

A Folk Theorem with Private Strategies

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Abstract

In this paper I prove a Folk theorem with T -private communication equilibria with an imperfect monitoring structure that may be public, private, and conditionally dependent or independent. I show that an efficient outcome is approachable as players become patient if every disobedience from efficiency is detectable by some player and some not necessarily efficient action profile. I also show that efficiency is approachable if and only if every profitable deviation from efficiency is uniformly and credibly detectable.

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1 Introduction

This is a very preliminary and incomplete draft on dynamic games. In this draft, I study repeated games with public or private monitoring and mediated communication. Specifically, I look at T -private communication equilibria. I begin with some examples and then I offer three main results. The first result equates the study of T -private equilibria as the discount factor δ tends to one to virtual enforcement in a contracting environment, by extending the idea of [Abreu et al. \(1990\)](#) of delayed information to repeated games without delay—including those with public monitoring. I do this by making failure and success regimes depend on the realization of uncertain behavior, thereby constructing T -private equilibria for any $T \in \mathbb{N}$. An implication is that I can get rid of pairwise identifiability to prove the Folk Theorem when such T -private equilibria are allowed, even in the case of public monitoring.

The second result establishes a “universal” Folk theorem. In this draft I restrict attention to just efficient outcomes, but the results generalize. I show that an efficient outcome can be approximated with T -private equilibria as $\delta \rightarrow 1$ if every disobedience from an efficient correlated strategy of the stage game is detectable by some action profile. By a result of [Lehrer \(1992\)](#), it follows that there is no discontinuity in the efficient equilibrium payoff correspondence with respect to δ at $\delta = 1$. This includes for instance the example of [Radner et al. \(1986\)](#). Thus, with T -private communication equilibria it is possible to restore continuity in their example. Moreover, since the condition only depends on the probability distribution of signals, it holds for all utility functions that lead to a well-behaved feasible, individually rational payoff set.

I show by example (specifically, see [Examples 4 and 5](#)) that there may be a discontinuity when the above condition fails. Then I present the last main result. I derive necessary and sufficient conditions jointly on the payoffs and probabilities in the stage games such that it is possible to approximate an efficient outcome with T -private communication equilibria.

At this point, I conjecture that the techniques developed below extend to general dynamic games. I intend to pursue this work in the next draft.

This paper is based on [Rahman \(2008\)](#), which studies a one-shot contracting problem. The main contribution here is to apply the techniques from that paper to the context of repeated games. Hence, [Theorem 1](#) is this paper’s main result. It shows that the Folk theorem is equivalent to a contracting problem that ignores self-generation.

In this paper I focus on communication equilibria and derive a Folk theorem with minimal observability. I show that as long as every disobedience from efficiency is detectable by someone after some perhaps inefficient action profile, patient players can come close to efficiency. I also show that every profitable deviation from efficiency is uniformly and credibly detectable if and only if efficiency is approachable. The techniques develop an algorithm based on that by [Fudenberg and Levine \(1994\)](#), to compute the equilibrium payoff set for any assumptions on the monitoring technology. I will explore this in more detail in the next draft.

Although some authors eschew mediated communication in dynamic games, others embrace it. In this paper I argue that mediated communication is helpful for several reasons. Firstly, of course, such communication enlarges the set of equilibria. As a result, this makes a Folk theorem possible when otherwise it would not (e.g., one-sided monitoring, as in [Example 2](#)). Without mediated communication, it is often necessary that everyone can (statistically) observe everyone's deviations. See [Fong et al. \(2007\)](#), for example. This seems too much to ask in some applications. Also, there is often an issue with conditionally dependent signals and private monitoring. See, e.g., [Sekiguchi \(1997\)](#), [Yamamoto \(2007\)](#) and [Sugaya \(2010\)](#). On the other hand, this point is moot with a mediator, as [Example 3](#) illustrates. Moreover, the literature often replaces mediated communication with arguably contrived equilibria where players effectively communicate through their actions. The communication equilibria studied here yield one side of a revelation principle for dynamic games.

Some forms of communication have been studied before in repeated games. See [Kandori and Matsushima \(1998\)](#), [Kandori \(2003\)](#), [Obara \(2009\)](#) and [Tomala \(2009\)](#). This paper is strictly more general. For instance, none of the papers above offer a solution to [Example 1](#) below. Mediated communication gives continuity of the equilibrium correspondence with respect to the discount factor δ at $\delta = 1$ a "fighting chance" when otherwise it might not have one (e.g., [Radner et al., 1986](#)). Finally, communication isolates the effect of the future for incentives. In determining when the folk theorem holds, for instance, mediated communication ensures that it is achieved by dynamic incentives: when it fails, it is because dynamic incentives fail, and not a lack of communication. Without a mediator, such a distinction is cannot be made.

In [Section 2](#), I present some motivating examples. In [Section 3](#), I describe the model. In [Section 4](#), I present results from [Rahman \(2008\)](#) that will be used to prove the Folk theorem. Finally, in [Section 5](#) I state and prove the main results of this paper.

2 Examples

In this section I present some motivating examples that illustrate the paper’s main results. I begin with the classic Prisoners’ Dilemma, followed by a variant of it.

2.1 Prisoners’ Dilemma with Public Monitoring

Let us begin with the Prisoner’s Dilemma with public monitoring.

Example 1. There are two players, each of whom may cooperate, C , or defect, D . Stage game payoffs are standard and given by the left bi-matrix below; signal probabilities are on the right:

	C	D
C	2, 2	-1, 3
D	3, -1	0, 0

Payoffs

	C	D
C	$\frac{3}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{3}{4}$
D	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{2}, \frac{1}{2}$

Probabilities

The stage game is repeated infinitely often. At each stage, players take an action and subsequently observe a public signal according to the conditional probabilities above. Players earn the average of discounted payoffs with discount factor $\delta \in [0, 1)$, as usual.

In this example, the set of symmetric perfect public equilibrium payoffs is bounded away from efficiency $(2, 2)$ uniformly in the discount factor δ . This result is standard, see [Green and Porter \(1984\)](#), [Radner et al. \(1986\)](#), and [Abreu et al. \(1990\)](#). One way to see this is with the usual observation of imperfect monitoring. Another way to see this is by noticing that pairwise identifiability fails in a binding way.

One might think that the [Kandori and Obara \(2006\)](#) approach to private strategies might be able to help with this example. However, it is not difficult to see that the [Kandori and Obara \(2006\)](#) approach also fails here, since they require that (i) the probability of “good news” decreases with the number of deviators, and (ii) the decrease is increasing. A more general approach is that suggested by ([Rahman and Obara, 2010](#), Example 1). However, this approach fails here, too, because they also require (i), which does not hold. Indeed, this is the challenge of this instance of the Repeated Prisoners’ Dilemma.

I will now show how cooperation may be sustained even when such monotonicity fails. Specifically, as $\delta \rightarrow 1$, I will construct a sequence of equilibria whose payoffs converge to $(2, 2)$, the symmetric efficient outcome of full cooperation.

Claim 1. *For any $\varepsilon > 0$ there exists $\underline{\delta} < 1$ and an equilibrium of the repeated game above such that each player's payoff lies within ε of full cooperation for every $\delta \geq \underline{\delta}$.*

I will now prove this claim. To do so, I will employ T -private equilibria and extend the approach of Abreu, Milgrom and Pearce (1991) to the current setting of public monitoring. But how can we do this if signals are publicly revealed every period? By delaying the announcement of (recommended or played) actions.¹

At every stage t , the mediator independently asks each player to play C with probability μ and D with probability $1 - \mu$. There are two relevant outcomes to be defined: "success" and "failure." The two outcomes obtain according to the following rule:

	CC	CD	DC	DD
g	success	failure	failure	success
b	failure	success	success	success

Now we can follow Abreu, Milgrom and Pearce's T -period arrangement by delaying the public announcement of behavior. During a T -period block, if a string of T consecutive failures occurs then the players will play (D, D) forever from then on with probability α to be determined. Otherwise, if the a string of consecutive failures did not occur, then the players will each revert to playing the mixed strategy $(\mu, 1 - \mu)$ for another T periods, and keep a tally of the next T failures or successes.

For any player i , let $\varphi_i(a_i, b_i)$ be the probability of failure conditional on a recommendation to play $a_i \in \{C, D\}$ and a decision to play $b_i \in \{C, D\}$. By Bayes' Rule,

$$\varphi_i(C, C) = 1/4, \quad \varphi_i(C, D) = \mu/4 + 1/2, \quad \varphi_i(D, C) = 3\mu/4, \quad \varphi_i(D, D) = \mu/4.$$

Therefore, when a player obeys his recommendations, the probability of failure is at least $\mu/4$ and at most $1/4$. Moreover, for any recommendation-contingent deviation plan, as long as μ is sufficiently close to 1, the probability of failure is at least $1/2$. Hence, if a player deviates from his recommended actions during any τ of the T periods, the likelihood of punishment is bounded below by

$$(1/2)^\tau (\mu/4)^{T-\tau} \alpha.$$

¹For now, I will focus on recommended actions, but by Rahman (2009, Theorem 10) it is possible to do away with the mediator. I will discuss this further later.

On the other hand, without deviations it is bounded above by $(1/4)^T \alpha$. Of course, after τ deviations, the utility gain is bounded above by $(1 - \delta)\tau\mu^\tau$.

To compute the cost, let v be the average discounted payoff from the game. The cost of τ deviations is at least

$$[(1/2)^\tau(\mu/4)^{T-\tau} - (1/4)^T]\alpha v.$$

Notice that $[(1/2)^\tau(\mu/4)^{T-\tau} - (1/4)^T] = (1/4)^T \tau [\mu^{T/\tau}(2/\mu) - 1] \geq (1/4)^T \tau [2\mu^{T-1} - 1]$.²

If we could choose α such that

$$1 - \delta = \delta^T (1/4)^T [2\mu^{T-1} - 1] \alpha v$$

then we could discourage any τ deviations. To see this, notice that

$$(1 - \delta)\mu^\tau \leq (1 - \delta)\tau = \tau \delta^T (1/4)^T [2\mu^{T-1} - 1] \alpha v \leq \tau (1/4)^T [2\mu^{T-1} - 1] \alpha v,$$

so τ deviations are indeed deterred as long as $\mu^{T-1} > 1/2$.

Let us now see what is achievable. Fix μ and T such that $\mu^{T-1} > 1/2$ and consider the following problem, to be explained in detail below:

$$\begin{aligned} V_{T,\mu}^*(\delta) &= \max_w v \text{ s.t.} \\ v &= (1 - \delta^T)2\mu + \delta^T[(1 - p)v + p(v - w)] && \text{(value recursion)} \\ 1 - \delta &\leq \delta^T (1/4)^T [2\mu^{T-1} - 1]w && \text{(incentive compatibility)} \\ 0 &\leq w \leq v, && \text{(self-generation)} \end{aligned}$$

where $p = \Pr(T \text{ failures}) = [\mu(1 + \mu)/4]^T$. The problem above maximizes the game's symmetric value subject to (i) value recursion, so the payoff is generated by the trigger strategy described above by interpreting w and αv ,³ (ii) incentive compatibility, since the previous argument showed that this constraint suffices to discourage all deviations, and (iii) self-generation, so that promised utility can be delivered with future play.

Clearly, the incentive constraint must bind at the optimum of this problem, hence

$$w = \frac{1 - \delta}{\delta^T (1/4)^T [2\mu^{T-1} - 1]}.$$

²This follows from $\mu/\lambda > 1 \Rightarrow (\mu/\lambda)^\tau - 1 \geq \tau[(\mu/\lambda) - 1]$.

³Notice that $2\mu = 2\mu^2 + (3 - 1)\mu(1 - \mu)$ is the flow payoff from the mixed strategies $(\mu, 1 - \mu)$.

Substituting this into the value recursion equation, it follows that

$$V_{T,\mu}^*(\delta) = 2\mu - \frac{1-\delta}{1-\delta^T} \frac{[\mu(1+\mu)]^T}{2\mu^{T-1}-1}.$$

If μ is sufficiently close to 1, $\mu(1+\mu) > 1$, therefore

$$V_{T,\mu}^*(\delta) \geq 2\mu - \frac{\delta}{\delta + \dots + \delta^T} \frac{2}{2\mu^{T-1}-1} \rightarrow 2\mu - \frac{2}{T[2\mu^{T-1}-1]} \quad \text{as } \delta \rightarrow 1.$$

Fix any small $\varepsilon \in (0, 1)$. For any $T \in \mathbb{N}$ there clearly exists $\mu_T < 1$ such that $2(\mu_T)^{T-1} \geq 1 + \varepsilon$, and of course $\mu_T \rightarrow 1$ as $T \rightarrow \infty$. If $V_T^*(\delta) = V_{T,\mu_T}^*(\delta)$ then clearly

$$\lim_{\delta \rightarrow 1} V_T^*(\delta) = 2\mu_T - \frac{2}{T[2(\mu_T)^{T-1}-1]} \geq 2\mu_T - \frac{2}{T\varepsilon} \rightarrow 2 \quad \text{as } T \rightarrow \infty.$$

Hence, the T -private equilibria above sustain virtual cooperation, proving the claim.

2.2 Prisoners' Dilemma with Private Monitoring

Now consider the Prisoners' Dilemma with private monitoring.

Example 2 (*One sided monitoring*). Just as in [Example 1](#) above, there are two players, each of whom may cooperate, C , or defect, D . Stage game payoffs are standard and given by the left bi-matrix below; signal probabilities are on the right:

	C	D
C	2, 2	-1, 3
D	3, -1	0, 0

Payoffs

	C	D
C	$\frac{3}{4}, \frac{1}{4}$	$\frac{1}{2}, \frac{1}{2}$
D	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$

Probabilities

The difference here is that now the signal is only observed by player 1. Player 2 observes nothing (apart from his own actions and recommendations). Also notice that the probabilities are slightly different. Player 1 playing D leads to an uninformative signal, but playing C is informative of player 2's action.

This example is interesting for several reasons. Firstly, one sided monitoring is completely excluded by the literature on repeated games. Secondly, as will be seen from the general model, every deviation from the efficient outcome is detectable, so it can be approximated by equilibria of the repeated game. However, for the same reasons as for [Example 1](#), T -private equilibria are needed to do so.

Example 3 (*Two sided monitoring*). Just as in [Example 1](#) above, there are two players, each of whom may cooperate, C , or defect, D . Stage game payoffs are standard and given by the left bi-matrix below; private signal *marginal* probabilities are on the middle for player 1 and on the right for player 2:

	C	D
C	2, 2	-1, 3
D	3, -1	0, 0
	Payoffs	

	C	D
C	$\frac{3}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{3}{4}$
D	$\frac{3}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{3}{4}$
	Player 1's Probs	

	C	D
C	$\frac{3}{4}, \frac{1}{4}$	$\frac{3}{4}, \frac{1}{4}$
D	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{4}, \frac{3}{4}$
	Player 2's Probs	

Of course, I have not specified how players' signals are correlated. But regardless of how correlated they are, it is still true that every disobedience from full cooperation is detectable, hence efficiency is attainable for sufficiently patient players.

2.3 Efficiency and Observability

The next examples show the limits of detectability when trying to sustain efficient behavior. On the one hand, Radner, Myerson and Maskin showed that there may be a discontinuity between what is attainable as $\delta \rightarrow 1$ and when $\delta = 1$, in some sense. [Example 1](#) suggests that this is not the case. Indeed, as long as every deviation is detectable, the example suggests that there is a sequence of (perhaps private) equilibria whose payoffs converge to efficiency, which Radner (1986) shows is attainable when $\delta = 1$. On the other hand, Lehrer (1992) shows that, when $\delta = 1$, an efficient outcome is attainable essentially if and only if every profitable deviation from the efficient outcome is detectable by some not necessarily efficient action profile.

The examples below show that this condition does not characterize attainability as $\delta \rightarrow 1$. In each of the examples below, the efficient outcome (D, C) cannot be approximated as $\delta \rightarrow 1$, yet every profitable deviation from that outcome is detectable by some action profile, not necessarily the efficient action profile.

Example 4 (*Detecting profitable deviations*). There are two players. Stage game payoffs appear in the left bi-matrix below; public signal probabilities are on the right:

	C	D	E
C	4, 2	1, 4	1, 5
D	4, 3	0, 4	0, 3
	Payoffs		

	C	D	E
C	$\frac{3}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{3}{4}, \frac{1}{4}$
D	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$
	Probabilities		

In this example there exists no equilibrium such that the efficient outcome (D, C) is attained even virtually as $\delta \rightarrow 1$. To see this, first notice that the profile (D, C) cannot be played with probability one. Otherwise, player 2 can profitably deviate to play D instead of C without being detected. Furthermore, the profile (D, C) cannot be played with positive probability. Otherwise, if player 1 plays C with positive probability then player 2 can play E which weakly dominates C and is also completely undetectable. On the other hand, for a deviation from (D, C) to be profitable, it must be the case that player 2 plays D with positive probability, but this is detectable when player 1 plays C . Therefore, although it is possible to sustain the payoff profile $(4, 3)$ in equilibrium when $\delta = 1$ in the sense of Lehrer (1992), it is impossible to do so as $\delta \rightarrow 1$.

For future reference, the same problem occurs even if we replace the probabilities given (C, D) with $(\frac{5}{6}, \frac{1}{6})$ instead. With these new probabilities, detection implies attribution in the sense of Rahman and Obara (2010). This is because deviations from (C, C) shift probability in different directions. As will be argued later, the set of unattributable deviations is important for understanding the set of payoffs attainable in a repeated game.

The same problem occurs in the next example, although for a slightly different reason.

Example 5 (*Dominated detection*). There are two players. Stage game payoffs appear in the left bi-matrix below; public signal probabilities are on the right:

	C	D	E
C	4, 2	1, 4	1, 5
D	4, 3	0, 4	0, 3
E	3, 2	-1, 4	-1, 3

Payoffs

	C	D	E
C	$\frac{3}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{3}{4}, \frac{1}{4}$
D	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, \frac{1}{2}$
E	$\frac{3}{4}, \frac{1}{4}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{3}{4}, \frac{1}{4}$

Probabilities

Again, although efficiency can be sustained when $\delta = 1$, it cannot be sustained as $\delta \rightarrow 1$. To see this, notice that although if player 1 plays E then the argument of the previous example fails, playing E is strictly dominated by C for player 1, and again is completely undetectable. Hence there is no way of getting player 1 to play E , and as a result (D, C) is not attainable.

These two examples reflect all that can go wrong in trying to sustain efficiency in a repeated game, as I argue later when I develop the general model.

3 Model

Consider a repeated game with private monitoring. The stage game consists of a finite set $I = \{1, \dots, n\}$ of players, a finite set A_i of actions for each player $i \in I$, where $A = \prod_i A_i$, utility profile $u : I \times A \rightarrow \mathbb{R}$, where $u_i(a)$ is the utility to player i from action profile a , and a finite set of signals S_i . Let $S = \prod_i S_i$ and $\Pr(s|a)$ be the probability that the profile s of signals is observed if the profile a of actions was played. Players have a common discount factor δ , where $\delta \in [0, 1)$. Given a sequence $a^\infty = (a_1, a_2, \dots)$ of action profiles, the utility to player i is given by

$$U_i(a^\infty) = [(1 - \delta)/\delta] \sum_{t=1}^{\infty} \delta^t u_i(a_t).$$

Let $U = \text{conv} \{u(a) \in \mathbb{R}^I : a \in A\}$ be the convex hull of possible payoff vectors in the stage game. I make the following relatively standard assumptions.

Assumption 1 (*Full support*). $\Pr(s|a) > 0$ for all (a, s) .

This assumption means that every signal is possible, regardless of what players play. It is largely made for simplicity. This way “the entire tree lights up,” in other words, nothing is off the path of play.

Assumption 2 (*Minimax*). $u_i(a) \leq 1$ for all (i, a) and

$$\min_{\mu \in \Delta(A)} \max_{\beta_i: A_i \rightarrow A_i} \sum_{a \in A} u_i(\beta_i(a_i), a_{-i}) \mu(a) = 0 \quad \forall i \in I.$$

The first part is just a normalization that will save some notation, but is otherwise without loss of generality. The second part says that the (correlated) minimax value for each player is normalized to 0. Let $U^* = \{u \in U : u \geq \mathbf{0}\}$ be the set of *feasible, individually rational payoffs*.

Assumption 3 (*Full dimensionality*). $\dim U^* = n$.

This is a technical way of saying that it is possible to punish and reward agents independently. It implies that we can use a version of the algorithm by [Fudenberg and Levine \(1994\)](#) to characterize equilibrium payoffs. Since we will be generalizing their algorithm, let us recall it formally next; then I will introduce my extension.

3.1 Public Sequential Equilibria

[To be completed.]

3.2 Public Communication Equilibria

[To be completed. In this draft, I will only study the algorithm for maximizing the unweighted sum of utilities.]

First, let us turn the repeated game into a contracting problem. For any player i , let $\rho_i : S_i \rightarrow S_i$ be a reporting strategy, where $\rho_i(s_i)$ is the planned report upon observing s_i . Let $\Delta u_i(a, b_i) = u_i(a_{-i}, b_i) - u_i(a)$ and $\Delta \Pr(s|a, b_i, \rho_i) = \Pr(s|a_{-i}, b_i, \rho_i) - \Pr(s|a)$, where

$$\Pr(s|a_{-i}, b_i, \rho_i) = \sum_{t_i \in \rho_i^{-1}(s_i)} \Pr(s_{-i}, t_i|a_{-i}, b_i)$$

is the probability that the signal profile s is *reported* if (a_{-i}, b_i) was played.

Consider the following *nonlinear* programming problem:

$$\begin{aligned} W &= \max_{\mu \geq 0, v, w} \sum_{i=1}^n v_i & \text{s.t.} & \sum_{a \in A} \mu(a) = 1, \\ v_i &= (1 - \delta) \sum_{a \in A} \mu(a) u_i(a) + \delta \sum_{(a, s)} \Pr(s|a) \mu(a) w_i(a, s) & \forall i \in I, \\ (1 - \delta) \sum_{a_{-i}} \mu(a) \Delta u_i(a, b_i) + \delta \sum_{(a_{-i}, s)} \mu(a) w_i(a, s) \Delta \Pr(s|a, b_i, \rho_i) &\leq 0 & \forall (i, a_i, b_i, \rho_i), \\ \sum_{i=1}^n w_i(a, s) \mu(a) &\leq \sum_{i=1}^n v_i \mu(a) & \forall (a, s). \end{aligned}$$

This problem is reminiscent of the equilibrium-characterizing problem of [Fudenberg and Levine \(1994\)](#), but there are some important differences. Regarding the similarities, I am maximizing the sum of lifetime utilities by choosing continuation values subject to incentive compatibility (the penultimate family of constraints) and self-generation (the last family). Applying different welfare weights to the objective and the self-generation constraint characterizes the set of public equilibria.

As for the differences, there are several. Firstly, the continuation values depend not only on (reported) signals, but also on recommended actions. Secondly, the problem chooses a correlated strategy subject to it being a *communication equilibrium*, i.e., honesty and obedience is incentive compatible.

I will now transform the problem in several ways to obtain a linear programming problem that will become my focus. I will do so in two steps. In the first step, I transform the problem from one of choosing transfers $w_i(a, s)$ to one of choosing

probability-weighted transfers $x_i(a, s)$.

$$\begin{aligned}
W &= \max_{\mu \geq 0, v, x} \sum_{i=1}^n v_i & \text{s.t.} & \quad \sum_{a \in A} \mu(a) = 1, \\
v_i &= (1 - \delta) \sum_{a \in A} \mu(a) u_i(a) + \delta \sum_{(a, s)} \Pr(s|a) x_i(a, s) & \forall i \in I, \\
(1 - \delta) \sum_{a_{-i}} \mu(a) \Delta u_i(a, b_i) + \delta \sum_{(a_{-i}, s)} x_i(a, s) \Delta \Pr(s|a, b_i, \rho_i) &\leq 0 & \forall (i, a_i, b_i, \rho_i), \\
\sum_{i=1}^n x_i(a, s) &\leq \sum_{i=1}^n v_i \mu(a) & \forall (a, s), \\
0 &\leq x_i(a, s) \leq \mu(a) & \forall (i, a, s).
\end{aligned}$$

The problem has changed in two ways. First, every entry of $w_i(a, s)\mu(a)$ has been replaced with $x_i(a, s)$, to be interpreted as a probability-weighted continuation value. Next, a new family of constraints (the last one) has been added to the problem. This family is there to ensure that the probability-weighted payments are *adapted* to μ . In other words, in order to avoid solutions to the problem with $x_i(a, s) \neq 0$ and $\mu(a) = 0$. If this were the case then the interpretation of probability-weighted transfers would not apply. The reason that the constraints capture this idea is due to [Assumption 2](#), which requires that continuation values $w_i(a, s)$ belong to $[0, 1]$. Therefore, probability-weighted continuation values $x_i(a, s)$ must belong to $[0, \mu(a)]$.

I now make the following change of variables. Let

$$\xi_i(a, s) = \frac{\delta}{1 - \delta} [v_i \mu(a) - x_i(a, s)].$$

Simple manipulations yield the following *linear* programming problem.

$$\begin{aligned}
W &= \max_{\mu \geq 0, \xi} \sum_{i=1}^n \sum_{a \in A} \mu(a) u_i(a) - \sum_{(a, s)} \Pr(s|a) \xi_i(a, s) & \text{s.t.} & \quad \sum_{a \in A} \mu(a) = 1, \\
\sum_{a_{-i}} \mu(a) \Delta u_i(a, b_i) &\leq \sum_{(a_{-i}, s)} \xi_i(a, s) \Delta \Pr(s|a, b_i, \rho_i) & \forall (i, a_i, b_i, \rho_i), \\
\sum_{i=1}^n \xi_i(a, s) &\geq 0 & \forall (a, s), \\
-\frac{\delta}{1 - \delta} \mu(a) &\leq \xi_i(a, s) \leq \frac{\delta}{1 - \delta} \mu(a) & \forall (i, a, s).
\end{aligned}$$

The only part of the transformation that is not immediate is the way that the last family of constraints changed. To obtain it, just use the fact that $v_i \in [0, 1]$, hence

$$-\frac{\delta}{1 - \delta} \mu(a) \leq \frac{\delta}{1 - \delta} [0 - x_i(a, s)] \leq \xi_i(a, s) \leq \frac{\delta}{1 - \delta} [v_i \mu(a) - 0] \leq \frac{\delta}{1 - \delta} \mu(a).$$

This last problem is clearly equivalent to the original one. (Although the adaptability constraints have changed slightly in the last transformation, this is without loss, as the only role they fill is in ensuring that no payments take place after zero-probability recommendations. Hence the problems are equivalent in terms of the limit as $\delta \rightarrow 1$.)

Definition 1. Call the linear program above the *primal*, and call its dual problem (you guessed it) the *dual*.

Notice how the change of variables has isolated the effect of δ to just the last family of constraints, where a higher δ means that there is a wider range of possible values for $\xi_i(a, s)$. As $\delta \rightarrow 1$, the constraints relax. This is also seen clearly from the dual, which I present next. Deriving the dual is long but straightforward, so I omit its derivation. Since the primal has a feasible solution (namely, a correlated equilibrium of the stage game together with 0 payments), the value of the primal equals the value of the dual. Here's the dual.

$$\begin{aligned}
W &= \min_{\lambda, \eta \geq 0, \kappa} \kappa \quad \text{s.t.} \\
\kappa &\geq \sum_{i=1}^n u_i(a) - \sum_{(b_i, \rho_i)} \lambda_i(a_i, b_i, \rho_i) \Delta u_i(a, b_i) \\
&+ \frac{\delta}{1-\delta} \sum_{s \in S} |\Pr(s|a) - \lambda_i(a_i, b_i, \rho_i) \Delta \Pr(s|a, b_i, \rho_i) - \eta(a, s)| \quad \forall a \in A.
\end{aligned}$$

Intuitively, think of λ_i as a deviation plan for player i . (See [Rahman \(2008\)](#).) The multiplier η is studied in some detail by [Rahman and Obara \(2010\)](#).

From the dual we can easily see that [Example 1](#) cannot have an approximately efficient public communication equilibrium, even as $\delta \rightarrow 1$. Consider the profile of deviation plans given by $\lambda_i(C, D) = 1$ and $\lambda_i(D, C) = 1/2$ for each player i . It is easy to see that there exists $\eta \geq 0$ with $\|\Pr(s|a) - \lambda_i(a_i, b_i, \rho_i) \Delta \Pr(s|a, b_i, \rho_i) - \eta(a, s)\| = 0$. Notice, furthermore, that for each a $\sum_i u_i(a) - \sum \lambda_i(a_i, b_i) \Delta u_i(a, b_i)$ is strictly less than 2. Therefore $W < 2$ regardless of δ . By strong duality, the result follows.

As $\delta \rightarrow 1$, it is tempting to think that the dual above converges to

$$\begin{aligned}
\min_{\lambda, \eta \geq 0, \kappa} \kappa \quad \text{s.t.} \quad \kappa &\geq \sum_{i=1}^n u_i(a) - \sum_{(b_i, \rho_i)} \lambda_i(a_i, b_i, \rho_i) \Delta u_i(a, b_i) \quad \forall a \in A, \\
\Pr(s|a) - \lambda_i(a_i, b_i, \rho_i) \Delta \Pr(s|a, b_i, \rho_i) &= \eta(a, s) \quad \forall (a, s),
\end{aligned}$$

but this would be wrong, as is easily seen from [Example 4](#). The difference between these two problems distinguishes the case $\delta \rightarrow 1$ from $\delta = 1$.

3.3 T -Private Communication Equilibria

I will now extend the argument of [Abreu et al. \(1990\)](#) to this general setting, as in [Example 1](#). Consider the repeated game whose stage game consists of T repetitions of the original stage game. We will use the duality above in this game; the primal associated with this reinterpreted repeated game is called the T -primal, and its dual the T -dual.

Fix a T -period stage game, where $T \in \mathbb{N}$. During every period $t \in \{1, \dots, T\}$, player i is confidentially asked by a mediator to choose $a_{it} \in A_i$ and decides $b_{it} \in A_i$, which may differ from A_i , then privately observes some signal $s_{it} \in S_i$, and submits a confidential report $r_{it} \in S_i$ to the mediator which may also differ from s_{it} . Let $H_i = A_i \times S_i$ be the set of all action-signal pairs, with typical elements (a_{it}, r_{it}) , (b_{it}, s_{it}) , etc. For now, assume that everyone is honest and obedient, so $b_{it} = a_{it}$ and $r_{it} = s_{it}$ for all (i, t) .⁴ Let $U_i(a^T) = [(1 - \delta)/\delta(1 - \delta^T)] \sum_{t=1}^T \delta^t u_i(a_t)$ be the utility accrued during the current T -period stage. Let $\Pr(s^T|a^T) = \prod_{t=1}^T \Pr(s_t|a_t)$.

A T -deviation for player i is any sequence of maps $(\beta_i, \rho_i) = (\beta_{i1}, \rho_{i1}, \dots, \beta_{iT}, \rho_{iT})$ such that $\beta_{it} : A_i^t \times S_i^{t-1} \rightarrow A_i$ and $\rho_{it} : A_i^t \times S_i^{t-1} \rightarrow S_i^t$, where β_{it} corresponds to possible disobedience contingent on past recommendations and observations, and ρ_{it} to possible dishonesty. Write $U_i(a^T|\beta_i) = [(1 - \delta)/\delta(1 - \delta^T)] \sum_{t=1}^T \delta^t u_i(a_{-it}, \beta_{it}(a_i^t, s_i^{t-1}))$ for the utility from disobeying according to β_i . Also, let

$$\Pr(s^T|a^T, \beta_i, \rho_i) = \prod_{t=1}^T \Pr(s_t|a_t, \beta_{it}(a_i^t, s_i^{t-1}), \rho_{it}(a_i^t, s_i^{t-1})),$$

where

$$\Pr(s_t|a_t, \beta_{it}(a_i^t, s_i^{t-1}), \rho_{it}(a_i^t, s_i^{t-1})) = \sum_{\tau_{it} \in \rho_{it}^{-1}(s_{it}|a_i^t, s_i^{t-1})} \Pr(s_{-it}, \tau_{it}|a_{-it}, \beta_{it}(a_i^t, s_i^{t-1}))$$

stands for the probability that s_t is reported.

A *communication mechanism* is any μ such that $\sum_{a_1} \mu(a_1) = 1$ and

$$\sum_{a_{t+1}} \mu(a^{t+1}, s^t) = \mu(a^t, s^{t-1}) \Pr(s_t|a_t) \quad \forall (a^t, s^t).$$

Given any such communication mechanism μ , let

$$\mu(a^t, s^{t-1}|\beta_i, \rho_i) = \mu(a^t, s^{t-1}) \frac{\Pr(s^t|a^t, \beta_i(a_i^t, s_i^{t-1}), \rho_{it}(a_i^t, s_i^{t-1}))}{\Pr(s^t|a^t)}.$$

⁴For any family $\{X_{it}\}$ of sets indexed by i and t , let $X_{i0} = \{\emptyset\}$, $X_t = \prod_i X_{it}$, $X_{-it} = \prod_{j \neq i} X_{jt}$, $X_i^t = \prod_{\tau=1}^t X_{i\tau}$, $X^t = \prod_i X_i^t$ and $X = \prod_{i,t} X_{it}$. Thus, S is the space of all signal profile histories.

Let $U_i(\mu) = \sum_{(a^T, s^{T-1})} U_i(a^T) \mu(a^T, s^{T-1})$ denote player i 's utility from the communication mechanism played at the current T -period stage when he is honest and obedient, and $U_i(\mu|\beta_i, \rho_i) = \sum_{(a^T, s^{T-1})} U_i(a^T|\beta_i) \mu(a^T, s^{T-1}|\beta_i, \rho_i)$ the utility when he deviates according to (β_i, ρ_i) . Write $\Delta U_i(\mu|\beta_i, \rho_i) = U_i(\mu|\beta_i, \rho_i) - U_i(\mu)$, and $\Delta \Pr_i(s^T|a^T, \beta_i, \rho_i) = \Pr(s^T|a^T, \beta_i, \rho_i) - \Pr(s^T|s^T)$.

Using the same transformations as for the case when $T = 1$ in the previous subsection, we end up with the following reduced-form T -primal:

$$\begin{aligned}
W_T(\delta) = \max_{\mu \geq 0, \xi} & \sum_{i=1}^n U_i(\mu) - \sum_{(a^T, s^T)} \Pr(s^T|a^T) \xi_i(a^T, s^T) \quad \text{s.t.} \quad \sum_{a_1} \mu(a_1) = 1, \\
& \sum_{a_{t+1}} \mu(a^{t+1}, s^t) = \mu(a^t, s^{t-1}) \Pr(s_t|a_t) \quad \forall (a^t, s^t), \\
& \sum_{a_{-i}} \mu(a) \Delta U_i(\mu|\beta_i, \rho_i) \leq \sum_{(a^T, s^T)} \xi_i(a^T, s^T) \Delta \Pr(s^T|a^T, \beta_i, \rho_i) \quad \forall (i, \beta_i, \rho_i), \\
& \sum_{i=1}^n \xi_i(a^T, s^T) \geq 0 \quad \forall (a, s), \\
-\frac{\delta^T}{1-\delta^T} \mu(a^T, s^{T-1}) & \leq \xi_i(a^T, s^T) \Pr(s^{T-1}|a^{T-1}) \leq \frac{\delta^T}{1-\delta^T} \mu(a^T, s^{T-1}) \quad \forall (i, a^T, s^T).
\end{aligned}$$

After much manipulation, the T -dual simplifies to the following problem. Denote by $\mathbf{a}_T : \bigcup_{t=1}^T S^{t-1} \rightarrow A$ any pure recommendation strategy for the mediator conditional on past reports, and let \mathcal{A}_T be the set of all such pure mediation strategies.

$$\begin{aligned}
W_T(\delta) = \min_{\lambda, \eta \geq 0, \kappa} & \kappa \quad \text{s.t.} \\
\kappa \geq & \sum_{i=1}^n \sum_{s^{T-1}} U_i(\mathbf{a}_T(s^{T-1})) - \sum_{(\beta_i, \rho_i)} \lambda_i(\beta_i, \rho_i) \Delta U_i(\mathbf{a}_T(s^{T-1})|\beta_i, \rho_i) + \\
\frac{\delta^T}{1-\delta^T} \sum_{s_t} & |\Pr(s^T|a^T) - \sum_{(\beta_i, \rho_i)} \lambda_i(\beta_i, \rho_i) \Delta \Pr(s^T|\mathbf{a}_T(s^{T-1}), \beta_i, \rho_i) - \eta(a^T, s^T)| \quad \forall \mathbf{a}_T \in \mathcal{A}_T.
\end{aligned}$$

At this point let me make three comments. First, the T -dual is structurally similar to the 1-dual from before. Secondly, the mediator's strategies in the T -period stage may vary with observed reports. Finally, an application of Kuhn's Theorem shows that we may rewrite the T -dual in terms of behavior strategies (i.e., with multipliers $\lambda_{it}(b_i^t, r_i^t|a_i^t, s_i^t)$ such that λ_{it} extends λ_{is} for $s < t$) rather than mixed strategies (i.e., λ_i defined as a mixture over all possible deviations).

4 Digression

The main goal of this paper is to establish a Folk theorem under minimal observability assumptions. First, though, let me digress and present some preliminary results taken from Rahman (2008). Consider a one-shot contracting problem, as in the 1-private equilibria considered above, but this time ignoring the self-generation constraint.

Let $\mu \in \Delta(A)$ be a correlated strategy and $\zeta : I \times A \times S \rightarrow \mathbb{R}$ be a payment scheme contingent on recommendations and reports for each player. A pair (μ, ζ) is called *incentive compatible* if honesty and obedience is optimal:

$$\sum_{a_{-i}} \mu(a) \Delta u_i(a, b_i) \leq \sum_{(a_{-i}, s)} \mu(a) \zeta_i(a, s) \Delta \Pr(s|a, b_i, \rho_i) \quad \forall (i, a_i, b_i, \rho_i).$$

In other words, (μ, ζ) is incentive compatible if μ is a *communication equilibrium* (Myerson, 1986; Forges, 1986) of the game induced by ζ .

Definition 2. A correlated strategy μ is *enforceable* if a payment scheme ζ exists such that (μ, ζ) is incentive compatible. Let E denote the set of enforceable correlated strategies. Call μ *virtually enforceable* if there exists a sequence $\{\mu^m\} \subset E$ such that $\mu^m \rightarrow \mu$. Let \bar{E} denote the set of virtually enforceable correlated strategies.

A *strategy* for agent i is a map $\sigma_i : A_i \rightarrow \Delta(A_i \times R_i)$, where $\sigma_i(b_i, \rho_i|a_i)$ is the probability that i plays (b_i, ρ_i) when recommended a_i . Call σ_i a *deviation* if it ever differs from honesty and obedience. The deviation σ_i is called *μ -profitable* if

$$\sum_{(a, b_i, \rho_i)} \mu(a) \Delta u_i(a, b_i) \sigma_i(b_i, \rho_i|a_i) > 0.$$

Let $\Pr(\mu)$ be the vector of report probabilities if everyone is honest and obedient, defined by $\Pr(s|\mu) = \sum_a \mu(a) \Pr(s|a)$ for each s . Let $\Pr(\mu, \sigma_i)$ be the vector of report probabilities if everyone is honest and obedient except for i , who plays σ_i instead, defined for each signal profile s by

$$\Pr(s|\mu, \sigma_i) = \sum_{a \in A} \mu(a) \sum_{(b_i, \rho_i)} \Pr(s|a_{-i}, b_i, \rho_i) \sigma_i(b_i, \rho_i|a_i).$$

Definition 3. For any $B \subset A$, σ_i is called *B -detectable* if $\Pr(s|a) \neq \Pr(s|a, \sigma_i)$ for some $a \in B$ and $s \in S$.⁵ Otherwise, σ_i is called *B -undetectable*. A strategy is simply *detectable* if it is A -detectable, and *undetectable* if it is A -undetectable.

⁵We abuse notation by identifying Dirac measure $[a] \in \Delta(A)$ with the action profile $a \in A$.

Intuitively, a strategy is detectable if there is a recommendation profile such that the report probabilities induced by the strategy differ from that induced by honesty and obedience, assuming that everyone else is honest and obedient. Crucially, different action profiles may be used to detect different strategies. To illustrate this difference, consider the next example, where, for any mixed (or correlated) strategy profile, individual full rank fails, yet every deviation is detectable.

Example 6. Two publicly verifiable signals and two players. Player 1 has two choices, $\{U, D\}$, and player 2 has three, $\{L, M, R\}$. Payoffs and probabilities are given in the bi-matrices below, as usual.

	L	M	R
U	1, 0	0, 1	0, 1
D	3, 2	1, 3	1, 3

Payoffs

	L	M	R
U	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{3}{4}, \frac{1}{4}$
D	$\frac{1}{3}, \frac{2}{3}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{3}{4}, \frac{1}{4}$

Probabilities

Since every deviation is detectable, by the main result of this paper there is a sequence of equilibria of the repeated game whose payoffs converge to the efficient level $(3, 2)$ as $\delta \rightarrow 1$. However, it is necessary to resort to T -private equilibria with recommendation-contingent continuation values.

The first result characterizes enforceability. For proofs, see [Rahman \(2008\)](#).

Lemma 1. *A correlated strategy μ is enforceable if and only if every μ -profitable deviation is supp μ -detectable.*

As this result shows, profitable deviations are important. These are necessarily disobediences. Formally, given $B \subset A$, a strategy σ_i is called B -disobedient if $\sigma_i(b_i, \rho_i | a_i) > 0$ for some $a_i \in B_i$ and $b_i \neq a_i$, where $B_i = \{b_i \in A_i : \exists b_{-i} \text{ s.t. } b \in B\}$ is the projection of B on A_i . Next, I pin down virtual enforcement with probabilities.

Lemma 2. *Fix any $\mu \in \Delta(A)$. Every supp μ -disobedience is detectable if and only if for any profile of utility functions, μ is virtually enforceable.*

This result characterizes when an action profile is virtually enforceable for every utility profile. To illustrate the characterizing condition, consider the following example.

Example 7. Two players, two public signals. Payoffs and probabilities appear below.

	L	M	R
U	3, 2	0, 3	0, 3
D	4, 0	0, 1	0, 1

Payoffs

	L	M	R
U	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{4}, \frac{3}{4}$
D	$\frac{1}{4}, \frac{3}{4}$	$\frac{3}{4}, \frac{1}{4}$	$\frac{3}{4}, \frac{1}{4}$

Probabilities

The efficient outcome here is (U, L) . Since every (U, L) -disobedience is detectable in this example, by [Lemma 2](#) it is virtually enforceable, therefore by the main result of this paper a Folk theorem applies, i.e., there is a sequence of equilibria of the repeated game whose payoffs converge to the efficient level $(3, 2)$ as $\delta \rightarrow 1$.

[Lemma 2](#) is a key result that will be used for the Folk theorem. It shows that a is virtually enforceable for every utility profile as long as every disobedience from a is detectable with some perhaps occasional behavior—call it “monitoring.” Crucially, there is no requirement on disobediences to behavior outside of a , so deviations from monitoring need not be detectable.

I now characterize virtual enforcement for a fixed utility profile. Although the previous lemma looks simple, a similar result for a fixed utility profile is not as simple. To see why, notice that, on the one hand, virtually enforcing some a does not require that every a -disobedience be detectable: strictly unprofitable a -disobediences may be undetectable, for instance. On the other hand, it is not enough that every strictly profitable a -disobedience be detectable, as [Example 4](#) shows. There, every strictly profitable (D, C) -disobedience is detectable, yet (D, C) is not virtually enforceable.

Detecting (D, C) -profitable deviations is not enough there because E weakly dominates C and is indistinguishable from it. Indeed, if E strictly dominated C then there would exist a (D, C) -profitable, undetectable strategy, rendering (D, C) virtually unenforceable. On the other hand, if player 1’s payoff from (D, E) was negative instead of zero then (D, C) would be virtually enforceable because playing E when asked to play C would be unprofitable if player 1 played C with low probability.

So what *is* required beyond detecting profitable deviations? Below, I will argue that profitable deviations must be *uniformly and credibly* detectable. To illustrate, note that if E is removed from [Example 4](#) then (D, C) is virtually enforceable, not just because every (D, C) -profitable deviation is detectable (this is true with or without *solitaire*), but also because the utility gains from every (D, C) -profitable deviation can be uniformly outweighed by monetary losses. To describe this formally, let us introduce some notation. For any strategy σ_i and any correlated strategy μ , write

$$\|\Delta \Pr(\mu, \sigma_i)\| = \sum_{s \in S} \left| \sum_{(a, b_i, \rho_i)} \mu(a) [\sigma_i(b_i, \rho_i | a_i) \Pr(s | a_{-i}, b_i, \rho_i) - \Pr(s | a)] \right|.$$

Intuitively, this norm describes the statistical difference between abiding by μ and deviating to σ_i . Thus, σ_i is *supp μ -undetectable* if and only if $\|\Delta \Pr(\mu, \sigma_i)\| = 0$.

Say that *every μ -profitable deviation is uniformly detectable* if $z \geq 0$ exists such that for every μ -profitable deviation σ_i there exists $\eta \in \Delta(A)$ (possibly different for different σ_i) such that σ_i is $\text{supp } \eta$ -detectable and $\Delta v_i(\eta, \sigma_i) < z \sum_a \eta(a) \|\Delta \Pr(a, \sigma_i)\|$. Uniform detectability implies that a bound $z \geq 0$ exists such that for every μ -profitable deviation σ_i , there exist (i) a correlated strategy η that detects σ_i , and (ii) an incentive scheme ζ satisfying $-z \leq \zeta_i(a, s) \leq z$, that strictly discourages σ_i .^{6,7} Intuitively, every μ -profitable deviation can be strictly discouraged with a correlated strategy and an incentive scheme bounded by the same amount. To see how uniform detectability fails in [Example 4](#) but holds without the action E , let α and β , respectively, be the probabilities that player 2 plays D and plays E after being asked to play C . Clearly, (D, C) -profitability requires $\alpha > 0$. For uniform detectability we need z such that given $\alpha > 0$ and $\beta \geq 0$, η exists with $(\alpha + \beta)\eta(C, C) + \alpha\eta(D, C) < 2z\alpha\eta(C, C)$. Therefore, $(\alpha + \beta)/\alpha < 2z$ is necessary for uniform detectability. However, no z satisfies this for all relevant (α, β) . Removing E restores uniform detectability: now $\beta = 0$, so any $z > 1/2$ works.

Uniform detectability is still not enough for virtual enforcement, as [Example 5](#) shows.

In [Example 5](#) every (D, C) -profitable deviation is uniformly detectable when player 1 plays E , but this action is not credible because it is strictly dominated by and indistinguishable from C . Hence, (D, C) is not virtually enforceable.

Definition 4. Say *every μ -profitable deviation is uniformly and credibly detectable* if there exists $z \geq 0$ such that for every μ -profitable deviation σ_i , there exists $\eta \in \Delta(A)$ satisfying (i) σ_i is $\text{supp } \eta$ -detectable, (ii) $\Delta v_i(\eta, \sigma_i) < z \sum_a \eta(a) \|\Delta \Pr(a, \sigma_i)\|$, and (iii) $\Delta v_j(\eta, \sigma_j) \leq z \sum_a \eta(a) \|\Delta \Pr(a, \sigma_j)\|$ for all other (j, σ_j) .

Intuitively, just as before, we may use different η to uniformly detect different σ_i , but these η must be credible in that incentives can be provided ([Footnote 6](#)) for everyone else to play η . This yields the characterization we sought, proved in [Rahman \(2008\)](#).

Lemma 3. *A given correlated strategy μ is virtually enforceable if and only if every μ -profitable deviation is uniformly and credibly detectable.*

⁶To find this payment scheme, invoking the Bang-Bang Principle, let $\zeta_i(a, s) = \pm z$ depending on the sign of the statistical change created by σ_i , namely $\sum_{(b_i, \rho_i)} \sigma_i(b_i, \rho_i | a_i) \Pr(s | a_{-i}, b_i, \rho_i) - \Pr(s | a)$.

⁷To see intuitively why I have a strict inequality, suppose that there is a μ -profitable deviation σ_i and a correlated strategy η for which $\Delta v_i(\eta, \sigma_i) = z \sum_a \eta(a) \|\Delta \Pr(a, \sigma_i)\|$. In this case, it would be impossible to discourage σ_i by playing η instead of μ with some probability and payments bounded within $\pm z$.

5 Results

In this section I will state the main results. The first main result claims that efficiency is attainable as $\delta \rightarrow 1$ if there exists an efficient outcome of the stage game such that every disobedience from the outcome is statistically detectable. The second claims that efficiency is attainable as $\delta \rightarrow 1$ if *and only if* there exists an efficient outcome of the stage game such that every disobedience is uniformly and credibly detectable. All these terms were defined in the previous section. To prove these claims I will use the results of Sections 3 and 4 together.

Of course, these results are subject to the restriction of T -private communication equilibria, for $T \in \mathbb{N}$. These equilibria need not be belief-free, and players may have strict incentives to play their recommended actions. At this point, I conjecture that the set of T -private equilibrium payoffs is dense in the set of private equilibria.

Let $\bar{U} = u(\bar{E}) = \{u(\mu) \in \mathbb{R}^I : \mu \in \bar{E}\}$ be the set of payoff profiles attainable with virtually enforceable correlated strategies, where $u(\mu) = \sum_a u(a)\mu(a)$ is the expected payoff profile from μ and $u(a) = (u_1(a), \dots, u_n(a))$ for each $a \in A$. Recall that U^* is the set of feasible, individually rational payoffs. We may now state the paper's main result, from which the Folk theorems below follow.

Theorem 1. *Let $u^* \in U^*$ be an efficient payoff profile. There exist $T(\delta)$ -private equilibria whose payoffs converge to u^* as $\delta \rightarrow 1$ if and only if $u^* \in \bar{U}$.⁸*

Theorem 1 leads to the following two Folk theorems.

Corollary 1. *Let $u^* \in U^*$ be an efficient payoff profile. If there is a correlated strategy μ such that $u(\mu) = u^*$ and every *supp* μ -disobedience is detectable then $T(\delta)$ -private equilibria exist whose payoffs converge to u^* as $\delta \rightarrow 1$.*

Interestingly, this Folk theorem gives conditions on the monitoring technology alone such that, for any profile of utility functions (satisfying Assumptions 1–3), an efficient outcome is approachable as players become increasingly patient. Also, by the Folk theorem of Lehrer (1992) for $\delta = 1$, it follows that *there is no discontinuity in the efficient equilibrium payoff correspondence with respect to δ at $\delta = 1$* when the detectability condition above is satisfied.

⁸In this preliminary and incomplete draft I focus on efficient outcomes, but in the next draft I will replace Theorem 1 with the following claim: the set of payoff profiles for which there is a sequence of approximating T -private equilibria is given by $\bar{U}^* = U^* \cap \bar{U}$.

Corollary 2. *Let $u^* \in U^*$ be efficient. There is a correlated strategy μ such that $u(\mu) = u^*$ and every μ -profitable deviation is uniformly and credibly detectable if and only if $T(\delta)$ -private equilibria exist whose payoffs converge to u^* as $\delta \rightarrow 1$.*

This result gives a joint condition on the monitoring technology and the utility functions such that an efficient outcome can be approached with T -private equilibria. In fact, this condition is necessary and sufficient for the Folk theorem. By Lemmas 2–3, [Theorem 1](#) above clearly implies [Corollaries 1 and 2](#), so proving [Theorem 1](#) is enough to prove both Folk theorems.

6 Proof of [Theorem 1](#)

Sufficiency is easy. Since u^* is efficient, it is on the boundary of U^* . Suppose that there exists a sequence of $T(\delta)$ -private equilibria whose payoffs converge to u^* as $\delta \rightarrow 1$. For each $T(\delta)$ -private equilibrium, take the expected correlated strategy played during the first $T(\delta)$ periods. Since less information relaxes incentive constraints, playing this correlated strategy, call it μ_δ , every period regardless of what players report is also a $T(\delta)$ -private equilibrium with the same continuation values. By hypothesis, taking a subsequence if necessary, $\mu_\delta \rightarrow \mu$ such that $u(\mu) = u^*$. Merging the payoffs from periods 2 to T to the continuation values finally implies that $u^* \in \bar{U}$.

For necessity, suppose that $u^* \in \bar{U}$ and u^* is efficient, so there exists μ such that $u(\mu) = u^*$ and a sequence $\{\mu^m\} \subset E$ such that $\mu^m \rightarrow \mu$. For any $\varepsilon > 0$, therefore, there exists M such that μ^m is within ε of μ for all $m \geq M$. Now, any such μ^m is enforceable, so by [Lemma 1](#) every μ^m -profitable deviation is $\text{supp } \mu^m$ -detectable. The problem of enforcing μ^m played for T consecutive periods (call the associated communication mechanism μ_T^m) with self-generated continuation values is given by

$$\begin{aligned}
 W_T(\mu_T^m) &= \max_{\xi} \sum_{i=1}^n U_i(\mu_T^m) - \sum_{(a^T, s^T)} \mu_T^m(a^T) \Pr(s^T | a^T) \xi_i(a^T, s^T) && \text{s.t.} \\
 \sum_{(a^T, s^T)} \mu^m(a^T) \Pr(s^T | a^T) \Delta U_i(\mu_T^m | \beta_i, \rho_i) &\leq \sum_{(a^T, s^T)} \mu_T^m(a^T) \xi_i(a^T, s^T) \Delta \Pr(s^T | a^T, \beta_i, \rho_i) && \forall (i, \beta_i, \rho_i), \\
 \sum_{i=1}^n \xi_i(a^T, s^T) &\geq 0 && \forall (a^T, s^T) \in \text{supp } \mu_m^T,
 \end{aligned}$$

where μ^m is fixed. The dual of this problem is

$$W_T(\mu^m) = \min_{\lambda, \eta \geq 0} \sum_{i=1}^n U_i(\mu_T^m) - \sum_{(\beta_i, \rho_i)} \lambda_i(\beta_i, \rho_i) \Delta U_i(\mu_T^m | \beta_i, \rho_i) \quad \text{s.t.}$$

$$\mu_T^m(a^T) [\Pr(s^T | a^T) - \sum_{(\beta_i, \rho_i)} \lambda_i(\beta_i, \rho_i) \Delta \Pr(s^T | a^T, \beta_i, \rho_i)] = \eta(a^T, s^T) \quad \forall i, (a^T, s^T) \in \text{supp } \mu_T^m.$$

Equivalently, we may write the dual constraints as

$$\Pr(s^T | a^T) - \sum_{(\beta_i, \rho_i)} \lambda_i(\beta_i, \rho_i) \Delta \Pr(s^T | a^T, \beta_i, \rho_i) = \eta(a^T, s^T) \quad \forall i, (a^T, s^T) \in \text{supp } \mu_T^m.$$

If there is no profile $\lambda \geq 0$ that satisfies these equations apart from honesty and obedience for every m then the self-generation constraint does not bind and the theorem is proved, so suppose not, i.e., there exists a profile of deviation that satisfies the dual constraints above. Again, if honesty and obedience solves the dual for every m then the self-generation constraint does not bind and we are done, so suppose not.

By Lemma 1, since μ_T^m is enforceable, it follows that every μ_T^m -profitable deviation is μ_T^m -detectable. Therefore, since the set of deviations is compact, it follows that

$$\sup_{\lambda \geq 0} \inf_{(a^T, s^T)} \frac{\sum_{(\beta_i, \rho_i)} \lambda_i(\beta_i, \rho_i) \Delta U_i(\mu_T^m | \beta_i, \rho_i)}{\sum_{(\beta_i, \rho_i)} \lambda_i(\beta_i, \rho_i) \Delta \text{Lr}(s^T | a^T, \beta_i, \rho_i)} = K_i^T < \infty,$$

where $(a^T, s^T) \in \text{supp } \mu_T^m$ in the inf above and

$$\Delta \text{Lr}(s^T | a^T, \beta_i, \rho_i) = \Delta \Pr(s^T | a^T, \beta_i, \rho_i) / \Pr(s^T | a^T).$$

Indeed, if $K_i^T = \infty$ then for every (a^T, s^T) the denominator is 0, but this implies that a μ_T^m -profitable, μ_T^m -undetectable deviation exists. Everything is finite, so the inf and sup above are attained at (\hat{a}^T, \hat{s}^T) and $\hat{\lambda}$, respectively. Let $\hat{\Lambda}_i = \sum_{(\beta_i, \rho_i)} \hat{\lambda}_i(\beta_i, \rho_i)$ and $\hat{\sigma}_i(\beta_i, \rho_i) = \hat{\lambda}_i(\beta_i, \rho_i) / \hat{\Lambda}_i$.

The dual above is solved by (\hat{a}^T, \hat{s}^T) and $\hat{\lambda}$ as above such that

$$\Pr(\hat{s}^T | \hat{a}^T) - \sum_{(\beta_i, \rho_i)} \lambda_i(\beta_i, \rho_i) \Delta \Pr(\hat{s}^T | \hat{a}^T, \beta_i, \rho_i) = 0.$$

To see this, for a given deviation profile λ to solve the dual (and not honesty and obedience), it is clearly necessary that $\sum_{(\beta_i, \rho_i)} \lambda_i(\beta_i, \rho_i) \Delta U_i(\mu_T^m | \beta_i, \rho_i) > 0$. If there is no (a^T, s^T) such that

$$\Pr(s^T | a^T) - \sum_{(\beta_i, \rho_i)} \lambda_i(\beta_i, \rho_i) \Delta \Pr(s^T | a^T, \beta_i, \rho_i) = 0 \quad (*)$$

then λ cannot solve the dual. Otherwise, it is possible to increase Λ_i for each i and further decrease the objective without violating the dual constraints.

Rearranging it follows that for any (a^T, s^T) such that (*) holds,

$$\Lambda_i = 1 / \sum_{(\beta_i, \rho_i)} \sigma_i(\beta_i, \rho_i) \Delta \text{Lr}(s^T | a^T, \beta_i, \rho_i).$$

Of course, given σ_i , as Λ_i is gradually increased from 0, the first (a^T, s^T) to hit the constraint (*) will be the pair (\hat{a}^T, \hat{s}^T) that solves

$$\max_{(a^T, s^T)} \sum_{(\beta_i, \rho_i)} \lambda_i(\beta_i, \rho_i) \Delta \text{Lr}(s^T | a^T, \beta_i, \rho_i).$$

Given λ that solves the dual and (\hat{a}^T, \hat{s}^T) as above, the objective becomes

$$W_T(\mu^m) = \sum_{i=1}^n U_i(\mu_T^m) - \frac{\sum_{(\beta_i, \rho_i)} \sigma_i(\beta_i, \rho_i) \Delta U_i(\mu_T^m | \beta_i, \rho_i)}{\sum_{(\beta_i, \rho_i)} \sigma_i(\beta_i, \rho_i) \Delta \text{Lr}(s^T | a^T, \beta_i, \rho_i)} = \sum_{i=1}^n U_i(\mu_T^m) - K_i^T,$$

as claimed.

The final step in the proof will be to show that $K_i^T \rightarrow 0$ as $T \rightarrow \infty$. Let $\hat{\lambda}$ and (\hat{a}^T, \hat{s}^T) satisfy

$$\frac{\sum_{(\beta_i, \rho_i)} \hat{\lambda}_i(\beta_i, \rho_i) \Delta U_i(\mu_T^m | \beta_i, \rho_i)}{\sum_{(\beta_i, \rho_i)} \hat{\lambda}_i(\beta_i, \rho_i) \Delta \text{Lr}(\hat{s}^T | \hat{a}^T, \beta_i, \rho_i)} = K_i^T \quad \forall i.$$

Since μ_T^m is a T -repetition of μ^m , without loss the solution $\hat{\sigma}$ above is the τ -repetition, where $\tau \leq T$, of a deviation in the stage game and (\hat{a}^T, \hat{s}^T) is also a T -repetition of some stage-game profile (a, s) . Let $\|\Delta u_i\| = \max_{(a, b_i)} \{u_i(a_{-i}, b_i) - u_i(a)\}$, so

$$K_i^T \leq \frac{\|(\tau/T) \Delta u_i\|}{\Delta^\tau \text{Lr}(s | a, \sigma_i)},$$

where

$$\Delta^\tau \text{Lr}(s | a, \sigma_i) = \left[\frac{\sum_{(b_i, \rho_i)} \sigma_i(a_i, b_i, \rho_i) \Pr(s | a_{-i}, b_i, \rho_i)}{\Pr(s | a)} \right]^\tau - 1.$$

Finally, since $[\sum_{(b_i, \rho_i)} \sigma_i(a_i, b_i, \rho_i) \Pr(s | a_{-i}, b_i, \rho_i) / \Pr(s | a)]^\tau > 1$ (otherwise (*) fails), it follows that $\Delta^\tau \text{Lr}(s | a, \sigma_i) \geq \tau \Delta \text{Lr}(s | a, \sigma_i)$, hence

$$K_i^T \leq \frac{\|\Delta u_i\|}{T \text{Lr}(s | a, \sigma_i)} \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

This completes the proof of [Theorem 1](#).

7 Extension

[In this section I intend to extend the previous results to arbitrary multistage games.]

8 Conclusion

[To be written.]

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