Network formation under institutional constraints*

By Norma Olaizola[†] and Federico Valenciano[‡]

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Abstract

We study the effects of institutional constraints on stability, efficiency and network formation. An exogenous "societal cover" consisting of a collection of possibly overlapping subsets that covers the set of players and no set in this collection is contained in another specifies the social organization in different groups or "societies". It is assumed that a player may initiate links only with players that belong to at least one society that s/he also belongs to, thus restricting the feasible networks. In this setting, we examine the impact of societal constraints on stable architectures and on dynamics, without and with decay.

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[†] *BRiDGE* group (http://www.bridgebilbao.es), Departamento de Fundamentos del Análisis Económico I, Universidad del País Vasco, Avenida Lehendakari Aguirre 83, 48015 Bilbao, Spain; norma.olaizola@ehu.es.

[‡] *BRiDGE* group (http://www.bridgebilbao.es), Departamento de Economía Aplicada IV, Universidad del País Vasco, Avenida Lehendakari Aguirre 83, 48015 Bilbao, Spain; federico.valenciano@ehu.es.

1 Introduction

In recent years the study of the economics of networks has attracted considerable attention from researchers and become one of the hottest topics of economic research[1]. The economics of networks is, in Goyal's words, "*an ambitious research program which combines aspects of markets (e.g., prices and competition) along with explicit patterns of connections between individual entities to explain economic phenomena*" (Goyal, 2007, p. 6).

Several seminal papers provide the basic models of strategic formation of networks paying special attention to stability and efficiency. In the simplest model links are formed unilaterally (Goyal (1993), Bala (1996)). In this setting Bala and Goyal (2000a) study Nash and strict Nash stability and provide a dynamic model. A model where links are formed on the basis of bilateral agreements is studied by Jackson and Wolinsky (1996), who introduce the notion of pairwise stability. In these seminal papers it is assumed that there is homogeneity across players and also that the current network is common knowledge to all node-players. These models have been extended in different directions. Bala and Goyal (2000b) introduce imperfect reliability of links. Galeotti et al. (2006) consider heterogeneous players, while Bloch and Dutta (2009) consider endogenous link strength. The common knowledge assumption may be unrealistic in many cases, and indeed is dropped by McBride (2006), who studies the effects of limited perception, namely, assuming that each node-player perceives the current network only up to a certain distance from the node.

In the seminal models networks provide a means for the flow of information or other benefits through the links, but the current network is assumed to be common knowledge to all players, who may unrestrictedly initiate links with any other players. In some cases this may be an unrealistic assumption, and in general the larger the number of agents and the network are the more unrealistic it will be. Due to what we generically refer here to as "institutional constraints" (social, cultural, linguistic, geographical, economic, etc.), often individuals may see only "part of the world" and initiate links only within that part or a part of that part. Thus it seems more realistic to assume that a set of possibly overlapping groups (family, tribe, clan, club, gender, age, linguistic community, nationality, professional association, department, etc., depending on the context) configures the social constraints within which individuals interact[2]. More precisely, we assume that each individual may initiate links only within the groups she belongs to. In a way this is an unorthodox approach if, as put by Goyal,

[1]Some recent books surveying this literature are Goyal (2007), Jackson (2008) and Vega-Redondo (2007).

[2]The importance of group membership is nowadays widely recognized. A vast literature in psychology deals with the relationship between identity and group membership since at least Tajfel and Turner (1979) (see also Brewer (1991) and Brewer and Gardner (1996)). In the economic literature Akerlof and Kranton (2000) introduce a model where group membership enters the definition of a utility function. In the experimental field see Chen and Li (2009). For a recent attempt to relate networks and identity formation see Dev (2010).

"*the theoretical research on network effects (..) is motivated by the idea that,* within the same group [in italics]*, individuals will have different connections and that this difference in connections will have a bearing on their behavior.*" (Goyal, 2007, p. 7). Nevertheless, this is the approach adopted here, and it is worth remarking that the orthodox single-group assumption is in fact a particular case of the more general setting adopted here. In particular, this allows Bala and Goyal's (2000a) "two-way flow" basic model, on which we concentrate in this paper, to be integrated into a wider model which sheds new light on various conclusions of their model, showing which prevail and up to which point, and which do not in this wider setting.

Based on this idea, in this paper we focus on the effects of such institutional constraints on stability, efficiency and network formation. More precisely, an exogenous "societal cover" specifies social organization in different groups or "societies". A societal cover is a collection of possibly overlapping subsets of the set of players or "societies" that covers the whole set (i.e., each player belongs to at least one set in this collection) such that no set in this collection is contained in another. It is assumed that a player may initiate links only with players that belong to one or more of the societies that she also belongs to, thus restricting her feasible strategies, and as a consequence the feasible networks.

Note that in this scenario only the players in the possibly empty "societal core", i.e., those that belong to all societies, may have direct access to all individuals. It is also assumed that only the part of the current network within each "component" *of the societal cover* (in a sense to be specified later) is common knowledge to all players in that "component". Note also that this model collapses to Bala and Goyal's (2000a) unrestricted setting for the particular case of the simplest societal cover consisting of a single society including all players. The notion of societal cover seems a rather natural constraining structure in the link-formation context. Moreover, we prove a somewhat confirming result relative to this naturalness: the societal cover model provides the most general *symmetric* link-formation constraint that can be considered. This in particular means that in the context of *bilateral* link formation (Jackson and Wolinsky, 1996), where only symmetric constraints make sense, the societal cover provides a general model of constraint. Of course, in the context of unilateral link-formation other (i.e., non symmetric) types of constraints can be considered.

For any given societal cover we constrain our attention to the admissible networks (i.e., those consistent with the cover) and first extend Bala and Goyal's (2000a) notion of a Nash network as those admissible networks where no player has an incentive to change her strategy, i.e., her choice of admissible links. We then easily extend their characterization of Nash networks as those among the admissible networks which are minimally connected. In this way the set of such Nash networks is a subset of the set of Bala and Goyal's unrestricted Nash networks. Then Bala and Goyal's (2000a) notion of strict Nash network is also naturally extended to this setting. Now a strict Nash network is a network consistent with the societal cover where no player may initiate and/or delete any admissible link(s) without loss. By contrast with Nash networks, things turn out to be more complicated with *strict* Nash networks. In Bala and Goyal's setting the center-sponsored star is the only (non empty) architecture of strict Nash networks, while in our setting the center-sponsored star architecture is feasible only when the societal core, i.e. the set of players belonging to all societies, is not empty. Moreover, even when the center-sponsored star architecture is feasible, this is not the only possible architecture of strict Nash networks. A variety of architectures of strict Nash networks appear for any non single-society cover, and the more complex the societal cover the greater this variety is. Nevertheless, some patterns are common to these architectures. Moreover, a full characterization of all strict Nash networks for a societal cover is provided by means of a condition that encapsulates synthetically the essence of the architecture of these networks, embodying a clear hierarchical principle. The main features of their architectures, where stars continue to play a prominent role, are studied. Particular attention is paid to the role of players who belong to more than one society, by means of whom different but overlapping societies can be connected. It turns out that the two-way flow model under societal constraints yields as strict Nash networks the paradigm of hierarchical structures: either oriented diverging trees (also called "arborescences" in graph theory) or a sort of "grafted" oriented trees. The latter are proved to be possible only when there are "hinge-players", i.e., players who are the *unique* common member of two or more societies.

We then apply Bala and Goyal's dynamic model, where starting from any initial network each player with some positive probability plays a best response or randomizes across them when there is more than one, otherwise the player exhibits inertia, i.e., keeps her links unchanged. In this way a Markov chain on the state space of all networks is defined. In Bala and Goyal's setting, the absorbing states are precisely the strict Nash networks and they prove that starting from any network the dynamic process converges to a strict Nash network (i.e., the empty network or a center-sponsored star) with probability 1. When adapted to our setting the best response dynamic model *does not* necessarily lead to strict Nash networks. The reason is that in our more complex setting this dynamic process may lead to the formation of partially stable "incomplete" strict Nash incompatible networks that cannot be part of the same strict Nash network, thus blocking the converging process. Therefore institutional constraints may hinder the way towards strict Nash networks. Nevertheless, best response dynamics lead to absorbing sets of minimally connected networks that we call "*quasi-strict* Nash networks" and characterize them. Thus, with probability 1, best response dynamics would lead either to a strict Nash network (whenever the set of quasi-strict Nash networks reached is a singleton) or one of these absorbing sets of quasi-strict Nash networks where the best response dynamics would oscillate for ever. Nevertheless *stability is reached in terms of payoffs* as it is proved that all quasi-strict Nash networks within each of these absorbing sets yield the same payoffs to all players.

Finally we examine the impact of decay in this setting. We first partially extend some of Bala and Goyal's results studying the relative robustness of different strict Nash networks in the presence of decay for certain societal covers. It turns out that when feasible, i.e., when the societal core is not empty, stars are the most robust strict Nash networks. More precisely, in the presence of decay stars remain strictly stable within a wider range of values for the parameters, while other strict Nash architectures remain stable only within narrower ranges. Then we study stochastically stable networks using Feri's (2007) dynamic model. We obtain similar conclusions about the relative robustness of different strict Nash architectures. We extend Ferri's (2007) showing in particular how when the societal core is not empty and for all the societies the number of individuals that belong to that society and only that one is sufficiently large, stars are the only stochastically stable architectures. As to efficiency in the presence of decay, a general conclusion arises: in the presence of decay efficiency and stability go hand in hand in the following sense: the greater the robustness of the stability is, the greater the efficiency, i.e. the greater the aggregated utility.

The rest of the paper is organized as follows. In section 2 the basic model is specified along with the necessary notation and terminology. Section 3 studies stability and efficiency under institutional constraints. In section 4 Bala and Goyal's dynamic model is extended to this setting. In section 5 we study the effects of introducing decay in the model. Finally, section 6 summarizes the main conclusions and points out some lines of further research.

2 The model

Let $N = \{1, 2, .., n\}$ denote the set of *nodes* or *players*. Players may initiate or delete *links* with other players. By $g_{ij} \in \{0, 1\}$ we denote the existence ($g_{ij} = 1$) or not ($g_{ij} = 0$) of a link connecting i and j initiated by i. Vector $g_i = (g_{ij})_{j \in N \backslash i} \in \{0, 1\}^{N \backslash i}$ specifies[3] the set of links initiated or supported[4] by i and will be referred to as an (unrestricted) *strategy* of player i. $G_i := \{0, 1\}^{N \backslash i}$ denotes the set of i's (unrestricted) strategies and $G_N = G_1 \times G_2 \times .. \times G_n$ the set of (unrestricted) strategy profiles. An unrestricted strategy profile $g \in G_N$ univocally determines a directed *network*[5] (N, Γ_g), where

$$\Gamma_g := \{(i, j) \in N \times N : g_{ij} = 1\},$$

that we identify with g and refer to as network g. If $M \subseteq N$ we denote by $g \mid_M$ the *subnetwork* $(M, \Gamma_{g|_M})$ with

$$\Gamma_{g|_M} := \{(i, j) \in M \times M : g_{ij} = 1\}.$$

We now consider the following situation. An exogenous "societal cover" specifies a set of possibly overlapping "societies" that represent a social constraint in the following

[3]We always drop the brackets "{..}" in expresions such as $N \backslash \{i\}$.

[4]We use indistinctly both terms, with preference for the first when dynamics considerations are involved and for the second otherwise.

[5]In graph theory this is called a "digraph" without loops, i.e., edges connecting a node with itself (see, for instance, Tutte (1984)).

sense: each player in N can initiate links with any other player as long as they belong to the same society. Formally, we have the following

Definition 1 *A "societal cover" of N is a collection of subsets of N (called "societies"), $\mathcal{K} \subseteq 2^N$, such that: (i) $\bigcup_{A \in \mathcal{K}} A = N$, and (ii) for all $A, B \in \mathcal{K}$ $(A \neq B)$, $A \nsubseteq B$.*

Condition *(i)* ensures that every player belongs to at least one society; while condition *(ii)* precludes superfluous societies: if $A \subseteq B$, A would be superfluous given the interpretation of societies.

We denote by $\mathcal{K}_i \subseteq \mathcal{K}$ the set of *societies* that i belongs to, and by $N(\mathcal{K}_i) \subseteq N$ the set of *nodes* that i may directly access, that is:

$$\mathcal{K}_i := \{A \in \mathcal{K} : i \in A\}$$

and

$$N(\mathcal{K}_i) := \bigcup_{A \in \mathcal{K}_i} A.$$

Two nodes i, j have *identical affiliation* if they belong to the same societies, i.e., $\mathcal{K}_i = \mathcal{K}_j$. Two nodes i, j have the *same reach* if $N(\mathcal{K}_i) = N(\mathcal{K}_j)$. Note that *identical affiliation* implies the *same reach*, but the converse is not true.

Example 1 If $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and

$$\mathcal{K} := \{\{1, 2, 3, 4, 5, 6\}, \{4, 5, 6, 7, 8, 9\}, \{1, 2, 4, 5, 7, 8\}, \{2, 3, 5, 6, 8, 9\}\},$$

then 2 and 4 have the *same reach*: $N(\mathcal{K}_2) = N(\mathcal{K}_4) = N$, but *different affiliations* as $\mathcal{K}_2 \neq \mathcal{K}_4$:

$$\mathcal{K}_2 = \{\{1, 2, 3, 4, 5, 6\}, \{1, 2, 4, 5, 7, 8\}, \{2, 3, 5, 6, 8, 9\}\}$$

and

$$\mathcal{K}_4 = \{\{1, 2, 3, 4, 5, 6\}, \{1, 2, 4, 5, 7, 8\}, \{4, 5, 6, 7, 8, 9\}\}.$$

Observe that we consider a particular type of a more general situation where an exogenous "link-constraining system" specifies for each player in N with which other players she can initiate links. Formally, we have the following

Definition 2 *A "link-constraining system" in N is a collection of subsets of N, $\mathcal{L} = \{\mathcal{L}_i\}_{i \in N}$, such that, for all i, $i \in \mathcal{L}_i$.*

With the interpretation anticipated: each player i is assumed to be able to initiate links with any player in $\mathcal{L}_i$ (different from himself, as it is only a matter of convenience to include i in set $\mathcal{L}_i$). Note that this allows for asymmetric situations, where it may be the case that a player i may initiate a link with j but j cannot initiate a link with i. In particular, a societal cover $\mathcal{K}$ imposes a link-constraining system, namely

$\{N(\mathcal{K}_i)\}_{i\in N}$, by limiting the reach of each player. This arises the reciprocal issue: under which conditions a link-constraining system $\mathcal{L}$ can be interpreted as associated with or imposed by a societal cover? The answer is given by the following condition: a link-constraining system $\mathcal{L}$ is *symmetric* if for all $i, j \in N$: $i \in \mathcal{L}_j$ if and only if $j \in \mathcal{L}_i$. Then we have the following

Proposition 1 *A link-constraining system $\mathcal{L}$ can be interpreted as associated with or imposed by a societal cover if and only if it is symmetric.*

Proof. *Necessity* ($\Rightarrow$): It follows immediately from the constraints imposed by a societal cover.

Sufficiency ($\Leftarrow$): Let $\mathcal{L}$ be a symmetric link-constraining system. Define $\mathcal{K}(\mathcal{L})$ as the set of nonempty subsets A of N s.t. (i) for all $i \in A$, $A \subseteq \mathcal{L}_i$, and (ii) no $A' \supsetneq A$ exists that satisfies condition (i). First note that $\mathcal{K}(\mathcal{L})$ is well-defined given that the set of subsets that satisfy condition (i) is partially ordered by inclusion, and consequently maximal elements do exist. $\mathcal{K}(\mathcal{L})$ consists of such maximal elements. Note also that for all i, $i \in A$ for some $A \in \mathcal{K}(\mathcal{L})$, given that $\{i\} \subseteq \mathcal{L}_i$, and for all $A, A' \in \mathcal{K}(\mathcal{L})$ s.t. $A \neq A'$, $A \nsubseteq A'$. Therefore $\mathcal{K}(\mathcal{L})$ is a societal cover. Only remains to be shown that for all i, $N(\mathcal{K}_i(\mathcal{L})) = \mathcal{L}_i$, i.e., that $\cup_{A:i\in A\in\mathcal{K}(\mathcal{L})}A = \mathcal{L}_i$. First, assume $j \in N(\mathcal{K}_i(\mathcal{L})) = \cup_{A:i\in A\in\mathcal{K}(\mathcal{L})}A$, i.e., $j \in A \in \mathcal{K}(\mathcal{L})$ for some A s.t. $i \in A$. Then by condition (i) in the definition of $\mathcal{K}(\mathcal{L})$, $A \subseteq \mathcal{L}_i$ and consequently $j \in \mathcal{L}_i$. Now reciprocally, assume $j \in \mathcal{L}_i$. Then, given that $\mathcal{L}$ is symmetric, $i \in \mathcal{L}_j$. Therefore $\{i, j\}$ satisfies condition (i) in the definition of $\mathcal{K}(\mathcal{L})$, and consequently $\{i, j\} \subseteq A$ for some $A \in \mathcal{K}(\mathcal{L})$. Thus $j \in \cup_{A:i\in A\in\mathcal{K}} = N(\mathcal{K}_i(\mathcal{L}))$. ■

Proposition 1 provides a different (but equivalent) interpretation: a societal cover means a *symmetric* link-constraining system[6]. It is worth remarking the generality of the type of constraint a societal cover imposes: all the results that follow apply to any symmetric link-constraining system[7]. Nevertheless we find the societal cover notion closer to, or at least embodying a more intuitive perception of, real world constraints, and it is in these terms that all results are presented.

The following terminology is used. A *component $\mathcal{C}$ of a societal cover $\mathcal{K}$* is a subset $\mathcal{C} \subseteq \mathcal{K}$ such that *(i)* for all $A, B \in \mathcal{C}$ there exist $A_1, .., A_k \in \mathcal{K}$ s.t. $A_1 = A$ and $B = A_k$, and $A_i \cap A_{i+1} \neq \varnothing$ for $i = 1, .., k-1$, and *(ii)* for all $B \in \mathcal{K}\backslash\mathcal{C}$, $B \cap (\cup_{A\in\mathcal{C}}A) = \varnothing$. The subset $\cup_{A\in\mathcal{C}}A$ of N covered by a component $\mathcal{C}$ is denoted by $N(\mathcal{C})$. For each i, $\mathcal{C}_i(\mathcal{K})$ denotes the component of $\mathcal{K}$ that contains $\mathcal{K}_i$. A societal cover is *connected* if it has a

[6] Different societal covers may yield the same link-constraining system. Nevertheless, for a link-constraining system $\mathcal{L}$ the cover $\mathcal{K}(\mathcal{L})$ constructed in the proof of Proposition 1 is the maximal one that yields it in the following sense: any society of any cover that yields that link-constraining system is contained in some society of $\mathcal{K}(\mathcal{L})$. For instance, the maximal societal cover that represents the link-constraining system imposed by the societal cover in Example 1 is $\{N\backslash\{1,3\}, N\backslash\{3,9\}, N\backslash\{7,9\}, N\backslash\{1,7\}\}$.

[7] In particular, in the context of bilateral link formation (Jackson and Wolinsky, 1996) only symmetric link-constraining make sense. In other words in the context of bilateral link formation the societal cover provides a general model of constraint.

unique component. The *societal core* of a societal cover is the set of nodes that belong to all societies

$$core(\mathcal{K}) := \bigcap_{A \in \mathcal{K}} A.$$

This set may be empty. Note that only the players in the societal core may have direct access to all individuals in N.

Let $\mathcal{K}$ be a societal cover of N, if $\mathcal{K}' \subseteq \mathcal{K}$ we say that $\mathcal{K}'$ is a *subcover* of $\mathcal{K}$ if $\mathcal{K}'$ is a societal cover of $N(\mathcal{K}') := \cup_{A \in \mathcal{K}'} A$ s.t. for all $A \in \mathcal{K}$, $A \subseteq N(\mathcal{K}')$ implies $A \in \mathcal{K}'$. In particular, a component of a a societal cover $\mathcal{K}$ is a (connected) subcover of $\mathcal{K}$.

The following definition constrains the structure of a network so as to be consistent with a given societal cover of N by ruling out links connecting individuals who are not members of at least one society in common.

Definition 3 *A network g is consistent with a societal cover $\mathcal{K}$ (or is a $\mathcal{K}$-network) if for every link $g_{ij} = 1$ there exists some $A \in \mathcal{K}$ s.t. $i, j \in A$ (i.e., $\mathcal{K}_i \cap \mathcal{K}_j \neq \varnothing$).*

A vector $g_i = (g_{ij})_{j \in N(\mathcal{K}_i) \setminus i} \in \{0,1\}^{N(\mathcal{K}_i) \setminus i}$ specifies a set of $\mathcal{K}$-*feasible* links initiated by i and is referred to as a $\mathcal{K}$-*admissible strategy* of player i, as we assume i's capacity to choose which links to initiate *in* $N(\mathcal{K}_i)$. $G_i(\mathcal{K}) := \{0,1\}^{N(\mathcal{K}_i) \setminus i}$ denotes the set of i's $\mathcal{K}$-admissible strategies and $G_{\mathcal{K}} = G_1(\mathcal{K}) \times G_2(\mathcal{K}) \times .. \times G_n(\mathcal{K})$ the set of $\mathcal{K}$-admissible strategy profiles. A $\mathcal{K}$-admissible strategy profile g univocally determines a $\mathcal{K}$-network that we identify with g.

Observe that this setting is *not* narrower than Bala and Goyal's standard one. It is in fact more general as the standard (i.e., unrestricted) notions of network, strategy and strategy profile correspond to the particular case of the simplest societal cover $\mathcal{K} = \{N\}$, where a single society includes all players and all links are feasible.

Given a network g, we denote $\bar{g}_{ij} := max\{g_{ij}, g_{ji}\}$. In this way a nondirected network $\bar{g}$ is defined[8]. $\bar{g}$ represents the effective communication provided by network g, which is independent of who initiated the existing links according to the assumptions of the model. We say that there is a *path* of *length* k from i to j in g if there exist $k+1$ players $j_0, j_1, .., j_k$, s.t. $i = j_0$, $j = j_k$, and for all $l = 1, .., k$, $\bar{g}_{j_{l-1}j_l} = 1$, and we say that such a path is i-*oriented* if for all $l = 1, .., k$, $g_{j_{l-1}j_l} = 1$ and $g_{j_l j_{l-1}} = 0$. A path (oriented or not) is $\mathcal{K}$-feasible if all its links are $\mathcal{K}$-feasible. The set of players with whom i supports a link is denoted by $N^d(i; g)$, and the set of players connected with i by a path (union $\{i\}$) by $N(i; g)$, and their cardinalities by $\mu_i^d(g) := \#N^d(i; g)$ and $\mu_i(g) := \#N(i; g)$. Note that if g is a $\mathcal{K}$-network then $N^d(i; g) \subseteq N(\mathcal{K}_i)$ and $N(i; g) \subseteq N(\mathcal{C}_i(\mathcal{K}))$. We say that a network g is an *oriented diverging tree* (*converging tree*) if there is a node i_0 such that for any other node j

there is a unique path connecting it with the *node root* i_0 and such path is i_0-*oriented* (j-*oriented*).

[8]In graph theory terms, $\bar{g}$ is the "underlying graph" of digraph g (see, e.g., Tutte, 1984).

It is assumed that each node contains valuable information and a link allows that information to flow in both directions without decay independently of who initiates it, so that each node receives the information from all nodes with which it is connected by a path. Let $v_{ij} > 0$ be the payoff that player i derives from connecting directly (by a link) or indirectly (by a path) with player j, and $c_{ij} > 0$ the cost for player i of initiating a link with j. Thus the payoff of player i in g is

$$\Pi_i(g) = \sum_{j \in N(i;g)} v_{ij} - \sum_{j \in N^d(i;g)} c_{ij}.$$

If we assume costs and benefits to be homogeneous across players (i.e., $v_{ij} = v$ and $c_{ij} = c$, for all i, j) and $v > c$, connections with *new* nodes are always profitable and[9]

$$\Pi_i(g) = v\mu_i(g) - c\mu_i^d(g). \tag{1}$$

A $\mathcal{K}$-network is *efficient* if it maximizes the aggregate payoff under the constraint of $\mathcal{K}$-feasible payoffs, that is, those that can be obtained by means of $\mathcal{K}$-networks.

A *component* of a network g is a subnetwork $g\,|_C$, where $C \subseteq N$, such that any two players in C are connected by a path, and no player in $N \setminus C$ is connected by a path with a player in C. We say that g is *connected* if g is the unique component of g. A network is *minimal* if for all i, j s.t. $g_{ij} = 1$, the number of components of g is smaller than the number of components of $g - ij$, where $g - ij$ is the network that results by replacing $g_{ij} = 1$ by $g_{ij} = 0$ in g (similarly, when $g_{ij} = 0$ we write $g + ij$ to represent the network that results by replacing $g_{ij} = 0$ by $g_{ij} = 1$ in g). A network is *minimally connected* if it is connected and minimal.

Remark: Note the relationship between the notions of connected component of a societal cover $\mathcal{K}$ of N and connected component of a $\mathcal{K}$-network: a connected component of a $\mathcal{K}$-network is always covered by a connected component of the societal cover $\mathcal{K}$.

We denote by g_{-i} the network where all links initiated by i in g are deleted, and by (g_{-i}, g_i') the strategy profile and network that results by replacing g_i by g_i' in g. In particular, $(g_{-i}, g_i) = g$.

We next discuss some notions of stability of networks consistent with a given societal cover $\mathcal{K}$.

3 Stability and efficiency

The following definitions are natural extensions of the notions of Nash stability and strict Nash stability due to Bala and Goyal (2000a) for a network in a scenario where

[9] Although the results presented here can easily be extended with some slight modifications to the case where payoffs are, as in Bala and Goyal (2000a), given by a function $\Phi(\mu_i(g), \mu_i^d(g))$, where $\Phi(x, y)$ is strictly increasing in x and strictly decreasing in y, we prefer this simpler assumption about payoffs so as to make the statements of the basic results simpler. The assumption $v > c$ is dropped in section 5 when we consider decay.

payoffs are given by (1) and: (i) a societal cover $\mathcal{K}$ allows only for links connecting individuals belonging to the same society, and (ii) all players in the same component $\mathcal{C}$ of $\mathcal{K}$, i.e., in $N(\mathcal{C})$, have common knowledge of the part of the current network connecting individuals of $N(\mathcal{C})$. The common knowledge assumption restricted to players in the same component of the cover can be justified by assuming that information about the current network propagates between overlapping societies. Note that this scenario yields the unconstrained and common-knowledge environment of Bala and Goyal (2000a) for the particular case of the simplest societal cover: $\mathcal{K} = \{N\}$.

Definition 4 *A Nash $\mathcal{K}$-network is a $\mathcal{K}$-network g that is stable under $\mathcal{K}$-admissible strategies, that is, for all $i \in N$:*

$$\Pi_i(g) \geq \Pi_i(g_{-i}, g_i') \quad \text{for all } g_i' \in G_i(\mathcal{K}). \tag{2}$$

When (2) holds we say that g_i is a *best (admissible) response* of i to g_{-i}. Thus, in a Nash $\mathcal{K}$-network every player is playing a best $\mathcal{K}$-admissible response to those played by the others. Note that for $\mathcal{K} = \{N\}$ a Nash $\mathcal{K}$-network is a Nash network in the standard setting.

The stability notion can be refined in the strict sense by extending Bala and Goyal's strict Nash networks.

Definition 5 *A strict Nash $\mathcal{K}$-network is a Nash $\mathcal{K}$-network g such that for all $i \in N$:*

$$\Pi_i(g) > \Pi_i(g_{-i}, g_i') \quad \text{for all } g_i' \in G_i(\mathcal{K}) \ (g_i' \neq g_i). \tag{3}$$

Thus (3) means that in a strict Nash $\mathcal{K}$-network every player is playing her *unique* best (admissible) response to those played by the others. Also note that for $\mathcal{K} = \{N\}$ a strict Nash $\mathcal{K}$-network is a strict Nash network in the standard setting.

Given the constraints on information, strategies and feasible networks that a societal cover imposes, the set of players $N(\mathcal{C})$ in each component $\mathcal{C}$ of the cover, where subcover $\mathcal{C}$ prescribes what links are feasible, form an entirely "separate world": no link with $N \backslash N(\mathcal{C})$ is possible and no information about it reaches $N(\mathcal{C})$. In particular we have the following straightforward result.

Proposition 2 *A $\mathcal{K}$-network g is a Nash (strict Nash) $\mathcal{K}$-network if and only if $g \mid_{N(\mathcal{C})}$ is a Nash (strict Nash) $\mathcal{C}$-network for each component $\mathcal{C}$ of $\mathcal{K}$.*

Remark: Note also that although societies consisting of a single individual are included in the model, such trivial societies are of no interest in this setting. Moreover, the only connected societal cover $\mathcal{K}$ that contains a society A s.t. $\#A = 1$ is $\mathcal{K} = \{A\}$.

Therefore, in view of Proposition 2 and the preceding remark, in what follows *we constrain our attention to connected societal covers and we always assume that all societies have at least two individuals* unless otherwise specified. The following proposition extends Bala and Goyal's result to this setting.

Proposition 3 *Given a connected societal cover $\mathcal{K}$ of N, a $\mathcal{K}$-network g is a Nash $\mathcal{K}$-network if and only if it is minimally connected.*

Proof. Let $\mathcal{K}$ be a connected societal cover of N, and g a $\mathcal{K}$-network. Assume g is not connected. Then there exist two nodes $i, j \in N$ not connected by a path in g. As cover $\mathcal{K}$ is connected, there exists a finite sequence of nodes $x_1, .., x_m$, such that $x_1 = i$, $x_m = j$ and for each $k = 1, .., m-1$, there is some $A \in \mathcal{K}$ s.t. $x_k, x_{k+1} \in A$. Then for at least two consecutive nodes among these m nodes, say x_k and x_{k+1}, there is no path in g connecting them. But then it is feasible and profitable for either of these two nodes to initiate a link with the other. Thus g must be connected. If it were not minimal there would be some superfluous link that could be eliminated and that would benefit the player that did so, and consequently g is not a Nash $\mathcal{K}$-network.

Reciprocally, assume that g is minimally connected. Let i be any player and g_i' be any strategy $g_i' \in G_i(\mathcal{K})$ $(g_i' \neq g_i)$. We show that $\Pi_i(g) \geq \Pi_i(g_{-i}, g_i')$. A new strategy $g_i' \neq g_i$ means deleting some links and initiating new ones. If g is minimally connected, then each deletion means disconnecting i with a set of nodes, and if there is more than one deletion any two of these sets of nodes disconnected from i must also be disconnected from each other (otherwise a deleted link would be redundant). Thus the number of links initiated should be at least equal to the number deleted, otherwise the payoff would decrease. But then i's payoff for (g_{-i}, g_i') cannot be greater than for g. Therefore if g is minimally connected no player has an incentive to make any $\mathcal{K}$-admissible change. ■

In Bala and Goyal (2000a) the following result is established (in our terminology and under the assumptions about costs and benefits made here[10]): a network is efficient if and only if it is minimally connected, and Nash networks are those minimally connected. In view of this, we have the following

Corollary 1 *When the societal cover $\mathcal{K}$ is connected the following conditions are equivalent for a network g:*
(i) g is a Nash $\mathcal{K}$-network.
(ii) g is a $\mathcal{K}$-consistent Nash network.
(iii) g is an efficient $\mathcal{K}$-network.

Therefore, for any given set of nodes N and any societal cover $\mathcal{K}$, the set of Nash $\mathcal{K}$-networks is a subset of the set of standard unrestricted Nash networks. In Figure 1 two minimally connected networks are represented[11]: (a) is a Nash $\mathcal{K}$-network, while (b) is not a Nash $\mathcal{K}$-network because one link connects two nodes that do not belong to the same society.

[10] In fact, given their weaker assumptions on the payoffs (see footnote 8), the empty network may also be Nash stable in their setting.

[11] As in all figures, nodes are represented by dots (without labels unless conveninet for the purpose of the illustration), links by segments between them, and a filled circle indicates the node that supports it.

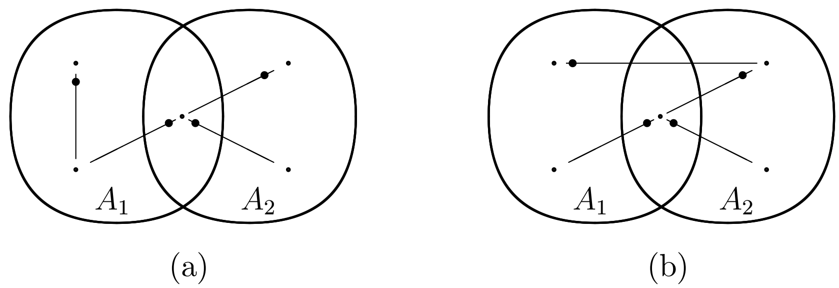

Figure 1: Nash networks and Nash $\mathcal{K}$-networks.

We now focus on strict Nash $\mathcal{K}$-networks. "Stars" of different types play an important role in network stability in different contexts (see, Bala and Goyal (2000a, 2006), Jackson and Wolinsky (1996), Bloch and Dutta (2009)), and, as we show below, they are also important in connection with strict Nash $\mathcal{K}$-networks. In this context the following variant of the notion of center-sponsored star proves useful.

Definition 6 *A set of players $M \subseteq N$ ($\#M \geq 2$) is said to be connected by a center-sponsored star s in a network g if $g\mid_M = s$ and there is a node $i \in M$ s.t. $N^d(i;g) = M \backslash i$ and $g_{jk} = 0$ for all $j \in M \backslash i$ and all $k \in M \backslash j$.*

Note that, according to this definition: (i) a center-sponsored star does *not* necessarily connect all players in N; (ii) its center i can be linked from other nodes different from those in the star; and (iii) the nodes in the periphery, i.e., those j in M s.t. $g_{ij} = 1$ can be connected with other nodes that do not belong to the star.

Re-stated in terms of the current setting, notation and terminology, and adapted to it, Bala and Goyal (2000a) establish the following result: *the only strict Nash networks are those consisting of a single center-sponsored star that connects all players*[12].

As we show below, the societal cover diversifies the stable/efficient networks as strict Nash $\mathcal{K}$-networks are not necessarily center-sponsored stars. A variety of constellations of interconnected stars emerges as possible strict Nash $\mathcal{K}$-networks depending on the structure of the societal cover; moreover, in general, several architectures appear as strict Nash for a given societal cover. Our next goal is to identify and characterize these networks.

In the characterization of strict Nash $\mathcal{K}$-networks the following binary relation on N associated with a network g plays an important role. Let $\xrightarrow{g}$ be the transitive closure of the binary relation L_g defined by

$$i \; L_g \; j \quad \Leftrightarrow (i = j \quad or \quad g_{ij} = 1)$$

That is to say, $i \xrightarrow{g} j$ if $i = j$ or there exists an *i-oriented* path from i to j. This relation is obviously transitive, but in general, for an arbitrary network g, is not com-

[12] Given their weaker assumptions on the payoffs (see footnotes 8 and 9), the empty network may also be strict Nash in their setting.

plete, antisymmetric or acyclic[13]. But if g is minimally connected, then $\xrightarrow{g}$ is certainly antisymmetric and acyclic (otherwise at least one link would be redundant). Thus, in view of Proposition 3, we have the following

Lemma 1 *For any Nash $\mathcal{K}$-network g, the binary relation $\xrightarrow{g}$ is a partial order on N.*

For any Nash $\mathcal{K}$-network g, we use the following terminology. We say that i is a *predecessor* of j (and that j is a *successor* of i) in g if $i \neq j$ and $i \xrightarrow{g} j$. We say that a node is *terminal* in g if it has no successors, and we say that a node is *maximal* in g if it has no predecessors.

As we presently prove, *strict* Nash $\mathcal{K}$-networks have a strongly hierarchical structure and the following terminology proves useful.

Definition 7 *A node j is "within hierarchical reach" of another i in a minimally connected $\mathcal{K}$-network g if j is within i's reach and j is not a predecessor of i nor there is a predecessor of i connected with j through a path not containing i.*

That is, j is within hierarchical reach of i in g if (i) $j \in N(\mathcal{K}_i)\backslash i$, (ii) $j \overset{g}{\nrightarrow} i$, (iii) there is no $k \neq i$ s.t. $k \xrightarrow{g} i$ and $j \in N(k; g\mid_{N\backslash i})$. Note that a necessary condition for for j to be within hierarchical reach of i in g it is that $g_{ji} = 0$, but it is *not* required that $g_{ij} = 1$. When this is required, so that every node supports links with every node within it's hierarchical reach, the network adopts a strongly hierarchical structure as we presently see. This motivates the following

Definition 8 *A minimally connected $\mathcal{K}$-network g is "hierarchical" if every node supports links with all those within it's hierarchical reach in g.*

Then we have the following characterization: strict Nash $\mathcal{K}$-networks are just hierarchical minimally connected $\mathcal{K}$-networks.

Theorem 1 *A network g is a strict Nash $\mathcal{K}$-network if and only if g is a hierarchical minimally connected $\mathcal{K}$-network.*

Proof. Necessity ($\Rightarrow$): Obviously, a $\mathcal{K}$-network g that is a strict Nash $\mathcal{K}$-network is also a Nash $\mathcal{K}$-network, and by Proposition 3, necessarily minimally connected, and by Lemma 1, $\xrightarrow{g}$ is a partial order. Now let i be a node in g and assume $g_{ij} = 0$ for some j within i's hierarchical reach, i.e., some $j \in N(\mathcal{K}_i)\backslash i$ that is not a predecessor of i and for which there is no k predecessor of i such that $j \in N(k; g\mid_{N\backslash i})$. As g is minimally connected, there must be a path connecting i and j, that then does not contain any

[13] A binary relation R on a set X is *antisymmetric* if, for all $x, y \in X$, xRy and yRx, implies $x = y$; and R is said to be *acyclic* if there is no finite chain $x_1, x_2, .., x_n$ in X s.t. $x_k R_{k+1}$ for $k = 1, 2, .., n-1$, and $x_n R x_1$, unless $x_k = x_{k+1}$ for $k = 1, 2, .., n-1$. In general the second condition is weaker than the first, but when the relation is transitive they are equivalent.

predecessor of i. In particular, on that path the first link must be a link initiated by i. But then i can delete that link and initiate a link with j without altering i's payoff, and consequently g is not a strict Nash $\mathcal{K}$-network.

Sufficiency ($\Leftarrow$): Assume that g is a minimally connected $\mathcal{K}$-network. By Proposition 3, g is a Nash $\mathcal{K}$-network. Let i be any node and any $g_i' \in G_i(\mathcal{K})$ s.t. $g_i' \neq g_i$. We show that $\Pi_i(g) > \Pi_i(g_{-i}, g_i')$ if g is hierarchical. Reasoning as in Proposition 3, as g is minimally connected, $g_i' \neq g_i$ involves deleting some links and initiating at least an equal number of new links for (g_{-i}, g_i') to be also minimally connected, otherwise i's payoffs would be smaller in (g_{-i}, g_i'), but in fact the number of links deleted and that of those newly initiated by i should be the same for the same reason. Let link ii' be one of the former (i.e., $g_{ii'} = 1$ and $g_{ii'}' = 0$) and let ij be one of the latter (i.e., $g_{ij} = 0$ and $g_{ij}' = 1$). If g is hierarchical, either j is a predecessor of i in g or there exists a k predecessor of i in g such that $j \in N(k; g\mid_{N\setminus i})$. But this implies a cycle in (g_{-i}, g_i'). The reason is this: evidently adding link $g_{ij}' = 1$ to g means a cycle in $(g - ii') + ij$, but it must be proved that this cycle is contained *in* (g_{-i}, g_i'). This is so because no link in the path in g connecting i and j can have been initiated by i (this would imply a cycle in g, which is assumed to be minimally connected). Therefore, no matter which other links in g_i are deleted in g_i', the cycle is entirely contained in (g_{-i}, g_i'). The same can be said about all new links in g_i' w.r.t. g_i, all new links are redundant in (g_{-i}, g_i'). Therefore necessarily $\Pi_i(g) > \Pi_i(g_{-i}, g_i')$. ■

This characterization allows in particular for a constructive proof of existence of strict Nash $\mathcal{K}$-networks for any societal cover $\mathcal{K}$: start at any node i_0 and initiate links with all nodes in $N(\mathcal{K}_{i_0})$, then extend the network by initiating new links from those nodes, always respecting hierarchical priority. In fact we have the following result:

Proposition 4 *For any connected societal cover $\mathcal{K}$ and any node $i_0 \in N$ there exists an oriented diverging tree g rooted at i_0 that is a strict Nash $\mathcal{K}$-network.*

Proof. Iterate the following procedure:

- Step 0: Initially let i_0 be any player in N, and g^0 the $\mathcal{K}$-network that results by i_0 initiating links with all players in $N(\mathcal{K}_{i_0})$.

- Step from k to $k+1$: If g^k is the current $\mathcal{K}$-network resulting form step k, take a terminal node, say i_{k+1}, in g^k, for which the set of nodes within hierarchical reach in g^k is *not empty*, and let i_{k+1} initiate links with all those players. If no such node exists, stop; otherwise, let g^{k+1} be the $\mathcal{K}$-network that results by adding to g^k all these links initiated by i_{k+1}.

It is clear that if $\mathcal{K}$ is connected this iterated process must stop in a finite number of steps and the resulting network will be an oriented diverging tree rooted in i_0 that is obviously hierarchical, thus forming a strict Nash $\mathcal{K}$-network connecting all players in N. If $\mathcal{K}$ were not connected the same iterated procedure could be applied within each component of the cover and by Proposition 2 a strict Nash $\mathcal{K}$-network would result. ■

As a corollary of Theorem 1, the following propositions establish some prominent features of the architecture of strict Nash $\mathcal{K}$-networks that help to form a clearer idea about these networks, which we later illustrate with some examples. The first one shows the role of stars in strict Nash $\mathcal{K}$-networks.

Proposition 5 *In a strict Nash $\mathcal{K}$-network g:*
(i) There is at least one node that is the center of a center-sponsored star that links with all players within its reach (i.e., for some $i \in N$, $N^d(i;g) = N(\mathcal{K}_i)/i$, and no player in N supports a link with i).
(ii) For each society $A \in \mathcal{K}$, either no link connects two nodes of that society or all or some of the members of that society are connected by center-sponsored stars and no other link exists connecting a pair of nodes in A (i.e., $g\mid_A$ consists of disjoint center-sponsored stars and/or isolated nodes).

Proof. *(i)* By Lemma 1, given that g is minimally connected, $\overset{g}{\to}$ is a partial order and necessarily exists at least one maximal element, i.e., with no predecessor. Let i_0 be a maximal element. As i_0 is maximal, by Theorem 1, necessarily $N^d(i_0;g) \cup \{i_0\} = N(\mathcal{K}_{i_0})$.

(ii) Let A be a society in the cover $\mathcal{K}$. Assume that for some $i, j \in A$, $g_{ij} = 1$. It is enough to show that the only other link that may exist connecting any $k \in A \backslash \{i,j\}$ with i or j is a link initiated by i. Assume that $g_{kj} = 1$. Then k can delete the link with j and initiate one with i and have the same payoff. Assume that $g_{jk} = 1$. Then i can delete the link with j and initiate one with k and have the same payoff. Finally, assume that $g_{ki} = 1$. Then k can delete the link with i and initiate one with j and have the same payoff. Thus the only remaining possibility of a link connecting any $k \in A \backslash \{i,j\}$ with i or j is a link $g_{ik} = 1$. ■

As an immediate corollary of part *(i)*, we have the following conclusion that yields Bala and Goyal's result as a particular case.

Corollary 2 *There exists a center-sponsored star that is a strict Nash $\mathcal{K}$-network if and only if the societal core is not empty and the center belongs to it.*

Observe the similarity of the proof of part *(ii)* with Bala and Goyal's proof of their result, and its differences: minimal connectedness and "strict Nash-ness" do *not* entail that *all* nodes in a society A are connected by a *single* star. Now the possibility of other center-sponsored stars within a society is left open, and even the possibility of some nodes being left outside these stars (but linked through nodes belonging to societies other than A). But the hierarchical arrangement of a strict Nash $\mathcal{K}$-network entails a maximum of *two* levels within each society: centers and spokes.

Thus we have in short that in a strict Nash $\mathcal{K}$-network g within each society either no pair of nodes is connected by a link or *some* center-sponsored stars connect *some* of the nodes in that society. But there is at least one center-sponsored star whose center connects *all* nodes of *all* societies to which the center belongs. The question now is: how do these stars interconnect in g? Evidently through overlapping societies. The following proposition answers this question more precisely by establishing the possible connections through overlapping societies: stars "hand in hand", i.e., interconnected through a free-rider player, are possible only if a *single* player belongs to both societies. Otherwise, if more than one player belongs to both societies, a player interconnecting them necessarily supports link(s) with players of one or both societies.

Proposition 6 *Let A, B be two overlapping societies in a societal cover $\mathcal{K}$ with $i \in A \cap B$, and g a strict Nash $\mathcal{K}$-network. If for some $j \in A \setminus (A \cap B)$ and $k \in B \setminus (A \cap B)$ it is $\bar{g}_{ij} = \bar{g}_{ik} = 1$, then $g_{ji} = g_{ki} = 1$ is possible only if $A \cap B = \{i\}$.*

Proof. Assume that $i \in A \cap B$ and for some $j \in A \setminus (A \cap B)$ and some $k \in B \setminus (A \cap B)$, $g_{ji} = g_{ki} = 1$. If $\{i\} \subsetneq A \cap B$ take $i' \in A \cap B$, $i \neq i'$. If i and i' were linked then j (or k) could delete the link with i and initiate a link with i' without loss. Thus we should have $\bar{g}_{ii'} = 0$. As g is minimally connected either there exists a path connecting i' and j and not containing k, or there exists a path connecting i' and k and not containing j. In the first case k can delete the link with i and initiate a link with i', and in the second j can delete the link with i and initiate a link with i'. In both cases this is without loss for the player who changes strategy, therefore proving that g is not a strict Nash $\mathcal{K}$-network. ■

The examples in Figure 2 illustrate the characterization and its corollaries and convey the logic of strict Nash $\mathcal{K}$-networks. Of course, the characterizing condition of respecting hierarchical priority holds in all cases, as the reader may check. Examples (a) and (b) represent societal covers with a nonempty core where a center-sponsored star is *one* of the possible architectures of strict Nash $\mathcal{K}$-networks: (d) and (c) represent other strict Nash $\mathcal{K}$-networks for the same covers. In examples (a), (b) and (d) a single center-sponsored star covers (partially) each society, while *two* center-sponsored stars cover society A_3 in (c) and society A_5 in (e), and in both cases no other link exists between pairs of individuals. In all cases a maximal node exists (represented by a white circle "$\circ$"), but *there may exist more than one,* as in examples (e), (f) and (g), which illustrate Proposition 6: stars connecting "hand in hand" by means of a "free rider" node are possible when a single player belongs to both societies. We have in fact the following conclusion: when no pair of societies in the societal cover $\mathcal{K}$ share a single player a strict Nash $\mathcal{K}$-network is an oriented diverging tree, as is proved by the following

Theorem 2 *Let $\mathcal{K}$ be a connected societal cover of N such that for all $A, B \in \mathcal{K}$, $A \cap B$ is empty or contains at least two nodes, then a strict Nash $\mathcal{K}$-network necessarily forms an oriented diverging tree.*

Proof. There is a unique path connecting any maximal node with each node. Assume that there are two maximal nodes i_0 and i_1. Then there is a path connecting i_0 and i_1, but then there must exist three nodes on that path i, j and k such that $g_{ij} = g_{kj} = 1$. Now if the intersection of any two societies in $\mathcal{K}$ is either empty or contains *more* than a single player, by Proposition 6, this is impossible. Therefore there can be only one

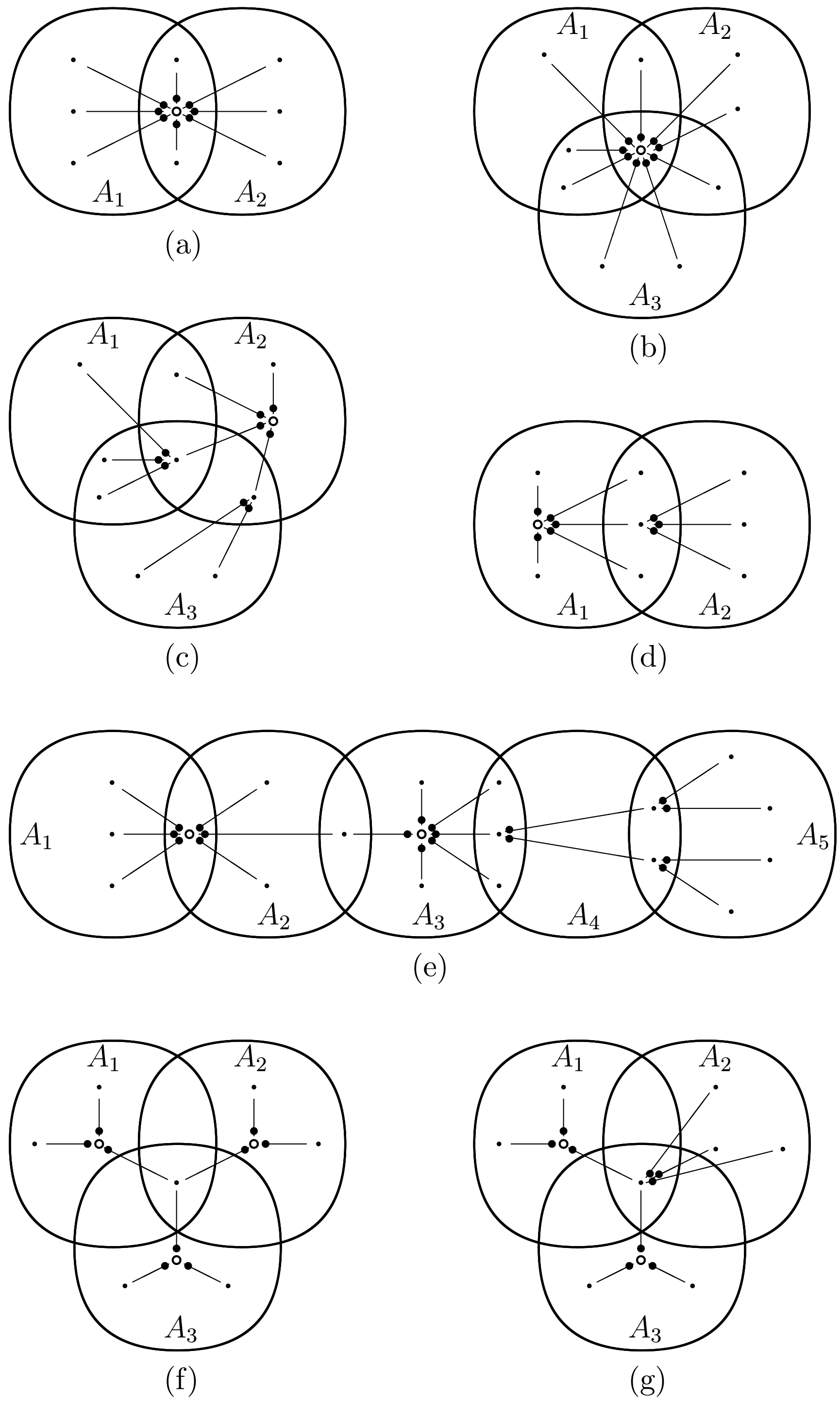

Figure 2: Strict Nash $\mathcal{K}$-networks.

maximal node connected with any other node by a unique path and consequently g is an oriented diverging tree. ■

But note that, as examples (e), (f) and (g) in Figure 2 show, when there are two or more societies to which a *single* player belongs several maximal nodes may exist. In such cases an oriented diverging tree does not result. In this case two or more "grafted" oriented diverging trees may emerge, so that any node is connected by an oriented diverging tree with at least one but possibly more maximal nodes. In this case several hierarchies overlap consistently.

Finally, in the spirit of the "community detection" problem (see, e.g., Jackson, 2009), we address an issue reciprocal to that considered so far. Given a network g, can it be interpreted as a strict Nash $\mathcal{K}$-network for any particular societal cover $\mathcal{K}$? Given the multiplicity of strict Nash $\mathcal{K}$-networks for a societal cover $\mathcal{K}$, it is easy to see that this question admits many answers: in general, an oriented diverging tree (or several grafted ones) can be seen as a strict Nash $\mathcal{K}$-network for different societal covers. Restricting attention to oriented diverging trees, the following associated covers are worth noting. Let g be an oriented diverging tree rooted at i_0. The *generational cover*, consisting of a minimal number of societies, each consisting of all nodes at the same distance from the root that are not terminal along with their "offspring"; the *family cover* where each node forms a society with its offspring; and the trivial *binary cover* where any two directly linked nodes form a society. For all the three societal covers the oriented diverging tree g is a strict Nash $\mathcal{K}$-network, and for the latter two it is the only one with maximal node i_0.

4 Dynamics

We now apply Bala and Goyal's (2000a) dynamic model to this setting. Namely, starting from any initial $\mathcal{K}$-network g each player i with some positive probability responds with a $\mathcal{K}$-admissible best response[14] to g_{-i} or randomizes across them when there are more than one, otherwise player i exhibits *inertia*, i.e., keeps her links unchanged. In this way a Markov chain on the state space of all $\mathcal{K}$-networks is defined. Bala and Goyal's prove that in their setting, i.e., for $\mathcal{K} = \{N\}$, starting from any network the dynamic process converges to a strict Nash network (i.e., the empty network or a center-sponsored star) with probability 1. In other words, the only absorbing sets are singletons consisting of strict Nash networks. The following example shows that this *is not* the case for the same dynamic model in the context of $\mathcal{K}$-networks.

Example 2 In Figure 3 (a) players in A_1 have no best response but keep their strategies, while player 1 is indifferent between initiating a link with 2 or 3 or 4, and consequently the best response dynamic process would oscillate forever within this three-element absorbing set. Similarly, in Figure 3 (b) all players in A_1 and players in A_3 keep

[14]Note that if g is a Nash $\mathcal{K}$-network any $\mathcal{K}$-admissible strategy g'_i of player i such that $\Pi_i(g) = \Pi_i(g_{-i}, g'_i)$, is a best response to g_{-i}.

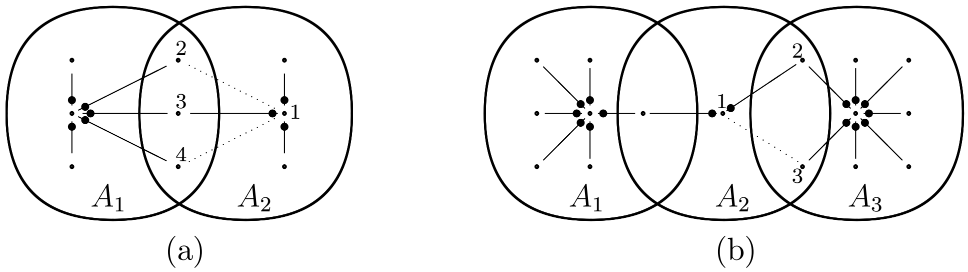

Figure 3: Dynamic deadlock towards a strict Nash $\mathcal{K}$-network.

their strategies, while player 1 is indifferent between initiating a link with 2 or 3, and consequently best response dynamics would oscillate forever among these two networks forming a two-element absorbing set. Note that in both examples the set of $\mathcal{K}$-networks among which the best response dynamics oscillates are minimally connected and yield the same payoffs to all players.

The example shows an interesting difference with respect to Bala and Goyal's setting. The same logic that in their setting leads to the absorbing strict Nash networks, in ours may also lead to the formation of interconnected center-sponsored stars, whose centers are fixed (i.e., immune to miscoordination), which are incompatible in any strict Nash $\mathcal{K}$-network. In this case the converging process is blocked. Thus in general the dynamic process leads to an *absorbing set*, that is, a minimal set of $\mathcal{K}$-networks closed under best response dynamics. This raises the question about what these absorbing sets consist of. We call *quasi*-strict Nash $\mathcal{K}$-networks to those that belong to any of these absorbing sets and explore their structure. For this purpose a clear understanding of the possibility of miscoordination in a *minimally connected* $\mathcal{K}$-network is needed.

Definition 9 *A minimally connected $\mathcal{K}$-network is "miscoordination-proof" if it cannot be disconnected by best response dynamics.*

Observe that in Figure 3 both examples consist of miscoordination-proof $\mathcal{K}$-networks. In a minimally connected $\mathcal{K}$-network miscoordination between two nodes can only occur if their reaches intersect and both support a link with the same node k. This occurs when both have best responses that consist of breaking these links with k and replacing them by initiating new ones with nodes connected by some path with the other that separately would not disconnect the network, but when they are simultaneous this would disconnect it. Moreover, even if two nodes do not support a link with the same node k, it may be the case that one or both have best responses consisting of linking the same node and we are back to the situation just discussed. The following lemma specifies in detail the conditions under which *none of these situations may occur in a minimally connected* $\mathcal{K}$-network, which is therefore miscoordination-proof.

Lemma 2 *A minimally connected $\mathcal{K}$-network g is miscoordination-proof if and only if for every society $A \in \mathcal{K}$, $g\mid_A$ consists of center-sponsored stars and/or isolated nodes* *and for any two nodes i, j either (i) $N(\mathcal{K}_i) \cap N(\mathcal{K}_j) = \varnothing$, or (ii) for all k, either (ii-1) $g_{ik} = g_{jk} = 1$ and*

$$N(\mathcal{K}_i) \cap N(j; g - jk) = \varnothing \quad \text{or} \quad N(\mathcal{K}_j) \cap N(i; g - ik) = \varnothing, \tag{4}$$

or (ii-2) $g_{ik} = 1$ and $g_{jk} = 0$ and for all k' s.t. $g_{jk'} = 1$ and it is a best response for j to delete link jk' and initiate jk, condition (4) holds for the resulting network, or (ii-3) $g_{ik} = g_{jk} = 0$ and for all k' s.t. $g_{ik'} = 1$ and it is a best response for i to delete ik' and initiate ik and all k'' s.t. $g_{jk''} = 1$ and it is a best response for j to delete jk'' and initiate jk, condition (4) holds for the network that results from both best responses.

Proof. Necessity ($\Rightarrow$): Let g be a minimally connected $\mathcal{K}$-network. First note that if for some society $A \in \mathcal{K}$, $g \mid_A$ does not consist of center-sponsored stars and/or isolated nodes miscoordination between nodes of that society can surely disconnect the network. Assume then that this condition holds. If for some pair of nodes i, j whose reaches intersect any of the other three conditions fails to hold it is easy to check that it is possible to disconnect the network by miscoordination in one best response step in case *(ii-1)* and in two steps in cases *(ii-2)* or *(ii-3)*.

Sufficiency ($\Leftarrow$): Let g be a minimally connected $\mathcal{K}$-network for which all conditions in the lemma hold. Then it is easy to check that no sequence of best response steps can disconnect the network. ■

We have then the following result that proves that *quasi*-strict Nash $\mathcal{K}$-networks are just miscoordination-proof minimally connected $\mathcal{K}$-networks.

Proposition 7 *Under a societal cover $\mathcal{K}$ the absorbing sets under best response dynamics consist of miscoordination-proof minimally connected $\mathcal{K}$-networks, and any miscoordination-proof minimally connected $\mathcal{K}$-network belongs to an absorbing set.*

Proof. First note that starting from any miscoordination-proof minimally connected $\mathcal{K}$-network best response dynamics cannot disconnect the network and can only yield another network satisfying the same conditions, i.e. another miscoordination-proof minimally connected $\mathcal{K}$-network. Therefore, any miscoordination-proof minimally connected $\mathcal{K}$-network along with all other that can be reached from it by best response dynamics form an absorbing set. Remains to be shown that there are no other absorbing sets. Starting from any $\mathcal{K}$-network, best response dynamics lead with probability 1 to a minimally connected $\mathcal{K}$-network g that for every society $A \in \mathcal{K}$, $g \mid_A$ consists of center-sponsored stars and/or isolated nodes[15]. If some of the conditions of Lemma 2 does not hold, miscoordination is possible (in one or two steps) in a way that the network is disconnected (i and j delete their links with k) and a cycle appears. In a new best response step, one of the involved nodes, say i, links k again and the other breaks the cycle. In this way a new minimally connected network results where the i-centered star has a new spoke and one of the possibilities of miscoordination has disappeared.

[15] The proof is similar to that of Theorem 4.1 in Bala and Goyal (2000a), just respecting $\mathcal{K}$-feasibilty.

In this way a sequence of best response steps that leads to a miscoordination-proof minimally connected $\mathcal{K}$-network is proved to exist. ■

As a corollary, we have the following result that shows that when an absorbing set is reached, in spite of the possibly perpetual oscillation, stability in terms of payoffs is reached given that all networks in the same absorbing set *yield the same payoffs to all players.*

Corollary 3 *For any two quasi-strict Nash $\mathcal{K}$-networks g, g' that belong to the same absorbing set and all $i \in N$, $\Pi_i(g) = \Pi_i(g')$.*

Proof. Let Q be a absorbing set and $g \in Q$. As g is a miscoordination-proof minimally connected $\mathcal{K}$-network, the number of links supported by each node is invariant under best response dynamics. Therefore the payoffs must remain unchanged for all players within Q. ■

In summary, quasi-strict Nash $\mathcal{K}$-networks, i.e. the constituent of the absorbing sets of best response dynamics, are not very different from strict Nash $\mathcal{K}$-networks. They are minimally connected $\mathcal{K}$-networks consisting of interconnected stars, one or several disjoint ones in each society, where nodes support links with all nodes within their hierarchical reach with the only possible exception of some nodes that support links with only one hinge-node among several between which best response dynamics can oscillate. Thus the architecture of quasi-strict Nash $\mathcal{K}$-networks is that of grafted trees, something that was only possible for strict Nash networks when a unique individual belonged to two different societies.

5 Decay

We consider now the case where the value that a player i receives from another player j is sensitive to the geodesic distance between them, i.e., the length of the path with the minimum number of links that connects them. Namely, if $d(i, j; g)$ denotes this distance in a network g, we assume, as in Bala and Goyal (2000a), that this value is discounted by $\delta^{d(i,j;g)}$, where $0 < \delta \leq 1$. Therefore, assuming homogeneity and, without loss of generality, that $v = 1$, the payoff of player i in network g is

$$\Pi_i(g) = 1 + \sum_{j \in N(i;g) \setminus i} \delta^{d(i,j;g)} - c\mu_i^d(g). \tag{5}$$

If $\delta = 1$ we have the linear case we have dealt with so far. In the sequel we assume there is actual decay, that is, $\delta < 1$. Now we have to deal with two parameters: c and δ.

5.1 Stability and decay

When a societal cover $\mathcal{K}$ constrains link formation, a natural extension of Bala and Goyal's notion of "tw-complete" network is the following: a *tw-complete $\mathcal{K}$-network* is a network g where $\bar{g}_{ij} = 1$ for all i, j s.t. $\mathcal{K}_i \cap \mathcal{K}_j \neq \varnothing$ (every node is at distance 1 from every other $\mathcal{K}$-reachable node) and $g_{ij} = 1 \Rightarrow\ g_{ji} = 0$ (no link is twice paid). In Bala and Goyal's setting, in the presence of decay a variety of *all-encompassing* ("mixed") stars become stable. That is to say, stars that (i) connect all other nodes to a center, and (ii) each link is either paid by the center or by the spoke agent, but never by both. An all-encompassing star is *periphery-sponsored* if all the links are paid by the spoke agents. In our setting such all-encompassing stars are feasible only when the societal core is not empty. Then, given that under a societal cover the feasible responses of any agent form a subset of her feasible responses without constraints, the following extension of Bala and Goyal's (2000a) Proposition 5.3 is straightforward:

Proposition 8 *Let the payoffs be given by (5) and $\mathcal{K}$ the societal cover that constrains link formation, then:*
(i) If $0 < c < \delta - \delta^2$, then tw-complete $\mathcal{K}$-networks are the only strict Nash $\mathcal{K}$-networks.
(ii) If $\delta - \delta^2 < c < \delta$ and the societal cover $\mathcal{K}$ has a nonempty core, then all all-encompassing stars centered at any point in the core are strict Nash $\mathcal{K}$-networks.
(iii) If $\delta < c < \delta + (n-2)\delta^2$ and the societal cover $\mathcal{K}$ has a nonempty core, then any periphery-sponsored star centered at any point in the core, but none of the other stars, is a strict Nash $\mathcal{K}$-network.
(iv) If $\delta < c$, then the empty network is strict Nash.

When a societal cover constrains link formation, the societal core may be empty and *(ii)-(iii)* parts of Proposition 8 do not apply in that case, but, as we have seen, even when it is not empty, non-star architectures may be strict Nash when there is no decay. Bala and Goyal (2000a) focus on the stability of different types of "mixed" stars under different ranges of cost and decay. Here the situation is more complicated, given the variety of strict Nash architectures even for relatively simple societal covers. In our setting a rather general analogous of the "mixed" stars whose stability Bala and Goyal deal with are $\mathcal{K}$-compatible "mixed" (i.e. not necessarily oriented) trees or grafted trees that result from a strict Nash $\mathcal{K}$-network without decay (i.e., an oriented diverging tree or several grafted diverging trees satisfying the hierarchical characterizing condition of Theorem 1) by just allowing each link to be paid by any one of the two agents it connects (but never by both). We call such $\mathcal{K}$-networks *mixed hierarchical.* We say that a mixed hierarchical $\mathcal{K}$-network is *periphery-sponsored* if every node that has only one node at distance 1 supports the link that connects it. Thus the question arises about the stability and efficiency of such architectures in the presence of decay and the comparison with all-encompassing stars when these are feasible. Some simple examples allow us to illustrate what seem to be the basic patterns. We first consider the case when the societal core is not empty. In order to make it easier the comparison with Bala and Goyal (2000a) we discuss the effect of decay for the same different intervals of values relating c and δ. In view of Proposition 8-*(i)*, we can start with c and δ in the interval of case *(ii)*. The following notation will de useful, for each $A \in \mathcal{K}$ we denote $\dot{A}$

the set of nodes in A that do not belong to any other society, i.e., $\dot{A} := A \backslash \cup_{A' \in \mathcal{K} \backslash \{A\}} A'$, we also denote $n_A = \#A$ and $n_{\dot{A}} = \#\dot{A}$.

1. Interval: $\delta - \delta^2 < c < \delta$. In this case it is worth initiating a link whose marginal contribution is that of connecting an isolated node ($c < \delta$) and it is not worth initiating a link whose marginal contribution is that of shortening form 2 to 1 the distance to just one node ($\delta - \delta^2 < c$), but this may be worth if that node is sufficiently well connected. As a simple term of reference let us consider a societal cover consisting of two intersecting societies $\mathcal{K} = \{A, B\}$. In this case an oriented diverging tree rooted at, say, $i_0 \in \dot{A}$, similar to the one represented in Fig. 2(d), where $i_1 \in A \cap B$ supports links with all nodes in $\dot{B}$, is strict Nash if there is no decay, but may fail to be stable if $\delta < 1$. In fact, if $c \leq \delta + (n_{\dot{B}} - 1)\delta^2 - n_{\dot{B}}\delta^3$ this network is *not* strict Nash because any individual in $\dot{A}$ different from i_0 would be better off (or at least as well) by initiating a link with i_1. Note that this number is surely greater than $\delta - \delta^2$, but it is within the interval considered (i.e., $\delta + (n_{\dot{B}} - 1)\delta^2 - n_{\dot{B}}\delta^3 < \delta$) *only if* $\delta > (n_{\dot{B}} - 1)/n_{\dot{B}}$. That is, for $n_{\dot{B}}$ sufficiently large, unless there is almost no decay, this network is *not* strict Nash for any value of c in the whole interval. If $\delta > (n_{\dot{B}} - 1)/n_{\dot{B}}$, this number divides the interval considered into two subintervals: only for high costs (i.e., above this number) this network is strict Nash. Now consider the mixed hierarchical variations of this $\mathcal{K}$-network. If $n_{\dot{B}} \geq n_A - 3$ and at least one node j in $\dot{A}$ supports her link with i_0, this $\mathcal{K}$-network is not strict Nash since j has a best response consisting of deleting the link with i_0 and replacing it by a link with i_1, and this does not depend on the value of c. If $n_{\dot{B}} < n_A - 3$ or there is no node in $\dot{A}$ that supports her link with i_0, the discussion and conclusions are entirely similar as those for the diverging tree rooted at i_0.

Similar conclusions are obtained for the oriented diverging tree (and all its mixed hierarchical variations) where *two or more* nodes in the intersection $A \cap B$ instead of only one support links with the reminder nodes in $\dot{B}$.

Now consider the case where $A \cap B$ contains a *unique* node i_0. As we have seen, in this case two center-sponsored stars "hand-in-hand", one centered at $\dot{A}$, the other at $\dot{B}$, connecting all nodes in A and B respectively and i_0 in particular, is a strict Nash $\mathcal{K}$-network. Assume $n_B \geq n_A$. The situation is again similar: this network is stable only if $c > \delta + (n_{\dot{B}} - 2)\delta^3 + (n_{\dot{B}} - 1)\delta^4$. Note that this number is greater than $\delta - \delta^2$, but it is within the interval considered (i.e., $\delta + (n_{\dot{B}} - 2)\delta^3 + (n_{\dot{B}} - 1)\delta^4 < \delta$) *only if* $\delta > (n_{\dot{B}} - 2)/(n_{\dot{B}} - 1)$. Therefore again *as $n_{\dot{B}}$ grows, unless there is almost no decay this network is not strict Nash in the whole interval.* If $\delta > (n_{\dot{B}} - 2)/(n_{\dot{B}} - 1)$, this number divides the interval considered into two subintervals: only for costs above this number this network is strict Nash. Now consider the mixed hierarchical variations of this $\mathcal{K}$-network. If at least one node j in $\dot{A}$ ($\dot{B}$) supports her link with the center of the star in A (B) and $(n_{\dot{A}} - 3) - (n_{\dot{A}} + n_{\dot{B}} - 4)\delta + (n_{\dot{B}} - 1)\delta^2 \leq 0$ ($(n_{\dot{B}} - 3) - (n_{\dot{A}} + n_{\dot{B}} - 4)\delta + (n_{\dot{A}} - 1)\delta^2 \leq 0$), this $\mathcal{K}$-network is not strict Nash since j has a best response consisting of deleting her link with the center of the star in A (B) and replacing it by a link with i_0, and this does not depend on the value of c. Otherwise, the discussion and conclusions are entirely similar as those for the case of two center-sponsored stars "hand-in-hand".

Thus, roughly speaking, for a social configuration like the one described, and c and δ within the interval considered, all-encompassing stars centered in the societal core are efficient and stable, while non efficient architectures as mixed hierarchical variations of strict Nash (without decay) $\mathcal{K}$-networks can be stable only for certain combinations involving a relatively small number of agents in either society outside the societal core, a relatively low decay and a relatively high cost.

Let us consider now the case where c and δ are in the interval of case *(iii)*.

2. Interval: $\delta < c < \delta + (n-2)\delta^2$. In this case it is *not* worth connecting an isolated node ($\delta < c$), but it would surely be worth for an isolated agent to initiate a link with the center of a star that connects all other nodes $\left(c < \delta + (n-2)\delta^2\right)$. In this case, given that $\delta < c$, only *periphery-sponsored* mixed hierarchical $\mathcal{K}$-networks can be stable. Let us consider then a tree as the first one we have considered in the first interval, *but* where the terminal agents support the links connecting them, while the link between nodes i_0 and i_1 could be supported by either of them. Given that $\delta < c$, restrictions on c are needed in order to ensure that no node finds profitable to delete the link she is supporting in the tree. If the link between nodes i_0 and i_1 is supported by i_1, it is necessary $c < \min\{\delta + (n_A - 2)\,\delta^2, \delta + n_{\dot{B}}\delta^2 + (n_A - 2)\,\delta^3\}$; and if the link between nodes i_0 and i_1 is supported by i_0, it is necessary $c < \min\{\delta + n_{\dot{B}}\delta^2, \delta + (n_A - 2)\,\delta^2 + n_{\dot{B}}\delta^3\}$. Furthermore in both cases, the same reason as for the interval $\delta - \delta^2 < c < \delta$ require $n_{\dot{B}} < n_A - 3$ and $c > \delta + (n_{\dot{B}} - 1)\delta^2 - n_{\dot{B}}\delta^3$ for these networks to be stable, which may still be actually a constraint as $\delta + (n_{\dot{B}} - 1)\delta^2 - n_{\dot{B}}\delta^3$ is in the interval now considered if $\delta < (n_{\dot{B}} - 1)/n_{\dot{B}}$. Again we see the same pattern: only for certain combinations involving a relatively small number of nodes in either society outside the societal core, a relatively low decay and a relatively high cost this architecture (as its *periphery-sponsored* mixed hierarchical variants) remains stable, while a periphery-sponsored star centered at any point of the societal core remains stable in this interval.

This range of cost-decay values has other implications. For instance, for the same cover, a tree where all nodes in A, but *two (or more)* nodes in the intersection $A \cap B$, support a link with $i_0 \in \dot{A}$, while the other two (or more), also connected either way with i_0, are linked by the nodes in $\dot{B}$ (as societies A_4 and A_5 in Fig. 2 (e), but with spoke-agents in A_5 paying their links) is *not* strict Nash whatever the cost in this range be. A strict Nash may only result if all nodes in $\dot{B}$ support links with only one and *same* node in $A \cap B$. Here we see another pattern: a tendency to concentrate inter-societal connections in the presence of decay.

Now consider the case where $A \cap B$ contains a *unique* node i_0. And consider two mixed stars "hand-in-hand", one centered at $i_1 \in \dot{A}$, the other at $i_2 \in \dot{B}$. Assume $n_B \geq n_A$. Given that $\delta < c$, only *periphery-sponsored* mixed hierarchical $\mathcal{K}$-networks can be stable, therefore the terminal agents support the links connecting them, while the link between nodes i_1 and i_0 could be supported by either of them, the same for the link between nodes i_2 and i_0. Given that $\delta < c$, restrictions on c are needed in order to ensure that no node finds profitable to delete the link she is supporting in the grafted tree. If i_0 supports both links with i_1 and i_2, it is necessary $c < \delta + (n_{\dot{A}} - 1)\,\delta^2$; if i_1 and i_2 both support their links with i_0, $c < \delta + \delta^2 + (n_{\dot{A}} - 1)\,\delta^3$; if i_0 supports the link with i_1 and i_2 supports the link with i_0, $c < \min\{\delta + (n_{\dot{A}} - 1)\,\delta^2, \delta + \delta^2 + (n_{\dot{A}} - 1)\,\delta^3\}$; finally if i_0 supports the link with i_2 and i_1 supports the link with i_0, $c < \min\{\delta + (n_B - 1)\,\delta^2, \delta + \delta^2 + (n_B - 1)\,\delta^3\}$. Furthermore, in the four cases, the same reason as for the interval $\delta - \delta^2 < c < \delta$ require $(n_{\dot{A}} - 3) - (n_{\dot{A}} + n_B - 4)\delta + (n_B - 1)\delta^2 > 0$ and $c > \delta + (n_B - 2)\delta^3 + (n_B - 1)\delta^4$ for these networks to be stable.

It would be long and tedious to discuss it in detail, but the case of a connected societal cover consisting of three societies with a nonempty societal core can be discussed case by case to obtain similar conclusions. In fact, whatever the number of societies, when the societal core is not empty we have similar conclusions: (i) the center-sponsored star is the most robust architecture among the strict Nash $\mathcal{K}$-networks without decay as it remains stable in the first interval ($\delta - \delta^2 < c < \delta$), while other architectures remain stable only for a relatively low decay, a relatively high cost and a relatively small number of agents in every society involved in the "graft" in the case of grafted trees, or relatively small number of agents in at least one society outside the societal core in the case of trees; (ii) in the second interval ($\delta < c < \delta + (n-2)\delta^2$), as spoke agents must pay the links that connect them to the network, neither center-sponsored stars nor any strict Nash network without decay is stable with decay, but some mixed trees and grafted trees may remain stable subject to similar limitations. In contrast, periphery-sponsored stars centered at any point of the societal core remain stable in this interval.

In sum, it is not a heavy societal core (i.e., with many nodes) that compels the society to organize itself as a star of one or other type when this is feasible, but a heavy core-periphery. More precisely, we have the following conclusion from the preceding discussion[16]:

Proposition 9 *Let the payoffs be given by (5) and $\mathcal{K}$ be a societal cover whose societal core is not empty, then:*
(i) If $\delta - \delta^2 < c < \delta$ and the number of nodes in $\dot{A}$ is sufficiently large for all $A \in \mathcal{K}$, then the only strict Nash $\mathcal{K}$-networks without decay that remain strict Nash with decay are the center-sponsored stars whose center is within the societal core; but every all-encompassing star with center in the core is also a strict Nash network.
(ii) If $\delta < c < \delta + (n-2)\delta^2$ and the number of nodes in $\dot{A}$ is sufficiently large for all $A \in \mathcal{K}$, then the only mixed hierarchical $\mathcal{K}$-networks that are strict Nash with decay are the periphery-sponsored stars whose center is within the societal core.

These results and the preceding discussion may make the reader think of Feri's (2007) results about stochastic stable networks. In the next subsection we explicitly

[16] Consider the following example: let $\mathcal{K}$ be a two-society cover $\mathcal{K} = \{A, B\}$ and $A \cap B \neq \varnothing$, and let g be the strict Nash $\mathcal{K}$-network without decay where some $i \in \dot{A}$ supports links with all other nodes in A, and a node in $A \cap B$ supports links with all other nodes in $\dot{B}$. If $\dot{A}$ is large but $\dot{B}$ is small, say, 1, this network remains strict Nash in the whole interval $\delta - \delta^2 < c < \delta$.

deal with Feri's dynamics and use these conclusions.

Let us consider now the case where the societal core is empty. The simplest case of a connected cover with an empty core is a three-society cover $\mathcal{K} = \{A, B, C\}$ with $A \cap B \cap C = \varnothing$. In this case, reasoning in similar terms to the case of a two-society cover, it can be concluded that *none* of those $\mathcal{K}$-networks which are strict Nash without decay (nor any mixed hierarchical variant of them) remains stable in the presence of decay if the number of agents that belong to each one, but only one, of the three societies is big enough[17]. Nevertheless, mixed stars, interlinked in a variety of ways, maybe redundant, sharing a number of spoke agents, appear as strict Nash for the different ranges of the parameters. Consider the case where $A \cap B \neq \varnothing$, $B \cap C \neq \varnothing$ and $A \cap C = \varnothing$. Let m be the cardinality of the smallest set of $\dot{A}$ and $\dot{C}$. Then if $\delta - \delta^2 < c < \delta + (m-1)\delta^2 - m\delta^3$, the network where a agent $i_1 \in A \cap B$ is linked with all agents in $\dot{A}$, a agent $i_2 \in B \cap C$ is linked with all agents in $\dot{C}$, all agents in $B \backslash \{i_1, i_2\}$ support links with both i_1 and i_2, and one of these two links the other[18], is a strict Nash network (note that as the upper bound of this interval can be in the second interval, i.e. if $\delta < \delta + (m-1)\delta^2 - m\delta^3$, in this case if c is in this second interval spoke agents should pay their links). Now if $\delta + (m-1)\delta^2 - m\delta^3 < c < \delta + m\delta^2 - \delta^3 - m\delta^4$, then the link between i_1 and i_2 should be eliminated, and doing so the remaining $\mathcal{K}$-network is a strict Nash where two stars share their spoke agents in B[19]..

In summary, in the presence of decay, when the core is not empty we have: (i) in the first interval center-sponsored stars are the most robust strict Nash among the those without decay, but also all mixed stars centered in the core become strict Nash; (ii) in the second interval periphery-sponsored stars centered in the core are the only stars which are strict Nash. These patterns corroborate, in a more complex context, Goyal diagnosis: *"Decay introduces incentives for players to reduce the lengths of paths between themselves. This means that the star network is even more attractive than before. However, the introduction of decay also means that cycles can be sustained in equilibrium."* (Goyal, 2007, p. 172). Moreover, even when the core is empty, stars, not any longer all-encompassing under institutional constraints, interlinked in possibly redundant ways seem a persisting feature in strict Nash networks under decay, while none of the strict Nash $\mathcal{K}$-networks without decay is robust in the presence of it if for all the societies the number of agents that belong to that society and only that one is big enough.

[17]Otherwise, in some particular cases a non decay strict Nash $\mathcal{K}$-network or a mixed tree variant remains stable with decay. For instance, if $\#(A \cap B) = \#(B \cap C) = 1$, and $\dot{B} = \varnothing$ then the oriented $\mathcal{K}$-tree rooted at the unique point in one intersection form a strict Nash $\mathcal{K}$-network in the first interval (in the second interval spoke nodes in $\dot{A} \cup \dot{C}$ should pay their links).

[18]Note this is a *quasi linked star* (qls) in Feri's (2007) terms.

[19]This is a *quasi linked star 2* (qls2) in Feri's (2007) terms.

5.2 Stochastic stability and decay

In Feri (2007) a different dynamic model consisting of unperturbed dynamic plus errors or mutations is considered[20]. Namely, at every period one agent is randomly chosen to revise her strategy by choosing a best response (or one of them at random when there are more than one). This is the *unperturbed dynamic*, but at every period the chosen agent with a probability $\varepsilon > 0$ makes a mistake consisting of choosing her strategy randomly. Thus an evolutionary process results which is an aperiodic and irreducible Markov chain, which consequently has a unique invariant probability distribution μ_ε. Then Feri studies the *stochastically stable networks*, i.e., those g for which $\hat{\mu}(g) > 0$, where $\hat{\mu} = \lim_{\varepsilon\to 0} \mu_\varepsilon$. This can be done applying the result according to which the stochastically stable states of such an evolutionary process are characterized as those that belong to an absorbing set of a recurrent set of the evolutionary process (Proposition 7.7 in Samuelson (1997)). A *recurrent set* is a set R of absorbing sets of the unperturbed dynamic s.t. (i) for any state within the recurrent set a mutation followed by unperturbed dynamics cannot end up in an absorbing set not belonging to R, and (ii) it is possible to reach any absorbing set in R from any other also in R by means of a sequence of *one-step mutations*, i.e., steps consisting of one mutation followed by unperturbed dynamics.

As is by now clear, when a societal cover $\mathcal{K}$ constrains link-formation things become rather complicated. Nevertheless, part of Theorem 1 in Feri (2007) can be easily extended. Denote by $\hat{G}(\mathcal{K})$ the set of stochastically stable $\mathcal{K}$-networks under Feri's dynamic, by $G^c(\mathcal{K})$ the set of all tw-complete $\mathcal{K}$-networks, by $G^s(\mathcal{K})$ ($G^{ps}(\mathcal{K})$) the set of all encompassing (periphery-sponsored) stars whose center belongs to $core(\mathcal{K})$ whenever it is not empty, and by $G^h(\mathcal{K})$ the set of all hierarchical minimally connected (i.e., in view of Theorem 1, all strict Nash $\mathcal{K}$-networks without decay). Then we have

Theorem 3 *Let the payoffs be given by (5) and $\mathcal{K}$ a societal cover, and let $0 < \delta < 1$:*

(i) If $c < \delta - \delta^2$, then $\hat{G}(\mathcal{K}) = G^c(\mathcal{K})$.

(ii) If $\delta - \delta^2 < c < \delta$ and $core(\mathcal{K}) \neq \varnothing$, then $\hat{G}(\mathcal{K}) \supseteq G^s(\mathcal{K})$; moreover, if $\delta - \delta^3 < c < \delta$, there exists $n(c,\delta)$ such that if $n_{\dot{A}} > n(c,\delta)$ for all $A \in \mathcal{K}$, then $\hat{G}(\mathcal{K}) = G^s(\mathcal{K})$.

(iii) If $\delta < c$ and $core(\mathcal{K}) \neq \varnothing$, there exists $n'(c,\delta)$ such that if $n_{\dot{A}} > n'(c,\delta)$ for all $A \in \mathcal{K}$, then $\hat{G}(\mathcal{K}) = G^{ps}(\mathcal{K}) \cup \{g^e\}$.

Proof. *(i)* The proof is an easy adaptation of Feri's proof of part (i) of his Theorem 1 that we omit.

(ii) The proof of the first part results from an easy adaptation of Feri's Lemmas 1 and 2. That is to say, within this interval of cost, from any $\mathcal{K}$-network an error followed by unperturbed dynamic is enough to reach a center-sponsored star whose

[20] Other papers dealing with dynamic models in the presence of decay are Watts (2001), Jackson and Watts (2002), Goyal and Vega-Redondo (2005), Hojman and Szeidl (2008) and Feri and Meléndez (2009).

center belongs to the core (Lemma 1 in Feri (2007)); and for any two all-encompassing stars whose centers are in the core a sequence of one-step mutations leads from one to the other (Lemma 2 in Feri (2007)). Then, using Proposition 7.7 in Samuelson (1997), one concludes that there is only one recurrent set, which contains $G^s(\mathcal{K})$. As to the second part, an overwhelmingly cumbersome detailed discussion of which the extension of Feri's Lemma 3 consists that we omit here, yields the conclusion.

(iii) We omit the details that again consist of an adaptation of Lemmas 4,5 and 6 in Feri (2007), from which the result follows. ■

A more detailed extension of Feri's Theorem 1 should be possible, but it does not seem feasible a general extension, given the variety of societal covers. In fact, there should be a precise extension for each particular cover. A difficulty now is that, even for very simple societal covers, there are several possible architectures for a strict Nash $\mathcal{K}$-network (with and without decay), in contrast with the only possible one in Bala and Goyal's setting, moreover, also quasi strict Nash $\mathcal{K}$-networks should be taken into account when dealing with recurrent sets. Thus, the complexity of extending Feri's Lemma 3 becomes explosive. To make things a bit more complicated, as we have seen in the preceding subsection, different architectures of strict or quasi strict Nash $\mathcal{K}$-networks are stable within different ranges of the parameters c and δ, which makes cumbersome a detailed formulation about which ones are stochastically stable within each subinterval.

Nevertheless, in order to gain some insight on how things go in this complex setting, we constrain our attention to a very simple example of a two-society connected cover and show how this extension can be done and study the impact of the cover in stochastic stability.

Example 3 Let $N = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{K} = \{A, B\}$, where $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{3, 4, 5, 6, 7\}$, so that $core(\mathcal{K}) = A \cap B = \{3, 4, 5, 6\}$. Up to isomorphism, there are four architectures for a strict Nash (without decay):

$(SN1)$ A center-sponsored star whose center is within $core(\mathcal{K})$;

$(SN2)$ An oriented tree rooted at $A \backslash B$ where a node in $A \cap B$ supports a link with node 7;

$(SN3)$ An oriented tree rooted at 7 where a node in $A \cap B$ supports a link with nodes 1 and 2;

$(SN4)$ An oriented tree rooted at 7 where one node in $A \cap B$ supports a link with node 1 and another node in $A \cap B$ supports a link with node 1 and 2.

There are also two quasi strict Nash architectures:

$(QSN5)$ Node 7 supports links with all other nodes in B and a node in A support links with the other in $A \backslash B$ and with one in $A \cap B$ (in best response dynamics the latter would oscillate between the four nodes in $A \cap B$).

$(QSN6)$ A node in $A \backslash B$ supports links with all other nodes in A and node 7 supports a link with a node in $A \cap B$ (in best response dynamics the latter would oscillate between the four nodes in $A \cap B$).

Now let us consider the interval $\delta - \delta^2 < c < \delta$. In view of the discussion in the preceding subsection, we can expect different subintervals within where these architectures and/or their mixed variants keep strictly stable (i.e., strict or quasi strict Nash). Let us denote by $M1$, $M2$ and $M3$ the architectures consisting of all mixed variants of $SN1$, $SN2$ and $SN3$; and by $M4$ and $M5$ the variants of $SN4$ and $QSN5$ where the links connecting 7 are supported by any one of the adjacent nodes, while the other links are initiated as in $SN4$ and $QSN5$. Note that the mixed variants of $QSN6$ are included in $M2$.

A similar discussion to that made in the preceding subsection leads to the following nested intervals where each of these architectures remain stable:

$M1$ remains strict Nash in the whole interval $\delta - \delta^2 < c < \delta$.

$M2$ remains strict Nash in the interval $m_2 := \delta - \delta^3 < c < \delta$.

$M3$ remains strict Nash in the interval $m_3 := \delta + \delta^2 - 2\delta^3 < c < \delta$ (note that this interval is not empty only for $\delta > 1/2$).

$M4$ remains strict Nash in the interval $m_4 := \delta + \delta^2 - \delta^3 - \delta^4 < c < \delta$ (not empty only for $\delta > 0.618$).

$M5$ remains *quasi* strict Nash in the interval $m_5 := \delta + \delta^2 - \delta^3 - 3\delta^4 < c < \delta$ (not empty only for $\delta > 0.7676$).

Thus we have that: (i) only for δ sufficiently high (always greater than 1/2) the architectures $M3$, $M4$, and $M5$, remain stable or quasi stable in a non empty subinterval; (ii) as $\delta - \delta^2 < m_2 < m_3 < m_4 < m_5$, the pattern is clear: as the cost goes down the set of stable architectures shrinks and below $m_2 = \delta - \delta^3$ only the stars in $M1$ remain.

As to stochastic stability things become much more complicated. It can be seen[21] that the following extensions of Feri's lemmas 1 and 2 hold: (i) a transition from any $\mathcal{K}$-network to star in $M1$ can be induced by a mutation followed by unperturbed dynamic; (ii) from any network in any of these five sets it is possible to reach any other in any other of these sets by a sequence of one-step mutations. Much more cumbersome as the reader may guess is the extension of Lemma 3 of Feri (2007). This needs to study all possible mutations in all of these architectures followed by unperturbed dynamic. A detailed discussion of cases leads to an upper bound for the cost in this interval in order to ensure that in the worst case unperturbed dynamic does not get stuck on its way towards some of these architectures. This upper bound is

$$c < b := \delta + 2\delta^2 - 3\delta^3.$$

Note that this number is between m_4 and m_5. In sum we have the following conclusions. If $\hat{G}(\mathcal{K})$ denotes the set of networks within the recurrent set then

-in the interval $\delta - \delta^2 < c < \delta - \delta^3$: $M1 \subseteq \hat{G}(\mathcal{K})$.

-in the interval $m_2 < c < m_3$: $\hat{G}(\mathcal{K}) = M1 \cup M2$.

-if $\delta > 1/2$, in the interval $m_3 < c < m_4$: $\hat{G}(\mathcal{K}) = M1 \cup M2 \cup M3$.

[21]We omit the details of the proofs of these extensions, easy for Lemma 1 and more tedious for Lemma 2.

-if $\delta > 0.618$, in the interval $m_4 < c < b$: $\hat{G}(\mathcal{K}) = M1 \cup M2 \cup M3 \cup M4$.

Note the shrinking set of stochastically stable architectures as the cost diminishes, and the wider interval for the mixed hierarchical architecture rooted in the set A of greater cardinality (i.e., $M2$) with respect to that rooted at the smaller set B (i.e., $M3$).

5.3 Efficiency and decay

As we have seen in section 3, efficiency and stability in the sense of Nash equlibrium are equivalent conditions: they are satisfied by all $\mathcal{K}$-networks minimally connected and only by them. Nevertheless, in the presence of decay this is not any longer the case. We have postponed any comment about efficiency so far to avoid a certain redundancy, given the parallel conclusions that we have about efficiency and stability, and about efficiency and stochastic stability. A general conclusion arises from the preceding discussion: in the presence of decay efficiency and stability go hand in hand. That is to say, the greater the stability the greater the efficiency, i.e. the greater the aggregated utility.

6 Concluding remarks

We have studied the impact of institutional constraints as modeled by a societal cover on Bala and Goyal's (2000a) benchmark two-way flow model. The notion of societal cover seems suitable for capturing in a formal and tractable way many factual constraints to which we refer generically as "institutional" that are to be observed in real world situations. Such constraints emerge due to social (cultural, economic, sociological, geographic, etc.) reasons and cannot be ignored in many contexts. Moreover, any symmetric link-constraining system is proved to be interpretable as the result of a societal cover.

In this paper we characterize and study in some detail the structure of stable and efficient networks under these constraints by extending Bala and Goyal's approach and results. In a nutshell, the conclusions when there is no decay can be synthesized by the equation:

$$\textit{Institutional constraints} + \textit{Strict stability} = \textit{Hierarchical organization.}$$

Namely, if there is no decay, center-sponsored star (when feasible) is no longer the only stable (in the strict Nash sense) architecture, but center-sponsored stars continue to be the basic building blocks of stable networks. Moreover, the architecture of such stable networks embodies a formal hierarchical principle that yields oriented diverging trees, the paradigm of hierarchical organization[22] or "grafted" trees adapted to the constraints imposed by the cover. It is also proved that simple best response dynamics "work"

[22] See, for instance, López et al. (2002).

basically well in this more complicated setting. They may fail to reach a strict Nash network if incompatible "incomplete" and almost stable hierarchical networks form, but a stable configuration of payoffs associated with an absorbing set of miscoordination proof networks is sure to be reached. Finally, it is studied the impact of decay on the stable architectures and stochastic stability. Although friction blurs the equation above, it is shown that when the societal core is not empty the star is the most robust architecture, although other stable and stochastically stable architectures emerge.

The results obtained with this approach suggest several lines of further research. In fact, this paper is a first step of a research project to explore the effects of institutional constraints. It may be interesting to further study: (i) an extension of the one-way flow model of Bala and Goyal (2000a) similar to the one achieved here for the two-way flow model; (ii) the impact of *asymmetric* link-constraining systems, which make sense for the one-way and two-way models; (iii) alternative assumptions about knowledge: here we have assumed that players within each component of the societal cover have common knowledge of the part of the network within that component, but it may be interesting to study the effects of further restricting information, which suggests an interesting scenario for interaction between network and knowledge; (iv) the effects of heterogeneity combined with institutional constraints. Finally, it could be interesting to see the impact of institutional constraints as modeled here on Jackson and Wolinsky's (1996) model and variants of it based on pairwise stability, given that in the context of bilateral link formation the societal cover notion provides the most general link-constraining system.

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