A Characterization of Separating Equilibrium in Multidimensional Signaling Games *

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Abstract

This paper studies the conditions for an equilibrium to be separating in signaling games when type and signaling spaces are multidimensional. While these conditions and the form of separating equilibrium (SE) in single dimensional games is well understood even in a fairly general setting (e.g., Mailath [2]), our knowledge of it in multidimensional signaling games is limited. The main obstacle is that though types are multidimensional, the only incentive device at our disposal is one-dimensional monetary payment. Despite this obstacle, putting some constraints on the signaling cost function, we obtain a characterization of SE in multidimensional games.

Our approach uses the known results in single dimensional games and the techniques from traditional consumer theory. We first consider a single dimensional subgame of the original game by which we mean that its type set is a linearly ordered subset of the original type set. From an SE in this subgame, which is easy to find due to the known results, we derive partial information about the signaling cost in SE of the original game. Exploiting this information and incentive compatibility between types that induce the same response of the receiver allows us to fully determine the signals in SE. This step is reminiscent of the derivation of the Hicksian demand function from the expenditure function. We observe the interesting phenomenon that an SE converges to a semi-pooling equilibrium as the complementarity between different attributes of type increases.

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1 Introduction and Setup

Most literature on signalling models has focused on one-dimensional signalling, i.e., a type is assumed to be a single number, and also signals are represented by a single scalar. But it is hard to imagine, say in a job market model, how the characteristics of a job applicant can be fully captured by a single scalar. For example, in addition to the education level, extracurricular activities can be a signal to reveal the ability of a job applicant. Therefore it is relevant and natural to extend signalling games to a setting with multi-dimensional types and signals, i.e., types and signals consisting of combinations of several scalars.

With this motivation, Quinzii and Rochet [3] studies a multidimensional extension of Spence's [4] job market model and gives an elegant characterization of separating equilibria as related to the convex solutions of some partial differential equation. But, they assume separable costs and therefore the question how to characterize separating equilibria when cost functions have different forms in multidimensional signaling games still remains far from fully answered.

We provide a characterization of a separating equilibrium in multidimensional signaling games. This extended abstract introduces the setting, provides the main theorem and shows one application.

1.1 Setup

In a signaling game, there are two players, the sender and the receiver. Let us denote by T and S the space of the sender's types and the space of signals, respectively. The space of the receiver's actions is A. We assume that the dimension of the type space is the same as that of the signal space. Since our interest lies in a separating equilibrium, on the one hand it can never be expected that an existence result will hold if the dimension of signals is strictly less than that of types. On the other hand, if the dimension of signals is larger, a proper subset of signals, with the same dimension as type, could serve the same role in the sense that it is large enough to convey the same informational content.

The game proceeds as follows. Nature decides the sender's type $t \in T$, which is revealed only to the sender. With the knowledge of her type t, the sender chooses a message $s \in S$, incurring corresponding cost. Upon observing the signal s, the receiver takes an action $a \in A$. The payoffs are realized and the game ends.

The payoff to the sender $U_S : S \times A \times T \to \mathbb{R}$ has the quasilinear form given as

$$U_S(s, a, t) = a - c(s, t)$$

where $c : S \times T \to \mathbb{R}$ is a signaling cost the sender of type t spends in delivering signal s. The payoff to the receiver is a mapping $U_R : A \times T \to \mathbb{R}$. Regarding the receiver's payoff, without specifying its form, we assume that for any $t \in T$, there exists a unique optimal action $\alpha(t)$ for the receiver, that is,

$$\{\alpha(t)\} = \arg\max_{a \in A} U_R(a, t).$$

We call the mapping α *the best response of the receiver*.

Definition 1. *A* signaling game $\mathbf{G}$ *is a game of two players, a sender and a receiver, and specifies a space of the sender's type T (with $t \in T$), a space of signals S (with $s \in S$), a space of the receiver's action A and the sender's signaling cost function $c(s,t)$ and the best response of the receiver $\alpha(t) \in A$. We write*

$$\mathbf{G} = (\{S, T, A\}, \{c, \alpha\}).$$

We assume that $S = T = \mathbb{R}^2_+$ and $A = \mathbb{R}_+$ and put the following structural assumptions on the mappings c and α.

Assumption 1. *The function α is continuous on $S \times T$ and strictly increasing in t_i ($i = 1, 2$). And, $\alpha(\mathbf{0}) = 0$.*

Assumption 2. *The cost function c is continuous on $S \times T$ and is strictly increasing in s_i and strictly decreasing in t_i ($i = 1, 2$).*

Assumption 3. *The cost function c is either quasiconvex or quasiconcave in s for all t.*

Assumption 4. *There exists a strictly decreasing and continuous function $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that for all $(s,t) \in S \times T$*

$$c(s, \lambda t) = g(\lambda) c(s,t)$$

for all $\lambda > 0$.

1.2 Incentive compatibility in separating equilibrium

A strategy of the sender is a mapping $\sigma\colon T \to S$. A strategy of the receiver is a function $\gamma\colon S \to A$. A pair of strategies (σ, γ) is a *separating equilibrium* (SE) if σ is one-to-one and each strategy is a best response to the other. When (σ, γ) is an SE, σ is called a *separating equilibrium strategy* (SE strategy). Note that when (σ, γ) is an SE, it holds $\gamma(\sigma(t)) = \alpha(t)$ because the receiver, upon observing $\sigma(t)$, infers the sender's type is t and chooses the best response $\alpha(t)$ against the sender of type t.

Lemma 1. *A strategy σ of the sender is an SE strategy if and only if $\sigma(\mathbf{0}) = \mathbf{0}$, σ is one-to-one and satisfies*

$$\alpha(t') - \alpha(t) \leq c(\sigma(t'), t) - c(\sigma(t), t) \tag{IC}$$

for all t, t.

The interpretation of (IC) is that when type t pretends to be t' by choosing the signal for type t', the marginal benefit is not larger than the marginal cost.

2 Main result

2.1 Characterization of SE

We provide a characterization of an SE strategy in $\mathbf{G} = (\{S, T, A\}, \{c, \alpha\})$. The main idea of our characterization is as follows. First, given t, we consider a subgame of which type set is given as the ray emanating from the origin and passing through t.

Since this game is one-dimensional, we can obtain an SE strategy in this subgame by using known results (e.g., Mailath [2] and Lee, Müller and Vermeulen [1]). But, an SE strategy in $\mathbf{G}$ restricted on the ray should also serve as an SE strategy in the subgame. Due to uniqueness of an SE strategy in the sub game, partial information about the signal for type t in SE of $\mathbf{G}$ is obtained. Then, by taking advantage of incentive compatibility with other types that induce the same response from the receiver, we can fully determine type t's signal in SE of $\mathbf{G}$.

Specifically, we fix t and consider the one-dimensional signaling game

$$\mathbf{g}_t = (\{\bar{S}, \bar{T}, A\}, \{\bar{c}, \bar{\alpha}\})$$

where $\bar{S} = \bar{T} = A = \mathbb{R}_+$ and $\bar{c}(x, \lambda) = xg(\lambda)$ and $\bar{\alpha}(\lambda) = \alpha(\lambda t)$.

Using the result from one dimensional signaling games, we derive a unique SE strategy in $\mathbf{g}_t$.

Lemma 2. *The mapping x_{SE} on $\bar{T}$ defined as*

$$\lambda \mapsto \int_0^\lambda \frac{t}{g(y)} \cdot \nabla\alpha(yt)dy$$

is the unique SE strategy in $\mathbf{g}_t$.

We define

$$F(s, t) := c(s, t) - \int_0^1 \frac{t}{g(y)} \cdot \nabla\alpha(yt)dy$$

and

$$U_t := \bigcap_{\{t' : \alpha(t') = \alpha(t)\}} \{s \in S : F(s, t') \geq 0\}.$$

The set U_t can be empty. Observe that $U_{t'} = U_t$ for any t' with $\alpha(t') = \alpha(t)$.

Theorem 3. *Suppose Assumptions 1-3 hold. If the mapping $\sigma : T \to S$ is an SE strategy in* $\mathbf{G}$, *it satisfies*

1. $\sigma(\mathbf{0}) = \mathbf{0}$*;*

2. σ *is one-to-one;*

3. $F(\sigma(t), t) = 0$ *for all* t*;*

4. $\sigma(t) = \arg\min_{s \in U_t} c(s, t)$ *for all* t.

If Assumption 4 holds, the reverse is also true.

Proof. We only sketch the main ideas.

($\Rightarrow$) It is straightforward to verify conditions 1 and 2. Condition 3 follows from the lemma below.

Lemma 4. *Suppose σ is an SE strategy in* $\mathbf{G}$. *Then, the function $x : \bar{T} \to \bar{S}$ defined as*

$$\lambda \mapsto c(\sigma(\lambda t), t)$$

is an SE strategy in $\mathbf{g}_t$.

Hence, $c(\sigma(\lambda t), t) = \int_0^\lambda \frac{t}{g(y)} \cdot \nabla\alpha(yt)dy$ by Lemma 2 and setting $\lambda = 1$ yields condition 3.

For condition 4, observe first that $c(s,t) \geq c(\sigma(t), t)$ for all $s \in U_t$. It is the case because if there exists $s \in U_t$ such that $c(s,t) < c(\sigma(t), t)$, then $F(s,t) < F(\sigma(t), t) = 0$, which is a contradiction to the assumption $s \in U_t$. It remains to show that $\sigma(t) \in U_t$.

Consider t' such that $\alpha(t) = \alpha(t')$. (IC) implies that

$$c(\sigma(t), t') \geq c(\sigma(t'), t').$$

By condition 3,

$$c(\sigma(t), t') \geq \int_0^1 \frac{t'}{g(y)} \cdot \nabla\alpha(yt')dy, \text{ or equivalently } F(\sigma(t), t') \geq 0.$$

Therefore, condition 4 follows.

($\Leftarrow$) It suffices to verify (IC) for all t and t' by Lemma 1. To this end, we choose $\lambda > 0$ such that $\alpha(\lambda t) = \alpha(t')$. Then, we use the fact that (IC) holds both for t and λt and for λt and t' to conclude that (IC) holds between t and t'. ■

2.2 Example

The theorem above is useful in the sense that it suggests the procedure to compute an SE strategy. The first step is to describe U_t and solve the optimization problem in condition 4. This step is reminiscent of the derivation of a Hicksian demand function. Then, the second step is to check whether the solution of the optimization problem satisfies the remaining conditions 1-3, which is a routine task. We clarify this point more through the following example.

Consider the signaling game $\mathbf{G} = (\{S, T, A\}, \{c, \alpha\})$ where the cost function is given as

$$c(s,t) = \frac{s_1}{t_1} + \frac{s_2}{t_2},$$

which is considered in Quinzii and Rochet [3], and the best response of the receiver is the CES (constant elasticity of substitution) function

$$\alpha(t) = (\frac{1}{2}t_1^\rho + \frac{1}{2}t_2^\rho)^{1/\rho} \text{ with } \rho \leq 1.$$

Note that such c and α satisfy Assumptions 1-4.

In order to find an SE strategy, we first describe the set U_t. Since α is homogeneous of degree 1, $t \cdot \nabla\alpha(yt) = \nabla\alpha(t)$ for all $y > 0$. Thus, noting that $g(y) = 1/y$, we have

$$\int_0^1 \frac{t}{g(y)} \cdot \nabla\alpha(yt)dy = \alpha(t) \int_0^1 ydy = \frac{\alpha(t)}{2}.$$

and so

$$U_t = \{s \in S | s_1^{\rho/(1+\rho)} + s_2^{\rho/(1+\rho)} \geq \frac{(t_1^\rho + t_2^\rho)^{2/(1+\rho)}}{2}\}.$$

The solution of the optimization problem $\min_{s\in U_t} c(s,t)$, coming from condition 4 in Theorem 3, is

$$\sigma(t) = \Big(\frac{1}{2^{(1+\rho)/\rho}}(t_1^\rho + t_2^\rho)^{(1-\rho)/\rho} \cdot t_1^{1+\rho}, \frac{1}{2^{(1+\rho)/\rho}}(t_1^\rho + t_2^\rho)^{(1-\rho)/\rho} \cdot t_2^{1+\rho}\Big).$$

It remains to examine whether σ satisfies conditions 1-3 indeed. Conditions 1 and 2 are easily verified. Condition 3 is also respected because for all t

$$\begin{aligned} F(\sigma(t),t) &= c(\sigma(t),t) - \int_0^1 \frac{t}{g(y)} \cdot \nabla\alpha(yt)dy \\ &= t_1^{-1} \cdot \frac{1}{2^{(1+\rho)/\rho}} \frac{t_1^{1+\rho}}{(t_1^\rho + t_2^\rho)^{1-1/\rho}} + t_2^{-1} \cdot \frac{1}{2^{(1+\rho)/\rho}} \frac{t_2^{1+\rho}}{(t_1^\rho + t_2^\rho)^{1-1/\rho}} - \frac{\alpha(t)}{2} = 0. \end{aligned}$$

Therefore, we conclude that σ is indeed an SE strategy by Theorem 3.

We perform comparative statics to see how an SE strategy changes according to the degree of the complementarity between different attributes of type:

$\rho = 1$: $\sigma(t) = \big(\frac{t_1^2}{4}, \frac{t_2^2}{4}\big)$;

$\rho \to 0$: $\sigma(t) = \frac{\sqrt{t_1 t_2}}{4}\big(t_1, t_2\big)$;

$$\rho \to -\infty\text{:}\ \sigma(t) = \begin{cases} \big(0, \frac{t_2^2}{2}\big) & \text{if } t_1 > t_2 \\ \big(\frac{t_1^2}{2}, 0\big) & \text{if } t_1 < t_2 \\ \big(\frac{t_1^2}{4}, \frac{t_2^2}{4}\big) & \text{if } t_1 = t_2. \end{cases}$$

As one may expect, when attributes of type are perfect substitutes ($\rho = 1$), the game under analysis is simply the sum of two independent one-dimensional signaling games. Thus, each attribute of signal only depends on its corresponding type. But, it becomes dependent on both dimensions of type as ρ increases, i.e., the complementarity increases.

Finally, consider the case that attributes of type are perfect complements ($\rho \to -\infty$). Intuitively, since the receiver only cares about which attribute of the sender's type is lower, in equilibrium it will suffice for the sender to only reveal his lower attribute. This intuition is indeed the case as shown above. Observe that we now have a semi-pooling equilibrium.

We also mention that Theorem 3 may help us to find the determinants on the primitives for the model for existence of SE.

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