# Correlated equilibria and communication equilibria in all-pay auctions*

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**Abstract**

We study cheap-talk pre-play communication in the static all-pay auctions. For the case of two bidders we show that all correlated and communication equilibria are payoff equivalent to the Nash equilibrium if there is no reserve price, or if it is commonly known that one bidder has a strictly higher value. Hence, in such environments the Nash equilibrium predictions are robust to pre-play communication between the bidders. If there are three or more symmetric bidders, or two symmetric bidders and a positive reserve price, then we show that there exist correlated and communication equilibria such that the bidders' payoffs are higher than in the Nash equilibrium. In these cases pre-play cheap talk may affect the outcomes of the game since the bidders have an incentive to coordinate on such equilibria.

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## 1 Introduction

In many situations it may be difficult to prevent the bidders from engaging in cheap talk before the auction. In this paper we study whether such pre-play communication can affect the outcomes

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of the static all-pay auctions. In particular, we are interested whether the predictions about the auction outcomes obtained using models that do not allow the bidders to talk are robust to allowing pre-play cheap-talk communication.

To study the all-pay auction with pre-play communication we use the solution concepts of correlated equilibrium (Aumann, 1974; 1987)) and communication equilibrium (Myerson, 1982), which are defined in Section 2. The significance of these solution concepts for studying games with communication is due to a version of the revelation principle for such games. Consider a Nash equilibrium outcome of a game that consists of some communication protocol followed by the all-pay auction. In the complete information environments such an outcome can be replicated as a correlated equilibrium of the all-pay auction, and in the incomplete information environments–as a communication equilibrium of the all-pay auction (Myerson, 1982).

In the all-pay auction models that we study there is either a unique Nash equilibrium, or all Nash equilibria result in the same payoffs for the bidders. If it happens that in a given environment all correlated (communication) equilibria are payoff equivalent to the Nash equilibrium, then we can say that the Nash equilibrium prediction is robust to pre-play communication between the bidders. However, if there exist correlated (communication) equilibria that are not payoff equivalent to the Nash equilibrium, then pre-play communication may affect the outcomes of the game. In particular, if in such correlated (communication) equilibria the bidders get higher payoffs than in the Nash equilibrium, then they have an incentive to coordinate on the former, possibly with the help of a mediator.[1]

In Section 3 we study correlated equilibria in the all-pay auctions with complete information. We show that with two bidders the correlated equilibria are payoff equivalent to the Nash equilibrium when there is no reserve price, or if the bidders are asymmetric (Proposition 2). In such cases the all-pay auction is "strategically equivalent" to a particular zero-sum game, and for the two-player zero-sum games the correlated and Nash equilibria are known to be payoff equivalent (Moulin and Vial, 1978). It turns out that this equivalence does not hold when there is a reserve price and the bidders are symmetric. For this case we construct correlated equilibria that are more profitable for the bidders than the Nash equilibrium (Proposition 3 and Example 1). When there are three or

[1] If there are sufficiently many bidders, then the outcome of any correlated (communication) equilibrium can be achieved as a Nash equilibrium of a game that consists of an unmediated communication protocol followed by the all-pay auction (Forges, 1990; Gerardi, 2004).

more symmetric bidders, such profitable correlated equilibria exist even when there is no reserve price (Proposition 4 and Example 2). The idea of the constructions is to introduce some negative correlation in the distribution of the bids. Say, when one of the bidders bids aggressively, then with a certain probability his opponents are "suggested" to bid zero, and thus save the cost of their bids.

In Section 4 we study communication equilibria in the all-pay auctions with independent private values. Similarly to the case of complete information, we show that with two bidders the communication equilibria are payoff equivalent to the Nash equilibrium when there is no reserve price (Proposition 6). That is, no payoff-consequential self-enforcing sharing of private information or correlation of play is possible in this case. However, in other cases there exist communication equilibria that are more profitable for the bidders than the Nash equilibrium. In a symmetric environment where each bidder has two possible valuations we demonstrate such possibilities for the case of two bidders and a positive reserve price (Proposition 8), and for the case of more than two bidders and no reserve price (Proposition 10). The constructions involve correlating the bidders' play in a way that is similar to the correlated equilibria in Section 3. The bidders share some private information, but only to a limited extent, because it is important to maintain enough uncertainty about the opponents' types and play for the construction to work.

Pre-play communication in auctions is typically studied in the context of bidder collusion.[2] Most of the studies of collusion in static auctions focus on a scenario when the bidders organize an explicit cartel that enforces coordinated behavior of the bidders in the action and facilitates exchange of side payments between the bidders.[3] The bidder collusion that is self-enforcing is for the most part considered in the context of repeated auctions.[4] In such models the enforcement of the desired bidder behavior is provided by the expectations of the future reaction of the opponents.

Only a few papers study collusion in static auctions when the behavior of the bidders in the auction cannot be directly controlled. Marshall and Marx (2007), and Lopomo et al. (2010) study collusion in the first- and second-price auctions under the following scenario. The bidders make reports to a "center"; based on these reports, the center privately recommends a bid to be made

[2] A separate strand of literature studies the bidders' individual decisions to disclose their private signals that are assumed to be verifiable. Hernando-Veciana and Troge (2005) and Benoit and Dubra (2006) study such a problem in auctions with interdependent valuations.

[3] For example, Graham and Marshall (1987) study collusion in second-price auctions, and McAfee and McMillan (1992) study collusion in first-price auctions.

[4] For example, Aoyagi (2003) studies self-enforcing collusion with pre-play communication in repeated auctions.

by each bidder, and requires payments from the bidders.[5] Note that if we remove the possibility of exchange of pre-auction side payments, then such a model of collusion is equivalent to assuming that the bidders play a communication equilibrium.

Lopomo et al. (2010) show that in the first-price auction with discrete bids such a collusion is completely ineffective: all collusive equilibria are payoff equivalent to the unique Nash equilibrium. An equivalent result for the communication equilibrium in such an environment follows directly from their result.[6] However, in the second-price auction such a collusion works as well as collusion in a model where the bidders behavior can be controlled by the cartel (Marshall and Marx, 2007). In fact, collusion in the second-price auctions continues to work quite well even when there is no possibility of side payments, i.e. there exist communication equilibria that are different from the Nash equilibria, and are more profitable for the bidders (Marshall and Marx, 2008).

The rest of the paper is organized as follows. The model and the definitions of correlated and communication equilibria are in Section 2. The all-pay auctions with complete information and incomplete information are studied in Sections 3 and 4, respectively. Concluding comments are in Section 5. The proofs are relegated to the Appendix unless stated otherwise.

## 2 Model

There are $n \geq 2$ bidders; each bidder $i$ chooses a bid $b_i$ from a set $A_i$. If there is no reserve price, then $A_i = [0, \infty)$. If there is a reserve price $r > 0$, then $A_i = \{0\} \cup [r, \infty)$, i.e., he can either submit an "active" bid $b_i \geq r$, or a "null" bid $b_i = 0$.[7]

We distinguish between complete and incomplete information environments as follows.

**Complete information**. Bidder $i$ has a valuation $v_i > 0$ for the good, and the bidders' values $(v_1, ..., v_n)$ are commonly known. If bidder $i$ bids $b_i$ and the other bidders bid $b_{-i}$, then his payoff is $u_i(b_i, b_{-i}) = v_i \rho_i(b_i, b_{-i}) - b_i$, where $\rho_i$ is the probability that bidder $i$ wins the good for given bids. If there is no reserve price, then

[5] Lopomo et al. (2005) and Garratt et al. (2008) study self-enforcing collusion without pre-auction side payments, but with a possibility of resale.

[6] See also Azacis and Vida (2010) for related results for the first-price auction with a continuum of bids.

[7] Alternatively one can keep the action set $A_i = [0, \infty)$, but this will result in an unnecessary multiplity of equilibria because there will be multiple possible "inactive" bids.

$$\rho_i(b_i, b_{-i}) = \begin{cases} 0 & \text{if} \quad b_i < \max_{j \neq i} b_j \\ 1 & \text{if} \quad b_i > \max_{j \neq i} b_j \\ \frac{1}{\#\{k:\, b_k = b_i\}} & \text{if} \quad b_i = \max_{j \neq i} b_j \end{cases}$$

If there is a reserve price $r > 0$, then

$$\rho_i(b_i, b_{-i}) = \begin{cases} 0 & \text{if} \quad b_i = 0 \text{ or } b_i \geq r \text{ and } b_i < \max_{j \neq i} b_j \\ 1 & \text{if} \quad b_i \geq r \text{ and } b_i > \max_{j \neq i} b_j \\ \frac{1}{\#\{k:\, b_k = b_i\}} & \text{if} \quad b_i \geq r \text{ and } b_i = \max_{j \neq i} b_j \end{cases}$$

Suppose there is a neutral trustworthy mediator who makes non-binding private recommendations (possibly stochastic) to each bidder of which bid to submit. The recommendations are made according to a *correlation rule* $\mu$, which is a probability measure over the bid profiles of the bidders ($A = \prod_{j=1}^{n} A_j$).[8] Each bidder $i$ then decides which bid to submit as a function of the mediator's recommendation, $\widehat{b}_i : A_i \to A_i$.[9]

**Definition 1** *A correlation rule $\mu$ is a* correlated equilibrium *if each bidder finds it optimal to obey the mediator's recommendations:*

$$\int_A u_i(b)\, \mu(db) \geq \int_A u_i\left(\widehat{b}_i(b_i), b_{-i}\right) \mu(db) \qquad \textit{for every } i, \textit{ and } \widehat{b}_i(\cdot).$$

The significance of the correlated equilibrium for studying all-pay auctions with communication is due to the revelation principle.[10] According to it any equilibrium outcome of a game that consists of some communication protocol followed by the all-pay auction can be replicated as a correlated equilibrium of the all-pay auction. There is no loss of generality in requiring that for each player it is optimal to obey the mediator's recommendation.

Let $\mu_i$ be the marginal probability measure of $\mu$ on $A_i$:

$$\mu_i(E_i) = \int_{E_i \times A_{-i}} \mu(db) \qquad \text{for every } E_i \subseteq A_i.$$

[8] All considered sets and functions are Borel measurable; all considered probability measures are Borel, with topology of weak convergence.

[9] It suffices to work with the pure strategy deviations for the definitions of correlated and communication equilibria.

[10] See Aumann (1974, 1987) and Myerson (1982). Cotter (1989) provides the revelation principle for settings with large action and type spaces.

A *Nash equilibrium* is a correlated equilibrium such that each bidder's behavior is independent from the actions of the opponents, i.e., $\mu^*$ is a product measure $\prod_{j=1}^n \mu_j^*$. When we encounter a Nash equilibrium, we write it as a profile of the individual strategies $(\mu_1^*, ..., \mu_n^*)$. Hence, both Nash and correlated equilibria are joint plans of actions that are individually self-enforcing, but correlated equilibrium allows for additional coordination by correlating recommendations to the bidders.

**Incomplete information**. Each bidder $i$ privately observes own value $v_i \in T_i = [\underline{v}_i, \overline{v}_i] \subset \mathbb{R}_+$. The value of bidder $i$ is distributed according to a probability measure $P_i$ on $T_i$, independently from the valuations of the other bidders. Let $P$ be a product measure $\prod_{j=1}^n P_j$, and $P_{-i} = \prod_{j \neq i} P_j$. This information structure is assumed to be common knowledge. The payoffs of the bidders are specified in a similar way as under the complete information: the payoff of bidder $i$ of type $v_i$ who bids $b_i$ while the other bidders bid $b_{-i}$ is $u_i(b_i, b_{-i}; v_i) = v_i \rho_i(b_i, b_{-i}) - b_i$.

Suppose the bidders first privately report their private types to a neutral trustworthy mediator, who then makes non-binding private recommendations (possibly stochastic) to each bidder of which bid to submit. The recommendations are made according to a *communication rule* $\mu$, which is a family of probability measures $\mu(\cdot|v)$ over the bid profiles of the bidders $(A)$, indexed by the profile of type reports submitted to the mediator $(v \in T = \prod_{j=1}^n T_j)$. Bidder $i$ with type $v_i$ decides which type $\widehat{v}_i \in T_i$ to report, and which bid to take as a function of the mediator's recommendation, $\widehat{b}_i : A_i \to A_i$.

**Definition 2** *A communication rule $\mu$ is a* communication equilibrium *if $P_i$-a.e. type of each bidder finds it optimal to report the true type and obey the mediator's recommendations:*

$$\int_{T_{-i}} \left( \int_A u_i(b; v_i)\, \mu(db|v) \right) P_{-i}(dv_{-i}) \geq \int_{T_{-i}} \left( \int_A u_i\left( \widehat{b}_i(b_i), b_{-i}; v_i \right) \mu(db|\widehat{v}_i, v_{-i}) \right) P_{-i}(dv_{-i})$$

$$\text{for every } i,\ P_i\text{-a.e. } v_i,\ \text{every } \widehat{v}_i,\ \text{and } \widehat{b}_i(\cdot).$$

As in the case of correlated equilibrium, the significance of communication equilibrium for studying all-pay auctions with communication is due to the revelation principle. Any equilibrium outcome of a game that consists of some communication protocol followed by the all-pay auction can be replicated as a communication equilibrium of the all-pay auction. There is no loss of generality in requiring that for each player reporting the true type and obeying the mediator's recommendation is optimal.

Let $\mu_i(\cdot|v_i)$ be the marginal probability measure of $\mu$ on $A_i$ conditional on $v_i$:

$$\mu_i(E_i|v_i) = \int_{T_{-i}} \left( \int_{E_i \times A_{-i}} \mu(db|v_i, v_{-i}) \right) P_{-i}(dv_{-i}) \qquad \text{for every } E_i \subseteq A_i.$$

A *Nash equilibrium* is a communication equilibrium such that each bidder's behavior is independent from the reports and the actions of the opponents: $\mu^*(\cdot|v)$ is a product measure $\prod_{j=1}^{n} \mu_j^*(\cdot|v_i)$. When we encounter a Nash equilibrium, we write it as a profile of individual strategies $(\mu_1^*, ..., \mu_n^*)$. Relative to the Nash equilibrium, communication equilibrium allows for self-enforcing sharing of private information between the bidders, as well as for coordination via correlation of recommended bids (as in the correlated equilibrium).

# 3 All-pay auctions with complete information

## 3.1 Two bidders

Here we consider the case of two bidders under complete information. Denote the difference in the bidders' valuations by $\Delta v = v_1 - v_2$, and without loss of generality assume $\Delta v \geq 0$. To avoid uninteresting cases we assume that the valuations of both bidders are strictly above the reserve price $r \geq 0$.

First, we review the existing results on Nash equilibrium.

**Proposition 1** *In a complete information environment with two bidders:*

*(i) If $r = 0$, then there is a unique Nash equilibrium. Bidder 1 bids uniformly on $[0, v_2]$; bidder 2 bids $0$ with probability $\frac{\Delta v}{v_1}$, and bids uniformly on $[0, v_2]$ otherwise. The bidders' payoffs are $U_1 = \Delta v$, $U_2 = 0$.*

*(ii) If $v_1 > v_2 > r > 0$, then there is a unique Nash equilibrium. Bidder 1 bids $r$ with probability $\frac{r}{v_2}$, and bids uniformly on $(r, v_2]$ otherwise; bidder 2 bids $0$ with probability $\frac{\Delta v + r}{v_1}$, and bids uniformly on $(r, v_2]$ otherwise. The bidders' payoffs are $U_1 = \Delta v$, $U_2 = 0$.*

*(iii) If $v_1 = v_2 = v > r > 0$, then there is a continuum of Nash equilibria. Bidder $i$ bids $0$ and $r$ with probabilities $\alpha \frac{r}{v}$ and $(1 - \alpha) \frac{r}{v}$ (where $\alpha \in [0, 1]$), respectively, and bids uniformly on $(r, v]$ otherwise; bidder $j$ bids $0$ with probability $\frac{r}{v}$, and bids uniformly on $(r, v]$ otherwise. The bidders'*

*payoffs are* $U_1 = U_2 = 0$.

**Proof.** Part (i) follows from Proposition 2 in Hillman and Riley (1989), part (ii) from Proposition 1 in Bertoletti (2008), and part (iii) from Proposition 7 in Section 4.1 below. ■

The Nash equilibria of the complete information all-pay auctions feature a phenomenon called "rent dissipation". The bidder with the lower valuation gets a zero payoff, while the bidder the higher valuation gets a payoff equal to the difference in the valuations. In the case of symmetric bidders the rents are fully dissipated, and each bidder gets a zero payoff.

We are interested in whether there exist correlated equilibria such that the bidders' payoffs are different from the Nash equilibrium payoffs. In games with a finite number of actions the correlated equilibria are easy to describe. In such a case for each player $i$ there are $|A_i| \cdot |A_i - 1|$ incentive obedience constraints, which are linear in the probabilities over the action profiles. However, if each player has a continuum of possible actions, then there is a double continuum of obedience constraints, which is difficult to work with. Though in general it is not known when the set of payoffs of correlated equilibria differs from the convex hull of payoffs of Nash equilibria, the results for certain classes of games exist. For example, these sets coincide in the two-player zero-sum games (Rosenthal, 1973), the two-player games that are "strategically equivalent" to zero-sum games (Moulin and Vial, 1978), and the concave potential games (Neyman, 1997; Ui, 2008).[11]

There exist some results on correlated equilibria in the sealed-bid first- and second-price auctions. The second-price auction has many Nash equilibria: the truthful equilibrium, and infinitely many equilibria involving weakly dominated strategies. If the bidders correlate their play, then it is possible, for example, to sustain the following collusive scheme. Before the auction a designated winner is randomly chosen; during the auction the bidders coordinate on the equilibrium where the designated winner obtains the good for free by submitting a very high bid while the other bidders submit zero bids.[12] Lopomo et al. (2010) study a model of collusion in the first-price auctions with pre-auction side payments and bid recommendations when there are two symmetric bidders. Their results imply that the correlated equilibria are payoff equivalent to the unique Nash equilibrium in a model with large discrete bid spaces.

[11] Moulin and Vial (1978) study this question for general games, but they use a different notion of correlated equilibrium that requires some commitment on the part of the players.

[12] See Marshall and Marx (2008).

We show that the correlated equilibria of the all-pay auction are payoff equivalent to the Nash equilibrium when $r = 0$ or $v_1 > v_2$.[13] The idea of the proof is similar to the one for the zero-sum games mentioned above.[14]

**Proposition 2** *In a complete information environment with two bidders, such that $r = 0$ or $v_1 > v_2$, every correlated equilibrium is payoff equivalent to the Nash equilibrium.*

**Proof.** Denote by $U_i^*$ and $U_i$ the expected payoffs of player $i$ in the Nash equilibrium $(\mu_1^*, \mu_2^*)$ and in the correlated equilibrium $\mu$, respectively. By the definition of Nash equilibrium

$$\begin{aligned} U_i^* &= v_i \int_{A_j} \int_{A_i} \rho_i(b)\, \mu_i^*(db_i) \mu_j^*(db_j) - \int_{A_i} b_i \mu_i^*(db_i) \\ &\geq v_i \int_{A_j} \int_{A_i} \rho_i(b)\, \widetilde{\mu}_i(db_i) \mu_j^*(db_j) - \int_{A_i} b_i \widetilde{\mu}_i(db_i) \end{aligned} \tag{1}$$

for every $i$, and $\widetilde{\mu}_i$. By the definition of correlated equilibrium

$$\begin{aligned} U_i &= v_i \int_{A} \rho_i(b)\, \mu(db) - \int_{A_i} b_i \mu_i(db_i) \\ &\geq v_i \int_{A_j} \int_{A_i} \rho_i(b)\, \widetilde{\mu}_i(db_i) \mu_j(db_j) - \int_{A_i} b_i \widetilde{\mu}_i(db_i) \end{aligned} \tag{2}$$

for every $i$, and $\widetilde{\mu}_i$.

Add (1) and (2), with $\widetilde{\mu}_i = \mu_i$ in (1), and $\widetilde{\mu}_i = \mu_i^*$ in (2), divide by $v_i$, and then sum up over $i$. The result is

$$\begin{aligned} & \int_{A_j} \int_{A_i} \left( \sum_{k=1,2} \rho_k(b) \right) \mu_i^*(db_i) \mu_j^*(db_j) + \int_{A} \left( \sum_{k=1,2} \rho_k(b) \right) \mu(db) \\ \geq & \int_{A_j} \int_{A_i} \left( \sum_{k=1,2} \rho_k(b) \right) \mu_i(db_i) \mu_j^*(db_j) + \int_{A_j} \int_{A_i} \left( \sum_{k=1,2} \rho_k(b) \right) \mu_i^*(db_i) \mu_j(db_j) \end{aligned} \tag{3}$$

If $r = 0$, then $\sum_{k=1,2} \rho_k(b) = 1$ for every $b$. If $v_1 > v_2$, the same is true even if $r > 0$, because bid 0 is not rationalizable for bidder 1. To see this note that no rational bidder bids above his value,

[13] One may conjecture that such a result is implied by the uniqueness of the Nash equilibrium. Moulin and Vial (1978) have shown that this is not true.

[14] Another way to prove this result is to show that the all-pay auction is "strategically equivalent" to the following game. Consider a game identical to the all-pay auction, except that the payoffs are obtained from the all-pay auction payoffs as follows: $w_i(b_i, b_j) = \frac{1}{v_i} u_i(b_i, b_j) - \frac{1}{2} + \frac{1}{v_j} b_j$. If $r = 0$ or $v_1 > v_2$, then this game is zero-sum, and thus the set of payoffs of correlated equilibria coincides with the set of Nash equilibria (Moulin and Vial, 1978).

and thus bidder 1 always prefers to bid slightly above $v_2$ to bidding 0. Hence, inequality (3) holds as equality. This implies that the following inequalities hold as equalities as well: (1) when $\widetilde{\mu}_i = \mu_i$, and (2) when $\widetilde{\mu}_i = \mu_i^*$.

Next we show that $\left(\mu_i^*, \mu_j\right)$ is a Nash equilibrium. By the above argument, bidder $j$'s payoff from $\left(\mu_i^*, \mu_j\right)$ is $U_j^*$, and $\mu_j$ is a best response to $\mu_i^*$. Bidder $i$'s payoff from $\left(\mu_i^*, \mu_j\right)$ is $U_i$. If $\mu_i^*$ is not a best response to $\mu_j$, then for some $\widetilde{\mu}_i$

$$\begin{aligned} U_i &= v_i \int_{A_j}\int_{A_i} \rho_i\left(b\right)\mu_i^*(db_i)\mu_j\left(db_j\right) - \int_{A_i} b_i\mu_i^*(db_i) \\ &< \; v_i \int_{A_j}\int_{A_i} \rho_i\left(b\right)\widetilde{\mu}_i(db_i)\mu_j\left(db_j\right) - \int_{A_i} b_i\widetilde{\mu}_i(db_i) \leq U_i \end{aligned}$$

where the second inequality follows from (2), which gives a contradiction.

By Proposition 1 the Nash equilibrium is unique when $r = 0$ or $v_1 > v_2$. This implies that $\mu_i = \mu_i^*$, and thus $U_i^* = U_i$ for every $i$. ■

The result shows that when there is no reserve price or if the bidders are asymmetric, then the prediction according to Nash equilibrium solution about the bidders payoffs (i.e., the rent dissipation property) is robust to pre-play communication between the bidders. The result also implies that in these cases no effective collusive agreement (mediated or unmediated) between the bidders is possible, if we restrict attention to collusive schemes that are self-enforcing, and do not allow side transfers or punishments.

When $v_1 = v_2 = v$ and $r > 0$ the same proof does not work. In particular, inequality (3) does not necessarily hold as an equality because the action profile $(b_1, b_2) = (0, 0)$ cannot be ruled out. Indeed, in some Nash equilibria both bidders submit null bids with positive probability. This means that there may exist correlated equilibria that are not payoff equivalent to Nash equilibria, and the next result confirms that this is indeed the case.

**Proposition 3** *In a complete information environment with two bidders, such that $v_i = v$ for $i = 1, 2$ and $v > r > 0$, for every $(U_1, U_2) \in co\left\{(0,0), \left(0, \frac{r^2(v-r)}{v^2}\right), \left(\frac{r^2(v-r)}{v^2+r^2}, \frac{r^2(v-r)}{v^2+r^2}\right), \left(\frac{r^2(v-r)}{v^2}, 0\right)\right\}$ there exists a correlated equilibrium that gives bidder $i$ payoff $U_i$.*

Quite surprisingly, complete rent dissipation can be avoided in this case if the bidders correlate their play. We illustrate the idea of the proof with the following example.

**Example 1** *Let $v_1 = v_2 = 1$, and $r \in (0,1)$. The bidders are given recommendations according to the following probability distribution:*

| *1's bid* \ *2's bid* | *bid* 0 | *bid uniformly on* $(r,1]$ |
|---|---|---|
| *bid* 0 | 0 | $r(1-r)$ |
| *bid* $r$ | $r^2$ | 0 |
| *bid uniformly on* $(r,1]$ | $(1-r)r$ | $(1-r)^2$ |

*If bidder 1 is suggested to bid 0, then he knows that the opponent bids aggressively, and thus he is content to stay out. If bidder 1 is suggested to bid $r$, then he knows that the opponent bids 0, and thus his best response is to bid $r$. If bidder 1 is suggested to above $r$, then his probability distribution over the opponent's bids is the same as in one of the Nash equilibria, and thus he is indifferent between all bids not higher than 1. Whether bidder 2 is suggested to bid 0 or to bid above $r$, he is indifferent between all bids not higher than 1.*

*Bidder 1 gets a payoff of $1-r$ when he is suggested to bid $r$, and a zero expected payoff otherwise. Hence, his ex ante payoff is $r^2(1-r)$. The expected payoff of bidder 2 is zero.*

The correlated equilibrium in this example results in higher bidders payoffs than the Nash equilibria summarized in part (iii) of Proposition 1. To see the reason, compare the Nash equilibrium where neither bidder stays out ($\alpha = 1$) with the presented correlated equilibrium. The only difference is that in the Nash equilibrium there is an event when bidder 1 bids $r$ and bidder 2 bids above $r$, while in the correlated equilibrium bidder 1 bids 0 in such a case. In both equilibria in this event bidder 1 loses the auction, but in the correlated equilibrium he saves the cost of the bid.

In Example 1 the bids are negatively correlated; say, if $r = \frac{1}{2}$, then correlation is approximately $-0.37$. Negative correlation is usually good for the bidders payoffs. In particular, note the bidders payoffs are maximized if the bids are perfectly negatively correlated: bidder $i$ bids the reserve and bidder $j$ stays out, and their roles are decided by a toss of a coin. However, such a correlation rule is not incentive compatible, because the bidder who is recommended to stay out has an incentive to deviate to an active bid. We conjecture that there exist no correlated equilibria with payoffs that Pareto dominate the payoffs described in Proposition 3.

## 3.2 Three or more bidders

Here we focus on the case of three or more symmetric bidders. Each bidder has a valuation $v$, which is assumed to be strictly above the reserve price $r \geq 0$. There are many Nash equilibria in this case.[15] In every Nash equilibrium complete rent dissipation takes place, i.e. each bidder gets a zero payoff. This fact is shown in Proposition 9 in Section 4.2.

The next result shows that there exist correlated equilibria where the bidders get positive payoffs. In contrast to the case of two bidders, such correlated equilibria exist even when there is no reserve price.

**Proposition 4** *In a complete information environment with $n \geq 3$ symmetric bidders, such that $v_i = v$ for every $i$ and $v > r \geq 0$, for every $U \in \left[0, \frac{2(v-r)}{n} \frac{(n-2)v^2+(n-2)vr+2nr^2}{(9n-14)v^2+(6n-8)vr+(n+6)r^2}\right]$ there exists a correlated equilibrium that gives each player payoff $U$.*

To illustrate the idea of the construction consider the following example.

**Example 2** *Let $n = 3$, $v = 1$, and $r = 0$. Consider the following symmetric correlation rule where each bidder is given one of three recommendations: bid 0; bid "low", i.e. uniformly on $\left(0, \frac{1}{2}\right]$; or bid "high", i.e. uniformly on $\left(\frac{1}{2}, 1\right]$. First, two bidders, say $i$ and $j$, are randomly chosen, and the third is recommended to bid 0. The chosen bidders are given recommendations according to the following probability distribution:*

| *i's bid \ j's bid* | *bid* 0 | *bid low* | *bid high* |
|---|---|---|---|
| *bid* 0 | 0 | $\frac{2}{26}$ | 0 |
| *bid low* | $\frac{2}{26}$ | $\frac{7}{26}$ | $\frac{5}{26}$ |
| *bid high* | 0 | $\frac{5}{26}$ | $\frac{5}{26}$ |

*If a bidder is suggested to bid high, then he knows that he has exactly one active opponent who is equally likely to bid low or high. The probability of winning with bid $b > 0$ is equal to $b$, and thus the bidder is willing to comply with the recommendation.*

*If a bidder is suggested to bid low, then he knows that he has exactly one active opponent who either bids 0, bids low, or bids high, with probabilities $\frac{1}{7}$, $\frac{1}{2}$, and $\frac{5}{14}$, respectively. The probability of*

[15] See, for example, Baye et al. (1996) for a characterization of Nash equilibria when there is no reserve.

*winning with bid $b > 0$ is equal to* $\min\left\{b + \frac{1}{7}, \frac{5}{7}b + \frac{2}{7}\right\}$*, and thus the payoff from any* $b \in \left(0, \frac{1}{2}\right]$ *is* $\frac{1}{7}$*, and the payoff from any* $b \in \left(\frac{1}{2}, 1\right]$ *is below* $\frac{1}{7}$*.*

*If a bidder is suggested to bid* $0$*, then he knows that either he was not chosen and thus faces two potentially active opponents, or that he was chosen but only his opponent was suggested to bid above* $0$*. It is possible to show that the probability of winning with bid* $b > 0$ *is equal to* $\min\left\{\frac{14}{15}b^2 + \frac{8}{15}b, \frac{2}{3}b^2 + \frac{1}{3}\right\}$*, and thus the payoff from any* $b \in (0, 1]$ *is nonpositive.*

*Each bidder gets an expected payoff of* $\frac{1}{7}$ *when he is suggested to bid low, and a zero expected payoff otherwise. This results in an ex ante payoff of* $\frac{2}{39}$*.*

Here is one way to see why this correlated equilibrium gives positive payoffs even though all Nash equilibria result in zero payoffs. Consider a Nash equilibrium such that two bidders bid uniformly on $(0, 1]$, and the third bidder bids 0. This Nash equilibrium can be viewed as a simple correlated equilibrium: to each of the first two bidders the mediator is equally likely to recommend to bid high or low (independently of each other); to the third bidder the mediator recommends to bid 0. Now suppose that in advance the mediator preforms a fair lottery that chooses two bidders who are to take active roles in the above Nash equilibrium, and each bidder is privately informed of his role. Finally, suppose that with a small probability a mediator "forgets" to inform one of the chosen bidders that he is to take an active role, and instead tells him to bid 0. If the probability of such a "mistake" is sufficiently small, then the bidders will choose to comply whenever they are recommended to bid 0. The introduction of such "mistakes" reduces the intensity of bidding, and thus raises the bidders payoffs.

While Proposition 4 is about symmetric correlated equilibria, it is possible to construct correlated equilibria with asymmetric payoffs as well. We do not claim that the upper bound on the payoff in the presented symmetric correlated equilibrium is the highest one could achieve.

# 4 All-pay auctions with incomplete information

## 4.1 Two bidders

In settings with incomplete information the players may wish to share their private information with each other. Communication equilibrium describes the information sharing schemes that can be sustained in a self-enforcing way. In addition, as in the case of correlated equilibrium, communication equilibrium allows for coordination between the players via correlation of recommended actions.

To provide a benchmark we first review the results on Nash equilibrium. The following result for the case when the bidders' valuations are distributed continuously and independently is due to Amann and Leininger (1996).[16]

**Proposition 5** *Suppose there are two bidders and no reserve price. Valuations $v_1$ and $v_2$ are independently distributed on $[0,1]$ according to $F_1$ and $F_2$, whose densities, $f_1$ and $f_2$, are continuously differentiable and positive on $(0,1)$. Then there exists a unique Nash equilibrium. Bidder $i$ bids according to a bidding function which has a strictly increasing inverse $\phi_i : \left(0, \overline{b}\right] \rightarrow [0,1]$ such that $f_i\left(\phi_i\left(b\right)\right)\phi_i'\left(b\right)\phi_j\left(b\right) = 1$ for every $b \in \left(0, \overline{b}\right]$.*

There exist some results on communication equilibria and related concepts in the first- and second-price auctions.[17] As discussed in Section 3.1, in the second-price auction the bidders may use a correlating device to randomize among the Nash equilibria. In fact, for some distributions of the bidders' valuations this is the optimal way to collude as long as side payments are not allowed.[18] Marshall and Marx (2008) discuss other communication equilibria in the second-price auction.

Lopomo et al. (2010) study a model of collusion with pre-auction side payments and bid recommendations in the first-price auctions with two bidders. They work with large discrete bid spaces, which allows them to use the linear programming techniques. Lopomo et al. (2010) show that in a symmetric environment with two possible types (with or without reserve) the collusive equilibrium is payoff equivalent to the unique Nash equilibrium.[19] This *a fortiori* implies an equivalent result for the communication equilibrium. Azacis and Vida (2010) study a similar environment in a model with continuum of bids. They show that several restricted versions of communication equilibrium are payoff equivalent to the Nash equilibrium, and they conjecture that the same is true for the canonical communication equilibrium.

[16] To obtain results for the case of general symmetric independent distribution one can adapt the results of Monteiro (2009) for the first price auctions. He provides an explicit formula for a symmetric Nash equilibrium, and shows that it is unique in the class of symmetric Nash equilibria.

[17] There are also some studies of communication equilibria in other trading environments. See, for example, Matthews and Postlewaite (1989) for analysis of double auctions in a bilateral trade model.

[18] See McAfee and McMillan (1992).

[19] Lopomo et al. (2010) check the robustness of the result by studying numerically other environments with two bidders.

In the all-pay auctions without a reserve the idea behind Proposition 2 can be extended to the case of two bidders under incomplete information. As in the case of complete information, the possibility to use correlated recommendations does not allow to achieve payoffs different from the Nash equilibrium payoffs. It turns out that in this case no payoff-consequential sharing of private information is possible either.

**Proposition 6** *Suppose that in a given incomplete information environment with two bidders and no reserve price there exists a unique Nash equilibrium. Then every communication equilibrium is interim payoff equivalent to the Nash equilibrium.*

A result analogous to Proposition 6 is likely to hold in other environments, even when the reserve price is strictly positive, if we can rule out the case when both bidders choose null bids. A sufficient condition for this to happen is when the supports for the bidders' valuations do not overlap, say, $\overline{v}_2 < \underline{v}_1$. Then bid 0 is not rationalizable for bidder 1 for any beliefs, because he prefers to bid slightly above $\overline{v}_2$ to bidding 0. It remains an open question, however, whether an analogous result holds for the case when the bidders have correlated and/or interdependent values.

If there is a positive reserve price, and it is not commonly known that one bidder has a higher value than the other bidder, then there may exist communication equilibria that are not payoff equivalent to Nash equilibria. We demonstrate this for a simple symmetric environment where each bidder has two possible valuations. First, we describe the Nash equilibria.

**Proposition 7** *Suppose there are two bidders, each bidder's value is equal to* $0$ *or* $v$ *with probabilities* $p$ *and* $1-p$*, independently of the opponent's value. There is a reserve price* $r \in (0, v)$*.*

*(i) If* $p \in \left[0, \frac{r}{v}\right)$ *then there is a continuum of Nash equilibria. Types* $0$ *of both bidders bid* $0$*. Type* $v$ *of bidder* $i$ *bids* $0$ *and* $r$ *with probabilities* $\alpha\frac{r-pv}{(1-p)v}$ *and* $(1-\alpha)\frac{r-pv}{(1-p)v}$ *(where* $\alpha \in [0,1]$*), respectively, and bids uniformly on* $(r, v]$ *otherwise; type* $v$ *of bidder* $j$ *bids* $0$ *with probability* $\frac{r-pv}{(1-p)v}$*, and bids uniformly on* $(r, v]$ *otherwise. In every Nash equilibrium every type gets a zero payoff.*

*(ii) If* $p \in \left[\frac{r}{v}, 1\right]$ *then there is a unique Nash equilibrium. Types* $0$ *of both bidders bid* $0$*. Type* $v$ *of both bidders* $i$ *bids uniformly on* $(r, v(1-p)+r]$*. Types* $0$ *of both bidders get a zero payoff, types* $v$ *of both bidders get a payoff of* $pv - r$*.*

Next we show that for some parameters there exists communication equilibria that result in bidder payoffs that are higher than in the Nash equilibrium.

**Proposition 8** *In the environment described in Proposition 7, for every $p \in \left[0, \frac{r}{v}\right)$ there exists a communication equilibrium that gives each bidder of type $v$ a positive payoff.*[20]

The construction of the communication equilibrium is quite involved. As in the correlated equilibrium one has to make sure that the bidders are willing to comply with the recommended bids. In addition, the bidders must be given incentives to report their types truthfully. Below are a few highlights of the construction.

First, the behavior of the bidders of type $v$ is coordinated in a way that is similar to the correlated equilibria in the complete information case. In fact, when $p = 0$, the construction used here is identical to the one in the proof of Proposition 3. Second, some information about the bidders' values is being shared. Consider, for example, an event when only one bidder has a value above the reserve price. In every Nash equilibrium there is a positive probability that this bidder submits a null bid and does not get the good. In the constructed communication equilibrium the bidder with the value above the reserve submits an active bid, and thus gets the good, with probability one.

However, in some cases it is important not too reveal too much information to the bidders about their opponents. Consider once again the event when only one bidder has (reported) a value above the reserve price. If this bidder learns that the opponent has value 0, then he will bid exactly the reserve price. But this implies that the bidder of type $v$ has a profitable deviation: report type 0, and then bid slightly above the reserve price. Thus, to make such a deviation unprofitable it is necessary that the bidder of type $v$ bids aggressively enough when the opponent has reported type 0. And to ensure that the bidder of type $v$ is willing to comply with recommendations for aggressive bidding it is important to maintain enough uncertainty about the opponent's type.

## 4.2 Three or more bidders

Here we continue to work with the symmetric independent case when each bidder's valuation can be either 0 or $v$. The Nash equilibrium payoffs when there are three or more bidders are as follows.

**Proposition 9** *Suppose there are $n \geq 3$ bidders, and each bidder's value is equal to 0 or $v$ with probabilities $p$ and $1 - p$, independently of the opponents' values. There is a reserve price $r \in [0, v)$.*

[20] We can prove an analogous result for the case when $p \in \left[\frac{r}{v}, 1\right]$ and $r$ is sufficiently high. The proof is long, and thus not included in the paper.

*In every Nash equilibrium types* 0 *of each bidder get a payoff of zero, types* $v$ *of each bidder get a payoff of* $\max\left\{0, p^{n-1}v - r\right\}$.

Note that there can be no communication equilibrium such that some bidder gets a payoff below his Nash equilibrium payoff. Bidding 0 guarantees a payoff of (at least) zero; bidding the reserve price $r$ leads to winning whenever all opponents have zero valuations, and thus guarantees a payoff of (at least) $p^{n-1}v - r$.

The next result demonstrates that there exist communication equilibria such that each bidder gets a payoff higher than in the Nash equilibrium. We focus on the case of no reserve price, but the construction can be extended to the case of positive reserve price as well.

**Proposition 10** *In the environment described in Proposition 9, if* $r = 0$*, then for* $p$ *sufficiently small there exists a communication equilibrium such that each bidder of type* $v$ *gets a strictly higher payoff than in the Nash equilibrium.*

# 5 Conclusion

We have studied cheap-talk pre-play communication in the static all-pay auctions. For the case of two bidders we have shown that all correlated and communication equilibria are payoff equivalent to the Nash equilibrium if there is no reserve price, or if it is commonly known that one bidder has a strictly higher value. It will be interesting to see if this equivalence result holds beyond the incomplete information environments with independent private values.

For the cases of three or more symmetric bidders, or two symmetric bidders and a positive reserve price, we have demonstrated that there may exist correlated and communication equilibria such that the bidders' payoffs are higher than in the Nash equilibrium. A characterization of all environments where such profitable correlated or communication equilibria exist remains an open question. It also may be interesting to compute the set of all payoffs that can be achieved by the communication or correlated equilibria in such cases. Finally, one may want to understand when the profitable correlated and communication equilibria can be achieved by unmediated cheap talk between the bidders.

We believe that many of the results of this paper can be extended to related models like the all-pay tournaments. We also hope an approach similar to the one used here will be useful for studying the effect of pre-play cheap talk communication on the outcomes of other games.

# 6 Appendix

## 6.1 Proofs of Section 3

**Proof of Proposition 3.** Denote $u_i = \frac{1}{v-r}U_i$ for $i = 1, 2$, and fix $(u_1, u_2) \in \mathbb{R}^2_+$ such that $v^2 u_i + r^2 u_j \leq r^2$. Consider the following correlation rule, where each bidder is given one of three recommendations: bid 0; bid $r$; or bid "high", i.e. uniformly on $(r, v]$. The recommendations are given according to the following probability distribution.[21]

| 1's bid \ 2's bid | bid 0 | bid $r$ | bid high |
|---|---|---|---|
| bid 0 | 0 | $u_2$ | $\frac{r}{v+r}(1-u_1-u_2)$ |
| bid $r$ | $u_1$ | 0 | 0 |
| bid high | $\frac{r}{v+r}(1-u_1-u_2)$ | 0 | $\frac{v-r}{v+r}(1-u_1-u_2)$ |

Suppose bidder 1 is suggested to bid 0 and bids $b \in (r, v]$ instead.[22] Then his payoff is

$$\begin{aligned}
&\left(\frac{u_2}{u_2 + \frac{r}{v+r}(1-u_1-u_2)} + \frac{\frac{r}{v+r}(1-u_1-u_2)}{u_2 + \frac{r}{v+r}(1-u_1-u_2)}\left(\frac{b-r}{v-r}\right)\right)v - b \\
= &\frac{r^2 u_1 + v^2 u_2 - r^2}{(v^2 - r^2)\left(u_2 + \frac{r}{v+r}(1-u_1-u_2)\right)}(v-b) \leq 0
\end{aligned}$$

where the inequality follows from $r^2 u_1 + v^2 u_2 \leq r^2$. If bidder 1 is suggested to bid $r$, then it is clearly optimal to comply, since the opponent bids 0 in such case. If bidder 1 is suggested to bid high, then he is indifferent between all bids since the payoff from bidding $b \in [r, v]$ is

$$\left(\frac{r}{v} + \left(1 - \frac{r}{v}\right)\left(\frac{b-r}{v-r}\right)\right)v - b = 0$$

[21] It is straightforward to verify that the entries in table: (i) sum up to one; and (ii) are nonnegative. The latter is because

$$u_1 + u_2 = \frac{1}{v^2 + r^2}\left((v^2 u_1 + r^2 u_2) + (r^2 u_1 + v^2 u_2)\right) \leq \frac{2r^2}{v^2 + r^2} < 1$$

where the first inequality follows from $v^2 u_i + r^2 u_j \leq r^2$.

[22] Bidding exactly $r$ is dominated by bidding slightly above $r$ if there is a positive probability that the opponent bids $r$.

To summarize, bidder 1 gets a payoff of $v-r$ when he is suggested to bid $r$, and zero payoff otherwise. Hence, his ex ante payoff is $u_1(v-r)=U_1$. Using a similar argument for bidder 2, we conclude that the considered correlation rule is a correlated equilibrium, and it achieves the desired payoffs. ■

**Proof of Proposition 4.** Fix $U \in \left[0,\overline{U}\right]$, where $\overline{U}=\frac{2(v-r)}{n}\frac{(n-2)v^2+(n-2)vr+2nr^2}{(9n-14)v^2+(6n-8)vr+(n+6)r^2}$. Consider the following symmetric correlation rule, where each bidder is given one of three recommendations: bid 0; bid "low", i.e. uniformly on $\left(r,\frac{1}{2}(v+r)\right]$; or bid "high", i.e. uniformly on $\left(\frac{1}{2}(v+r),v\right]$. First, two out of $n$ bidders are chosen by a fair lottery, say bidders $i$ and $j$, and all the other bidders are recommended to bid 0. The chosen bidders are given recommendations according to the following probability distribution, where $x=\frac{1}{v+r}\left(\frac{v-r}{4}-\frac{3v+r}{v}\frac{n}{8}U\right)$.[23]

| $i$'s bid \ $j$'s bid | bid 0 | bid low | bid high |
|---|---|---|---|
| bid 0 | $0$ | $\pi_l=\frac{2r}{v-r}x+\frac{v+r}{v-r}\frac{n}{2v}U$ | $\pi_h=\frac{2r}{v-r}x$ |
| bid low | $\pi_l=\frac{2r}{v-r}x+\frac{v+r}{v-r}\frac{n}{2v}U$ | $\pi_{ll}=x+\frac{n}{2v}U$ | $\pi_{hl}=x$ |
| bid high | $\pi_h=\frac{2r}{v-r}x$ | $\pi_{hl}=x$ | $\pi_{hh}=x$ |

If a bidder is suggested to bid 0, then he knows that either he was not chosen (which happens with probability $\frac{n-2}{n}$), or that he was chosen but only his opponent was suggested to bid above 0 (which happens with probability $\frac{2}{n}(\pi_l+\pi_h)$).

If this bidder bids $b\in\left(r,\frac{1}{2}(v+r)\right]$ instead, then he has a chance to win only if none of his opponents bid high. In particular, bidder $i$ could win if (i) he was not chosen, and one chosen bidder bids low (which happens with probability $\frac{n-2}{n}(2\pi_l)$); (ii) he was not chosen, and two chosen bidders bid low (which happens with probability $\frac{n-2}{n}\pi_{ll}$); (iii) he was chosen, and his opponent bids low (which happens with probability $\frac{2}{n}\pi_l$). The expected payoff this bidder is then

$$\left(\frac{\frac{n-2}{n}(2\pi_l)+\frac{2}{n}\pi_l}{\frac{n-2}{n}+\frac{2}{n}(\pi_l+\pi_h)}\left(\frac{b-r}{\frac{1}{2}(v+r)-r}\right)+\frac{\frac{n-2}{n}\pi_{ll}}{\frac{n-2}{n}+\frac{2}{n}(\pi_l+\pi_h)}\left(\frac{b-r}{\frac{1}{2}(v+r)-r}\right)^2\right)v-b \qquad (4)$$

[23] It is straightforward to verify that the entries in table: (i) sum up to one; and (ii) are nonnegative. The latter is because

$$\frac{v-r}{4}-\frac{3v+r}{v}\frac{n}{8}U\geq\frac{v-r}{4}-\frac{3v+r}{v}\frac{n}{8}\overline{U}=\frac{(v-r)^2(v+r)((3n-4)v+nr)}{2v\left((9n-14)v^2+(6n-8)vr+(n+6)r^2\right)}\geq 0$$

where the equality is by definition of $\overline{U}$.

Note that (4) is equal to $-r$ if $b=r$. If $b=\frac{1}{2}(v+r)$, then (4) becomes

$$\frac{\frac{n-2}{n}(2\pi_l+\pi_{ll})+\frac{2}{n}\pi_l}{\frac{n-2}{n}+\frac{2}{n}(\pi_l+\pi_h)}v-\frac{1}{2}(v+r)=\frac{n}{8}\frac{(9n-14)v^2+(6n-8)vr+(n+6)r^2}{(v-r)((n-2)v+nr+nU)}\left(U-\overline{U}\right)\leq 0 \quad (5)$$

where the inequality holds since $U\leq\overline{U}$. Since (4) is convex in $b$, this implies that it is nonpositive for every $b\in\left[r,\frac{1}{2}(v+r)\right]$.

If this bidder bids $b\in\left(\frac{1}{2}(v+r),v\right]$ instead, then he wins for sure if none of his opponents bid high, and has a chance to win otherwise. In particular, bidder $i$ wins for sure if (i) he was not chosen, and none of the chosen bidders bid high (which happens with probability $\frac{n-2}{n}(2\pi_l+\pi_{ll})$); (ii) he was chosen, and his opponent does not bid high (which happens with probability $\frac{2}{n}\pi_l$). Also bidder $i$ could win if (i) he was not chosen, and one chosen bidder bids high (which happens with probability $\frac{n-2}{n}(2\pi_h+2\pi_{hl})$); (ii) he was not chosen, and two chosen bidders bid high (which happens with probability $\frac{n-2}{n}\pi_{hh}$); (iii) he was chosen, and his opponent bids high (which happens with probability $\frac{2}{n}\pi_h$). The expected payoff of this bidder is then

$$\begin{aligned}&\left(\frac{\frac{n-2}{n}(2\pi_l+\pi_{ll})+\frac{2}{n}\pi_l}{\frac{n-2}{n}+\frac{2}{n}(\pi_l+\pi_h)}+\frac{\frac{n-2}{n}(2\pi_h+2\pi_{hl})+\frac{2}{n}\pi_h}{\frac{n-2}{n}+\frac{2}{n}(\pi_l+\pi_h)}\left(\frac{b-\frac{1}{2}(v+r)}{v-\frac{1}{2}(v+r)}\right)+\right. \\ &\left.+\frac{\frac{n-2}{n}\pi_{hh}}{\frac{n-2}{n}+\frac{2}{n}(\pi_l+\pi_h)}\left(\frac{b-\frac{1}{2}(v+r)}{v-\frac{1}{2}(v+r)}\right)^2\right)v-b\end{aligned} \quad (6)$$

Note that (6) is equal to (5) if $b=\frac{1}{2}(v+r)$, and (6) is equal to zero if $b=v$. Since (6) is convex in $b$, this implies that it is nonpositive for every $b\in\left(\frac{1}{2}(v+r),v\right]$.

If a bidder is suggested to bid low, then he knows that he is chosen, and faces exactly one chosen opponent. This opponent bids 0, low, or high with probabilities $\frac{\pi_l}{\pi_l+\pi_{ll}+\pi_{hl}}$, $\frac{\pi_{ll}}{\pi_l+\pi_{ll}+\pi_{hl}}$, and $\frac{\pi_{hl}}{\pi_l+\pi_{ll}+\pi_{hl}}$, respectively. The expected payoff of this bidder from bidding any $b\in\left(r,\frac{1}{2}(v+r)\right]$ is

$$\left(\frac{\pi_l}{\pi_l+\pi_{ll}+\pi_{hl}}+\frac{\pi_{ll}}{\pi_l+\pi_{ll}+\pi_{hl}}\left(\frac{b-r}{\frac{1}{2}(v+r)-r}\right)\right)v-b=\frac{U}{\frac{2}{n}(\pi_l+\pi_{ll}+\pi_{hl})}\geq 0$$

If he bids $b \in \left(\frac{1}{2}(v+r), v\right]$ instead, then his payoff is

$$\begin{aligned}
&\left(\frac{\pi_l + \pi_{ll}}{\pi_l + \pi_{ll} + \pi_{hl}} + \frac{\pi_{hl}}{\pi_l + \pi_{ll} + \pi_{hl}}\left(\frac{b - \frac{1}{2}(v+r)}{v - \frac{1}{2}(v+r)}\right)\right) v - b \\
= \quad &\frac{U}{\frac{2}{n}(\pi_l + \pi_{ll} + \pi_{hl})}\frac{2(v-b)}{v-r} < \frac{U}{\frac{2}{n}(\pi_l + \pi_{ll} + \pi_{hl})}
\end{aligned}$$

If a bidder is suggested to bid high, then he knows that he is chosen, and faces exactly one chosen opponent. This opponent bids 0, low, or high with probabilities $\frac{\pi_h}{\pi_h+\pi_{hl}+\pi_{hh}}$, $\frac{\pi_{hl}}{\pi_h+\pi_{hl}+\pi_{hh}}$, and $\frac{\pi_{hh}}{\pi_h+\pi_{hl}+\pi_{hh}}$, respectively. The expected payoff of this bidder from bidding any $b \in \left(\frac{1}{2}(v+r), \frac{1}{2}v\right]$ is

$$\begin{aligned}
&\left(\frac{\pi_h + \pi_{hl}}{\pi_h + \pi_{hl} + \pi_{hh}} + \frac{\pi_{hh}}{\pi_h + \pi_{hl} + \pi_{hh}}\left(\frac{b - \frac{1}{2}(v+r)}{v - \frac{1}{2}(v+r)}\right)\right) v - b \\
= \quad &\left(\frac{v+r}{2v} + \frac{v-r}{v}\left(\frac{b - \frac{1}{2}(v+r)}{v-r}\right)\right) v - b = 0.
\end{aligned}$$

If he bids $b \in \left(r, \frac{1}{2}(v+r)\right]$ instead, then his payoff is

$$\left(\frac{\pi_h}{\pi_h + \pi_{hl} + \pi_{hh}} + \frac{\pi_{hl}}{\pi_h + \pi_{hl} + \pi_{hh}}\left(\frac{b-r}{\frac{1}{2}(v+r) - r}\right)\right) v - b = \left(\frac{r}{v} + \frac{v-r}{v}\left(\frac{b-r}{v-r}\right)\right) v - b = 0$$

Each bidder gets a payoff of $\frac{U}{\frac{2}{n}(\pi_l+\pi_{ll}+\pi_{hl})}$ when he is suggested to bid low, and zero payoff otherwise. Hence, his ex ante payoff is $U$. Thus the considered correlation rule is a correlated equilibrium, and it achieves the desired payoffs. ■

## 6.2 Proofs of Section 4

**Proof of Proposition 6.** Denote by $U_i^*(v_i)$ and $U_i(v_i)$ the interim expected payoffs of player $i$ of type $v_i$ in the Nash equilibrium $(\mu_1^*, \mu_2^*)$ and in the communication equilibrium $\mu$, respectively. By the definition of Nash equilibrium

$$\begin{aligned}
U_i^*(v_i) \quad &= \quad v_i \int_{T_j}\left(\int_{A_j}\int_{A_i}\rho_i(b)\,\mu_i^*(db_i|v_i)\mu_j^*(db_j|v_j)\right)P_j(dv_j) - \int_{A_i} b_i\mu_i^*(db_i|v_i) \qquad (7) \\
&\geq \quad v_i \int_{T_j}\left(\int_{A_j}\int_{A_i}\rho_i(b)\,\widetilde{\mu}_i(db_i|v_i)\mu_j^*(db_j|v_j)\right)P_j(dv_j) - \int_{A_i} b_i\widetilde{\mu}_i(db_i|v_i)
\end{aligned}$$

for every $i$, $P_i$-a.e. $v_i$, and every $\widetilde{\mu}_i(\cdot|v_i)$. By the definition of communication equilibrium

$$
\begin{aligned}
U_i(v_i) &= v_i \int_{T_j} \left( \int_A \rho_i(b)\, \mu(db|v_i, v_j) \right) P_j(dv_j) - \int_{A_i} b_i \mu_i(db_i|v_i) \\
&\geq v_i \int_{T_j} \left( \int_A \left( \int_{A_i} \rho_i\left(\widehat{b}_i, b_j\right) \widetilde{\mu}_i(d\widehat{b}_i|v_i) \right) \int_{T_j} \mu(db|\widehat{v}_i, v_j) dP_i(d\widehat{v}_i) \right) P_j(dv_j) - \int_{A_i} \widehat{b}_i \widetilde{\mu}_i(d\widehat{b}_i|v_i) \\
&= v_i \int_{T_j} \left( \int_{A_j} \int_{A_i} \rho_i\left(\widehat{b}_i, b_j\right) \widetilde{\mu}_i(d\widehat{b}_i|v_i) \mu_j(db_j|v_j) \right) P_j(dv_j) - \int_{A_i} \widehat{b}_i \widetilde{\mu}_i(d\widehat{b}_i|v_i)
\end{aligned}
\tag{8}
$$

for every $i$, $P_i$-a.e. $v_i$, and every $\widetilde{\mu}_i(\cdot|v_i)$. Add (7) and (8), with $\widetilde{\mu}_i = \mu_i$ in (7), and $\widetilde{\mu}_i = \mu_i^*$ in (8), and divide by $v_i$ for every $v_i \neq 0$

$$
\begin{aligned}
&\int_{T_j} \left( \int_{A_j} \int_{A_i} \rho_i(b)\, \mu_i^*(db_i|v_i) \mu_j^*(db_j|v_j) + \int_A \rho_i(b)\, \mu(db|v_i, v_j) \right) P_j(dv_j) \\
\geq\ &\int_{T_j} \left( \int_{A_j} \int_{A_i} \rho_i(b)\, \mu_i(db_i|v_i) \mu_j^*(db_j|v_j) + \int_{A_j} \int_{A_i} \rho_i(b)\, \mu_i^*(db_i|v_i) \mu_j(db_j|v_j) \right) P_j(dv_j)
\end{aligned}
\tag{9}
$$

If $v_i = 0$, then such bidder bids 0 with probability 1 both in the Nash equilibrium and in the communication equilibrium. This implies that inequality (9) holds with equality for $v_i = 0$.

Next integrate (9) with respect to $P_i$, and sum up over $i$:

$$
\begin{aligned}
&\int_T \left( \int_{A_j} \int_{A_i} \left( \sum_{k=1,2} \rho_k(b) \right) \mu_i^*(db_i|v_i) \mu_j^*(db_j|v_j) + \int_A \left( \sum_{k=1,2} \rho_k(b) \right) \mu(db|v_i, v_j) \right) P(dv) \\
\geq\ &\int_T \left( \int_{A_j} \int_{A_i} \left( \sum_{k=1,2} \rho_k(b) \right) \mu_i(db_i|v_i) \mu_j^*(db_j|v_j) + \int_{A_j} \int_{A_i} \left( \sum_{k=1,2} \rho_k(b) \right) \mu_i^*(db_i|v_i) \mu_j(db_j|v_j) \right) P(dv)
\end{aligned}
\tag{10}
$$

Since $\sum_{k=1,2} \rho_k(b) = 1$ for every $b$, the inequality (10) holds as an equality. This implies that the following inequalities hold as equalities as well for $P_i$-a.e. $v_i$: (7) when $\widetilde{\mu}_i = \mu_i$, and (8) when $\widetilde{\mu}_i = \mu_i^*$.

Next we show that $\left(\mu_i^*, \mu_j\right)$ is a Nash equilibrium. By the above argument, the payoff of bidder $j$ of type $v_j$ from $\left(\mu_i^*, \mu_j\right)$ is $U_j^*(v_j)$, and $\mu_j$ is a best response to $\mu_i^*$. The payoff of bidder $i$ of type

$v_i$ from $\left(\mu_i^*, \mu_j\right)$ is $U_i\left(v_i\right)$. If $\mu_i^*$ is not a best response to $\mu_j$, then for some $\widetilde{\mu}_i\left(\cdot|v_i\right)$

$$
\begin{aligned}
& v_i \int_{T_j}\left(\int_{A_j} \int_{A_i} \rho_i\left(b_i, b_j\right) \mu_i^*(db_i|v_i) \mu_j(db_j|v_j)\right) P_j\left(dv_j\right)-\int_{A_i} b_i \mu_i^*(db_i|v_i) \\
< \; & v_i \int_{T_j}\left(\int_{A_j} \int_{A_i} \rho_i\left(b_i, b_j\right) \widetilde{\mu}_i(db_i|v_i) \mu_j(db_j|v_j)\right) P_j\left(dv_j\right)-\int_{A_i} b_i \widetilde{\mu}_i(db_i|v_i) \leq U_i\left(v_i\right)
\end{aligned}
$$

where the second inequality follows from (8), which gives a contradiction.

Since by assumption the Nash equilibrium is unique, we have $\mu_i = \mu_i^*$, and thus $U_i^*\left(v_i\right) = U_i\left(v_i\right)$ for every $i$, and $P_i$-a.e. $v_i$. ■

**Proof of Proposition 7.** First, note that in any Nash equilibrium types 0 of both bidders bid 0. For bidder $i$ of type $v$, let $G_i : \{0\} \cup [r, \infty) \to [0,1]$ be the equilibrium distribution function, $\underline{b}_i$ and $\overline{b}_i$ be the infimum and the supremum of the support of the equilibrium bids, and $U_i \geq 0$ be the equilibrium payoff.

Note that it is not possible to have $U_i < U_j$, since this implies $U_j \leq v - \overline{b}_j$, and thus bidder $i$ has a profitable deviation to $\overline{b}_j + \varepsilon$, for $\varepsilon > 0$ small: $v - \left(\overline{b}_j + \varepsilon\right) \geq U_j - \varepsilon > U_i$. Thus $U_i = U_j = U$. Also note that by bidding $r$ (or slightly above) each bidder of type $v$ can secure a payoff of $pv - r$. Thus $U \geq \max\{0, pv - r\}$.

If $U > \max\{0, pv - r\}$, then neither bidder bids 0 with positive probability, and thus $\underline{b}_i \geq r$. Also note that to get such a payoff each bidder of type $v$ with positive probability must win against the opponent of type $v$. Hence, we cannot have $\underline{b}_i \neq \underline{b}_j$; but if $\underline{b}_i = \underline{b}_j = \underline{b}$, then both bidders must bid $\underline{b}$ with positive probability, which cannot happen in equilibrium. Thus $U = \max\{0, pv - r\}$.

By a standard argument, $\overline{b}_i = \overline{b}_j = v - U$, and there can be no gaps and no atoms in the distribution of bids on $(r, v - U]$. Thus $G_i\left(b\right) = \frac{1}{(1-p)v}\left(b + U - pv\right)$ on $[r, v - U]$. If $p \in \left[0, \frac{r}{v}\right)$, then $G_i\left(r\right) = \frac{1}{(1-p)v}\left(r - pv\right)$ on $[r, v]$, and $G_i\left(r\right) - G_i\left(0\right)$ can be positive for at most one bidder. It is straightforward to see that every such possibility constitutes an equilibrium of the game. If $p \in \left[\frac{r}{v}, 1\right]$, then $G_i\left(b\right) = \frac{1}{(1-p)v}\left(b - r\right)$ on $[r, (1-p)v - r]$, and $G_i\left(r\right) = 0$. ■

**Proof of Proposition 8.** We show that for $p \in \left[0, \frac{r}{v}\right)$ there exists a communication equilibrium such that each bidder of type $v$ gets a payoff of $\frac{\left(r^2 - p^2 v^2\right)(v - r)}{v^2 + r^2 - 2pv^2}$. Consider the following symmetric communication rule, where each bidder is given one of three recommendations: bid 0; bid $r$; or bid "high", i.e. uniformly on $(r, v]$. If a bidder reports type 0, then he is suggested to bid 0. If a bidder reports type $v$ and his opponent reports type 0, then this bidder is suggested to bid $r$ or high, with probabilities $\widehat{\pi} = \frac{(r-pv)(v+r)}{v^2+r^2-2pv^2}$ and $1-\widehat{\pi}$, respectively. If both bidders report type $v$, then they are given recommendations according to the following probability distribution.[24]

| 1's bid \ 2's bid | bid 0 | bid $r$ | bid high |
|---|---|---|---|
| bid 0 | 0 | $\pi_r = \frac{(r-pv)r}{v^2+r^2-2pv^2}$ | $\pi_h = \frac{(r-pv)(v-r)}{v^2+r^2-2pv^2}$ |
| bid $r$ | $\pi_r = \frac{(r-pv)r}{v^2+r^2-2pv^2}$ | 0 | 0 |
| bid high | $\pi_h = \frac{(r-pv)(v-r)}{v^2+r^2-2pv^2}$ | 0 | $\pi_{hh} = \frac{(v-r)^2}{v^2+r^2-2pv^2}$ |

We need to check the incentives to tell the truth and to comply with the recommendations only for the bidders of type $v$, since the bidders of type 0 have no incentive to lie or to disobey.

If a bidder of type $v$ has reported $v$ and is suggested to bid 0, then he knows that his opponent must be of type $v$ and bids $r$ or high, with probabilities $\frac{\pi_r}{\pi_r+\pi_h}$ and $\frac{\pi_h}{\pi_r+\pi_h}$, respectively. If this bidder bids $b \in (r, v]$ instead, then his payoff is[25]

$$\left(\frac{\pi_r}{\pi_r+\pi_h} + \frac{\pi_h}{\pi_r+\pi_h}\left(\frac{b-r}{v-r}\right)\right)v - b = \left(\frac{r}{v} + \frac{v-r}{v}\left(\frac{b-r}{v-r}\right)\right)v - b = 0$$

If a bidder of type $v$ has reported $v$ and is suggested to bid $r$, then it is clearly optimal to comply, since the opponent bids 0, regardless of the type, in such case.

If a bidder of type $v$ has reported $v$ and is suggested to bid above $r$, then he knows that either his opponent is of type 0 and thus bids 0, or his opponent is of type $v$ and bids 0 or high, with probabilities $\frac{p(1-\widehat{\pi})}{p(1-\widehat{\pi})+(1-p)(\pi_h+\pi_{hh})}$, $\frac{(1-p)\pi_h}{p(1-\widehat{\pi})+(1-p)(\pi_h+\pi_{hh})}$, and $\frac{(1-p)\pi_{hh}}{p(1-\widehat{\pi})+(1-p)(\pi_h+\pi_{hh})}$, respectively. The expected payoff of this bidder from bidding any $b \in [r, v]$ is

$$\begin{aligned} & \left(\frac{p\left(1-\widehat{\pi}\right)+\left(1-p\right)\pi_h}{p\left(1-\widehat{\pi}\right)+\left(1-p\right)\left(\pi_h+\pi_{hh}\right)} + \frac{\left(1-p\right)\pi_{hh}}{p\left(1-\widehat{\pi}\right)+\left(1-p\right)\left(\pi_h+\pi_{hh}\right)}\left(\frac{b-r}{v-r}\right)\right)v - b \\ = & \left(\frac{r}{v} + \frac{v-r}{v}\left(\frac{b-r}{v-r}\right)\right)v - b = 0 \end{aligned}$$

To summarize, if a bidder of type $v$ truthfully reports his type and follows the recommendations, then he gets a payoff of $v-r$ when he is suggested to bid low, and zero payoff otherwise. Hence,

[24] It is straightforward to verify that the entries in table: (i) sum up to one; and (ii) are nonnegative (since $r > pv$).

[25] Bidding exactly $r$ is dominated by bidding slightly above $r$ if there is a positive probability that the opponent bids $r$.

his ex ante payoff is $\left(p\widehat{\pi}+\left(1-p\right)\pi_r\right)\left(v-r\right)=\frac{\left(r^2-p^2v^2\right)\left(v-r\right)}{v^2+r^2-2pv^2}$.

If a bidder of type $v$ has reported 0, then he is suggested to bid 0. He knows that either his opponent is of type 0 and thus bids 0, or his opponent is of type $v$ and bids $r$ or high, with probabilities $p$, $\left(1-p\right)\widehat{\pi}$, and $\left(1-p\right)\left(1-\widehat{\pi}\right)$, respectively. If this bidder bids $b\in(r,v]$, then his payoff is

$$\left(\left(p+\left(1-p\right)\widehat{\pi}\right)+\left(1-p\right)\left(1-\widehat{\pi}\right)\left(\frac{b-r}{v-r}\right)\right)v-b\leq\max\left\{\left(p+\left(1-p\right)\widehat{\pi}\right)v-r,0\right\}=\frac{\left(r^2-p^2v^2\right)\left(v-r\right)}{v^2+r^2-2pv^2}$$

where the first equality follows from the fact that payoff is a linear function of $b$ and is thus maximized either at $b=r$ or at $b=v$.

Thus the considered communication rule is a communication equilibrium, and it achieves the desired payoffs. ■

**Proof of Proposition 9.** First, note that in any Nash equilibrium types 0 of both bidders bid 0. For bidder $i$ of type $v$, let $G_i:\{0\}\cup[r,\infty)\to[0,1]$ be the equilibrium distribution function, $\underline{b}_i$ and $\overline{b}_i$ be the infimum and the supremum of the support of the equilibrium bids, and $U_i\geq0$ be the equilibrium payoff.

Note that it is not possible to have $U_i<U_j$, since this implies $U_j\leq\prod_{k\neq j}\left(p+\left(1-p\right)G_k\left(\overline{b}_j\right)\right)v-\overline{b}_j$, and thus bidder $i$ has a profitable deviation to $\overline{b}_j+\varepsilon$, for $\varepsilon>0$ small:

$$\textstyle\prod_{k\neq j,i}\left(p+\left(1-p\right)G_k\left(\overline{b}_j+\varepsilon\right)\right)v-\left(\overline{b}_j+\varepsilon\right)\geq U_j-\varepsilon>U_i$$

Thus $U_i=U$ for every $i$. Also note that by bidding $r$ (or slightly above) each bidder of type $v$ can secure a payoff of $p^{n-1}v-r$. Thus $U\geq\max\left\{0,p^{n-1}v-r\right\}$. ■

If $U>\max\left\{0,p^{n-1}v-r\right\}$, then neither bidder bids 0 with positive probability, and thus $\underline{b}_i\geq r$. Also note that to get such a payoff each bidder of type $v$ with positive probability must win against the opponents of type $v$. Hence, we cannot have $\underline{b}_i\neq\underline{b}_j$; but if $\underline{b}_i=\underline{b}$ for every $i$, then all bidders must bid $\underline{b}$ with positive probability, which cannot happen in equilibrium. Thus, $U=\max\left\{0,p^{n-1}v-r\right\}$.

**Proof of Proposition 10.** We show for sufficiently small $p>0$ there exists a communication equilibrium such that each bidder of type $v$ gets a payoff of $2p^{n-1}v$. Consider the following symmetric communication rule, where each bidder is given one of three recommendations: bid 0; bid "low", i.e.

uniformly on $\left(0, \frac{1}{2}v\right]$; or bid "high", i.e. uniformly on $\left(\frac{1}{2}v, v\right]$. If all bidders report 0, then everyone is suggested to bid 0. If exactly one bidder reports $v$, then this bidder is suggested to bid low, and the others to bid 0. If $m \geq 1$ bidders report $v$ and the others report 0, then two out $m$ bidders are chosen by a fair lottery, say bidders $i$ and $j$, and all the other bidders are recommended to bid 0. The chosen bidders are given recommendations according to the probability distribution in the table below, where

$$g = \sum_{k=1}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} (1-p)^k p^{n-1-k} \left(\frac{2}{k+1}\right) = 2\left(\frac{1}{n}\frac{1-p^n}{1-p} - p^{n-1}\right)$$

is the probability that a bidder who submitted report $v$ was not the only one who submitted $v$ and was chosen.[26]

| $i$'s bid \ $j$'s bid | bid 0 | bid low | bid high |
|---|---|---|---|
| bid 0 | 0 | $\pi_l = \frac{1}{g}p^{n-1}$ | 0 |
| bid low | $\pi_l = \frac{1}{g}p^{n-1}$ | $\pi_{ll} = \frac{1}{4} + \frac{1}{g}p^{n-1}$ | $\pi_{hl} = \frac{1}{4} - \frac{1}{g}p^{n-1}$ |
| bid high | 0 | $\pi_{hl} = \frac{1}{4} - \frac{1}{g}p^{n-1}$ | $\pi_{hh} = \frac{1}{4} - \frac{1}{g}p^{n-1}$ |

We need to check the incentives to tell the truth and to comply with the recommendations only for the bidders of type $v$, since the bidders of type 0 have no incentive to lie or to disobey.

If a bidder of type $v$ has reported $v$ and is suggested to bid 0, then he knows that either he was not chosen (which happens with probability $1 - p^{n-1} - g$), or that he was chosen but only his opponent is suggested to bid above 0 (which happens with probability $g\pi_l$).

If this bidder bids $b \in \left(0, \frac{1}{2}v\right]$ instead, then he has a chance to win only if none of his opponents bid high. In particular, bidder $i$ could win if (i) he was not chosen, and one chosen bidder bids low (which happens with probability $\left(1 - p^{n-1} - g\right) 2\pi_l$); (ii) he was not chosen, and two chosen bidders bid low (which happens with probability $\left(1 - p^{n-1} - g\right) \pi_{ll}$); (iii) he was chosen, and his opponent bids low (which happens with probability $g\pi_l$). The expected payoff of this bidder is then

$$\left(\frac{\left(1 - p^{n-1} - g\right) 2\pi_l + g\pi_l}{1-g}\left(\frac{b}{\frac{1}{2}v}\right) + \frac{\left(1 - p^{n-1} - g\right)\pi_{ll}}{1-g}\left(\frac{b}{\frac{1}{2}v}\right)^2\right) v - b \qquad (11)$$

[26] It is straightforward to verify that the entries in table: (i) sum up to one; and (ii) are nonnegative for $p$ sufficiently small (since $g = \frac{2}{n}$ when $p = 0$).

Note that (11) is equal to 0 if $b = 0$. If $b = \frac{1}{2}v$, then (11) becomes

$$\left(\frac{\left(1 - p^{n-1} - g\right)\left(2\pi_l + \pi_{ll}\right) + g\pi_l}{1 - g}\right) v - \frac{1}{2}v = \left(\frac{\left(3\frac{1-p^{n-1}}{g} - \frac{9}{4}\right)p^{n-1}}{1 - g} - \frac{1}{4}\right) v \tag{12}$$

Note that (12) is equal to $-\frac{1}{4}v$ if $p = 0$, and thus it is nonpositive for $p$ small enough. Since (11) is convex in $b$, this implies that it is nonpositive for every $b \in \left(0, \frac{1}{2}v\right]$.

If this bidder bids $b \in \left(\frac{1}{2}v, v\right]$ instead, then he wins for sure if none of his opponents bid high, and has a chance to win otherwise. In particular, bidder $i$ wins for sure if (i) he was not chosen, and none of the chosen bidders bid high (which happens with probability $\left(1 - p^{n-1} - g\right)\left(2\pi_l + \pi_{ll}\right)$); (ii) he was chosen, and his opponent does not bid high (which happens with probability $g\pi_l$). Also bidder $i$ could win if (i) he was not chosen, and one chosen bidder bids high (which happens with probability $\left(1 - p^{n-1} - g\right)2\pi_{hl}$); (ii) he was not chosen, and two chosen bidders bid high (which happens with probability $\left(1 - p^{n-1} - g\right)\pi_{hh}$). The expected payoff of this bidder is then

$$\begin{aligned}&\left(\frac{\left(1 - p^{n-1} - g\right)\left(2\pi_l + \pi_{ll}\right) + g\pi_l}{1 - g} + \frac{\left(1 - p^{n-1} - g\right)2\pi_{hl}}{1 - g}\left(\frac{b - \frac{1}{2}v}{\frac{1}{2}v}\right) + \right. \\ &\left. + \frac{\left(1 - p^{n-1} - g\right)\pi_{hh}}{1 - g}\left(\frac{b - \frac{1}{2}v}{\frac{1}{2}v}\right)^2\right) v - b\end{aligned} \tag{13}$$

Note that (13) is equal to (12) if $b = \frac{1}{2}v$, and (13) is equal to zero if $b = v$. Since (13) is convex in $b$, this implies that it is nonpositive for every $b \in \left(\frac{1}{2}v, v\right]$.

If a bidder of type $v$ has reported $v$ and is suggested to bid low, then he knows that either he faces no opponents (with probability $p^{n-1}$), or that he was chosen and faces one chosen opponent who bids 0, low, or high with probabilities $g\pi_l$, $g\pi_{ll}$, and $g\pi_{hl}$, respectively. The expected payoff of this bidder from bidding any $b \in \left(0, \frac{1}{2}v\right]$ is

$$\left(\frac{p^{n-1} + g\pi_l}{p^{n-1} + g\left(\pi_l + \pi_{ll} + \pi_{hl}\right)} + \frac{g\pi_{ll}}{p^{n-1} + g\left(\pi_l + \pi_{ll} + \pi_{hl}\right)}\frac{b}{\frac{1}{2}v}\right) v - b = \frac{2p^{n-1}v}{2p^{n-1} + \frac{1}{2}g} \geq 0$$

If he bids $b \in \left(\frac{1}{2}v, v\right]$ instead, then his payoff is

$$\begin{aligned}\left(\frac{p^{n-1}+g\left(\pi_l+\pi_{ll}\right)}{p^{n-1}+g\left(\pi_l+\pi_{ll}+\pi_{hl}\right)}+\frac{g\pi_{hl}}{p^{n-1}+g\left(\pi_l+\pi_{ll}+\pi_{hl}\right)}\left(\frac{b-\frac{1}{2}v}{\frac{1}{2}v}\right)\right)v-b &= \frac{2p^{n-1}v}{2p^{n-1}+\frac{1}{2}g}\left(\frac{v-b}{\frac{1}{2}v}\right) \\ &< \frac{2p^{n-1}v}{2p^{n-1}+\frac{1}{2}g}\end{aligned}$$

If a bidder of type $v$ has reported $v$ and is suggested to bid high, then he knows that he was chosen and faces one chosen opponent who bids low or high, with equal probabilities. Thus the expected payoff of this bidder from bidding any $b \in (0, v]$ is equal to zero.

To summarize, if a bidder of type $v$ truthfully reports his type and follows the recommendations, then he gets a payoff of $\frac{2p^{n-1}v}{2p^{n-1}+\frac{1}{2}g}$ when he is suggested to bid low, and zero payoff otherwise. Hence, his ex ante payoff is $2p^{n-1}v$.

If a bidder of type $v$ has reported 0, then he is suggested to bid 0. He knows that he faces no active opponents (with probability $p^{n-1}$); one active opponent who bids on low (with probability $(n-1)(1-p)p^{n-2}+d2\pi_l$, where $d=\left(1-p^{n-1}-(n-1)(1-p)p^{n-2}\right)$); two active opponents who both bid low, both bid high, or one bids low and another high with probabilities $d\pi_{ll}$, $d\pi_{hh}$, and $d2\pi_{hl}$, respectively.

If this bidder bids $b \in \left(0, \frac{1}{2}v\right]$, then his payoff is

$$\left(p^{n-1}+\left((n-1)(1-p)p^{n-2}+d2\pi_l\right)\left(\frac{b}{\frac{1}{2}v}\right)+d\pi_{ll}\left(\frac{b}{\frac{1}{2}v}\right)^2\right)v-b \tag{14}$$

Note that if $b=0$, then (14) is equal $p^{n-1}v$ which is smaller than the payoff from truthtelling $2p^{n-1}v$. If $b=\frac{1}{2}v$ then (14) becomes

$$\left(p^{n-1}+(n-1)(1-p)p^{n-2}+d\left(2\pi_l+\pi_{ll}\right)\right)v-\frac{1}{2}v \tag{15}$$

Note that (15) is equal to $-\frac{1}{4}v$ if $p=0$, and thus it is nonpositive for $p$ small enough. Since (14) is convex in $b$, this implies that it is nonpositive for every $b \in \left(0, \frac{1}{2}v\right]$.

If this bidder bids $b \in \left(\frac{1}{2}v, v\right]$, then his payoff is

$$\left(p^{n-1} + (n-1)(1-p)\,p^{n-2} + d\,(2\pi_l + \pi_{ll}) + d2\pi_{hl}\left(\frac{b - \frac{1}{2}v}{\frac{1}{2}v}\right) + d\pi_{hh}\left(\frac{b - \frac{1}{2}v}{\frac{1}{2}v}\right)^2\right) v - b \quad (16)$$

Note that (16) is equal to (15) if $b = \frac{1}{2}v$, and (16) is equal to zero if $b = v$. Since (16) is convex in $b$, this implies that it is nonpositive for every $b \in \left(\frac{1}{2}v, v\right]$.

Thus the considered communication rule is a communication equilibrium, and it achieves the desired payoffs. ■

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