_i^{\theta} l[p, h_i] \text{ and } l[p, h_i] \succ_i^{\theta'} a.$$

Thus, p–monotonicity requires that whenever an alternative a drops out of the SCR in going from θ to θ', then there must exist a player i for whom there is reversal of her preferences over lotteries around a (i.e., there is a lottery that she weakly prefers to a in θ

but this is not true in θ'), where we consider only those lotteries that are induced by $[p, h_i]$ for any $h_i : \mathcal{N} \to A$ with the property that $h_i(S) = a$ for all $S \in \mathcal{N} \setminus \mathcal{N}(i)$ with $p(S) > 0$.

The following theorem shows that p–monotonicity is necessary for p–implementation.

Theorem 3.2. *If F is p–implementable, then F is p–monotonic.*

Proof. Let $\Gamma = ((M_i)_{i \in N}, p, (g^S)_{S \in \mathcal{N}})$ be the mechanism that p–implements F. Suppose there exist a, θ and θ' such that $a \in F(\theta)$ but $a \notin F(\theta')$. Then there must exist $m \in NE(\Gamma, \theta)$ with $l[\Gamma, m] = a$, i.e., $g^S(m^S) = a$ for all $S \in \mathcal{N}$ such that $p(S) > 0$. Since $a \notin F(\theta')$, it must be that $m \notin NE(\Gamma, \theta')$. Hence, there must exist $i \in N$ and $m_i' \in M_i$ such that

$$a \succeq_i^{\theta} l[\Gamma, (m_i', m_{-i})] \text{ and } l[\Gamma, (m_i', m_{-i})] \succ_i^{\theta'} a.$$

Define function $h_i : \mathcal{N} \to A$ such that $h_i(S) = g^S((m_i', m_{-i})^S)$ for all $S \in \mathcal{N}$. Then $h_i(S) = a$ for all $S \in \mathcal{N} \setminus \mathcal{N}(i)$ such that $p(S) > 0$. Moreover, lottery $l[p, h_i] = l[\Gamma, (m_i', m_{-i})]$. Hence, player i and function h_i satisfy the required condition in p–monotonicity. □

The proof clarifies why p–monotonicity requires the particular reversal of preferences over lotteries. Under the political process p, the message of player i is considered only if she is selected as the senator. Hence, by changing her message, player i can in principle change only those alternatives that are implemented under senates $S \in \mathcal{N}(i)$. As a result, player i is only able to generate a lottery that is induced by $[p, h_i]$, where $h_i : \mathcal{N} \to A$ is constant over all $S \in \mathcal{N} \setminus \mathcal{N}(i)$ that can be selected with positive probability.

If the political process is p_D, then for any $h_i : \mathcal{N} \to A$, the lottery $l[p_D, h_i]$ assigns probability 1 to the alternative $h_i(N)$. Hence, if player i has a reversal of her preferences over lottery $l[p_D, h_i]$ around a (i.e., $a \succeq_i^{\theta} l[p_D, h_i]$ and $l[p_D, h_i] \succ_i^{\theta'} a$), then it is equivalent to player i having a reversal of her preferences over alternative $h_i(N)$ around a (i.e., $a \succeq_i^{\theta} h_i(N)$ and $h_i(N) \succ_i^{\theta'} a$). This observation implies that Maskin monotonicity is equivalent to p_D–monotonicity. We note this result below (proof is omitted):

Proposition 3.3. *F is Maskin monotonic if and only if it is p_D–monotonic.*

On the other hand, there are political processes p such that p–monotonicity is weaker than Maskin monotonicity. In particular, if p is such that every player has a positive probability of being selected in some senate, then any Maskin monotonic SCR is also p–monotonic. Intuitively, whenever a drops out of the SCR, Maskin monotonicity requires that some player has a reversal of her preferences over alternatives around a, which is stronger than that player having a reversal of her preferences over lotteries around a. Formally, we have:

Lemma 3.4. *If F is Maskin monotonic, then F is p–monotonic for any $p \in \mathcal{P}$ such that for all $i \in N$, there exists an $S \in \mathcal{N}(i)$ with $p(S) > 0$.*

Proof. Suppose $a \in F(\theta)$ but $a \notin F(\theta')$. By Maskin monotonicity, there exist player i and alternative a' such that $a \succeq_i^\theta a'$ and $a' \succ_i^{\theta'} a$.

Define the mapping $h_i : \mathcal{N} \to A$ such that $h_i(S) = a', \forall S \in \mathcal{N}(i)$, and $h_i(S) = a, \forall S \in \mathcal{N} \setminus \mathcal{N}(i)$. By assumption, there exists an $S \in \mathcal{N}(i)$ such that $p(S) > 0$. Therefore, lottery $l[p, h_i]$ puts a positive probability on a'. Moreover, a is the only other alternative that could be in the support of $l[p, h_i]$. Since preferences over lotteries are monotone,

$$a \succeq_i^\theta a' \implies a \succeq_i^\theta l[p, h_i] \quad \text{while} \quad a' \succ_i^{\theta'} a \implies l[p, h_i] \succ_i^{\theta'} a.$$

Hence, F is p–monotonic. $\square$

In fact, there are p–implementable – and hence, p–monotonic – SCRs that are not Maskin monotonic. Consider the following example:

Example 3.5 (**King Solomon's Dilemma**)**.** Two women, Ann (1) and Beth (2), approach King Solomon both claiming to be the mother of a baby. There are two states, $\{\theta, \theta'\}$, with Ann (Beth) being the true mother in state θ (θ'). There are four alternatives, $\{a, b, c, d\}$ where a is to give the baby to Ann, b is to give the baby to Beth, c is to cut the baby in half, and d is to kill both women and child. The women's preferences over alternatives are:

$$\begin{array}{cc} \theta & \theta' \\ a \succ_1^\theta b \succ_1^\theta c \succ_1^\theta d & a \succ_1^{\theta'} c \succ_1^{\theta'} b \succ_1^{\theta'} d \\ b \succ_2^\theta c \succ_2^\theta a \succ_2^\theta d & b \succ_2^{\theta'} a \succ_2^{\theta'} c \succ_2^{\theta'} d \end{array}$$

King Solomon's SCR is to give the baby to the true mother, i.e., $F(\theta) = a$ and $F(\theta') = b$. However, this SCR is not Maskin monotonic and hence, not Nash implementable.

Now, suppose there exists a lottery $l \in \Delta A$ such that:

- $l(c) = 0$ and
- for Ann, $l \succ_1^\theta b$ but $b \succ_1^{\theta'} l$, and symmetrically for Beth, $l \succ_2^{\theta'} a$ but $a \succ_2^\theta l$.

Since preferences over lotteries are monotone and $l \succ_1^\theta b$, we have $l(a) > 0$; similarly, since $l \succ_2^{\theta'} a$, we have $l(b) > 0$; and finally, since $b \succ_1^{\theta'} l$ and $a \succ_2^\theta l$, it must be that $l(d) > 0$. Intuitively, when a woman is the true mother, she is willing to risk death for all to obtain the baby for herself rather than give the baby to the other woman but when she is not the true mother, she is not willing to take the same risk. As an example, suppose both Ann and Beth have expected-utility preferences with the following Bernoulli utilities: $u_1^{\theta}(a) = u_2^{\theta'}(b) = 3000$, $u_1^{\theta}(b) = u_2^{\theta'}(a) = 50$, $u_1^{\theta}(c) = u_2^{\theta'}(c) = 1$, $u_1^{\theta}(d) = u_2^{\theta'}(d) = 0$, $u_1^{\theta'}(a) = u_2^{\theta}(b) = 50$, $u_1^{\theta'}(c) = u_2^{\theta}(c) = 2$, $u_1^{\theta'}(b) = u_2^{\theta}(a) = 1$ and $u_1^{\theta'}(d) = u_2^{\theta}(d) = 0$. Then the lottery l such that $l(a) = l(b) = \frac{1}{52}$, $l(c) = 0$ and $l(d) = \frac{50}{52}$ satisfies the above requirement.

Now, consider the following mechanism Γ:

- $M_i = \{\theta, \theta'\}$ for all $i \in \{1, 2\}$.
- $p \in \mathcal{P}$ is such that $p(\{1\}) = l(a)$, $p(\{2\}) = l(b)$ and $p(\{1, 2\}) = l(d)$.
- $g^{\{1\}}(\theta) = a$ and $g^{\{1\}}(\theta') = b$; $g^{\{2\}}(\theta) = a$ and $g^{\{2\}}(\theta') = b$; and $g^{\{1,2\}}(\theta, \theta) = a$, $g^{\{1,2\}}(\theta', \theta') = b$ and $g^{\{1,2\}}(\theta, \theta') = g^{\{1,2\}}(\theta', \theta) = d$.

In this mechanism, each woman announces whether the state is θ or θ'. According to p, King Solomon collects only Ann's message with probability $l(a)$, only Beth's message with probability $l(b)$ and both women's messages with probability $l(d)$. If only one woman's message is collected, then King Solomon gives the baby to that woman who is the true mother in the announced state. If both women's messages are collected, then King Solomon faces two possibilities: if both women agree on the state, then he gives the baby to that woman who is true mother in agreed state whereas if they disagree, then he kills everyone.

We show that (θ, θ) is the only Nash equilibrium of Γ in state θ. Fix state θ and consider the message profile (θ, θ). In this profile, a is implemented in every senate. Ann would not deviate to θ' because then she does not obtain her baby with a positive probability. On the other hand, if Beth deviates to θ', then the outcome is lottery l. But $a \succ_2^{\theta} l$. Hence, (θ, θ) is an equilibrium in state θ and $l[\Gamma, (\theta, \theta)] = a$. Now, let us show that any other message profile is not an equilibrium in state θ. First, (θ', θ') with outcome b in every senate is not an equilibrium because if Ann deviates to θ, then the outcome will be lottery l and $l \succ_1^{\theta} b$. Second, (θ, θ') with outcome l is not an equilibrium because if Beth deviates to θ, then a is implemented in every senate and $a \succ_2^{\theta} l$. Finally, (θ', θ) is not an equilibrium because Ann does not obtain her baby with a positive probability but by deviating to θ, Ann would obtain her baby for sure. Similarly, (θ', θ') is the only Nash equilibrium in state θ' and $l[\Gamma, (\theta', \theta')] = b$. Hence, F is p–implementable. $\diamond$

Here we note that for certain other preferences of the women, King Solomon's SCR is not implementable by any political process p.

Finally, there are also political processes p such that p–monotonicity is stronger than Maskin monotonicity. This is the case for S oligarchy p_S (see Example 4.7) since some players are never selected as senators.

3.2 Sufficient Conditions

Theorem 3.6. *Let $n \geq 3$ and $\mathcal{E}$ be an economic environment. If F is p–monotonic for $p \in \mathcal{P}$ such that $p(N) > 0$ and $p(S) > 0$ only if $|S| \geq 3$, then F is p–implementable.*

Sketch of the proof: Consider mechanism $\Gamma = ((M_i)_{i\in N}, p, (g^S)_{S\in\mathcal{N}})$ in which for all $i \in N$,

$$M_i = \Theta \times \underbrace{A \times \cdots \times A}_{|\mathcal{N}(i)|+1} \times \mathbb{Z}_+,$$

where $\mathbb{Z}_+$ is the set of nonnegative integers. Note that since $|\mathcal{N}(i)| = |\mathcal{N}(j)|$ for all i, j, the numbers of components in the players' messages are equal. Let a typical message m_i be of the form $(\theta_i, a_i^1, (a_i^{S'})_{S'\in\mathcal{N}(i)}, z_i)$.[13] For each $S \in \mathcal{N}$, the outcome function g^S is as follows:

(i) If for every player $i \in S$, $m_i = (\theta, a_i^1, a, \ldots, a, 0)$ and $a \in F(\theta)$, then $g^S\big((m_i)_{i\in S}\big) = a$.

(ii) If for $|S| - 1$ players $i \neq j$ in S, $m_i = (\theta, a_i^1, a, \ldots, a, 0)$ and $a \in F(\theta)$, but $m_j = (\theta_j, a_j^1, (a_j^{S'})_{S'\in\mathcal{N}(j)}, z_j) \neq (\theta, a_j^1, a, \ldots, a, 0)$, then

$$g^S\big((m_i)_{i\in S}\big) = \begin{cases} a & \text{if } l[p, h_j] \succ_j^\theta a \\ a_j^S & \text{if } a \succeq_j^\theta l[p, h_j], \end{cases}$$

where $h_j : \mathcal{N} \to A$ is such that

$$h_j(S') = \begin{cases} a_j^{S'} & \text{if } S' \in \mathcal{N}(j) \\ a & \text{if } S' \in \mathcal{N} \setminus \mathcal{N}(j). \end{cases}$$

(iii) In all other cases, $g^S\big((m_i)_{i\in S}\big) = a_j^1$ where $j \in S$ is the player with the lowest index among those who announce the highest integer in $(m_i)_{i\in S}$.[14]

[13] The size of the message space for each player can be reduced by replacing $(a_i^{S'})_{S'\in\mathcal{N}(i)}$, i.e., one alternative for each senate in $\mathcal{N}(i)$, with $(a_i^{S'})_{S'\in\mathcal{N}(i):p(S')>0}$, i.e., one alternative for only those senates $S' \in \mathcal{N}(i)$ with $p(S') > 0$. However, this will make the exposition cumbersome.

[14] The advantage of the integer game in our framework is that by announcing a high enough integer, a player can win the integer game in every senate. Notice that this cannot be achieved by using the modulo game in which each player announces a single integer. A particular integer might make the player win the modulo game in one senate but at the same time she could lose the modulo game in some other senate. Hence, in order to replace the integer game with the modulo game, we need each player to announce as many integers between 1 and n as the number of senates she can be a member of, $|\mathcal{N}(i)|$. All proofs, except that of Theorem 6.5, can be altered in this fashion without changing the results. In contrast, the proof of Theorem 6.5 relies heavily on the existence of one player who has the lowest index among those who announce the highest integer in the whole population. Thus, replacing the integer game with the modulo game seems difficult in that proof. However, the use of both these games in implementation has been criticized; see Jackson (1992).

Interpret $a_i^{S'}$ as the alternative player i wants to implement under senate $S' \in \mathcal{N}(i)$. Thus, each player announces a state θ_i, an alternative a_i^1, alternatives $a_i^{S'}$ that she wants to implement under each senate $S' \in \mathcal{N}(i)$ and an integer z_i. For any senate S, the outcome function g^S is defined like in the canonical mechanism of Maskin (1999) with three rules. The first rule is used when all the messages received by the planner agree on the state, want a unique socially desirable alternative to be implemented under each senate and announce integer 0 (notice that the planner ignores the second component a_i^1 in this rule). In this case, the planner implements the commonly agreed alternative. Thus, if each player announces the true state θ, a fixed $a \in F(\theta)$ for each senate and integer 0, then the first rule ensures that a is implemented under each senate. The second rule guarantees that such a message profile is an equilibrium. If player j unilaterally deviates in this situation (unilateral deviations can be identified since $p(S) > 0$ only if $|S| \geq 3$), then in each $S' \in \mathcal{N}(j)$, the planner implements the alternative $a_j^{S'}$ wanted by player j under senate S' if and only if by doing so in every such senate (which generates lottery $l[p, h_j]$) would not make player j better-off.

The second rule also allows the planner to use p–monotonicity of the SCR to eliminate the possibility that when the true state is θ, players agree in their messages but on a "bad" alternative, i.e., all the players announce that the state is θ', alternative $a \in F(\theta')$ for each senate and integer 0 but $a \notin F(\theta)$. In such a situation, the second rule gives any player j the opportunity to unilaterally deviate and implement a strictly preferred lottery $l[p, h_j]$ by using the reversal of her preferences over such lotteries around the "bad" alternative a. This is done by implementing whatever player j wants under each senate $S' \in \mathcal{N}(j)$.

Finally, the third rule eliminates the possibility of any equilibria other than those in which all players agree in their messages. In these situations, at least $n-1$ players are such that each of them can implement her most-preferred alternative (proof takes care of the case when this does not exist) under senate N by announcing such an alternative in the second component of her message and a high enough integer. However, such a deviation will be improving if it does not cause a worse alternative to be implemented under other senates $S \neq N$. This is obviously the case if the senate does not include the deviating player. On the other hand, if the senate includes the deviating player and the third rule is used in that senate, then the deviating player still gets her most-preferred alternative. Finally, even in those senates that include the deviating player but in which the second rule is used, the deviating player ensures that a worse alternative is not implemented since the second rule does not depend on the altered components (i.e., the second and last) of her message. Thus, in such equilibria, the alternative implemented under senate N must be most-preferred by at least $n-1$ players, which contradicts with the assumption of economic environment. These arguments are formalized in the proof presented in the Appendix. $\diamond$

It is worth emphasizing the role of each assumption in the above result. Firstly, p–monotonicity ensures that any agreement on a "bad" alternative is avoided by giving each player the option to announce one alternative for each senate in which she could be a senator. Secondly, $p(S) > 0$ only if $|S| \geq 3$ ensures that the planner can identify unilateral deviations in each senate. Finally, a positive chance of referendum, i.e., $p(N) > 0$, in conjunction with economic environment ensure that any disagreement is avoided.

Since any Maskin monotonic SCR is also p–monotonic for any p with $p(N) > 0$ (see Lemma 3.4), we obtain the following corollary:

Corollary 3.7. *Let $n \geq 3$ and $\mathcal{E}$ be an economic environment. If F is Maskin monotonic, then F is p–implementable for any p such that $p(N) > 0$ and $p(S) > 0$ only if $|S| \geq 3$.*

Hence, in economic environments, any Nash implementable SCR can be implemented using any political process that always selects at least three senators and has a positive chance – no matter how small – of referendum. However, as we will show later in Example 5.8, this result is not true in noneconomic environments.

4 Implementation by Oligarchy

Consider any set of oligarchs $S \subset N$. Since p_S puts probability 1 on the senate S, for any $h_i : \mathcal{N} \to A$, the lottery $l[p_S, h_i]$ assigns probability 1 to the alternative $h_i(S)$. Hence, if player i has a reversal of her preferences over lottery $l[p_S, h_i]$ around a, then it is equivalent to player i having a reversal of her preferences over alternative $h_i(S)$ around a. Furthermore, since messages of the commoners are ignored, only the preference reversals of the oligarchs matter. These observations imply the following result (proof is omitted):

Proposition 4.1. *Let $S \subset N$. F is p_S–monotonic if and only if whenever $a \in F(\theta)$ and $a \notin F(\theta')$ for some a, θ and θ', there exist player $i \in S$ and $a' \in A$ such that*

$$a \succeq_i^{\theta} a' \text{ and } a' \succ_i^{\theta'} a.$$

Theorem 3.2 implies the following necessary condition for implementation by S oligarchy:

Corollary 4.2. *If F is implementable by S oligarchy, then F is p_S–monotonic.*

Clearly, p_S–monotonicity is stronger than Maskin monotonicity.

Definition 4.3. Let $S \subset N$. F satisfies *S-no-veto power* if for any $\theta \in \Theta$ and $S' \subseteq S$ such that $|S'| \geq |S| - 1$, we have

$$\bigcap_{i \in S'} A_i^*(\theta) \subseteq F(\theta).$$

Notice that an S oligarchy in environment $\mathcal{E} = \left\langle N, A, \left((\succeq_i^{\theta})_{i\in N}\right)_{\theta\in\Theta}\right\rangle$ is equivalent to a direct democracy in the restricted environment $\mathcal{E}(S) = \left\langle S, A, \left((\succeq_i^{\theta})_{i\in S}\right)_{\theta\in\Theta}\right\rangle$. Hence, the sufficient conditions for implementation by S oligarchy in environment $\mathcal{E}$ will be equivalent to the corresponding conditions for Nash implementation in environment $\mathcal{E}(S)$. We thus obtain the following corollary from Theorem 2.13:

Corollary 4.4. *If $3 \leq |S| < n$, then any SCR that satisfies p_S–monotonicity and S-no-veto power is implementable by S oligarchy.*

Since S-no-veto power is vacuously true if $\mathcal{E}(S)$ is an economic environment, we also have the following result:

Corollary 4.5. *Suppose $3 \leq |S| < n$ and $\mathcal{E}(S)$ is an economic environment. F is implementable by S oligarchy if and only if it is p_S–monotonic.*

4.1 Comparison with Nash Implementation

Proposition 4.6. *If F is implementable by S oligarchy for some $S \subset N$, then F is Nash implementable.*

Proof. Let $\Gamma_S = ((M_i)_{i\in N}, p_S, (g^{S'})_{S'\in\mathcal{N}})$ be the S-oligarchic mechanism that p_S–implements F. Define the direct democracy $\Gamma_D = ((M_i)_{i\in N}, p_D, (\hat{g}^{S'})_{S'\in\mathcal{N}})$ with $\hat{g}^N(m) = g^S(m^S), \forall m \in M$, and $\hat{g}^{S'} = g^{S'}, \forall S' \neq N$. It is easy to show that for all θ, we have $m \in NE(\Gamma_S, \theta) \iff m \in NE(\Gamma_D, \theta)$, and hence, the result follows. □

Thus, in any environment, any SCR that is implementable by an oligarchy is also implementable by direct democracy (i.e., Nash implementable). However, the converse is not true. That is, there are SCRs that are Nash implementable but not implementable by S oligarchy for any $S \subset N$. Consider the following example:

Example 4.7. Let $N = \{1, 2, 3, 4\}$, $A = \{a, b, c, d\}$ and $\Theta = \{\theta, \theta'\}$. The players' preferences over alternatives are:

θ	θ'
$a \succ_1^{\theta} b \succ_1^{\theta} c \succ_1^{\theta} d$	$b \succ_1^{\theta'} a \succ_1^{\theta'} c \succ_1^{\theta'} d$
$b \succ_2^{\theta} c \succ_2^{\theta} d \succ_2^{\theta} a$	$c \succ_2^{\theta'} b \succ_2^{\theta'} d \succ_2^{\theta'} a$
$c \sim_3^{\theta} d \succ_3^{\theta} a \succ_3^{\theta} b$	$d \succ_3^{\theta'} c \succ_3^{\theta'} a \succ_3^{\theta'} b$
$a \succ_4^{\theta} d \succ_4^{\theta} b \succ_4^{\theta} c$	$d \sim_4^{\theta'} a \succ_4^{\theta'} b \succ_4^{\theta'} c$

Let F be such that $F(\theta) = \{a, b, c\}$ and $F(\theta') = \{d\}$. Player 1 is the only player i for whom there exists an alternative $\hat{a}$ such that $a \succeq_i^{\theta} \hat{a}$ and $\hat{a} \succ_i^{\theta'} a$. Player 2 is the only player i for whom there exists an alternative $\hat{a}$ such that $b \succeq_i^{\theta} \hat{a}$ and $\hat{a} \succ_i^{\theta'} b$. Player 3 is the only player i for whom there exists an alternative $\hat{a}$ such that $c \succeq_i^{\theta} \hat{a}$ and $\hat{a} \succ_i^{\theta'} c$. Finally, player 4 is the only player i for whom there exists an alternative $\hat{a}$ such that $d \succeq_i^{\theta'} \hat{a}$ and $\hat{a} \succ_i^{\theta} d$. Hence, F is Maskin monotonic but not p_S–monotonic for any $S \subset N$. Since the environment is economic, it follows that F is Nash implementable but not implementable by any S oligarchy. $\diamond$

5 Implementation by Oligarchic Democracy

5.1 Necessary Condition

Definition 5.1. Let $S \subset N$. F is *weak p_S–monotonic* if there exists a sequence of (S, ϵ_k)-oligarchic democracies $\left(p_{(S,\epsilon_k)}\right)_{k=1}^{\infty}$ such that

(i) $\lim_{k\to\infty} \epsilon_k = 0$ and

(ii) for all k, F is $p_{(S,\epsilon_k)}$–monotonic.

Weak p_S–monotonicity can alternatively be characterized as follows:

Proposition 5.2. *Let $S \subset N$. F is weak p_S–monotonic if and only if there exists a sequence $(\epsilon_k)_{k=1}^{\infty}$ with $\epsilon_k \in (0,1)$, for all k, and $\lim_{k\to\infty} \epsilon_k = 0$ such that whenever $a \in F(\theta)$ and $a \notin F(\theta')$ for some a, θ and θ' then for all k either:*

(i) there exist player $i_k \notin S$ and alternative a'_k such that $a \succeq_{i_k}^{\theta} a'_k$ and $a'_k \succ_{i_k}^{\theta'} a$ or

(ii) there exist player $i_k \in S$, lottery l_k, and alternatives $\hat{a}_k$ and $\tilde{a}_k$ such that

$$l_k(\hat{a}_k) = \epsilon_k, \quad l_k(\tilde{a}_k) = 1 - \epsilon_k, \quad a \succeq_{i_k}^{\theta} l_k \text{ and } l_k \succ_{i_k}^{\theta'} a.$$

Thus, weak p_S–monotonicity requires that there exists a sequence of positive probabilities ϵ_k converging to 0 such that whenever an alternative a drops out of the SCR in going from state θ to θ', then for all k there must exist either a commoner with a reversal of preferences over alternatives around a or an oligarch with a reversal of preferences over lotteries around a, where we consider only those lotteries that have at most two alternatives in their supports and assign probability of ϵ_k to one of those alternatives. This difference between the commoners and oligarchs is because in any (S, ϵ)-oligarchic democracy, the senate can be either S with probability $1 - \epsilon$ or N with probability ϵ, and the messages of the commoners are considered only under senate N whereas the messages of the oligarchs are considered under both senates.

As a corollary of Theorem 3.2, we obtain that weak p_S–monotonicity is a necessary condition for implementation by S-oligarchic democracy.

Corollary 5.3. *If F is implementable by S-oligarchic democracy, then F is weak p_S–monotonic.*

Proof. If F is implementable by S-oligarchic democracy, then there exists a sequence of (S, ϵ_k)-oligarchic democracies $\left(p_{(S,\epsilon_k)}\right)_{k=1}^{\infty}$ such that (i) $\lim_{k\to\infty} \epsilon_k = 0$ and (ii) for all k, F is $p_{(S,\epsilon_k)}$–implementable. Theorem 3.2 implies that for all k, F is $p_{(S,\epsilon_k)}$–monotonic. □

Consider any (S, ϵ)-oligarchic democracy $p_{(S,\epsilon)}$. Since $p_{(S,\epsilon)}(N) > 0$, any Maskin monotonic SCR is also $p_{(S,\epsilon)}$–monotonic (Lemma 3.4). Hence, we obtain the following corollary:

Corollary 5.4. *If F is Maskin monotonic, then F is weak p_S-monotonic for all $S \subset N$.*

5.2 Economic Environments

5.2.1 Sufficient Condition

It turns out, if there are at least three oligarchs in S, then weak p_S–monotonicity of an SCR is also a sufficient condition for its implementation by S-oligarchic democracy in economic environments. This follows as a corollary of Theorem 3.6.

Corollary 5.5. *Suppose $3 \leq |S| < n$ and $\mathcal{E}$ is an economic environment. If F is weak p_S–monotonic, then F is implementable by S-oligarchic democracy.*

Proof. Let $\left(p_{(S,\epsilon_k)}\right)_{k=1}^{\infty}$ be the sequence of (S, ϵ_k)-oligarchic democracies that make F weak p_S–monotonic. By definition of (S, ϵ_k)-oligarchic democracy, $\epsilon_k \in (0, 1), \forall k$. We also know that $\lim_{k\to\infty} \epsilon_k = 0$ and for each k, F is $p_{(S,\epsilon_k)}$–monotonic. Now, $p_{(S,\epsilon_k)}(S) = 1 - \epsilon_k$ and $p_{(S,\epsilon_k)}(N) = \epsilon_k$. But $|S| \geq 3$ and $\mathcal{E}$ is an economic environment. So Theorem 3.6 implies that F is $p_{(S,\epsilon_k)}$–implementable for all k. Thus, F is implementable by S-oligarchic democracy. □

5.2.2 Comparison with Nash Implementation

Any Nash implementable SCR is Maskin monotonic (Theorem 2.12) and hence, weak p_S–monotonic for all $S \subset N$ (Corollary 5.4). Therefore, we obtain the following result from Corollary 5.5:

Corollary 5.6. *Suppose $3 \leq |S| < n$ and $\mathcal{E}$ is an economic environment. If F is Nash implementable, then F is implementable by S-oligarchic democracy.*

Thus, in economic environments, any SCR that is implementable by direct democracy (i.e., Nash implementable) is also implementable by oligarchic democracy that has three – or more – oligarchs. Notice that the identities of these oligarchs do not matter for this result.

The next example shows that there are SCRs that are implementable by oligarchic democracies but not Nash implementable. The SCR in this example is weak p_S–monotonic for some $S \subset N$ but not Maskin monotonic.

Example 5.7 (Strong Pareto Correspondence). Let $N = \{1, 2, 3, 4\}$, $A = \{a, b, c, d\}$ and $\Theta = \{\theta, \theta'\}$. The players' preferences over alternatives are:

$$\begin{array}{cc} \theta & \theta' \\ a \succ_1^\theta b \succ_1^\theta c \succ_1^\theta d & a \succ_1^{\theta'} b \sim_1^{\theta'} c \succ_1^{\theta'} d \\ b \succ_2^\theta c \succ_2^\theta d \succ_2^\theta a & b \sim_2^{\theta'} c \sim_2^{\theta'} d \sim_2^{\theta'} a \\ c \succ_3^\theta d \succ_3^\theta a \succ_3^\theta b & d \succ_3^{\theta'} c \succ_3^{\theta'} a \succ_3^{\theta'} b \\ a \succ_4^\theta c \sim_4^\theta d \succ_4^\theta b & c \succ_4^{\theta'} a \sim_4^{\theta'} d \succ_4^{\theta'} b \end{array}$$

We also assume that players' preferences over lotteries are represented by expected utility.

Let F be the strong Pareto correspondence, i.e., for all $\theta'' \in \Theta$,

$$F(\theta'') = \{\hat{a} \in A : \nexists a' \in A \text{ s.t. } \forall i \in N, a' \succeq_i^{\theta''} \hat{a} \text{ and } \exists j \in N \text{ with } a' \succ_j^{\theta''} \hat{a}\}$$

It is easy to see that $F(\theta) = \{a, b, c\}$ and $F(\theta') = \{a, c, d\}$. F is not Maskin monotonic since $b \in F(\theta)$, $b \notin F(\theta')$ but there does not exist any player i and alternative $\hat{a}$ such that $b \succeq_i^\theta \hat{a}$ and $\hat{a} \succ_i^{\theta'} b$. Therefore, F is not Nash implementable.

We argue that F is weak p_S–monotonic for all $S \subset N$ such that $1 \in S$. Since player 1 has expected-utility preferences, $a \succ_1^\theta b \succ_1^\theta c$ and $a \succ_1^{\theta'} b \sim_1^{\theta'} c$ imply that there exists a small enough $\epsilon' \in (0, 1)$ such that for all $\epsilon \in (0, \epsilon']$, we have

$$b \succeq_1^\theta l_\epsilon \text{ and } l_\epsilon \succ_1^{\theta'} b,$$

where l_ϵ is the lottery with $l_\epsilon(a) = \epsilon$ and $l_\epsilon(c) = 1 - \epsilon$. Moreover, for alternative d, which is such that $d \in F(\theta')$ but $d \notin F(\theta)$, we have player 3 with $d \succ_3^{\theta'} c$ and $c \succ_3^\theta d$. Therefore, if $1 \in S$, then for any sequence $(\epsilon_k)_{k=1}^\infty$ such that $\epsilon_k \in (0, \epsilon']$ and $\lim_{k\to\infty} \epsilon_k = 0$, F satisfies the conditions for weak p_S–monotonicity in Proposition 5.2.

It is easy to see that the environment is economic. Hence, Corollary 5.5 implies that F is implementable by S-oligarchic democracy for any S with $1 \in S$ and $|S| = 3$. $\diamond$

Here we note that the strong Pareto correspondence is not necessarily weak p_S–monotonic in every economic environment. Still, as the above example illustrates, compared to Nash implementation, we can implement the strong Pareto correspondence by S-oligarchic democracy in a larger set of economic environments.

5.3 Noneconomic Environments

Interestingly, as illustrated below, in noneconomic environments, there can exist SCRs that are Nash implementable but not implementable by any S-oligarchic democracy.

Example 5.8. Let $N = \{1, 2, 3, 4\}$, $A = \{a, b\}$ and $\Theta = \{\theta, \theta', \theta'', \hat{\theta}, \hat{\theta}', \hat{\theta}''\}$. The players' preferences over alternatives are:

θ	θ'	θ''	$\hat{\theta}$	$\hat{\theta}'$	$\hat{\theta}''$
$a \succ_1^{\theta} b$	$a \succ_1^{\theta'} b$	$b \sim_1^{\theta''} a$	$b \sim_1^{\hat{\theta}} a$	$a \succ_1^{\hat{\theta}'} b$	$a \succ_1^{\hat{\theta}''} b$
$a \sim_2^{\theta} b$	$b \succ_2^{\theta'} a$	$b \succ_2^{\theta''} a$	$b \succ_2^{\hat{\theta}} a$	$b \succ_2^{\hat{\theta}'} a$	$a \sim_2^{\hat{\theta}''} b$
$a \sim_3^{\theta} b$	$b \succ_3^{\theta'} a$	$b \succ_3^{\theta''} a$	$b \succ_3^{\hat{\theta}} a$	$a \sim_3^{\hat{\theta}'} b$	$b \succ_3^{\hat{\theta}''} a$
$a \succ_4^{\theta} b$	$b \sim_4^{\theta'} a$	$a \succ_4^{\theta''} b$	$b \sim_4^{\hat{\theta}} a$	$a \succ_4^{\hat{\theta}'} b$	$a \succ_4^{\hat{\theta}''} b$

This environment is not economic. For instance, in state θ, $a \succeq_i^{\theta} b$ for all $i \in N$.

Let F be such that $F(\theta) = F(\hat{\theta}') = F(\hat{\theta}'') = \{a\}$ and $F(\theta') = F(\theta'') = F(\hat{\theta}) = \{b\}$. F is Maskin monotonic and hence, weak p_S–monotonic for any $S \subset N$ (Corollary 5.4). Furthermore, F satisfies no-veto power. Hence, F is Nash implementable.

Suppose F is $p_{(S,\epsilon)}$–implementable for some $S \subset N$ and $\epsilon \in (0, 1)$. Consider $m \in NE(\Gamma_{(S,\epsilon)}, \theta')$. It must be that $g^S(m^S) = g^N(m) = b$. Moreover, for any $m_1' \in M_1$, we must have $g^S((m_1', m_{-1})^S) = g^N(m_1', m_{-1}) = b$ because otherwise, player 1 has an incentive to deviate to m_1' in state θ'. This further implies that there must exist $m_4' \in M_4$ such that either $g^S((m_4', m_{-4})^S) = a$ or $g^N(m_4', m_{-4}) = a$ because otherwise, $m \in NE(\Gamma_{(S,\epsilon)}, \theta)$.

If $4 \notin S$, then $g^S((m_4', m_{-4})^S) = g^S(m^S) = b$. Hence, it must be that $g^N(m_4', m_{-4}) = a$. We claim that $(m_4', m_{-4}) \in NE(\Gamma_{(S,\epsilon)}, \theta)$. Clearly, players 2, 3 and 4 do not have any improving unilateral deviations. On the other hand, if player 1 were to deviate to any m_1', then $g^S((m_1', m_4', m_{-\{1,4\}})^S) = g^S((m_1', m_{-1})^S) = b$. Hence, player 1 also does not have an improving unilateral deviation. So $(m_4', m_{-4}) \in NE(\Gamma_{(S,\epsilon)}, \theta)$. But then we have a contradiction since $g^S((m_4', m_{-4})^S) = b$. Therefore, $4 \in S$.

The above argument involved θ' and θ. Similarly, using θ'' and θ we can argue that $1 \in S$; using $\hat{\theta}'$ and $\hat{\theta}$ we can argue that $3 \in S$, and finally, using $\hat{\theta}''$ and $\hat{\theta}$ we can argue that $2 \in S$. Hence, we obtain a contradiction to the fact that $S \subset N$.

Thus, F is not $p_{(S,\epsilon)}$–implementable for any $S \subset N$ and $\epsilon \in (0, 1)$. As a result, F is not implementable by S-oligarchic democracy for any $S \subset N$. $\diamond$

5.3.1 Sufficient Conditions

The environment in the previous example does not satisfy DTA-S for all $S \subset N$. The problem for implementation by S-oligarchic democracy in noneconomic environments that do not satisfy DTA-S can be intuitively explained as follows. Suppose the players have expected-utility preferences. Let a be in the SCR in state θ and consider the equilibrium m of an (S,ϵ)-oligarchic-democratic mechanism that $p_{(S,\epsilon)}$–implements a. It must be that by unilaterally changing her message, no oligarch can implement an alternative under senate S that she strictly prefers to a in state θ. If this were not true, and if ϵ is small enough, then the oligarch will prefer to deviate. Now, consider another state θ' such that in going from state θ to θ', alternative a drops out of the SCR but none of the oligarchs have a reversal of their respective preferences over alternatives around a (i.e., for all $i \in S$, $a \succeq_i^{\theta} a' \implies a \succeq_i^{\theta'} a', \forall a'$). Pick any message profile $m' = ((m_i)_{i\in S}, (m'_j)_{j\notin S})$, i.e., the messages of the oligarchs in m' are the same as in m. Since $m'^S = m^S$, alternative a is still implemented under senate S. But a is not in the SCR in state θ' and hence, m' must not be an equilibrium in state θ'. Since none of the oligarchs have a reversal of their preferences over alternatives around a, it follows from the previous argument that no oligarch can implement a strictly preferred alternative under senate S. Thus, an oligarch will deviate from m' only if she can implement a strictly preferred alternative under senate N. However, now suppose that m' is such that the alternative implemented under senate N is unanimously most-preferred by all the players, which is possible since the environment it not economic. Then clearly, no player will have an incentive to deviate from m' and hence, we will not be able to implement the SCR.

As the next result shows, this problem can be avoided if we restrict attention to environments satisfying DTA-S.

Theorem 5.9. *Suppose $3 \leq |S| < n$ and $\mathcal{E}$ satisfies DTA-S. Any Maskin monotonic F that satisfies no-veto power is implementable by S-oligarchic democracy.*

Remark 5.10. Stronger results can be established in environments satisfying unique-top condition and DTA-S (note that this class of environments is not a subset of economic environments). In particular, we can show the following (proofs upon request):

1. *Suppose $3 \leq |S| < n$ and $\mathcal{E}$ satisfies DTA-S and unique-top condition. Any Maskin monotonic F is implementable by S-oligarchic democracy.*

As an application of this result, consider an environment with single-peaked preferences. For instance, suppose a large society faces the problem of implementing a tax rate. A state describes the "ideological" biases of each individual. The "left-wing" individuals have the peaks of their preferences at higher tax rates whereas "right-wing" individuals have the peaks of their preferences at lower tax rates. Furthermore, suppose that there are two individuals, say i_l and i_r, whose "ideological" biases never coincide, i.e., the peaks of their preferences are different in every state. Then this result says that we can implement any Maskin monotonic SCR by oligarchic democracy if we designate individuals i_l, i_r and any other third individual

j as oligarchs.

2. *Suppose* $3 \leq |S| < n$ *and* $\mathcal{E}$ *satisfies DTA-S and unique-top condition. Any weak* p_S*–monotonic* F *that satisfies no-veto power is implementable by* S*-oligarchic democracy.* $\diamond$

6 Implementation by Random Sampling

6.1 Necessary Condition

Recall the definition of $\mathcal{N}_{\bar{n}}$ and let $\mathcal{N}_{\bar{n}}(i)$ be the set of all senates $S \in \mathcal{N}_{\bar{n}}$ such that $i \in S$. Under $p_{\bar{n}}$, a senate S is selected with a positive probability if and only if $S \in \mathcal{N}_{\bar{n}}$. Thus, $p_{\bar{n}}$–monotonicity can alternatively be characterized as follows (proof is omitted):

Proposition 6.1. *F is $p_{\bar{n}}$–*monotonic *if and only if whenever* $a \in F(\theta)$ *and* $a \notin F(\theta)$ *for some* a, θ *and* θ', *then there exist player* $i \in N$ *and function* $h_i : \mathcal{N} \to A$ *such that*

$$h_i(S) = a, \forall S \in \mathcal{N}_{\bar{n}} \setminus \mathcal{N}_{\bar{n}}(i), \quad a \succeq_i^{\theta} l[p_{\bar{n}}, h_i] \text{ and } l[p_{\bar{n}}, h_i] \succ_i^{\theta'} a.$$

The following corollary of Theorem 3.2 says that $p_{\bar{n}}$–monotonicity is necessary for implementation by $\bar{n}$-random sampling.

Corollary 6.2. *If F is implementable by $\bar{n}$-random sampling, then F is $p_{\bar{n}}$–monotonic.*

Under random sampling, there is a positive probability of selecting each player as a senator. Hence, we have the following corollary of Lemma 3.4:

Corollary 6.3. *If F is Maskin monotonic, then F is $p_{\bar{n}}$–monotonic for all positive $\bar{n} < n$.*

6.2 Sufficient Conditions

We cannot apply Theorem 3.6 to $\bar{n}$-random sampling since $p_{\bar{n}}(N) = 0$. However, suppose we were to use the same mechanism as defined in the proof of Theorem 3.6 except that we replace the political process p by $p_{\bar{n}}$. The sole purpose of assumption $p(N) > 0$ in Theorem 3.6 is to ensure that whenever there is disagreement in the messages in equilibrium, the alternative implemented under senate N is most-preferred by at least $n-1$ players, contradicting economic environment. Is there a way to obtain such a contradiction under random sampling?

Although $p_{\bar{n}}(N) = 0$, if the sample size $\bar{n} \geq 4$, then each triple of players meets every other player in some senate. Suppose there exist three players $\{i_1, i_2, i_3\}$ whose messages disagree in equilibrium. There are two possible cases. First, assume that all three disagree amongst each other. Then for any other player j, consider any senate $S(j) \supseteq \{j, i_1, i_2, i_3\}$. In $S(j)$, every player can implement her most-preferred alternative (proofs take care of the case when this does not exist) by inducing the integer game. Thus, the alternative implemented under $S(j)$, say a_j, must be most-preferred for all players in $\{j, i_1, i_2, i_3\}$. Similarly, if we replace player j with player j' to obtain senate $S(j')$, then the alternative implemented under $S(j')$, say $a_{j'}$, must be most-preferred for all players in $\{j', i_1, i_2, i_3\}$. Now, if any of the players in $\{i_1, i_2, i_3\}$ has a unique most-preferred alternative, say a, then $a_j = a_{j'} = a, \forall j, j' \notin \{i_1, i_2, i_3\}$. Then a is most-preferred by all n players, contradicting economic environment. Second, assume that there is a unique disagreeing player in $\{i_1, i_2, i_3\}$, say i_2. Again, for any other player j, consider any senate $S(j) \supseteq \{j, i_1, i_2, i_3\}$. In $S(j)$, players i_1, i_3 and j can implement their most-preferred alternatives by inducing the integer game. Thus, the alternative implemented under $S(j)$, say a_j, must be most-preferred for all players in $\{j, i_1, i_3\}$. If we replace player j with player j' to obtain senate $S(j')$, then player i_2 will still disagree with players i_1 and i_3.[15] Hence, the alternative implemented under $S(j')$, say $a_{j'}$, must be most-preferred for all players in $\{j', i_1, i_3\}$. Now, if any of the players in $\{i_1, i_3\}$ has a unique most-preferred alternative, say a, then $a_j = a_{j'} = a, \forall j, j' \notin \{i_1, i_2, i_3\}$. Then a is most-preferred by at least $n-1$ players – everyone except i_2 –, contradicting economic environment. Furthermore, since $\bar{n} \geq 4$, $p_{\bar{n}}$ satisfies the second requirement ($p(S) > 0$ only if $|S| \geq 3$) in Theorem 3.6. Therefore, other arguments used to prove Theorem 3.6 apply in this case as well, giving us the following result:

Theorem 6.4. *Let $4 \leq \bar{n} < n$ and $\mathcal{E}$ be an economic environment satisfying unique-top condition. If F is $p_{\bar{n}}$–monotonic, then F is implementable by $\bar{n}$-random sampling.*

Without the unique-top condition, it becomes difficult to find an alternative that is most-preferred by at least $n-1$ players in case of disagreement in equilibrium. This is especially a problem when there is a single player, say i_1, who disagrees and in every senate S' that includes i_1, the alternative wanted by i_1, $a_{i_1}^{S'}$, is implemented. Of course, every other player in S' can implement her most-preferred alternative under S' by inducing the integer game. Thus, $a_{i_1}^{S'}$ must be most-preferred by all players in $S' \setminus \{i_1\}$. The same argument holds for any $S'' \in \mathcal{N}(i_1) \setminus S'$. But if $a_{i_1}^{S'} \neq a_{i_1}^{S''}$, then it is not possible to argue that there exists an alternative that is most-preferred by every player other than i_1. Nevertheless,

[15] This is where the problem comes with sample size of 3. If i_1 and i_2 disagree, then it could be that i_2 is the unique disagreeing player in $\{j, i_1, i_2\}$ while i_1 is the unique disagreeing player in $\{j', i_1, i_2\}$. In the former senate, players i_1 and j can implement their most-preferred alternatives while in the latter senate players i_2 and j' can implement their most-preferred alternatives. Unless both players i_1 and i_2 have the same unique most-preferred alternative, the implemented alternatives under $\{j, i_1, i_2\}$ and $\{j', i_1, i_2\}$ can be different. Hence, we will be unable to find an alternative that is most-preferred by at least $n-1$ players. As a result, we will not contradict economic environment. This problem is illustrated in Example 6.7.

this suggests a solution to the problem: let each player ask for only one alternative instead of one alternative for each senate $(a_i^{S'})_{S'\in\mathcal{N}(i)}$. However, recall that the purpose of letting each player ask for one alternative for each senate in the proof of Theorem 3.6 was to avoid agreements on "bad" alternatives using p–monotonicity. Thus, if we restrict each player to ask only one alternative, then we need to replace $p_{\bar{n}}$–monotonicity with a stronger condition. As the next result shows, replacing $p_{\bar{n}}$–monotonicity with Maskin monotonicity is sufficient for implementation by $\bar{n}$-random sampling in a class of environments that is *weaker* than the class of economic environments.

Theorem 6.5. *Let $4 \leq \bar{n} < n$ and $\mathcal{E}$ satisfy DTA-N and top-coincidence condition. If F is Maskin monotonic, then F is implementable by $\bar{n}$-random sampling.*

6.3 Comparison with Nash Implementation

Since every economic environment satisfies both DTA-N and top-coincidence condition, the following result easily follows from Theorems 2.12 and 6.5:

Corollary 6.6. *Suppose $4 \leq \bar{n} < n$ and $\mathcal{E}$ is an economic environment. If F is Nash implementable, then F is implementable by $\bar{n}$-random sampling.*

Thus, in economic environments, any SCR that is implementable by direct democracy (i.e., Nash implementable) is also implementable by randomly sampling only 4 – or more – messages of the players.

The next example proves that 4 is the minimal sample size that guarantees the implementation of Maskin monotonic SCRs by random sampling.

Example 6.7. Let $N = \{1, 2, 3, 4, 5\}$, $A = \{a, b\}$ and $\Theta = \{\theta, \theta'\}$. The players' preferences over alternatives are:

$$\begin{array}{cc} \theta & \theta' \\ a \succ_1^{\theta} b & a \succ_1^{\theta'} b \\ a \succ_2^{\theta} b & a \succ_2^{\theta'} b \\ a \succ_3^{\theta} b & b \succ_3^{\theta'} a \\ b \succ_4^{\theta} a & b \succ_4^{\theta'} a \\ b \succ_5^{\theta} a & b \succ_5^{\theta'} a \end{array}$$

Let F be such that $F(\theta) = \{a, b\}$ and $F(\theta') = \{b\}$. The environment is economic and F is Maskin monotonic. Hence, F is Nash implementable.

We argue that F is not implementable by $\bar{n}$-random sampling for any $\bar{n} \leq 3$. Suppose there exists a $\bar{n}$-sampling mechanism $\Gamma_{\bar{n}}$ such that $\left\{l[p_{\bar{n}}, m] : m \in NE(\Gamma_{\bar{n}}, \tilde{\theta})\right\} = F(\tilde{\theta})$ for all $\tilde{\theta} \in \Theta$.

Consider $m \in NE(\Gamma_{\bar{n}}, \theta)$ such that $l[p_{\bar{n}}, m] = a$. It must be that $g^S(m^S) = a, \forall S \in \mathcal{N}_{\bar{n}}$. Moreover, for $i \in \{4, 5\}$, there must not exist any m'_i such that for some $S \in \mathcal{N}_{\bar{n}}(i)$, we have $g^S\big((m'_i, m_{-i})^S\big) = b$; otherwise, player i has an incentive to deviate to m'_i in state θ. Consider $\hat{m} \in NE(\Gamma_{\bar{n}}, \theta)$ such that $l[p_{\bar{n}}, \hat{m}] = b$. It must be that $g^S(\hat{m}^S) = b, \forall S \in \mathcal{N}_{\bar{n}}$. Moreover, for $i \in \{1, 2, 3\}$, there must not exist any m'_i such that for some $S \in \mathcal{N}_{\bar{n}}(i)$, we have $g^S\big((m'_i, \hat{m}_{-i})^S\big) = a$; otherwise, player i has an incentive to deviate to m'_i in state θ.

Consider the message profile $\tilde{m} = (m_1, m_2, m_3, \hat{m}_4, \hat{m}_5)$.

- Suppose $\bar{n} = 3$. If $S \in \{1, 4, 5\} \bigcup \{2, 4, 5\} \bigcup \{3, 4, 5\}$, then $g^S(\tilde{m}^S) = b$ whereas if $S \in \mathcal{N}_{\bar{n}} \setminus \{1, 4, 5\} \bigcup \{2, 4, 5\} \bigcup \{3, 4, 5\}$, then $g^S(\tilde{m}^S) = a$. Then $\tilde{m} \in NE(\Gamma_{\bar{n}}, \theta)$ since players 1, 2 and 3 cannot unilaterally change the alternative implemented for any $S \in \{1, 4, 5\} \bigcup \{2, 4, 5\} \bigcup \{3, 4, 5\}$ whereas players 4 and 5 cannot unilaterally change the alternative implemented for any $S \in \mathcal{N}_{\bar{n}} \setminus \{1, 4, 5\} \bigcup \{2, 4, 5\} \bigcup \{3, 4, 5\}$. But both a and b are in the support of lottery $l[p_{\bar{n}}, \tilde{m}]$, a contradiction.

- Suppose $\bar{n} = 2$. Then for $S = \{3, 4\}$, we have $g^S(\tilde{m}^S) = g^S(m_3, \hat{m}_4)$. We have already argued that for $S = \{3, 4\}$, there does not exist any m'_4 such that $g^S(m_3, m'_4) = b$. Hence, $g^S(m_3, \hat{m}_4) = a$. For $S = \{3, 4\}$, we have also argued that there does not exist any m'_3 such that $g^S(m'_3, \hat{m}_4) = a$. Hence, $g^S(m_3, \hat{m}_4) = b$, a contradiction.

- Suppose $\bar{n} = 1$. We have already argued that for $S = \{4\}$, there does not exist any m'_4 such that $g^S\big((m'_4, m_{-4})^S\big) = g^S(m'_4) = b$. But that contradicts the fact that $g^S(\hat{m}_4) = b$. $\diamond$

The next two examples show that the sufficiency result of Theorem 6.5 need not hold outside the class of environments satisfying DTA-N and top-coincidence condition.

Example 6.8. Let $N = \{1, 2, 3, 4, 5\}$, $A = \{a, b\}$ and $\Theta = \{\theta, \theta'\}$. The players' preferences over alternatives are:

$$\begin{array}{cc}
\theta & \theta' \\
a \sim_1^{\theta} b & b \succ_1^{\theta'} a \\
a \sim_2^{\theta} b & b \succ_2^{\theta'} a \\
a \sim_3^{\theta} b & b \succ_3^{\theta'} a \\
a \succ_4^{\theta} b & a \succ_4^{\theta'} b \\
a \succ_5^{\theta} b & b \succ_5^{\theta'} a
\end{array}$$

Let F be such that $F(\theta) = \{a\}$ and $F(\theta') = \{b\}$. Since F is Maskin monotonic and satisfies no-veto power, F is Nash implementable.

The environment satisfies top-coincidence condition but not DTA-N since $a \in \bigcap_{i \in N} A_i^*(\theta)$. We argue that F is not implementable by $\bar{n}$-random sampling for $\bar{n} = 4$. Suppose there exists a $\bar{n}$-sampling mechanism $\Gamma_{\bar{n}}$ such that $\left\{ l[p_{\bar{n}}, m] : m \in NE(\Gamma_{\bar{n}}, \tilde{\theta}) \right\} = F(\tilde{\theta})$ for all $\tilde{\theta} \in \Theta$.

Consider $m \in NE(\Gamma_{\bar{n}}, \theta')$ such that $l[p_{\bar{n}}, m] = b$. Thus, $g^S(m^S) = b, \forall S \in \mathcal{N}_{\bar{n}}$. Moreover, there must not exist any m_4' such that $g^S\big((m_4', m_{-4})^S\big) = a$ for some $S \in \mathcal{N}_{\bar{n}}(4)$; otherwise, player 4 has an incentive to deviate to m_4' in state θ'. Hence, in particular, b will be implemented under senate $\{1, 2, 3, 4\}$ as long as players 1, 2 and 3 announce messages m_1, m_2 and m_3, respectively.

Since $b \notin F(\theta)$, it must be that $m \notin NE(\Gamma_{\bar{n}}, \theta)$. Now, $b \in A_i^*(\theta)$ for all $i \in \{1, 2, 3\}$. Moreover, given m_{-4}, player 4 cannot change the alternative implemented under any senate by changing her message. Hence, it must be player 5 for whom there exists m_5' such that $g^S\big((m_5', m_{-5})^S\big) = a$ for some $S \in \mathcal{N}_{\bar{n}}(5)$. Without loss of generality, let m_5' be the best response to m_{-5} in state θ. The message profile $m' = (m_1, m_2, m_3, m_4, m_5') \notin NE(\Gamma_{\bar{n}}, \theta)$ because b is implemented under senate $\{1, 2, 3, 4\}$. Since m_5' is a best response to $m_{-5} = m'_{-5}$, and players 1, 2, and 3 are indifferent between a and b, it must be that player 4 has an improving unilateral deviation. Let $\hat{m}_4$ be the best response to m'_{-4}. Since m' is such that $g^S(m'^S) = a$ for at least one $S \in \mathcal{N}_{\bar{n}}$, the new message profile $\hat{m} = (m_1, m_2, m_3, \hat{m}_4, m_5')$ must be such that $g^S(\hat{m}^S) = a$ for at least two $S \in \mathcal{N}_{\bar{n}}$; otherwise, player 4 will not strictly improve with her deviation to $\hat{m}_4$. The message profile $\hat{m} \notin NE(\Gamma_{\bar{n}}, \theta)$ because b is implemented under senate $\{1, 2, 3, 4\}$. Since $\hat{m}_4$ is a best response to $m'_{-4} = \hat{m}_{-4}$, and players 1, 2, and 3 are indifferent between a and b, it must be that player 5 has an improving unilateral deviation. Let $\tilde{m}_5$ be the best response to $\hat{m}_{-5}$. Since $\hat{m}$ was such that $g^S(\hat{m}^S) = a$ for at least two $S \in \mathcal{N}_{\bar{n}}$, the new message profile $\tilde{m} = (m_1, m_2, m_3, \hat{m}_4, \tilde{m}_5)$ must be such that $g^S(\tilde{m}^S) = a$ for at least three $S \in \mathcal{N}_{\bar{n}}$; otherwise, player 5 will not strictly improve with her deviation to $\tilde{m}_5$. The message profile $\tilde{m} \notin NE(\Gamma_{\bar{n}}, \theta)$ because b is implemented under senate $\{1, 2, 3, 4\}$. By repeating the above argument, player 4 will switch to her best response $\hat{m}_4'$ to $\tilde{m}_{-4}$, and the new message profile $\hat{m}' = (m_1, m_2, m_3, \hat{m}_4', \tilde{m}_5)$ will be such that $g^S(\hat{m}'^S) = a$ for at least four $S \in \mathcal{N}_{\bar{n}}$. In message profile $\hat{m}'$, b is implemented under senate $\{1, 2, 3, 4\}$. Hence, it must be that $g^S(\hat{m}'^S) = a$ for all $S \in \mathcal{N}_{\bar{n}}(5)$. Then $\hat{m}' \in NE(\Gamma_{\bar{n}}, \theta)$ because players 1, 2, and 3 are indifferent between a and b, player 4 is already playing a best response to $\tilde{m}_{-4} = \hat{m}'_{-4}$ and player 5 cannot change the alternative implemented under $S = \{1, 2, 3, 4\}$. However, since b is implemented under senate $\{1, 2, 3, 4\}$, we have a contradiction. $\diamond$

Example 6.9. Let $N = \{1, 2, 3, 4, 5\}$, $A = \{a, b, c\}$ and $\Theta = \{\theta, \theta'\}$. The players' preferences over alternatives are:

$$\begin{array}{cc}
\theta & \theta' \\
c \succ_1^{\theta} a \sim_1^{\theta} b & c \succ_1^{\theta'} a \succ_1^{\theta'} b \\
a \sim_2^{\theta} b \succ_2^{\theta} c & a \sim_2^{\theta'} b \succ_2^{\theta'} c \\
a \sim_3^{\theta} b \succ_3^{\theta} c & a \sim_3^{\theta'} b \succ_3^{\theta'} c \\
a \sim_4^{\theta} b \succ_4^{\theta} c & a \sim_4^{\theta'} b \succ_4^{\theta'} c \\
a \sim_5^{\theta} b \succ_5^{\theta} c & a \sim_5^{\theta'} b \succ_5^{\theta'} c
\end{array}$$

Let F be such that $F(\theta) = \{a, b\}$ and $F(\theta') = \{a\}$. F is Maskin monotonic. However, F does not satisfy no-veto power since in θ', b is most-preferred alternative for four players but $b \notin F(\theta')$. Still, F is Nash implementable. To show this, first we argue that F is implementable by S oligarchy, where $S = \{1\}$. Define an S-oligarchic mechanism such that $M_1 = \{a, b\}$ and $g^S(m_1) = m_1$ for all $m_1 \in M_1$. Since only the message of player 1 is transmitted to the planner, in state θ, player 1 is indifferent between announcing messages a and b whereas in state θ', player 1 will announce a. Thus, $\{g^S(m^S) : m \in NE(\Gamma_S, \theta)\} = \{a, b\}$ and $\{g^S(m^S) : m \in NE(\Gamma_S, \theta')\} = \{a\}$. Hence, F is implementable by $S = \{1\}$ oligarchy. It follows from Proposition 4.6 that F is Nash implementable.

The environment satisfies DTA-N but not top-coincidence condition since both a and b are most-preferred alternatives for four players in state θ. We argue that F is not implementable by $\bar{n}$-random sampling for $\bar{n} = 4$. Suppose there exists a $\bar{n}$-sampling mechanism $\Gamma_{\bar{n}}$ such that $\left\{l[p_{\bar{n}}, m] : m \in NE(\Gamma_{\bar{n}}, \tilde{\theta})\right\} = F(\tilde{\theta})$ for all $\tilde{\theta} \in \Theta$.

Consider $m \in NE(\Gamma_{\bar{n}}, \theta)$ such that $l[p_{\bar{n}}, m] = b$. It must be that $g^S(m^S) = b, \forall S \in \mathcal{N}_{\bar{n}}$. Observe that for any m_1'' and $S \in \mathcal{N}_{\bar{n}}$, $g^S\big((m_1'', m_{-1})^S\big) \neq c$; otherwise, player 1 would unilaterally deviate to m_1'' from m in state θ. Moreover, because $b \notin F(\theta')$, it must be that $b \notin NE(\Gamma_{\bar{n}}, \theta')$. Hence, there must exist player i, $m_i' \in M_i$ and $S \in \mathcal{N}_{\bar{n}}$ such that $g^S\big((m_i', m_{-i})^S\big) \succ_i^{\theta'} b$. Because $b \in A_i^*(\theta')$ for all $i \in \{2, 3, 4, 5\}$, it must be that $i = 1$. We already know $g^S\big((m_1', m_{-1})^S\big) \neq c$. Consequently, $g^S\big((m_1', m_{-1})^S\big) = a$.

Consider $\big(m_1', m_{-1}\big)$. As argued, a is implemented under some $S \in \mathcal{N}_{\bar{n}}$. However, b is implemented when $S = \{2, 3, 4, 5\}$ since $g^S\big((m_1', m_{-1})^S\big) = g^S(m^S) = b$ as $(m_1', m_{-1})^S = m^S$. Hence, to reach the desired contradiction, it suffices to show $\big(m_1', m_{-1}\big) \in NE(\Gamma_{\bar{n}}, \theta)$.

If $1 \notin S \in \mathcal{N}_{\bar{n}}$, then $g^S\big((m_1', m_{-1})^S\big) = b$ because $(m_1', m_{-1})^S = m^S$. In addition, if $1 \in S \in \mathcal{N}_{\bar{n}}$, then $g^S\big((m_1', m_{-1})^S\big)$ is either a or b. Since $\{a, b\} = A_i^*(\theta)$ for all $i \in \{2, 3, 4, 5\}$, none of these players has an incentive to deviate from (m_1', m_{-1}). We already know that for any m_1'' and $S \in \mathcal{N}_{\bar{n}}$, $g^S\big((m_1'', m_{-1})^S\big) \neq c$. Therefore, for any unilateral deviation by player 1 from (m_1', m_{-1}), either a or b is implemented. But player 1 is indifferent between a and b in state θ. Hence, player 1 also has no incentive to deviate from (m_1', m_{-1}) in state θ. Thus, $\big(m_1', m_{-1}\big) \in NE(\Gamma_{\bar{n}}, \theta)$. $\diamond$

7 Implementation by a Political Process

In this section, we consider the situation when the social planner has the flexibility to use any political process. Thus, the relevant notion is implementation by a political process.

7.1 Necessary Condition

The following result follows as a corollary of Theorem 3.2:

Corollary 7.1. *If F is implementable by a political process, then F is p–monotonic for some $p \in \mathcal{P}$.*

In any environment, the set of all Maskin monotonic SCRs is a subset of the set of SCRs that are p–monotonic for some $p \in \mathcal{P}$ (using Proposition 3.3). The former can also be a strict subset of the latter (Examples 3.5 and 5.7).

Palfrey and Srivastava (1991) have shown that property Q is necessary for implementation in undominated Nash equilibrium.[16] The set of SCRs that satisfy property Q and the set of SCRs that are p–monotonic for some $p \in \mathcal{P}$ are two distinct sets, i.e., there are SCRs that satisfy property Q but are not p–monotonic for any $p \in \mathcal{P}$ (Example 7.2) and there are SCRs that are p–monotonic for some $p \in \mathcal{P}$ but do not satisfy property Q (Example 7.3).

Example 7.2. Let $A = \{a, b, c, d\}$ and $\Theta = \{\theta, \theta'\}$. Suppose only player i's preference changes between the two states. Moreover, player i has expected-utility preference with following Bernoulli utilities: $u_i^\theta(a) = u_i^{\theta'}(a) = 5$, $u_i^\theta(b) = u_i^{\theta'}(b) = 3$, $u_i^\theta(c) = 1$, $u_i^{\theta'}(c) = -1$, and $u_i^\theta(d) = u_i^{\theta'}(d) = 0$. Let $F(\theta) = \{a, b\}$ and $F(\theta') = \{a\}$. F satisfies property Q but is not p–monotonic for any $p \in \mathcal{P}$ since for any $l \in \Delta A$, $b \succeq_i^\theta l \implies b \succeq_i^{\theta'} l$.

Example 7.3. Reconsider the previous example except that player i's Bernoulli utilities are as follows: $u_i^\theta(a) = u_i^{\theta'}(a) = 5$, $u_i^\theta(b) = u_i^{\theta'}(b) = 3$, $u_i^\theta(c) = 1$, $u_i^{\theta'}(c) = 2$, and $u_i^\theta(d) = u_i^{\theta'}(d) = 0$. Now, F does not satisfy property Q but is p–monotonic for some $p \in \mathcal{P}$.[17]

7.2 Sufficient Condition

For any $\epsilon \in (0, 1)$ and two lotteries $l, l' \in \Delta A$, let $(1-\epsilon)l + \epsilon l'$ be the lottery obtained by the convex combination l and l'. Until now we have assumed that preferences over lotteries are monotone, which is weaker than imposing the independence axiom on players' preferences.

[16] F satisfies property Q if whenever $a \in F(\theta)$ and $a \notin F(\theta')$, then there exist player i and two alternatives a', a'' such that (i) player i's preference ordering over $\{a', a''\}$ is different in the two states θ and θ', and (ii) player i is not indifferent between all alternatives in state θ'.

[17] Consider p such that $p(S) = p(S') = 0.5$ for any two senates $S, S' \in \mathcal{N}(i)$. Let $h_i : \mathcal{N} \to A$ be such that $h_i(S) = a$ and $h_i(S') = c$. Then $b \succeq_i^\theta l[p, h_i]$ and $l[p, h_i] \succ_i^{\theta'} b$.

Definition 7.4. *Independence Axiom*: Player i's preference satisfies the independence axiom if for all $\theta \in \Theta$, lotteries $l, l', l'' \in \Delta A$, and $\epsilon \in (0,1)$, we have

$$l \succeq_i^\theta l' \iff (1-\epsilon)l + \epsilon l'' \succeq_i^\theta (1-\epsilon)l' + \epsilon l''.$$

Recall p_D, the political process that selects N as the senate with probability 1. For any $\epsilon \in (0,1)$ and political process $p \in \mathcal{P}$, let $(1-\epsilon)p + \epsilon p_D$ denote the political process that selects each senate $S \in \mathcal{N}$ with probability $(1-\epsilon)p(S) + \epsilon p_D(S)$.

Lemma 7.5. *Suppose $\mathcal{E}$ is such that every player's preference satisfies the independence axiom. If F is p–monotonic for some $p \in \mathcal{P}$ such that $p(N) = 0$, then F is $(1-\epsilon)p + \epsilon p_D$–monotonic for all $\epsilon \in (0,1)$.*

Proof. Suppose $a \in F(\theta)$ but $a \notin F(\theta')$. Since F is p–monotonic for some $p \in \mathcal{P}$ such that $p(N) = 0$, there exist $i \in N$ and function $h_i : \mathcal{N} \to A$ such that

$$h_i(S) = a, \forall S \in \mathcal{N} \setminus \mathcal{N}(i) \text{ with } p(S) > 0, \quad a \succeq_i^\theta l[p, h_i] \text{ and } l[p, h_i] \succ_i^{\theta'} a.$$

Since players' preferences satisfy the independence axiom, for any $\epsilon \in (0,1)$, we have $a \succeq_i^\theta (1-\epsilon)l[p, h_i] + \epsilon a$ and $(1-\epsilon)l[p, h_i] + \epsilon a \succ_i^{\theta'} a$.[18]

Let $\hat{h}_i : \mathcal{N} \to A$ be such that $\hat{h}_i(N) = a$ and $\hat{h}_i(S) = h_i(S), \forall S \neq N$. Since $p(N) = 0$, lottery $l[(1-\epsilon)p + \epsilon p_D, \hat{h}_i] = (1-\epsilon)l[p, h_i] + \epsilon a$. Clearly, $\hat{h}_i(S) = a, \forall S \in \mathcal{N} \setminus \mathcal{N}(i)$ with $(1-\epsilon)p(S) + \epsilon p_D(S) > 0$. Hence, F is $(1-\epsilon)p + \epsilon p_D$–monotonic. $\square$

Lemma 7.5 and Theorem 3.6 give us the following corollary:

Corollary 7.6. *Let $n \geq 3$ and $\mathcal{E}$ be an economic environment such that every player's preference satisfies the independence axiom. If F is p–monotonic for some $p \in \mathcal{P}$ such that $p(S) > 0$ only if $|S| \geq 3$, then F is implementable by a political process.*

Proof. Suppose F is p–monotonic for some $p \in \mathcal{P}$ such that $p(S) > 0$ only if $|S| \geq 3$. If $p(N) > 0$, then F is p–implementable (Theorem 3.6). On the other hand, if $p(N) = 0$, then for any $\epsilon \in (0,1)$, F is $(1-\epsilon)p + \epsilon p_D$–monotonic (Lemma 7.5) and hence, $(1-\epsilon)p + \epsilon p_D$–implementable (Theorem 3.6). $\square$

Thus, in economic environments satisfying the independence axiom, any F that is p–monotonic for a political process p that always selects at least three senators is implementable by a political process. Moreover, if p is such that there is a positive chance of referendum,

[18] This implication cannot be obtained under the weaker assumption that preferences over lotteries are monotone unless the support of $l[p, h_i]$ is singleton or contains two elements with one of them being a.

i.e., $p(N) > 0$, then F can be implemented using p itself. On the other hand, if there is no chance of referendum under p, then F can be implemented using a perturbed process $(1-\epsilon)p + \epsilon p_D$ in which there is a small but positive chance of referendum.

8 Conclusion

Within the large class of economic environments, our results suggest that costs and time involved in collecting opinions of individuals in a community in order implement Maskin monotonic SCRs can be substantially reduced by using alternative political processes. For simplicity, let's assume that there is a fixed cost c of collecting each individual's message. Then the cost incurred in direct democracy is nc. On the other hand, our result on oligarchic democracy implies that the expected cost of collecting individuals' messages can be reduced to arbitrarily close to $3c$ in economic environments. However, there is a positive though very small chance of referendum in oligarchic democracy, in which case the realized cost will be nc. If we are concerned with the ex-post cost of collecting individuals' messages, then random sampling with sample size of 4 can be used in economic environments, guaranteeing an ex-post cost of only $4c$.

Our positive results, however, need not carry over to incomplete information environments. For instance, if players have private values, then the state of the world cannot be known even if $n-1$ individuals truthfully report their types. Thus, SCRs that are not "measurable" with respect to the information of the selected subset of individuals cannot be implemented in such an environment. Nevertheless, we expect that similar positive results can be obtained for incomplete information environments in which some subsets of the individuals are "better" informed than others. We leave these issues for future research.

9 Appendix

Proof of Theorem 3.6: Consider the mechanism Γ defined in the sketch of the proof.
Step 1. For any $\theta \in \Theta$, $F(\theta) \subseteq \Big\{ l[\Gamma, m] : m \in NE(\Gamma, \theta) \Big\}$.

Pick any $a \in F(\theta)$ and consider $m \in M$ such that $m_i = (\theta, a, a, \ldots, a, 0)$ for all $i \in N$. Then $l[\Gamma, m] = a$. Suppose player i deviates from m_i to $m'_i = (\theta_i, a_i^1, (a_i^{S'})_{S' \in \mathcal{N}(i)}, z_i)$. Pick any $S \in \mathcal{N}$ with $p(S) > 0$. First, suppose $S \in \mathcal{N} \setminus \mathcal{N}(i)$. Then rule (i) will be used under senate S and hence, a will be implemented. Second, suppose $S \in \mathcal{N}(i)$. Because $p(S) > 0$ implies $|S| \geq 3$, rule (ii) will be used under senate S. In that case, a is implemented if $l[p, h_i] \succ_i^\theta a$ and $h_i(S) = a_i^S$ is implemented if $a \succeq_i^\theta l[p, h_i]$. Therefore, if $l[p, h_i] \succ_i^\theta a$, then

$l[\Gamma, (m_i', m_{-i})] = a$. On the other hand, if $a \succeq_i^\theta l[p, h_i]$, then $l[\Gamma, (m_i', m_{-i})] = l[p, h_i]$. Hence, in either case, player i does not improve by her deviation. So, $m \in NE(\Gamma, \theta)$.

Step 2. For any $\theta \in \Theta$, $\Big\{ l[\Gamma, m] : m \in NE(\Gamma, \theta) \Big\} \subseteq F(\theta)$.

Consider any $\hat{m} \in NE(\Gamma, \theta)$. For any i, let $\hat{m}_i = (\hat{\theta}_i, \hat{a}_i^1, (\hat{a}_i^{S'})_{S' \in \mathcal{N}(i)}, \hat{z}_i)$.

First, suppose $\hat{m}$ is such that $g^N(\hat{m})$ follows rule (iii). We argue that $g^N(\hat{m}) \in A_i^*(\theta), \forall i \in N$. Suppose this is not true for some player i. Define a_i^1 as follows: if $A_i^*(\theta) \neq \emptyset$, then let a_i^1 be any alternative in $A_i^*(\theta)$; whereas if $A_i^*(\theta) = \emptyset$, then let a_i^1 be any alternative such that $a_i^1 \succ_i^\theta a, \forall a \in A(l[\Gamma, \hat{m}])$. Now, consider player i's deviation to m_i that differs from $\hat{m}_i$ only in the second and last components with these being, respectively, a_i^1 and $z_i > \max_{j \neq i} \hat{z}_j$. If the selected senate is some $S \in \mathcal{N}(i)$, then one of the following will hold:

- $g^S((m_i, \hat{m}_{-i})^S)$ follows rule (iii). Then $g^S((m_i, \hat{m}_{-i})^S) = a_i^1$. Moreover, this will be the case under senate N.

- $g^S((m_i, \hat{m}_{-i})^S)$ follows rule (ii). There are two possibilities: $g^S(\hat{m}^S)$ followed either rule (i) or (ii). However, in either case $g^S((m_i, \hat{m}_{-i})^S) = g^S(\hat{m}^S)$ since player i has only changed the second and last components of her message and the outcome under rule (ii) does not depend on these components.

On the other hand, if the selected senate is $S \in \mathcal{N} \setminus \mathcal{N}(i)$, then $g^S((m_i, \hat{m}_{-i})^S) = g^S(\hat{m}^S)$. Since preferences over lotteries are monotone, player i will be better-off after the deviation to m_i, a contradiction. Hence, $g^N(\hat{m}) \in A_i^*(\theta), \forall i \in N$, which contradicts $\mathcal{E}$ being an economic environment.

Second, suppose $g^N(\hat{m})$ follows rule (ii). It must be that for $n-1$ players $i \neq j$, $\hat{m}_i = (\hat{\theta}, \hat{a}_i^1, \hat{a}, \ldots, \hat{a}, 0)$ and $\hat{a} \in F(\hat{\theta})$, but $\hat{m}_j = (\hat{\theta}_j, \hat{a}_j^1, (\hat{a}_j^{S'})_{S' \in \mathcal{N}(j)}, \hat{z}_j) \neq (\hat{\theta}, \hat{a}_j^1, \hat{a}, \ldots, \hat{a}, 0)$. Using a similar argument as in the previous case, $g^N(\hat{m}) \in A_i^*(\theta)$ for all $i \neq j$, which again contradicts $\mathcal{E}$ being an economic environment.

Therefore, $g^N(\hat{m})$ follows rule (i). Hence, each player $i's$ message $\hat{m}_i = (\hat{\theta}, \hat{a}_i^1, \hat{a}, \ldots, \hat{a}, 0)$, where $\hat{a} \in F(\hat{\theta})$. If $\hat{a} \in F(\theta)$, then we are done. If $\hat{a} \notin F(\theta)$ then, by p–monotonicity, there must exist player i and function $h_i : \mathcal{N} \to A$ such that

$$h_i(S) = \hat{a}, \forall S \in \mathcal{N} \setminus \mathcal{N}(i) \text{ with } p(S) > 0, \quad \hat{a} \succeq_i^{\hat{\theta}} l[p, h_i] \text{ and } l[p, h_i] \succ_i^\theta \hat{a}.$$

Thanks to rule (ii), player i has an incentive to deviate to $(\hat{\theta}, \hat{a}_i^1, (h_i(S))_{S \in \mathcal{N}(i)}, 0)$. □

Proof of Proposition 5.2: Suppose F is weak p_S–monotonic. Let $\big(p_{(S,\epsilon_k)}\big)_{k=1}^\infty$ be the sequence of (S, ϵ_k)-oligarchic democracies which make F weak p_S–monotonic. We argue that $(\epsilon_k)_{k=1}^\infty$ is the sequence that satisfies the condition in the lemma. By definition of (S, ϵ_k)-oligarchic democracy, $\epsilon_k \in (0, 1), \forall k$. We also know that $\lim_{k \to \infty} \epsilon_k = 0$. Suppose $a \in F(\theta)$

but $a \notin F(\theta')$. F is $p_{(S,\epsilon_k)}$–monotonic for all k. Hence, for all k, there exist player $i_k \in N$ and function $h_{i_k} : \mathcal{N} \to A$ such that

$$h_{i_k}(S') = a, \forall S' \in \mathcal{N} \backslash \mathcal{N}(i_k) \text{ with } p_{(S,\epsilon_k)}(S') > 0, \quad a \succeq^{\theta}_{i_k} l[p_{(S,\epsilon_k)}, h_{i_k}] \text{ and } l[p_{(S,\epsilon_k)}, h_{i_k}] \succ^{\theta'}_{i_k} a.$$

Now, $p_{(S,\epsilon_k)}$ puts probability $1-\epsilon_k$ on S and ϵ_k on N. Hence, if $i_k \notin S$, then $h_{i_k}(S) = a$. In this case, $l[p_{(S,\epsilon_k)}, h_{i_k}]$ is a lottery that puts probability $1-\epsilon_k$ on a and ϵ_k on $h_{i_k}(N) \equiv a'_k$. Since preferences over lotteries are monotone,

$$a \succeq^{\theta}_{i_k} l[p_{(S,\epsilon_k)}, h_{i_k}] \implies a \succeq^{\theta}_{i_k} a'_k \quad \text{while} \quad l[p_{(S,\epsilon_k)}, h_{i_k}] \succ^{\theta'}_{i_k} a \implies a'_k \succ^{\theta'}_{i_k} a.$$

On the other hand, if $i_k \in S$, then $l[p_{(S,\epsilon_k)}, h_{i_k}]$ is a lottery that puts probability $1-\epsilon_k$ on $h_{i_k}(S) \equiv \tilde{a}_k$ and ϵ_k on $h_{i_k}(N) \equiv \hat{a}_k$. Hence, $l[p_{(S,\epsilon_k)}, h_{i_k}]$ is the required lottery l_k.

To argue the opposite implication, suppose F satisfies the condition in the lemma with respect to the sequence $(\epsilon_k)_{k=1}^{\infty}$. Since $\epsilon_k \in (0,1)$, we can define (S,ϵ_k)-oligarchic democracy $p_{(S,\epsilon_k)}$. Now, consider the sequence $\left(p_{(S,\epsilon_k)}\right)_{k=1}^{\infty}$. Clearly, $\lim_{k\to\infty} \epsilon_k = 0$. Suppose $a \in F(\theta)$ but $a \notin F(\theta')$. Fix k and consider the player i_k. If $i_k \notin S$, then pick any $h_{i_k} : \mathcal{N} \to A$ such that $h_{i_k}(S) = a$ and $h_{i_k}(N) = a'_k$. Since preferences over lotteries are monotone,

$$a \succeq^{\theta}_{i_k} a'_k \implies a \succeq^{\theta}_{i_k} l[p_{(S,\epsilon_k)}, h_{i_k}] \quad \text{while} \quad a'_k \succ^{\theta'}_{i_k} a \implies l[p_{(S,\epsilon_k)}, h_{i_k}] \succ^{\theta'}_{i_k} a.$$

On the other hand, if $i_k \in S$, then pick any $h_{i_k} : \mathcal{N} \to A$ such that $h_{i_k}(S) = \tilde{a}_k$ and $h_{i_k}(N) = \hat{a}_k$. Then lottery $l[p_{(S,\epsilon_k)}, h_{i_k}]$ is the equal to lottery l_k and hence $a \succeq^{\theta}_{i_k} l[p_{(S,\epsilon_k)}, h_{i_k}]$ and $l[p_{(S,\epsilon_k)}, h_{i_k}] \succ^{\theta'}_{i_k} a$. Thus, F is $p_{(S,\epsilon_k)}$–monotonic for all k. □

Proof of Theorem 5.9: For any $\epsilon \in (0,1)$, define $\Gamma_{(S,\epsilon)} = ((M_i)_{i\in N}, p_{(S,\epsilon)}, (g^{S'})_{S'\in\mathcal{N}})$ such that for all $i \in N$,

$$M_i = \Theta \times A \times A \times \mathbb{Z}_+.$$

Let a typical message m_i be of the form $(\theta_i, a_i^1, a_i^2, z_i)$. For all $S' \in \{N, S\}$, the outcome function $g^{S'}$ is as follows:

(i) If for every player $i \in S'$, $m_i = (\theta, a_i^1, a, 0)$ and $a \in F(\theta)$, then $g^{S'}\left((m_i)_{i\in S'}\right) = a$.

(ii) If for $|S'|-1$ players $i \neq j$ in S', $m_i = (\theta, a_i^1, a, 0)$ and $a \in F(\theta)$, but $m_j = (\theta_j, a_j^1, a_j^2, z_j) \neq (\theta, a_j^1, a, 0)$, then

$$g^{S'}\left((m_i)_{i\in S'}\right) = \begin{cases} a & \text{if } a_j^2 \succ^{\theta}_j a. \\ a_j^2 & \text{if } a \succeq^{\theta}_j a_j^2. \end{cases}$$

(iii) In all other cases, $g^{S'}\big((m_i)_{i\in S'}\big) = a_j^1$ where $j \in S'$ is the player with the lowest index among those players in S' who announce the highest integer in the profile $(m_i)_{i\in S'}$.

Finally, for all $S' \in \mathcal{N} \setminus \{N, S\}$, the outcome function $g^{S'}$ can be arbitrary.

Step 1. For all $\epsilon \in (0,1)$ and $\theta \in \Theta$, $F(\theta) \subseteq \Big\{ l[\Gamma_{(S,\epsilon)}, m] : m \in NE(\Gamma_{(S,\epsilon)}, \theta) \Big\}$.

Fix $\epsilon \in (0,1)$ and θ. Pick any $a \in F(\theta)$. Consider $m \in M$ such that $m_i = (\theta, a, a, 0)$ for all $i \in N$. Then $l[\Gamma_{(S,\epsilon)}, m] = a$. Suppose player i deviates from m_i to $m_i' = (\theta_i, a_i^1, a_i^2, z_i)$.

First, suppose $i \in S$. Then rule (ii) will be used under both senates S and N. In either case, a_i^2 is implemented if $a \succeq_i^\theta a_i^2$ whereas a is implemented if $a_i^2 \succ_i^\theta a$. Thus, player i does not improve by her deviation.

Second, suppose $i \notin S$. Then rule (ii) will be used under senate N. In that case, a_i^2 is implemented if $a \succeq_i^\theta a_i^2$ whereas a is implemented if $a_i^2 \succ_i^\theta a$. On the other hand, a is implemented under senate S. Since the preferences over lotteries are monotone, player i does not improve by her deviation. Therefore, $m \in NE(\Gamma_{(S,\epsilon)}, \theta)$.

Step 2. For all $\epsilon \in (0,1)$ and $\theta \in \Theta$, $\Big\{ l[\Gamma_{(S,\epsilon)}, m] : m \in NE(\Gamma_{(S,\epsilon)}, \theta) \Big\} \subseteq F(\theta)$.

Fix $\epsilon \in (0,1)$ and θ. Pick any $\hat{m} \in NE(\Gamma_{(S,\epsilon)}, \theta)$. For any i, let $\hat{m}_i = (\hat{\theta}_i, \hat{a}_i^1, \hat{a}_i^2, \hat{z}_i)$.

First, suppose $\hat{m}$ is such that $g^N(\hat{m})$ follows rule (iii). We argue that $g^N(\hat{m}) \in A_i^*(\theta), \forall i \in N$. Suppose this is not true for some player i. Define a_i^1 as follows: if $A_i^*(\theta) \neq \emptyset$, then let a_i^1 be any alternative in $A_i^*(\theta)$; whereas if $A_i^*(\theta) = \emptyset$, then let a_i^1 be any alternative such that $a_i^1 \succ_i^\theta a, \forall a \in A(l[\Gamma_{(S,\epsilon)}, \hat{m}])$. Now, consider player i's deviation to m_i that differs from $\hat{m}_i$ only in the second and last components with these being, respectively, a_i^1 and $z_i > \max_{j\neq i} \hat{z}_j$. After this deviation, $g^N(m_i, \hat{m}_{-i})$ follows rule (iii) and hence, $g^N(m_i, \hat{m}_{-i}) = a_i^1$. On the other hand, one of the following in true under senate S:

- $i \notin S$ and $g^S((m_i, \hat{m}_{-i})^S) = g^S(\hat{m}^S)$.
- $i \in S$ and $g^S((m_i, \hat{m}_{-i})^S)$ follows rule (iii). Then $g^S((m_i, \hat{m}_{-i})^S) = a_i^1$.
- $i \in S$ and $g^S((m_i, \hat{m}_{-i})^S)$ follows rule (ii). Then $g^S((m_i, \hat{m}_{-i})^S) = g^S(\hat{m}^S)$.

Since preferences over lotteries are monotone, player i will be better-off after the deviation to m_i, a contradiction. Hence, $g^N(\hat{m}) \in A_i^*(\theta), \forall i \in N$, which contradicts DTA-$S$.

Second, suppose $\hat{m}$ is such that $g^N(\hat{m})$ follows rule (ii). It must be that for $n-1$ players $i \neq j$, $\hat{m}_i = (\hat{\theta}, \hat{a}_i^1, \hat{a}, 0)$ and $\hat{a} \in F(\hat{\theta})$, but $\hat{m}_j = (\hat{\theta}_j, \hat{a}_j^1, \hat{a}_j^2, \hat{z}_j) \neq (\hat{\theta}, \hat{a}_j^1, \hat{a}, 0)$. Using a similar argument as in the previous case, $g^N(\hat{m}) \in A_i^*(\theta)$ for all $i \neq j$. Since F satisfies no-veto power, we have $g^N(\hat{m}) \in F(\theta)$. If $j \notin S$, then that contradicts DTA-S. Hence, $j \in S$. It follows from rule (ii) that $g^S(\hat{m}^S) = g^N(\hat{m}) \in F(\theta)$.

Finally, suppose $g^N(\hat{m})$ follows rule (i). Hence, for all i, $\hat{m}_i = (\hat{\theta}, \hat{a}_i^1, \hat{a}, 0)$, where $\hat{a} \in F(\hat{\theta})$. If $\hat{a} \in F(\theta)$, then we are done. If $\hat{a} \notin F(\theta)$, then since F is Maskin monotonic, there exist player i and alternative a' such that $\hat{a} \succeq_i^{\hat{\theta}} a'$ and $a' \succ_i^{\theta} \hat{a}$. Then, because of rule (ii) and preferences over lotteries being monotone, player i has an incentive to deviate to $(\hat{\theta}, \hat{a}_i^1, a', 0)$.

Since the above argument was made for all $\epsilon \in (0,1)$, it follows that F is implementable by S-oligarchic democracy. $\square$

Proof of Theorem 6.4: Consider the $\bar{n}$-sampling mechanism $\Gamma_{\bar{n}} = ((M_i)_{i\in N}, p_{\bar{n}}, (g^S)_{S\in\mathcal{N}})$ in which for all $i \in N$, the message space M_i and for all $S \in \mathcal{N}$, the outcome function g^S are the same as in the proof of Theorem 3.6.

Step 1. For any $\theta \in \Theta$, $F(\theta) \subseteq \Big\{ l[\Gamma_{\bar{n}}, m] : m \in NE(\Gamma_{\bar{n}}, \theta) \Big\}$. This can be argued like Step 1 in the proof of Theorem 3.6.

Step 2. For any $\theta \in \Theta$, $\Big\{ l[\Gamma_{\bar{n}}, m] : m \in NE(\Gamma_{\bar{n}}, \theta) \Big\} \subseteq F(\theta)$.

Let $\hat{m} \in NE(\Gamma_{\bar{n}}, \theta)$ be such that for any i, $\hat{m}_i = (\hat{\theta}_i, \hat{a}_i^1, (\hat{a}_i^{S'})_{S'\in\mathcal{N}(i)}, \hat{z}_i)$. We show that $l[\Gamma_{\bar{n}}, \hat{m}] \in F(\theta)$.

First, suppose $\hat{m}$ is such that there exist at least three players, i_1, i_2 and i_3 such that either

1. $(\hat{\theta}_{i_1}, (\hat{a}_{i_1}^{S'})_{S'\in\mathcal{N}(i_1)}, \hat{z}_{i_1}) \neq (\hat{\theta}_{i_2}, (\hat{a}_{i_2}^{S'})_{S'\in\mathcal{N}(i_2)}, \hat{z}_{i_2}) \neq (\hat{\theta}_{i_3}, (\hat{a}_{i_3}^{S'})_{S'\in\mathcal{N}(i_3)}, \hat{z}_{i_3})$ or
2. $(\hat{\theta}_{i_1}, (\hat{a}_{i_1}^{S'})_{S'\in\mathcal{N}(i_1)}, \hat{z}_{i_1}) = (\hat{\theta}_{i_2}, (\hat{a}_{i_2}^{S'})_{S'\in\mathcal{N}(i_2)}, \hat{z}_{i_2}) = (\hat{\theta}_{i_3}, (\hat{a}_{i_3}^{S'})_{S'\in\mathcal{N}(i_3)}, \hat{z}_{i_3})$ and any one of the following holds: (a) $\hat{z}_{i_1} > 0$, (b) $\hat{a}_{i_1}^{S'} \notin F(\hat{\theta}_{i_1})$ for some $S' \in \mathcal{N}(i_1)$, or (c) $\hat{a}_{i_1}^{S} \neq \hat{a}_{i_1}^{S'}$ for some $S \neq S'$.

Notice that the second components of these players' messages are not being considered. Consider any $j \in N \setminus \{i_1, i_2, i_3\}$ and pick any $S(j) \in \mathcal{N}_{\bar{n}}(j)$ such that $\{j, i_1, i_2, i_3\} \subseteq S(j)$ (this is possible since $\bar{n} \geq 4$). If $S(j)$ is selected as the senate, rule (iii) will be used. Let a_j be the alternative implemented under $S(j)$. We claim that $a_j \in A_i^*(\theta)$ for all $i \in \{j, i_1, i_2, i_3\}$. Suppose this is not the case for some $i \in \{j, i_1, i_2, i_3\}$. Define a_i^1 as follows: if $A_i^*(\theta) \neq \emptyset$, then let a_i^1 be any alternative in $A_i^*(\theta)$; whereas if $A_i^*(\theta) = \emptyset$, then let a_i^1 be any alternative such that $a_i^1 \succ_i^{\theta} a, \forall a \in A(l[\Gamma_{\bar{n}}, \hat{m}])$. Now, consider player i's deviation to m_i that differs from $\hat{m}_i$ only in the second and last components with these being, respectively, a_i^1 and $z_i > \max_{i'\neq i} \hat{z}_{i'}$. If the selected senate is some $S \in \mathcal{N}_{\bar{n}}(i)$, then one of the following will hold:

- $g^S((m_i, \hat{m}_{-i})^S)$ follows rule (iii). Then $g^S((m_i, \hat{m}_{-i})^S) = a_i^1$. Moreover, this will be the case when $S = S(j)$.
- $g^S((m_i, \hat{m}_{-i})^S)$ follows rule (ii). There are two possibilities: $g^S(\hat{m}^S)$ followed either rule (i) or (ii). However, in either case $g^S((m_i, \hat{m}_{-i})^S) = g^S(\hat{m}^S)$ since player i has only changed the second and last components of her message and the outcome under rule (ii) does not depend on these components.

On the other hand, if the selected senate is $S \in \mathcal{N}_{\bar{n}} \setminus \mathcal{N}_{\bar{n}}(i)$, then $g^S((m_i, \hat{m}_{-i})^S) = g^S(\hat{m}^S)$. Since preferences over lotteries are monotone, player i will be better-off after the deviation to m_i, a contradiction.

Now, consider the set of alternatives $\{a_j : j \in N \setminus \{i_1, i_2, i_3\}\}$. Every alternative in this set is most-preferred in state θ by all players i_1, i_2 and i_3. Hence, unique-top condition implies that this set of alternatives is singleton, i.e., there is an a such that $a_j = a, \forall j \in N \setminus \{i_1, i_2, i_3\}$. Then $a \in \bigcap_{i \in N} A_i^*(\theta)$, which contradicts $\mathcal{E}$ being an economic environment.

Second, suppose $\hat{m}$ is such that there exist at least two players, i_1 and i_2 such that $(\hat{\theta}_{i_1}, (\hat{a}_{i_1}^{S'})_{S' \in \mathcal{N}(i_1)}, \hat{z}_{i_1}) \neq (\hat{\theta}_{i_2}, (\hat{a}_{i_2}^{S'})_{S' \in \mathcal{N}(i_2)}, \hat{z}_{i_2})$ – again, the second components of these players' messages are not being considered. If there exists a j such that $(\hat{\theta}_j, (\hat{a}_j^{S'})_{S' \in \mathcal{N}(j)}, \hat{z}_j) \neq (\hat{\theta}_{i_1}, (\hat{a}_{i_1}^{S'})_{S' \in \mathcal{N}(i_1)}, \hat{z}_{i_1}) \neq (\hat{\theta}_{i_2}, (\hat{a}_{i_2}^{S'})_{S' \in \mathcal{N}(i_2)}, \hat{z}_{i_2})$, then we are back in the first case. Hence, for every player j there exists a player in $\{i_1, i_2\}$, denoted by $i(j)$, such that $(\hat{\theta}_j, (\hat{a}_j^{S'})_{S' \in \mathcal{N}(j)}, \hat{z}_j) = (\hat{\theta}_{i(j)}, (\hat{a}_{i(j)}^{S'})_{S' \in \mathcal{N}(i(j))}, \hat{z}_{i(j)})$. Let $J_1 = \{j \in N : i(j) = i_1\}$ and $J_2 = \{j \in N : i(j) = i_2\}$. Without loss of generality, suppose $|J_1| \geq 2$. Let $i_3 \in J_1$ such that $i_3 \neq i_1$.

For any $j \in N \setminus \{i_1, i_2, i_3\}$, let $S(j) \in \mathcal{N}_{\bar{n}}(j)$ be any senate such that $\{j, i_1, i_2, i_3\} \subseteq S(j)$. Let a_j be the alternative implemented under $S(j)$. If $j \in J_2$, then rule (iii) will be used in senate $S(j)$. Like above, we can argue that $a_j \in A_i^*(\theta)$ for all $i \in \{j, i_1, i_2, i_3\}$. On the other hand, if $j \in J_1$, then either rule (ii) or (iii) will be used in senate $S(j)$. We argue that $a_j \in A_i^*(\theta)$ for all $i \in \{j, i_1, i_3\}$. Suppose this is not the case for some $i \in \{j, i_1, i_3\}$. Let player i deviate to m_i that differs from $\hat{m}_i$ only in the second and last components with these being, respectively, a_i^1 (as defined in the first case) and $z_i > \max_{i' \neq i} \hat{z}_{i'}$. If the selected senate is $S \in \mathcal{N}_{\bar{n}}(i)$, then one of the following will hold:

- $g^S((m_i, \hat{m}_{-i})^S)$ follows rule (iii). Then $g^S((m_i, \hat{m}_{-i})^S) = a_i^1$. Moreover, this will be the case when $S = S(j)$.

- $g^S((m_i, \hat{m}_{-i})^S)$ follows rule (ii). There are two possibilities: $g^S(\hat{m}^S)$ followed either rule (i) or (ii). However, in either case $g^S((m_i, \hat{m}_{-i})^S) = g^S(\hat{m}^S)$ since player i has only changed the second and last components of her message and the outcome under rule (ii) does not depend on these components.

On the other hand, if the selected senate is $S \in \mathcal{N}_{\bar{n}} \setminus \mathcal{N}_{\bar{n}}(i)$, then $g^S((m_i, \hat{m}_{-i})^S) = g^S(\hat{m}^S)$. Since preferences over lotteries are monotone, player i will be better-off after the deviation to m_i, a contradiction.

Now, consider the set of alternatives $\{a_j : j \in N \setminus \{i_1, i_2, i_3\}\}$. Every alternative in this set is most-preferred in state θ by players i_1 and i_3. Hence, unique-top condition implies that this set of alternatives is singleton, i.e., there is an a such that $a_j = a, \forall j \in N \setminus \{i_1, i_2, i_3\}$.

Then a is most-preferred alternative in state θ for at least $n-1$ players (all except i_2), which contradicts $\mathcal{E}$ being an economic environment.

Therefore, $\hat{m}$ is such that for all $i \in N$, $(\hat{\theta}_i, (\hat{a}_i^{S'})_{S' \in \mathcal{N}(i)}, \hat{z}_i) = (\hat{\theta}, \hat{a}, \ldots, \hat{a}, 0)$ with $\hat{a} \in F(\hat{\theta})$. Then because of rule (i), $\hat{a}$ is implemented under any senate $S \in \mathcal{N}_{\bar{n}}$. If $\hat{a} \in F(\theta)$, then we are done. If $\hat{a} \notin F(\theta)$, then due to $p_{\bar{n}}$–monotonicity, there exist player $i \in N$ and function $h_i : \mathcal{N} \to A$ such that

$$h_i(S) = \hat{a}, \forall S \in \mathcal{N}_{\bar{n}} \setminus \mathcal{N}_{\bar{n}}(i), \quad \hat{a} \succeq_i^{\hat{\theta}} l[p_{\bar{n}}, h_i] \text{ and } l[p_{\bar{n}}, h_i] \succ_i^{\theta} \hat{a}.$$

Thanks to rule (ii), player i has an incentive to deviate to $(\hat{\theta}, \hat{a}_i^1, (h_i(S))_{S \in \mathcal{N}(i)}, 0)$. □

Proof of Theorem 6.5: For each player $j \in N$, let

$$\mathbb{S}(j) = \begin{cases} \{S \in \mathcal{N}_{\bar{n}}(j) : \{1,2\} \subset S\}, & \text{if } j > 2. \\ \{S \in \mathcal{N}_{\bar{n}}(j) : \{3,4\} \subset S\}, & \text{if } j \in \{1,2\}. \end{cases}$$

Define the $\bar{n}$-sampling mechanism $\Gamma_{\bar{n}} = ((M_i)_{i \in N}, p_{\bar{n}}, (g^S)_{S \in \mathcal{N}})$ in which for all $i \in N$,

$$M_i = \Theta \times A \times A \times \mathbb{Z}_+.$$

Let a typical message m_i be of the form $(\theta_i, a_i^1, a_i^2, z_i)$.

For each $S \in \mathcal{N}_{\bar{n}}$, the outcome function g^S is as follows:

(i) If for every player $i \in S$, $m_i = (\theta, a_i^1, a, 0)$ and $a \in F(\theta)$, then $g^S\big((m_i)_{i \in S}\big) = a$.

(ii) If for $|S|-1$ players $i \neq j$ in S, $m_i = (\theta, a_i^1, a, 0)$ and $a \in F(\theta)$, but $m_j = (\theta_j, a_j^1, a_j^2, z_j) \neq (\theta, a_j^1, a, 0)$, then

$$g^S\big((m_i)_{i \in S}\big) = \begin{cases} a & \text{if } a_j^2 \succ_j^{\theta} a \text{ or } S \notin \mathbb{S}(j) \\ a_j^2 & \text{if } a \succeq_j^{\theta} a_j^2 \text{ and } S \in \mathbb{S}(j). \end{cases}$$

(iii) In all other cases, $g^S\big((m_i)_{i \in S}\big) = a_j^1$ where $j \in S$ is the player with the lowest index among those players in S who announce the highest integer in the profile $(m_i)_{i \in S}$.

Finally, for all $S \in \mathcal{N} \setminus \mathcal{N}_{\bar{n}}$, the outcome function g^S can be arbitrary.

Step 1. For any $\theta \in \Theta$, $F(\theta) \subseteq \Big\{ l[\Gamma_{\bar{n}}, m] : m \in NE(\Gamma_{\bar{n}}, \theta) \Big\}$.

Pick any $a \in F(\theta)$ and consider $m \in M$ such that $m_i = (\theta, a, a, 0)$ for all $i \in N$. Then $l[p_{\bar{n}}, m] = a$. Suppose player i deviates from m_i to $m_i' = (\theta_i, a_i^1, a_i^2, z_i)$. Pick any $S \in \mathcal{N}_{\bar{n}}$. First, suppose $S \in \mathcal{N}_{\bar{n}} \setminus \mathcal{N}_{\bar{n}}(i)$. Then rule (i) will be used under senate S and hence, a will be implemented. Second, suppose $S \in \mathcal{N}_{\bar{n}}(i)$. Since $|S| = \bar{n} \geq 4$, rule (ii) will be used under senate S. In that case, a is implemented if $a_i^2 \succ_i^\theta a$ or $S \notin \mathbb{S}(i)$ and a_i^2 is implemented if $a \succeq_i^\theta a_i^2$ and $S \in \mathbb{S}(i)$. Since preferences over lotteries are monotone, player i does not improve by her deviation. So, $m \in NE(\Gamma_{\bar{n}}, \theta)$.

Step 2. For any $\theta \in \Theta$, $\left\{l[p_{\bar{n}}, m] : m \in NE(\Gamma_{\bar{n}}, \theta)\right\} \subseteq F(\theta)$.

Let $\hat{m} \in NE(\Gamma_{\bar{n}}, \theta)$ be such that for any i, $\hat{m}_i = (\hat{\theta}_i, \hat{a}_i^1, \hat{a}_i^2, \hat{z}_i)$. Define i_1 as the player with the lowest index amongst the players who announce the highest integer in $\hat{m}$.

First, suppose $\hat{m}$ is such that there exist at least three players, i_2, i_3 and i_4 such that $(\hat{\theta}_{i_2}, \hat{a}_{i_2}^2, \hat{z}_{i_2}) \neq (\hat{\theta}_{i_3}, \hat{a}_{i_3}^2, \hat{z}_{i_3}) \neq (\hat{\theta}_{i_4}, \hat{a}_{i_4}^2, \hat{z}_{i_4})$ – the second components of these players' messages are not being considered. Let $I = \left\{i \in \{i_2, i_3, i_4\} : (\hat{\theta}_i, \hat{a}_i^2, \hat{z}_i) = (\hat{\theta}_{i_1}, \hat{a}_{i_1}^2, \hat{z}_{i_1})\right\}$. Clearly, either I is empty or singleton. Consider any $j \in N$ and pick any $S(j) \in \mathcal{N}_{\bar{n}}(j)$ such that (a) if $i_1 \in I$, then $\{i_2, i_3, i_4\} \subset S(j)$, (b) if $i_1 \notin I$ but $I \neq \emptyset$, then $\{i_1, i_2, i_3, i_4\} \setminus I \subset S(j)$, and (c) if $I = \emptyset$, then $\{i_1, i_2, i_3\} \subset S(j)$ (this is possible since $\bar{n} \geq 4$). Now, rule (iii) will be used under senate $S(j)$. Moreover, since player $i_1 \in S(j)$, the alternative $\hat{a}_{i_1}^1$ will be implemented under $S(j)$. We claim that $\hat{a}_{i_1}^1 \in \bigcap_{j \in N} A_j^*(\theta)$ for all $j \in N$. Suppose not and let $j \in N$ be such that $\hat{a}_{i_1}^1 \notin A_j^*(\theta)$. Define a_j^1 as follows: if $A_j^*(\theta) \neq \emptyset$, then let a_j^1 be any alternative in $A_j^*(\theta)$; whereas if $A_j^*(\theta) = \emptyset$, then let a_j^1 be any alternative such that $a_j^1 \succ_j^\theta a, \forall a \in A(l[\Gamma_{\bar{n}}, \hat{m}])$. Now, consider player j's deviation to m_j that differs from $\hat{m}_j$ only in the second and last components with these being, respectively, a_j^1 and $z_j > \hat{z}_{i_1}$. If the selected senate is some $S \in \mathcal{N}_{\bar{n}}(j)$, then one of the following will hold:

- $g^S((m_j, \hat{m}_{-j})^S)$ follows rule (iii). Then $g^S((m_j, \hat{m}_{-j})^S) = a_j^1$. Moreover, this will be the case when $S = S(j)$.

- $g^S((m_j, \hat{m}_{-j})^S)$ follows rule (ii). There are two possibilities: $g^S(\hat{m}^S)$ followed either rule (i) or (ii). However, in either case $g^S((m_j, \hat{m}_{-j})^S) = g^S(\hat{m}^S)$ since player j has only changed the second and last components of her message and the outcome under rule (ii) does not depend on these components.

On the other hand, if the selected senate is $S \in \mathcal{N}_{\bar{n}} \setminus \mathcal{N}_{\bar{n}}(j)$, then $g^S((m_j, \hat{m}_{-j})^S) = g^S(\hat{m}^S)$. Since preferences over lotteries are monotone, player j will be better-off after the deviation to m_j, a contradiction. But $\hat{a}_{i_1}^1 \in \bigcap_{j \in N} A_j^*(\theta)$ for all $j \in N$ contradicts DTA-N.

Second, suppose $\hat{m}$ is such that there exist at least two players, i_2 and i_3 such that $(\hat{\theta}_{i_2}, \hat{a}_{i_2}^2, \hat{z}_{i_2}) \neq (\hat{\theta}_{i_3}, \hat{a}_{i_3}^2, \hat{z}_{i_3})$ – again, the second components of these players' messages are not being considered. If there exists a player j such that $(\hat{\theta}_j, \hat{a}_j^2, \hat{z}_j) \neq (\hat{\theta}_{i_2}, \hat{a}_{i_2}^2, \hat{z}_{i_2}) \neq (\hat{\theta}_{i_3}, \hat{a}_{i_3}^2, \hat{z}_{i_3})$, then we are back in the first case. Hence, for every player j there exists a player in $\{i_2, i_3\}$, denoted by $i(j)$, such that $(\hat{\theta}_j, \hat{a}_j^2, \hat{z}_j) = (\hat{\theta}_{i(j)}, \hat{a}_{i(j)}^2, \hat{z}_{i(j)})$. Let $J_2 = \{j \in N : i(j) = i_2\}$ and $J_3 = \{j \in N : i(j) = i_3\}$. As before, i_1 is the player with the lowest index amongst the players who announce the highest integer in $\hat{m}$. Without loss of generality, suppose $i_1 \in J_2$.

- Suppose there exist $j_2 \neq i_1$ and $j_3 \neq i_3$ such that $j_2 \in J_2$ and $j_3 \in J_3$. Consider any $j \in N$ and define $S(j)$ as follows. If $j \in \{i_1, j_2, i_3, j_3\}$, then let $S(j)$ be any set in $\mathcal{N}_{\bar{n}}(j)$ such that $\{i_1, j_2, i_3, j_3\} \subseteq S(j)$. If $j \in J_2 \setminus \{i_1, j_2\}$, then let $S(j)$ be any set in $\mathcal{N}_{\bar{n}}(j)$ such that $\{i_1, j, i_3, j_3\} \subseteq S(j)$. Finally, if $j \in J_3 \setminus \{i_3, j_3\}$, then let $S(j)$ be any set in $\mathcal{N}_{\bar{n}}(j)$ such that $\{i_1, j_2, i_3, j\} \subseteq S(j)$. In defining $S(j)$, we have ensured that $i_1 \in S(j)$ and it contains at least two players each from J_2 and J_3, which is possible since $\bar{n} \geq 4$. Hence, rule (iii) will be used and alternative $\hat{a}^1_{i_1}$ will be implemented under $S(j)$. As before, we can show that $\hat{a}^1_{i_1} \in \bigcap_{j \in N} A^*_j(\theta)$ for all $j \in N$, which contradicts DTA-N.

- Suppose there exists $j_2 \neq i_1$ such that $j_2 \in J_2$ but $J_3 = \{i_3\}$. If $\hat{z}_{i_1} \neq 0$ or $\hat{a}^2_{i_1} \notin F(\hat{\theta}_{i_1})$, then for any $j \in N$, let $S(j)$ be any set in $\mathcal{N}_{\bar{n}}(j)$ such that $\{i_1, j_2, i_3\} \subset S(j)$. This is possible since $\bar{n} \geq 4$. Rule (iii) is used and hence, $\hat{a}^1_{i_1}$ is implemented under $S(j)$. As before, we can show that $\hat{a}^1_{i_1} \in \bigcap_{j \in N} A^*_j(\theta)$ for all $j \in N$, which contradicts DTA-N.

  On the other hand, if $\hat{z}_{i_1} = 0$ and $\hat{a}^2_{i_1} \in F(\hat{\theta}_{i_1})$, then rule (ii) is used under any $S \in \mathcal{N}_{\bar{n}}(i_3)$ (since i_3 is the only player in S who "disagrees") whereas rule (i) is used under any $S \in \mathcal{N}_{\bar{n}} \setminus \mathcal{N}_{\bar{n}}(i_3)$ (since all players in S "agree"). Therefore, $\hat{a}^2_{i_1}$ is implemented under all $S \in \mathcal{N}_{\bar{n}} \setminus \mathbb{S}(i_3)$.

  - Suppose $\hat{a}^2_{i_3} \succ^{\hat{\theta}_{i_1}}_{i_3} \hat{a}^2_{i_1}$. Then $\hat{a}^2_{i_1}$ is also implemented under all $S \in \mathbb{S}(i_3)$. If $\hat{a}^2_{i_1} \in F(\theta)$, then we are done. If $\hat{a}^2_{i_1} \notin F(\theta)$, then by Maskin monotonicity, there exist player i and a' such that $\hat{a}^2_{i_1} \succeq^{\hat{\theta}_{i_1}}_i a'$ but $a' \succ^{\theta}_i \hat{a}^2_{i_1}$. Let player i deviate to $m'_i = (\hat{\theta}_i, a', a', z')$, where $z' > 0 = \hat{z}_{i_1} \geq \hat{z}_{i_3}$. If the selected senate is some $S \in \mathcal{N}_{\bar{n}}(i)$, then one of the following will hold:

    * $g^S((m'_i, \hat{m}_{-i})^S)$ follows rule (iii), which happens if $i \neq i_3$ and $i_3 \in S$. Then $g^S((m'_i, \hat{m}_{-i})^S) = a'$.
    * $g^S((m'_i, \hat{m}_{-i})^S)$ follows rule (ii), which happens if $i \neq i_3$ and $i_3 \notin S$ or $i = i_3$. Then $g^S((m'_i, \hat{m}_{-i})^S) = a'$ if $S \in \mathbb{S}(i)$ and $g^S((m'_i, \hat{m}_{-i})^S) = g^S(\hat{m}^S) = \hat{a}^2_{i_1}$ if $S \notin \mathbb{S}(i)$.

    Since preference over lotteries are monotone, player i has an incentive to deviate, which is a contradiction.

  - Suppose $\hat{a}^2_{i_1} \succeq^{\hat{\theta}_{i_1}}_{i_3} \hat{a}^2_{i_3}$. Then $\hat{a}^2_{i_3}$ is implemented under all $S \in \mathbb{S}(i_3)$. If $\hat{a}^2_{i_1} = \hat{a}^2_{i_3} \in F(\theta)$, then we are done. If $\hat{a}^2_{i_1} = \hat{a}^2_{i_3} \notin F(\theta)$, then we can obtain a contradiction using Maskin monotonicity as in the previous case. So suppose $\hat{a}^2_{i_1} \neq \hat{a}^2_{i_3}$. We argue that $\{\hat{a}^2_{i_1}, \hat{a}^2_{i_3}\} \in A^*_j(\theta)$ for all $j \neq i_3$. If not, then let player j deviate to m_j that differs from $\hat{m}_j$ only in the second and last components with these being, respectively, a_j^1 (as defined in the first case) and $z_j > 0 = \hat{z}_{i_1} \geq \hat{z}_{i_3}$. If the selected senate is some $S \in \mathcal{N}_{\bar{n}}(j)$, then one of the following will hold:

* $g^S((m_j, \hat{m}_{-j})^S)$ follows rule (iii). Then $g^S((m_j, \hat{m}_{-j})^S) = a_j^1$. This will be the case for all $S \in \mathcal{N}_{\bar{n}}(j)$ such that $i_3 \in S$. Furthermore, there exist $S_1 \in \mathbb{S}(i_3)$ and $S_2 \in \mathcal{N}_{\bar{n}} \setminus \mathbb{S}(i_3)$ such that $\{i_3, j\} \subset S_1$ and $\{i_3, j\} \subset S_2$. This is because $n > \bar{n} \geq 4$. Before the deviation, $g^{S_1}(\hat{m}^{S_1}) = \hat{a}_{i_3}^2$ while $g^{S_2}(\hat{m}^{S_2}) = \hat{a}_{i_1}^2$ with at least one alternative out of these two being strictly worse for j than a_j^1. On the other hand, after the deviation, player j will be able to implement a_j^1 under both senates S_1 and S_2.
* $g^S((m_j, \hat{m}_{-j})^S)$ follows rule (ii). This will be the case for all $S \in \mathcal{N}_{\bar{n}}(j)$ such that $i_3 \notin S$. Hence, $g^S(\hat{m}^S)$ followed rule (i). However, $g^S((m_j, \hat{m}_{-j})^S) = g^S(\hat{m}^S)$ since player j has only changed the second and last components of her message.

On the other hand, if the selected senate is $S \in \mathcal{N}_{\bar{n}} \setminus \mathcal{N}_{\bar{n}}(j)$, then $g^S((m_j, \hat{m}_{-j})^S) = g^S(\hat{m}^S)$. Since preferences over lotteries are monotone, player j will be better-off after the deviation to m_j, a contradiction. But $\{\hat{a}_{i_1}^2, \hat{a}_{i_3}^2\} \in A_j^*(\theta)$ for all $j \neq i_3$ such that $\hat{a}_{i_1}^2 \neq \hat{a}_{i_3}^2$ contradicts top-coincidence condition.

- Suppose there exists $j_3 \neq i_3$ such that $j_3 \in J_3$ but $J_2 = \{i_1\}$. This case can be argued like the previous case.

Therefore, $\hat{m}$ is such that $(\hat{\theta}_i, \hat{a}_i^2, \hat{z}_i) = (\hat{\theta}, \hat{a}, \hat{z})$ for all $i \in N$. Then $i_1 = 1$.

- Suppose either $\hat{z} > 0$ or $\hat{a} \notin F(\hat{\theta})$. Consider any $j \in N$ and let $S(j)$ be any set in $\mathcal{N}_{\bar{n}}(j)$ such that $1 \in S(j)$. Rule (iii) is used and hence, $\hat{a}_1^1$ is implemented under $S(j)$. As before, we can show that $\hat{a}_{i_1}^1 \in \bigcap_{j \in N} A_j^*(\theta)$ for all $j \in N$, which contradicts DTA-N.

- Suppose $\hat{z} = 0$ and $\hat{a} \in F(\hat{\theta})$. Then because of rule (i), alternative $\hat{a}$ is implemented under any $S \in \mathcal{N}_{\bar{n}}$. If $\hat{a} \in F(\theta)$, then we are done. If $\hat{a} \notin F(\theta)$, then there exist player i and a' such that $\hat{a} \succeq_i^{\hat{\theta}} a'$ but $a' \succ_i^{\theta} \hat{a}$. Thanks to rule (ii), player i has an incentive to deviate to $(\hat{\theta}, \hat{a}_i^1, a', 0)$. □

References

Benoît, J-P., Ok, E.A., 2008. Nash implementation without no-veto power. Games and Economic Behavior 64, 51-67.

Bergemann, D., Morris, S., 2008. Ex post implementation. Games and Economic Behavior 63, 527-566.

Bochet, O., 2007. Nash implementation with lottery mechanisms. Social Choice and Welfare 28, 111-125.

Danilov, V., 1992. Implementation via Nash equilibria. Econometrica 60, 43-56.

Jackson, M.O., 1992. Implementation in undominated strategies: A look at bounded mechanisms. Review of Economic Studies 59, 757-775.

Maskin, E., 1999. Nash equilibrium and welfare optimality. Review of Economic Studies 66, 23-38.

Maskin, E., Sjöström, T., 2002. Implementation theory, in: Arrow, K.J., Sen, A.K., Suzumura, K., (Eds.), Handbook of Social Choice and Welfare 1. North-Holland, Amsterdam, pp 237288.

McKelvey, R.D., 1989. Game forms for Nash implementation of general social choice correspondences. Social Choice and Welfare 6, 139-156.

Moore, J., Repullo, R., 1990. Nash implementation: A full characterization. Econometrica 58, 1083-1099.

Palfrey, T.R., Srivastava, S., 1991. Nash implementation using undominated strategies. Econometrica 59, 479-501.

Saijo, T., 1988. Strategy space reduction in Maskin's theorem: Sufficient conditions for Nash implementation. Econometrica 56, 693-700.

Serrano, R., 2004. The Theory of implementation of social choice rules. SIAM Review 46, 377-414.

Sjöström, T., 1991. On the necessary and sufficient conditions for Nash implementation. Social Choice and Welfare 8, 333-340.

Yamato, T., 1992. On Nash implementation of social choice correspondences. Games and Economic Behavior 4, 484-492.