Comparison of Distribution Procedures for Few Indivisible Goods among Two Players

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Extended Abstract

The so-called 'cake cutting' procedure for dividing a continuous but inhomogenous cake among two players, introduced by H. Steinhaus in 1949 [S 1949], is now well-known. The problem of how to best divide a fixed number of *indivisible* goods among players was already mentioned in Steinhaus' paper (see also the description in [S 1998]), and has been discussed in a number of papers since. However, in most of these papers the models considered are slightly different to ours, mostly with respect to the following:

- Papers like [S 1949] or [ADG 1991] assume that monetary side payments are allowed. We don't allow that.
- Some more recent papers, like [BEF 2003], [HP 2002], and [BK 2005] use preferences that are ordinal. We use cardinal preferences.
- In [BS 2006] the 'Santa Claus' optimization problem is investigated. This point of view is also taken in [LMMS 2004] and part of [BD 2005].
- Some papers, like [HP 2002] or part of [BD 2005] assume total privacy of preferences, i.e. incomplete information. We adopt this assumption in only part of our paper.

In the model described in our paper, we have two players, Ann and Beth, and n indivisible goods, with $n \in \{5, 6, 7, 8\}$. Let $A(1), A(2), \dots A(n)$ denote the values Ann assigns to the goods, and $B(1), B(2), \dots B(n)$ the values they have for Beth. We will disucss both cases where each player knows all preferences—complete information— and also the case where each player knows only her private preferences—incomplete information. Ann and Beth play a game by which the goods are distributed to Ann and Beth. The payoff for each player is the sum of her values of the goods she got. This excludes the possibilities for complementary goods or substitute goods.

1 The Games

Maybe the most natural distribution game is that, starting with Ann, Ann and Beth alternate to select one item each. In variants of this game, the players do not strictly alternate, but follow a certain pattern of who selects when, see [BT 1999] . The name of such a game is a sequence of letters 'A' and 'B', where an 'A' respectively 'B' at position i indicates that the ith selection is made by Ann respectively Beth. We only consider games where the number of selections each player has, the number of goods each player gets, is as equal as possible, and where no players selects three goods in a row. For $n = 5$, we consider the games **ABABA**, **ABABB**, **ABBAB**, for $n = 6$ we consider **ABABAB**, **ABABBA**, **ABBABA**, and for $n = 7$ we consider **ABABABA**, **ABABBA**, and **ABBABAB**.

Cut and Choose (**C&C**) is the obvious discrete version of the cake cutting procedure. Ann divides the items into two heaps. Then Beth decides which heap she wants, and Ann gets the remaining heap.

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A variant of this procedure works as follows: Ann makes a proposal what goods she wants to get. Beth either accepts, and gets the remaining ones, or claims the heap of goods Ann wanted, in which case she has to give one of these goods to Ann.

Finally we consider a few exchange procedures, which usually start with a random distribution of about the same number of goods for both players. Then the players are supposed to exchange items.
In Random & Exchange (**R&E(**n**)**), after the initial random distribution of the goods, n rounds are played, where one player can (but doesn't have to) propose an exchange of two pralines. The other player either agrees or disagrees to the exchange. The role of the proposer alternates. Random & Forced Exchange (**R&FE(**n**)**) works similar, except that in each round the proposer must make an exchange proposal. The next procedure is similar, but allows more flexibility by breaking the exchange proposals into two subproposals made by both players: After the initial random distribution n rounds are also played in Random & Double Exchange **(R&DE(**n**))**. In each round the proposer tells the other player which one of the other goods she would like to have. This other player tells the proposer which one of her goods she would like to have in exchange for that. If both agree the pieces are exchanged. Again, the proposer role alternates in each round. Instead of starting with a random distribution, one could also start with a distribution obtained from playing some of the other games.

2 Features of the Distributions

It is clear what the two players expect from the games—a payoff as large as possible. However, we, as the authority choosing and organizing the selection game, may have different interests. Above all, we might want a fair and efficient distribution. In this section we discuss a few objectives that the procedures, or rather the outcome they produce, should obey.
First we define a level of satisfaction between 0% and 100% for each player, 100% being the sum of the values of all six goods for that player. Therefore the **satisfaction** for a player would be the ratio of total value received and total possible value for that player. For a different formulation, we look at the relative values defined by $a(i) = A(i)/(A(1) + \cdots A(n))$ and $b(i) = B(i)/(B(1) + \cdots B(n))$. Since the sum of all relative values of each player equals 1. the satisfaction of a player is the sum of the player's relative values of all received goods.
First, we would like to see the sum of both satisfactions as high as possible. But to evaluate the quality of the outcome, more important than the value of this sum of satisfactions would be how much more could be achieved. Let us call the difference between possible total satisfaction that can be achieved with the given preferences and the actual total satisfaction the **inefficiency** of the outcome. The lower it is, the better.
Next we call the absolute value of the difference of both satisfactions the **inequity** of the outcome. If fairness of the distribution is an aim, the lower thew inequity is, the better.
A third interesting measure is the **minimum satisfaction** of the two players. Having a high such minimum would imply that both are to some extend satisfied. This measure, which is also discussed in [BS 2006] and [BD 2005], could be seen as a blend of efficiency and equity, since among different outcomes with the same efficiency, higher minimum satisfaction means higher equity. And among different outcomes with the same equity higher minimum satisfaction means higher efficiency.
An outcome is **envy-free** for a player if she prefers her share over what the other player gets. Envy-freeness for a player means that the player's satisfaction is at least 50%. Depending on the data, envy-freeness may be not achievalble.
Another requirement of an outcome is **Pareto-maximality**—meaning that one cannot increase the satisfaction of one player without lowering the satisfaction of the other.

3 Computer Simulations

We implemented optimal play for all games considered, for $n = 5, 6, 7, 8$. For each one of these n, 8000 data values were randomly created, where the preferences for each good were uniformly distributed inside the interval $[0, 1]$, independently. We calculated the averages of the features considered, for all of the games. The results are rather strong, although different for different values n.
By ordering the data according to how closely the values for both players coincide, we also get results on how the games behave in case of similar preferences and in case of opposite preferences. The results also imply an relationship between the five features considered. Some seem to be to be closely correlated with some others.
For the incomplete information situation, where none of the players knows the payoffs of the other, many of these games are much more difficult to analyze. But for some of these games, optimal strategies could also be formulated then. For these games we also ran these simulations and calculated the averages of the features. Interestingly enough, some of the games yield slightly better results for incomplete information than for complete information. This may imply that mutual openness is not always desirable for society aiming towards efficiency and fairness.
The complete information data for $n = 5$ has already been discussed in the paper [P*].

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