# A Continuous Time Approach for the Asymptotic Value in Two-Person Zero-Sum Repeated Games

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**Abstract**

We consider the asymptotic value of two person zero-sum repeated games. We extend results due to Laraki (2001) (2010), obtained for incomplete information games, splitting games and absorbing games in the discounted case using comparison arguments. The technique of proof consists in embedding the discrete repeated game into a continuous time one and to use viscosity solution tools. The results extend to general decreasing evaluations of the stream of stage payoffs.

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**JEL classification:** C73, D82.

**Keywords:** stochastic games, repeated games, incomplete information, asymptotic value, comparison principle, variational inequalities, viscosity solutions, continuous time.

## 1 Introduction

In this paper we are interested in the asymptotic value of two person zero-sum repeated games. Our aim is to show that techniques which are typical of continuous time games ("viscosity solution") can be used to prove the convergence of the discounted value of such games as the discount factor tends to 0, as well as the convergence of the value of the $n-$stage games as $n \to +\infty$. The originality of our approach is that it provides the *same* proof for both classes of problems. It also allows to handle general decreasing evaluations of the stream of stage payoffs, as well as situations in which the payoff varies "slowly" in time. We illlustrate our purpose through three typical problems: repeated games with incomplete information on both sides, first analyzed by Mertens-Zamir (1971) [11], splitting games, introduced by Laraki (2001) [6] and absorbing games, studied in particular by Kohlberg (1974) [5]. For the splitting games, we show that the value of the $n-$stage game has a limit, which was not known yet. In order to better explain our approach, let us first recall the definition of Shapley operator for stochastic games, and its adaptation to games with imperfect information. Then we briefly describe the operator approach and its link with the viscosity solution techniques used in this paper.

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### 1.1 Discounted stochastic games and Shapley operator

A stochastic game is a repeated game where the state changes from stage to stage according to a transition depending on the current state and the moves of the players. We consider the two person zero-sum case.

The game is specified by a state space $\Omega$, move sets $I$ and $J$, a transition probability $\rho$ from $I \times J \times \Omega \to \Delta(\Omega)$ and a payoff function $g$ from $I \times J \times \Omega \to I\!R$. All sets $A$ under consideration are finite and $\Delta(A)$ denotes the set of probabilities on $A$.

Inductively, at stage $n = 1, ...,$ knowing the past history $h_n = (\omega_1, i_1, j_1, ...., i_{n-1}, j_{n-1}, \omega_n)$, player 1 chooses $i_n \in I$, player 2 chooses $j_n \in J$. The new state $\omega_{n+1} \in \Omega$ is drawn according to the probability distribution $\rho(i_n, j_n, \omega_n)$. The triplet $(i_n, j_n, \omega_{n+1})$ is publicly announced and the situation is repeated. The payoff at stage $n$ is $g_n = g(i_n, j_n, \omega_n)$ and the total payoff is the discounted sum $\sum_n \lambda(1-\lambda)^{n-1} g_n$.

This discounted game has a value $v_\lambda$ (Shapley, 1953 [16]).

The Shapley operator $\mathbf{T}(\lambda, \cdot)$ associates to a function $f$ in $I\!R^\Omega$ the function:

$$\mathbf{T}(\lambda, f)(\omega) = \mathtt{val}_{\Delta(I)\times\Delta(J)}[\lambda g(x, y, \omega) + (1-\lambda)\sum_{\tilde{\omega}} \rho(x, y, \omega)(\tilde{\omega}) f(\tilde{\omega})] \tag{1}$$

where $g(x, y, \omega) = \mathsf{E}_{x,y} g(i, j, \omega) = \sum_{i,j} x_i y_j g(i, j, \omega)$ is the multilinear extension of $g(., ., \omega)$ and similarly for $\rho(., ., \omega)$, and $\mathtt{val}$ is the value operator

$$\mathtt{val}_{\Delta(I)\times\Delta(J)} = \max_{x\in\Delta(I)} \min_{y\in\Delta(J)} = \min_{y\in\Delta(J)} \max_{x\in\Delta(I)} .$$

The Shapley operator $\mathbf{T}(\lambda, \cdot)$ is well defined from $I\!R^\Omega$ to itself. Its unique fixed point is $v_\lambda$ (Shapley, 1953 [16]).

### 1.2 Extension: repeated games

A recursive structure leading to an equation similar to the previous one (1) holds in general for repeated games described as follows:

$M$ is a parameter space and $g$ a function from $I \times J \times M$ to $I\!R$. For each $m \in M$ this defines a two person zero-sum game with action spaces $I$ and $J$ for Player 1 and 2 respectively and payoff function $g(m, .)$. The initial parameter $m_1$ is chosen at random and the players receive some initial information about it, say $a_1$ (resp. $b_1$) for player 1 (resp. player 2). This choice is performed according to some initial probability $\pi$ on $A \times B \times M$, where $A$ and $B$ are the signal sets of both players. At each stage $n$, player 1 (resp. 2) chooses an action $i_n \in I$ (resp. $j_n \in J$). This determines a stage payoff $g_n = g(i_n, j_n, m_n)$, where $m_n$ is the current value of the parameter. Then a new value of the parameter is selected and the players get some information. This is generated by a map $\rho$ from $I \times J \times M$ to probabilities on $A \times B \times M$. Hence at stage $n$ a triple $(a_{n+1}, b_{n+1}, m_{n+1})$ is choosen according to the distribution $\rho(i_n, j_n, m_n)$. The new parameter is $m_{n+1}$, and the signal $a_{n+1}$ (resp. $b_{n+1}$) is transmitted to player 1 (resp. player 2). Note that each signal may reveal some information about the previous choice of actions $(i_n, j_n)$ and both the previous $(m_n)$ and the new $(m_{n+1})$ values of the parameter.

**Stochastic games** correspond to public signals including the parameter.

**Incomplete information games** correspond to an absorbing transition on the parameter (which thus remains fixed) and no further information (after the initial one) on the parameter.

Mertens, Sorin and Zamir (1994) [12] Section IV.3, associate to each such repeated game $G$ an auxiliary stochastic game $\Gamma$ having the same values that satisfy a recursive equation of the type (1). However the play, hence the strategies in both games differ. More precisely, in games with incomplete information on both sides, $M$ is a product space $K \times L$, $\pi$ is a product probability $p \otimes q$ with $p \in P = \Delta(K)$, $q \in Q = \Delta(L)$ and in addition $a_1 = k$ and $b_1 = \ell$. Given the

parameter $m = (k, \ell)$, each player knows his own component and holds a prior on the other player's component. From stage 1 on, the parameter is fixed and the information of the players after stage $n$ is $a_{n+1} = b_{n+1} = \{i_n, j_n\}$.
The auxiliary stochastic game $\Gamma$ corresponding to the recursive structure can be taken as follows: the "state space" $\Omega$ is $P \times Q$ and is interpreted as the space of beliefs on the true parameter.
$\mathbf{X} = \Delta(I)^K$ and $\mathbf{Y} = \Delta(J)^L$ are the type-dependent mixed action sets of the players; $g$ is extended on $\mathbf{X} \times \mathbf{Y} \times M$ by $g(x, y, p, q) = \sum_{k,\ell} p^k q^\ell g(x^k, y^\ell, k, \ell)$.
Given $(x, y, p, q)$, let $x(i) = \sum_k x_i^k p^k$ be the probability of action $i$ and $p(i)$ be the conditional probability on $K$ given the action $i$, explicitely $p^k(i) = \frac{p^k x_i^k}{x(i)}$ (and similarly for $y$ and $q$).
In this framework the Shapley operator is defined on the set $\mathcal{F}$ of continuous concave-convex fonctions on $P \times Q$ :

$$\mathbf{T}(\lambda, f)(p, q) = \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\{\lambda g(p, q, x, y) + (1 - \lambda) \sum_{i,j} x(i)y(j)f(p(i), q(j))\} \tag{2}$$

and $v_\lambda(p, q)$ is the unique fixed point of $\mathbf{T}(\lambda, .)$ on $\mathcal{F}$. These relations are due to Aumann and Maschler (1966) [1] and Mertens and Zamir (1971) [11].

### 1.3 Extension: general evaluation

The basic formula expressing the discounted value as a fixed point of the Shapley operator

$$v_\lambda = \mathbf{T}(\lambda, v_\lambda) \tag{3}$$

can be extended for values of games with the same plays but alternative evaluations of the stream of payoffs $\{g_n\}$.
For example the $n$-stage game with payoff defined by the Cesaro mean $\frac{1}{n}\sum_{m=1}^n g_m$ has a value $v_n$ and the recursive formula for these values is obtained similarly as

$$v_n = \mathbf{T}(\frac{1}{n}, v_{n-1})$$

with obviously $v_0 = 0$.
Consider now an arbitrary evaluation probability $\mu$ on $I\!N^\star$. The corresponding payoff in the game is $\sum_n \mu_n g_n$. Note that $\mu$ induces a partition $\Pi = \{t_n\}$ of $[0, 1]$ with $t_0 = 0, t_n = \sum_{m=1}^n \mu_m, \ldots$ and thus the repeated game is naturally represented as a game played between times 0 and 1, where the actions are constant on each subinterval $(t_{n-1}, t_n)$ which length $\mu_n$ is the weight of stage $n$ in the original game. Let $v_\Pi$ be its value. The corresponding recursive equation is now

$$v_\Pi = \mathtt{val}\{t_1 g_1 + (1 - t_1)\mathsf{E} v_{\Pi_{t_1}}\}$$

where $\Pi_{t_1}$ is the normalization on $[0, 1]$ of the trace of the partition $\Pi$ on the intervall $[t_1, 1]$.
If one defines $V_\Pi(t_n)$ as the value of the game starting at time $t_n$, i.e. with evaluation $\mu_{n+m}$ for the payoff $g_m$ at stage $m$, one obtains the alternative recursive formula

$$V_\Pi(t_n) = \mathtt{val}\{(t_{n+1} - t_n)g_{n+1} + \mathsf{E}V_\Pi(t_{n+1})\}. \tag{4}$$

The stationarity properties of the game form in terms of payoffs and dynamics induce time homogeneity

$$V_\Pi(t_n) = (1 - t_n)V_{\Pi_{t_n}}(0) \tag{5}$$

where, as above, $\Pi_{t_n}$ stands for the normalization of $\Pi$ restricted to the intervall $[t_n, 1]$.
By taking the linear extension of $V_\Pi(t_n)$ we define for every partition $\Pi$, a function $V_\Pi(t)$ on $[0, 1]$.

**Lemma 1** *Assume that the sequence $\mu_n$ is decreasing. Then $V_\Pi$ is $C$-Lipschitz in $t$, where $C$ is a uniform bound on the payoffs in the game.*

**Proof.** Given a pair of strategies $(\sigma, \tau)$ in the game $G$ with evaluation $\Pi$ starting at time $t_n$ the total payoff can be written in the form

$$E_{\sigma,\tau}[\mu_{n+1} g_1 + ... + \mu_{n+k} g_k + ...]$$

where $g_k$ is the payoff at stage $k$. Assume now that $\sigma$ is optimal in the game $G$ with evaluation $\Pi$ starting at time $t_{n+1}$, then the alternative evaluation of the stream of payoffs satisfies, for all $\tau$

$$E_{\sigma,\tau,}[\mu_{n+2} g_1 + ... + \mu_{n+k+1} g_k + ...] \geq V_\Pi(t_{n+1}, p, q).$$

It follows that

$$V_\Pi(t_n, p, q) \geq V_\Pi(t_{n+1}, p, q) - |E_{\sigma,\tau}[(\mu_{n+1} - \mu_{n+2}) g_1 + ... + (\mu_{n+k} - \mu_{n+k+1}) g_k + ...]|$$

hence $\mu_n$ being decreasing

$$V_\Pi(t_n, p, q) \geq V_\Pi(t_{n+1}, p, q) - \mu_{n+1} C.$$

This and the dual inequality imply that the linear interpolation $V_\Pi(., p, q)$ is a $C$ Lipschitz function. ∎

### 1.4 Asymptotic analysis: previous results

We consider now the asymptotic behavior of $v_n$ as $n$ goes to $\infty$, or of $v_\lambda$ as $\lambda$ goes to 0.

For games with incomplete information on one side, the first results proving the existence of $\lim_{n\to\infty} v_n$ and $\lim_{\lambda\to 0} v_\lambda$ are due to Aumann and Maschler (1966) [1], including in addition an identification of the limit as $\mathtt{Cav}_{\Delta(K)} u$. Here $u(p) = \mathtt{val}_{\Delta(I)\times\Delta(J)} \sum_k p^k g(x, y, k)$ is the value of the one shot non revealing game, where the informed player does not use his information and $\mathtt{Cav}_C$ is the concavification operator: given $\phi$, a real bounded function defined on a convex set $C$, $\mathtt{Cav}_C(\phi)$ is the smallest function greater than $\phi$ and concave, on $C$.
Extensions of these results to games with lack of information on both sides were achieved by Mertens and Zamir (1971) [11]. In addition they identified the limit as the only solution of the system of implicit functional equations with unknown $\phi$:

$$\phi(p, q) = \mathtt{Cav}_{p\in\Delta(K)} \min\{\phi, u\}(p, q), \tag{6}$$

$$\phi(p, q) = \mathtt{Vex}_{q\in\Delta(L)} \max\{\phi, u\}(p, q) \tag{7}$$

Here again $u$ stands for the value of the non revealing game:

$$u(p, q) = \mathtt{val}_{\Delta(I)\times\Delta(J)} \sum_{k,\ell} p^k q^\ell g(x, y, k, \ell)$$

and we will write **MZ** for the corresponding operator

$$\phi = \mathbf{MZ}(u). \tag{8}$$

As for stochastic games, the existence of $\lim_{\lambda\to 0} v_\lambda$ in the finite case ($\Omega, I, J$ finite) is due to Bewley and Kohlberg (1976) [3] using algebraic arguments: the Shapley fixed point equation can be written as a finite set of polynomial equalities and inequalities involving the variables $\{\lambda, x_\lambda(\omega), y_\lambda(\omega), v_\lambda(\omega); \omega \in \Omega\}$ thus it defines a semi-algebraic set in some euclidean space $\mathbb{R}^N$, hence by projection $v_\lambda$ has an expansion in Puiseux series of $\lambda$.
The existence of $\lim_{n\to\infty} v_n$ is obtained by an algebraic comparison argument, Bewley and Kohlberg (1976) [4].
The asymptotic values for specific classes of absorbing games with incomplete information are studied in Sorin (1984), (1985) [17], [18], see also Mertens, Sorin and Zamir (1994) [12].

### 1.5 Asymptotic analysis: operator approach and comparison criteria

Starting with Rosenberg and Sorin (2001) [14] several existence results for the asymptotic value have been obtained, based on the Shapley operator: continuous absorbing and recursive games, games with incomplete information on both sides, and for absorbing games with incomplete information on one side, Rosenberg (2000) [13].
We describe here an approach that was initially introduced by Laraki (2001) [6] for the discounted case. The analysis of the asymptotic behavior for the discounted games is simpler because of its stationarity: $v_\lambda$ is a fixed point of (3). Many discounted game models have been solved using a variational approach (see Laraki [6], [7] and [10]).
Our work is the natural extension of this analysis to more general evaluations of the stream of stage payoffs including the limit of Cesaro mean. Recall that each evaluation of the stream of payoffs is interpreted as a discretization of an underlying continuous time game. We prove for several classes of games (incomplete information, splitting, absorbing) the existence of a (uniform) limit of the values of the discretized continuous time game as the mesh of the discretization goes to zero. The basic recursive structure is used to formulate variational inequalities that have to be satisfied by any accumulation point of the sequences of values. Then an ad-hoc comparison principle allows to prove uniqueness, hence convergence. Note that this technique is a simple transposition to discrete games of the numerical schemes used to approximate the value function of differential games via viscosity solution arguments, as developed in Barles-Souganidis [2]. The main difference is that, in our case, the limit equation is singular and does not satisfy the conditions usualy required to apply the comparison principles.
To sum up, the paper unifies tools used in discrete and continuous time approaches by dealing with functions defined on the product state $\times$ time space, in the spirit of Vieille (1992) [21] for weak approachability or Laraki (2002) [8] for the dual game with lack of information on one side, see also Sorin (2005) [20].

## 2 Repeated Games with Incomplete Information

Let us briefly recall the structure of repeated games with incomplete information: at the beginning of game the pair $(k, \ell)$ is chosen at random according to some product probability $p \otimes q$ where $p \in P = \Delta(K)$ and $q \in Q = \Delta(L)$. Player 1 knows $k$ while player 2 knows $\ell$. At each stage $n$ of the game, player 1 (resp. player 2) choses a mixed strategy $x_n \in \mathbf{X} = (\Delta(I))^K$ (resp. $y_n \in \mathbf{Y} = (\Delta(J))^K$). This determines a payoff $g(x_n, y_n, p, q)$. In the discounted case, the total payoff is given by $\sum_n \lambda(1-\lambda)^n g(x_n, y_n, p, q)$ and we denote by $v_\lambda(p, q)$ the corresponding value. In this framework the Shapley operator is defined on the set $\mathcal{F}$ of continuous concave-convex fonctions on $P \times Q$ :

$$\mathbf{T}(\lambda, f)(p, q) = \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\{\lambda g(p, q, x, y) + (1-\lambda)\sum_{i,j} x(i)y(j)f(p(i), q(j))\} \tag{9}$$

where, given $(x, y, p, q)$, $x(i) = \sum_k x_i^k p^k$ is the probability of action $i$ and $p(i)$ be the conditional probability on $K$ given the action $i$, namely: $p^k(i) = \frac{p^k x_i^k}{x(i)}$ (and similarly for $y$ and $q$). Recall that $v_\lambda(p, q)$ is the unique fixed point of $\mathbf{T}(\lambda, .)$ on $\mathcal{F}$ ([1], [11]). In particular, $v_\lambda$ is concave in $p$ and convex in $q$.

### 2.1 The discounted game

We now describe the analysis in the discounted case. We follow here Laraki (2001) [6].
Note that the family of functions $\{v_\lambda(p, q)\}$ is $C-$Lipschitz continuous, where $C$ is an uniform bound on the payoffs, hence relatively compact. To prove convergence it is enough to show that

there is only one accumulation point (for the uniform convergence on $P \times Q$).
Note that by (3) any accumulation point $w$ of the family $\{v_\lambda\}$ will satisfy

$$w = \mathbf{T}(0, w)$$

i.e. is a fixed point of the projective operator, see Sorin [19], appendix C.
Explicitly here: $\mathbf{T}(0,w) = \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\{\sum_{i,j} x(i)y(j)w(p(i),q(j))\} = \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\mathsf{E}_{x,y,p,q}w(\tilde{p},\tilde{q})$, where $\tilde{p} = (p^k(i))$ and $\tilde{q} = (q^l(j))$.
Let $\mathcal{S}$ be the set of fixed points of $\mathbf{T}(0,\cdot)$ and $\mathcal{S}_0 \subset \mathcal{S}$ the set of accumulation points of the family $\{v_\lambda\}$ . Given $w \in \mathcal{S}_0$, we denote by $\mathbf{X}(p,q,w) \subseteq \Delta(I)^K = \mathbf{X}$ the set of optimal strategies for player 1 (resp. $\mathbf{Y}(p,q,w) \subseteq \Delta(J)^L = \mathbf{Y}$ for player 2) in the projective game with value $\mathbf{T}(0,w)$ at $(p,q)$. A strategy $x \in \mathbf{X}$ of player 1 is called non-revealing at $p$, $x \in NR_{\mathbf{X}}(p)$ if $\tilde{p} = p$ a.s. (i.e. $p(i) = p$ for all $i \in I$ with $x(i) > 0$) and similarly for $y \in \mathbf{Y}$. Note that the value of the non revealing game satisfies

$$u(p,q) = \mathtt{val}_{NR_{\mathbf{X}}(p)\times NR_{\mathbf{Y}}(q)}g(x,y,p,q)\ . \tag{10}$$

A subset of strategies is non-revealing if *all* its elements are non-revealing.

**Lemma 2** *Let $w \in \mathcal{S}_0$ and $\mathbf{X}(p,q,w) \subset NR_{\mathbf{X}}(p)$ then*

$$w(p,q) \leq u(p,q).$$

**Proof.** Consider a family $\{v_{\lambda_n}\}$ converging to $w$ and $x_n \in \mathbf{X}$ optimal for $\mathbf{T}(\lambda_n, v_{\lambda_n})(p,q)$, see (2). Fix $j \in J$. Jensen's inequality applied to (2) leads to

$$v_{\lambda_n}(p,q) \leq \lambda_n g(p,q,x_n,j) + (1-\lambda_n)v_{\lambda_n}(p,q), \qquad \forall j \in J\ .$$

Thus

$$v_{\lambda_n}(p,q) \leq g(p,q,x_n,j).$$

If $\bar{x} \in \mathbf{X}$ is an accumulation point of the family $\{x_n\}$, then $\bar{x}$ is still optimal in $\mathbf{T}(0,w)(p,q)$. Since, by assumption $\mathbf{X}(p,q,w) \subset NR_{\mathbf{X}}(p)$, $\bar{x}$ is non revealing and therefore one obtains as $\lambda_n$ goes to 0:

$$w(p,q) \leq g(\bar{x},j,p,q), \qquad \forall j \in J\ .$$

So, by (10),

$$w(p,q) \leq \max_{x\in NR_{\mathbf{X}}(p)} \min_{j\in J} g(x,j,p,q) = u(p,q)\ .$$

■

Consider now $w_1$ and $w_2$ in $\mathcal{S}$ and let $(p_0,q_0)$ be an extreme point of the (convex hull of) the compact set in $P \times Q$ where the difference $(w_1 - w_2)(p,q)$ is maximal (this argument goes back to Mertens Zamir (1971) [11]).

**Lemma 3**

$$\mathbf{X}(p_0,q_0,w_1) \subset NR_{\mathbf{X}}(p_0), \qquad \mathbf{Y}(p_0,q_0,w_2) \subset NR_{\mathbf{Y}}(q_0).$$

**Proof.** By definition, if $x \in \mathbf{X}(p_0,q_0,w_1)$ and $y \in \mathbf{Y}(p_0,q_0,w_2)$,

$$w_1(p_0,q_0) \leq \mathsf{E}_{x,y,p_0,q_0}w_1(\tilde{p},\tilde{q})$$

and

$$w_2(p_0,q_0) \geq \mathsf{E}_{x,y,p_0,q_0}w_2(\tilde{p},\tilde{q}).$$

Hence $(\tilde{p},\tilde{q})$ belongs a.s. to the argmax of $w_1 - w_2$ and the result follows from the extremality of $p_0, q_0$. ■

**Proposition 4** $\lim_{\lambda \to 0} v_\lambda$ *exists.*

**Proof.** Let $w_1$ and $w_2$ be two different elements in $\mathcal{S}_0$. To fix the ideas we suppose that $\max w_1 - w_2 > 0$. Let $(p_0, q_0)$ be an extreme point of the (convex hull of) the compact set in $P \times Q$ where the difference $(w_1 - w_2)(p, q)$ is maximal. Then Lemmas 2 and 3 imply $w_1(p_0, q_0) \leq u(p_0, q_0) \leq w_2(p_0, q_0)$, hence a contradiction. The convergence of the family $\{v_\lambda\}$ follows. ∎

Given $w \in \mathcal{S}$ let $\mathcal{E}w(., q)$ be the set of $p \in P$ such that $(p, w(p, q))$ is an extreme point of the epigraph of $w(., q)$.

**Lemma 5** *Let* $w \in \mathcal{S}$. *Then* $p \in \mathcal{E}w(., q)$ *implies* $\mathbf{X}(p, q, w) \subset NR_{\mathbf{X}}(p)$.

**Proof.** Use the fact that if $x \in \mathbf{X}(p, q, w)$ and $y \in NR_{\mathbf{Y}}(q)$

$$w(p, q) \leq \mathsf{E}_{x,y,p,q} w(\tilde{p}, \tilde{q}) = \mathsf{E}_{x,p} w(\tilde{p}, q).$$

∎

Hence one recovers the characterization through the variational inequalities of Mertens and Zamir (1971) [11] and one identifies the limit as **MZ** (u).

**Proposition 6** $\lim_{\lambda \to 0} v_\lambda = \mathbf{MZ}(u)$

**Proof.** Use Lemma 5 and the characterization of Laraki (2001) [7] or Rosenberg and Sorin (2001) [14]. ∎

## 2.2 The finitely repeated game

We now turn to the finitely repeated game: recall that the payoff at stage $n$ is given by $\frac{1}{n} \sum_{k=1}^{n} g(x_k, y_k, p, q)$. We denote by $v_n$ the value of this game. We have the recursive formula:

$$v_n\,(p, q) = \max_{x \in \mathbf{X}} \min_{y \in \mathbf{Y}} \left[ \frac{1}{n} g(x, y, p, q) + (1 - \frac{1}{n}) \sum_{i,j} x(i) y(j) v_{n-1}(p(i), q(j)) \right] = \mathbf{T}(\frac{1}{n}, v_{n-1}). \quad (11)$$

Given an integer $n$, let $\Pi$ be the uniform partition of $[0, 1]$ with mesh $\frac{1}{n}$ and write simply $W_n$ for the associate function $V_\Pi$. Hence $W_n(1, p, q) := 0$ and for $m = 0, ..., n - 1$, $W_n(\frac{m}{n}, p, q)$ satisfies:

$$W_n\left(\frac{m}{n}, p, q\right) = \max_{x \in \Delta(I)^K} \min_{y \in \Delta(J)^L} \left[ \frac{1}{n} g(x, y, p, q) + \sum_{i,j} x(i) y(j) W_n(\frac{m+1}{n}, p(i), q(j)) \right] \quad (12)$$

Note that $W_n(\frac{m}{n}, p, q, \omega) = \left(1 - \frac{m}{n}\right) v_{n-m}(p, q, \omega)$ and if $W_n$ converges uniformly to $W$, $v_n$ converges uniformly to some function $v$ with $W(t, p, q) = (1 - t)\, v(p, q)$.
Let $\mathcal{T}$ be the set of real continuous functions $W$ on $[0, 1] \times P \times Q$ such that for all $t \in [0, 1]$, $W(t, ., .) \in \mathcal{S}$. $\mathbf{X}(t, p, q, W)$ is the set of optimal strategies for Player 1 in $\mathbf{T}(0, W(t, ., .))$ and $\mathbf{Y}(t, p, q, W)$ is defined accordingly.
Let $\mathcal{T}_0$ be the set of accumulation points of the family $\{W_n\}$ for the uniform convergence.

**Lemma 7** $\mathcal{T}_0 \neq \emptyset$ *and* $\mathcal{T}_0 \subset \mathcal{T}$.

**Proof.** $W_n(t, ., .)$ is $C-$Lipschitz continuous in $(p, q)$ for the $L^1$ norm since the payoff, given the strategies $(\sigma, \tau)$ of the players, is of the form $\sum_{k,\ell} p^k q^\ell A^{k\ell}(\sigma, \tau)$. Using Lemma 1 it follows that the family $\{W_n\}$ is uniformly Lipschitz on $[0, 1] \times P \times Q$ hence is relatively compact for the uniform norm. Note finally using (11) that $\mathcal{T}_0 \subset \mathcal{T}$. ∎

We now define two properties for a function $W \in \mathcal{T}$ and a $C^1$ test function $\phi : [0, 1] \to I\!\!R$.

- **P1:** If $t \in [0,1)$ is such that $\mathbf{X}(t,p,q,W)$ is non-revealing and $W(\cdot,p,q)-\phi(\cdot)$ has a global maximum at $t$, then $u(p,q)+\phi'(t) \geq 0$.
- **P2:** If $t \in [0,1)$ is such that $\mathbf{Y}(t,p,q,W)$ is non-revealing and $W(\cdot,p,q)-\phi(\cdot)$ has a global minimum at $t$ then $u(p,q)+\phi'(t) \leq 0$.

**Lemma 8** *Any* $W \in \mathcal{T}_0$ *satisfies* **P1** *and* **P2**.

Note that this result is the variational counterpart of Lemma 2.

**Proof.** Let $t \in [0,1)$, $p$ and $q$ be such that $\mathbf{X}(t,p,q,W)$ is non-revealing and $W(\cdot,p,q)-\phi(\cdot)$ admits a global maximum at $t$. Adding the function $s \mapsto (s-t)^2$ to $\phi$ if necessary, we can assume that this global maximum is strict.
Let $W_{\varphi(n)}$ be a sequence converging uniformly to $W$. Define $\theta(n) \in \{0,\dots,\varphi(n)-1\}$ such that $\frac{\theta(n)}{\varphi(n)}$ is a global maximum of $W_{\varphi(n)}(\cdot,p,q)-\phi(\cdot)$ on the set $\{0,\dots,\varphi(n)-1\}$. Since $t$ is a strict maximum, one has $\frac{\theta(n)}{\varphi(n)} \to t$, as $n \to \infty$. From (12):

$$W_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)},p,q\right) = \max_{x\in\mathbf{X}}\min_{y\in\mathbf{Y}}\left[\frac{1}{\varphi(n)}g(x,y,p,q)+\sum_{i,j}x(i)y(j)W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)},p(i),q(j))\right]$$

Let $x_n \in \mathbf{X}$ be optimal for player 1 in the above formula and let $j \in J$ be any (non-revealing) pure action of player 2. Then:

$$W_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)},p,q\right) \leq \frac{1}{\varphi(n)}g(x_n,j,p,q)+\sum_i x_n(i)W_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)},p_n(i),q\right)$$

By concavity of $W_{\varphi(n)}$ with respect to $p$, we have

$$\sum_{i\in I}x_n(i)W_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)},p_n(i),q\right) \leq W_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)},p,q\right)\ ,$$

hence:

$$0 \leq g(x_n,j,p,q)+\varphi(n)\left[W_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)},p,q\right)-W_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)},p,q\right)\right].$$

Since $\frac{\theta(n)}{\varphi(n)}$ is a global maximum of $W_{\varphi(n)}(\cdot,p,q)-\phi(\cdot)$ on $\{0,\dots,\varphi(n)-1\}$ one has:

$$W_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)},p,q\right)-W_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)},p,q\right) \leq \phi\left(\frac{\theta(n)+1}{\varphi(n)}\right)-\phi\left(\frac{\theta(n)}{\varphi(n)}\right)$$

so that:

$$0 \leq g(x_n,j,p,q)+\varphi(n)\left[\phi\left(\frac{\theta(n)+1}{\varphi(n)}\right)-\phi\left(\frac{\theta(n)}{\varphi(n)}\right)\right].$$

Since $\mathbf{X}$ is compact, one can assume without loss of generality that $\{x_n\}$ converges to some $x$. Note that $x$ belongs to $\mathbf{X}(t,p,q,W)$ by upper semicontinuity using the uniform convergence of $W_{\varphi(n)}$ to $W$. Hence $x$ is non-revealing. Thus, passing to the limit one obtains:

$$0 \leq g(x,j,p,q)+\phi'(t).$$

Since this inequality holds true for every $j \in J$, we also have:

$$\min_{j\in J} g(x,j,p,q)+\phi'(t) \geq 0\ .$$

Taking the maximum with respect to $x \in NR_{\mathbf{X}}(p)$ gives the desired result:

$$u(p,q) + \phi'(t) \geq 0 \ .$$

■

The comparison principle in this case is given by the next result.

**Lemma 9** *Let $W_1$ and $W_2$ in $\mathcal{T}$ satisfying* **P1**, **P2** *and*

- ***P3:*** $W_1(1,p,q) \leq W_2(1,p,q)$ *for any* $(p,q) \in \Delta(K) \times \Delta(L)$.

*Then* $W_1 \leq W_2$ *on* $[0,1] \times \Delta(K) \times \Delta(L)$.

**Proof.** We argue by contradiction, assuming that

$$\max_{t\in[0,1],p\in P,q\in Q} [W_1(t,p,q) - W_2(t,p,q)] = \delta > 0 \ .$$

Then, for $\varepsilon > 0$ sufficiently small,

$$\delta(\varepsilon) := \max_{t\in[0,1],s\in[0,1],p\in P,q\in Q} [W_1(t,p,q) - W_2(s,p,q) - \frac{(t-s)^2}{2\varepsilon} + \varepsilon s] \ > 0 \ . \tag{13}$$

Moreover $\delta(\varepsilon) \to \delta$ as $\varepsilon \to 0$.
We claim that there is $(t_\varepsilon, s_\varepsilon, p_\varepsilon, q_\varepsilon)$, point of maximum in (13), such that $\mathbf{X}(t_\varepsilon, p_\varepsilon, q_\varepsilon, W_1)$ is non-revealing for player 1 and $\mathbf{Y}(s_\varepsilon, p_\varepsilon, q_\varepsilon, W_2)$ is non-revealing for player 2. The proof of this claim is like Lemma 3 and follows again Mertens Zamir (1971) [11]. Let $(t_\varepsilon, s_\varepsilon, p'_\varepsilon, q'_\varepsilon)$ be a maximum point of (13) and $C(\varepsilon)$ be the set of maximum points in $P \times Q$ of the function: $(p,q) \mapsto W_1(t_\varepsilon,p,q) - W_2(s_\varepsilon,p,q)$. This is a compact set. Let $(p_\varepsilon, q_\varepsilon)$ be an extreme point of the convex hull of $C(\varepsilon)$. By Caratheodory's theorem, this is also an element of $C(\varepsilon)$. Let $x_\varepsilon \in \mathbf{X}(t_\varepsilon, p_\varepsilon, q_\varepsilon, W_1)$ and $y_\varepsilon \in \mathbf{Y}(s_\varepsilon, p_\varepsilon, q_\varepsilon, W_2)$. Since $W_1$ and $W_2$ are in $\mathcal{T}$, we have:

$$W_1(t_\varepsilon, p_\varepsilon, q_\varepsilon) - W_2(s_\varepsilon, p_\varepsilon, q_\varepsilon) \leq \sum_{i,j} x_\varepsilon(i) y_\varepsilon(j) \left[W_1(t_\varepsilon, p_\varepsilon(i), q_\varepsilon(j)) - W_2(s_\varepsilon, p_\varepsilon(i), q_\varepsilon(j))\right].$$

By optimality of $(p_\varepsilon, q_\varepsilon)$, one deduces that, for every $i$ and $j$ with $x_\varepsilon(i) > 0$ and $y_\varepsilon(j) > 0$, $(p_\varepsilon(i), q_\varepsilon(j)) \in C(\varepsilon)$. Since $(p_\varepsilon, q_\varepsilon) = \sum_{i,j} x_\varepsilon(i) y_\varepsilon(j)(p_\varepsilon(i), q_\varepsilon(j))$ and $(p_\varepsilon, q_\varepsilon)$ is an extreme point of the convex hull of $C(\varepsilon)$ one concludes that $(p_\varepsilon(i), q_\varepsilon(j)) = (p_\varepsilon, q_\varepsilon)$ for all $i$ and $j$: $x_\varepsilon$ and $y_\varepsilon$ are non-revealing. Therefore we have constructed $(t_\varepsilon, s_\varepsilon, p_\varepsilon, q_\varepsilon)$ as claimed.
Finally we note that $t_\varepsilon < 1$ and $s_\varepsilon < 1$ for $\varepsilon$ sufficiently small, because $\delta(\varepsilon) > 0$ and $W_1(1,p,q) \leq W_2(1,p,q)$ for any $(p,q) \in P \times Q$ by **P3**.
Since the map $t \mapsto W_1(t, p_\varepsilon, q_\varepsilon) - \frac{(t-s_\varepsilon)^2}{2\varepsilon}$ has a global maximum at $t_\varepsilon$ and since $\mathbf{X}(t_\varepsilon, p_\varepsilon, q_\varepsilon, W_1)$ is non-revealing for player 1, condition **P1** implies that

$$u(p_\varepsilon, q_\varepsilon) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} \geq 0 \ . \tag{14}$$

In the same way, since the map $s \mapsto W_2(s, p_\varepsilon, q_\varepsilon) + \frac{(t_\varepsilon - s)^2}{2\varepsilon} - \varepsilon s$ has a global minimum at $s_\varepsilon$ and since $\mathbf{Y}(s_\varepsilon, p_\varepsilon, q_\varepsilon, W_2)$ is non-revealing for player 2, we have by condition **P2** that

$$u(p_\varepsilon, q_\varepsilon) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \varepsilon \leq 0 \ .$$

This latter inequality contradicts (14). ■

We are now ready to prove the convergence result for $\lim_{n\to\infty} v_n$.

**Proposition 10** *$W_n$ converges uniformly to the unique point $W \in \mathcal{T}$ that satisfies the variational inequalities* **P1** *and* **P2** *and the terminal condition $W(0,p,q) = 0$.*
*Consequently, $v_n(p,q)$ converges uniformly to $v(p,q) = W(0,p,q)$ and $W(t,p,q) = (1-t)v(p,q)$, where $v = \mathbf{MZ}(u)$.*

**Proof.** Let $W \in \mathcal{T}_0$. From Lemma 8, $W$ satisfies the variational inequalities **P1** and **P2**. Moreover, $W(1,p,q) = 0$. Since, from Lemma 9, there is at most one function fulfilling these conditions, we obtain convergence of the family $\{W_n\}$.
Consequently, $v_n(p,q)$ converges uniformly to $v(p,q) = W(0,p,q)$ and $W(t,p,q) = (1-t)v(p,q)$. In particular if one considers $\phi(t) = W(t,p,q)$ as test function, then $\phi'(t) = -v(p,q)$. Now **P1** and **P2** reduce to Lemma 2 hence via Lemma 5 to the variational characterization of $\mathbf{MZ}(u)$. ∎

### 2.3 General evaluation

Consider now an arbitrarly evaluation probability $\mu$ on $I\!N^*$ with $\mu_n \geq \mu_{n+1}$ inducing the partition $\Pi$. Let $V_\Pi(t_k,p,q)$ be the value of the game starting at time $t_k$. One has $V_\Pi(1,p,q) := 0$ and

$$V_\Pi(t_n,p,q) = \max_{x\in\mathbf{X}} \min_{y\in\mathbf{Y}} \left[ \mu_{n+1} g(x,y,p,q) + \sum_{i,j} x(i)y(j)V_\Pi(t_{n+1},p(i),q(j)) \right]. \tag{15}$$

Moreover $V_\Pi$ belongs to $\mathcal{F}$ and is $C$ Lipschitz in $(p,q)$.

Lemma 1 then implies that any family of values $V_{\Pi(m)}$ associated to partitions $\Pi(m)$ with $\mu_1(m) \to 0$ as $m \to \infty$ has an accumulation point. Denote by $\mathcal{T}_1$ the set of those. Then $\mathcal{T}_1 \subset \mathcal{T}$ by (15) and lemma 8 extends in a natural way: let $\bar{V} \in \mathcal{T}_1$ and $V_{\Pi(m)} \to \bar{V}$ uniformly. Let $t_n^m$ be a global maximum of $V_{\Pi(m)}(.,p,q) - \phi(.)$ on $\Pi(m)$. Then $t_n^m \to t$ and one has

$$0 \leq g(x_n,j,p,q) + \frac{1}{\mu_n(m)} \left[ V_{\Pi(m)}\left(t_{n+1}^m,p,q\right) - V_{\Pi(m)}\left(t_n^m,p,q\right) \right]$$

hence

$$0 \leq g(x_n,j,p,q) + \frac{1}{\mu_n(m)} \left[ \phi(t_{n+1}^m) - \phi\left(t_n^m\right) \right]$$

and letting $n \to \infty$ the result follows.
Using Lemma 9 this implies the convergence. Thus:

**Proposition 11** *$V_{\Pi(m)}$ converges uniformly to the unique point $V \in \mathcal{T}$ that satisfies the variational inequalities* **P1** *and* **P2**.
*Consequently, $v_{\Pi(k)}(p,q)$ converges uniformly to $v(p,q) = V(0,p,q)$ and $V(t,p,q) = (1-t)v(p,q)$. Moreover $v = \mathbf{MZ}(u)$.*

In particular the convergence of $\{V_{\Pi(m)}\}$ for any family of decreasing partitions allows to use $\lim_{\lambda\to 0} v_\lambda$ to characterize the limit.

## 3 Splitting games

We consider now the framework of splitting games, Sorin (2002) [19], p. 78. Let $P$ and $Q$ be two simplexes (or product of simplexes) of some finite dimensional spaces, and $H$ a $C$-Lipschitz function from $P \times Q$ to $I\!R$. The corresponding Shapley operator is defined on continuous real functions $f$ on $P \times Q$ by

$$\mathbf{T}(\lambda,f)(p,q) = \mathtt{val}_{\mu\in M_p^P \times \nu\in M_q^Q} \int_{P\times Q} [(\lambda H(p',q') + (1-\lambda)f(p',q')]\mu(dp')\nu(dq')$$

where $M_p^P$ stands for the set of Borel probabilities on $P$ with expectation $p$ (and similarly for $M_q^Q$).
The associated repeated game is played as follows: at stage $n+1$ knowing the state $(p_n, q_n)$ player 1 (resp. player 2) chooses $\mu_{n+1} \in M_{p_n}^P$ (resp. $\nu \in M_{q_n}^Q$). A new state $(p_{n+1}, q_{n+1})$ is selected according to these distributions and the stage payoff is $H(p_{n+1}, q_{n+1})$. We denote by $V_\lambda$ the value of the discounted game and by $v_n$ the value of the discounted game.
A procedure analogous to the previous study of discounted games with incomplete information has been developed by Laraki [6], [7], [9].

### 3.1 The discounted game

The next properties are established in Laraki (2001) [7].
Let $\mathcal{G}$ be the set of C-Lipschitz functions that are concave convex on $P \times Q$.

**Lemma 12** *The Shapley operator* $\mathbf{T}(\lambda, \cdot)$ *maps* $\mathcal{G}$ *to itself and* $V_\lambda(p,q)$ *is the only fixed point of* $T(\lambda, .)$ *in* $\mathcal{G}$.

The corresponding projective operator is the splitting operator $\Psi$:

$$\Psi(f)(p,q) = \mathtt{val}_{M_p^P \times \nu M_q^Q} \int_{P\times Q} f(p',q')\mu(dp')\nu(dq') \tag{16}$$

and we denote again by $\mathcal{S}$ its set of fixed points. Given $W \in \mathcal{S}$, $\mathbf{P}(p,q,W) \subset M_p^P$ denotes the set of optimal strategies of player 1 in (16) for $\Psi(W)(p,q)$. We say that $\mathbf{P}(p,q,W)$ is non-revealing if it is reduced to $\delta_p$, the Dirac mass at $p$. We use the symmetric notation $\mathbf{Q}(p,q,W)$ and terminology for player 2.

We define two properties for functions in $\mathcal{S}$.

- **PP1:** If $\mathbf{P}(p,q,W)$ is non-revealing, then $W(p,q) \le H(p,q)$.
- **PP2:** If $\mathbf{Q}(p,q,W)$ is non-revealing, then $W(p,q) \ge H(p,q)$

**Proposition 13** $V_\lambda$ *converges uniformly to the unique point* $V \in \mathcal{S}$ *that satisfies the variational inequalities* **PP1** *and* **PP2**.

The link with the **MZ** operator is as follows: as in Lemma 5 one defines:

- **QQ1:** If $p \in \mathcal{E}W(.,q)$, then $W(p,q) \le H(p,q)$.
- **QQ2:** If $q \in \mathcal{E}W(p,.)$, then $W(p,q) \ge H(p,q)$

(where, as before, $\mathcal{E}V$ denotes the set of extreme points of a convex or concave map $V$). Then one has

**Proposition 14** *Let* $G \in \mathcal{G}$. *Then* $G$ *satisfies* **QQ1** *and* **QQ2** *iff* $G = \mathbf{MZ}(H)$.

### 3.2 The finitely repeated game

Recall the recursive formula defining by induction the value of the $n$ stage game $v_n \in \mathcal{G}$, using Lemma 12:

$$v_n(p,q) = \mathtt{val}_{M_p^P \times M_q^Q} \int_{P\times Q} [\frac{1}{n} H(p',q') + (1-\frac{1}{n}) v_{n-1}(p',q')]\mu(dp')\nu(dq') = \mathbf{T}(\frac{1}{n}, V_{n-1}). \tag{17}$$

For each integer $n$, let $W_n(1,p,q) := 0$ and for $m = 0, ..., n-1$ define $W_n(\frac{m}{n}, p, q)$ inductively as follows:

$$W_n\left(\frac{m}{n}, p, q\right) = \mathtt{val}_{M_p^P \times M_q^Q} \int_{P\times Q} [\frac{1}{n} H(p', q') + W_n(\frac{m+1}{n}, p', q')] \mu(dp')\nu(dq') \; . \tag{18}$$

By induction we have $W_n(\frac{m}{n}, p, q) = \left(1 - \frac{m}{n}\right) v_{n-m}(p,q)$. Note that $W_n$ is the function on $[0,1] \times P \times Q$ associated to the uniform partition of mesh $\frac{1}{n}$.

**Lemma 15** *$W_n$ is Lipschitz continuous uniformly in $n$ on $\{\frac{m}{n},\ m \in \{0,\dots,n\}\} \times P \times Q$.*

**Proof.** By Lemma 12 $W_n(t,.,.)$ belongs to $\mathcal{G}$ for any $t$. As for Lipschitz continuity with respect to $t$, we have, if $\mu$ is optimal in (18) and by Jensen inequality:

$$\begin{aligned} W_n(\frac{m}{n}, p, q) &\leq \int_{P\times Q} \tfrac{1}{n} H(p', q) + W_n(\tfrac{m+1}{n}, p', q) d\mu(p') \\ &\leq \tfrac{\|H\|_\infty}{n} + W_n(\tfrac{m+1}{n}, p, q) \; . \end{aligned}$$

On gets the reverse inequality $W_n(\frac{m}{n}, p, q) \geq -\frac{\|H\|_\infty}{n} + W_n(\frac{m+1}{n}, p, q)$ with the symmetric arguments. Therefore $W_n(\cdot, p, q)$ is $\|H\|_\infty-$Lipschitz continuous. ■

Let $\mathcal{T}$ be the set of real continuous functions $W$ on $[0,1] \times P \times Q$ such that for all $t \in [0,1], W(t,.,.) \in \mathcal{S}$. $\mathbf{P}(t,p,q,W)$ is defined as $\mathbf{P}(p,q,W(t,.,.))$ and $\mathbf{Q}(t,p,q,W)$ as $\mathbf{Q}(p,q,W(t,.,.))$. Let $\mathcal{T}_0$ be the set of accumulation points of the family $W_n$. Using (18), we have that $\mathcal{T}_0 \subset \mathcal{T}$.

We introduce two properties for a function $W \in \mathcal{T}$ and any $C^1$ test function $\phi : [0,1] \to I\!R$.

- **PS1:** If, for some $t \in [0,1)$, $\mathbf{P}(t,p,q,W)$ is non-revealing and $W(\cdot,p,q) - \phi(\cdot)$ has a global maximum at $t$, then $H(p,q) + \phi'(t) \geq 0$.
- **PS2:** If, for some $t \in [0,1)$, $\mathbf{Q}(t,p,q,W)$ is non-revealing and $W(\cdot,p,q) - \phi(\cdot)$ has a global minimum at $t$ then $H(p,q) + \phi'(t) \leq 0$.

**Lemma 16** *Any $W \in \mathcal{T}_0$ satisfies* **PS1** *and* **PS2**.

**Proof.** The proof is very similar to the proof of Lemma 8.
Let $t \in [0,1)$, $p$ and $q$ be such that $\mathbf{P}(t,p,q,W)$ is non-revealing and $W(\cdot,p,q) - \phi(\cdot)$ admits a global maximum at $t$. Adding $(\cdot - t)^2$ to $\phi$ if necessary, we can assume that this global maximum is strict.
Let $W_{\varphi(n)}$ be a sequence converging uniformly to $W$. Define $\theta(n) \in \{0,\dots,\varphi(n)-1\}$ such that $\frac{\theta(n)}{\varphi(n)}$ is a global maximum of $W_{\varphi(n)}(\cdot,p,q) - \phi(\cdot)$ on $\{0,\dots,\varphi(n)-1\}$. Since $t$ is a strict maximum, we have $\frac{\theta(n)}{\varphi(n)} \to t$. By (18) we have that:

$$W_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)}, p, q\right) = \mathtt{val}_{M_p^P \times M_q^Q} \int_{P\times Q} [\frac{1}{\varphi(n)} H(p', q') + W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)}, p', q')] \mu(dp')\nu(dq').$$

Let $\mu_n$ be optimal for player 1 in the above formula and let $\nu = \delta_q$ be the Dirac mass at $q$. Then:

$$W_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)}, p, q\right) \leq \int_P \frac{1}{\varphi(n)} H(p', q)\mu_n(dp') + \int_P W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)}, p', q)\mu_n(dp').$$

By concavity of $W_{\varphi(n)}$ with respect to $p$, we have

$$\int_P W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)}, p', q)\mu_n(dp') \leq W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)}, p, q)$$

Hence:

$$0 \le \int_P H(p',q)\mu_n(dp') + \varphi(n)\left[W_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)},p,q\right) - W_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)},p,q\right)\right] .$$

Since $\frac{\theta(n)}{\varphi(n)}$ is a global maximum of $W_{\varphi(n)}(\cdot,p,q) - \phi(\cdot)$ on $\{0,\dots,\varphi(n)-1\}$ one has:

$$W_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)},p,q\right) - W_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)},p,q\right) \le \phi\left(\frac{\theta(n)+1}{\varphi(n)}\right) - \phi\left(\frac{\theta(n)}{\varphi(n)}\right)$$

So that

$$0 \le \int_P H(p',q)\mu_n(dp') + \varphi(n)\left[\phi\left(\frac{\theta(n)+1}{\varphi(n)}\right) - \phi\left(\frac{\theta(n)}{\varphi(n)}\right)\right] \tag{19}$$

Since $M_p^P$ is compact, one can assume without loss of generality that $\{\mu_n\}$ converges to some $\mu$. Note that $\mu$ belongs to $\mathbf{P}(t,p,q,W)$ by upper semicontinuity and uniform convergence of $W_{\varphi(n)}$ to $W$. Hence $\mu$ is non-revealing: $\mu = \delta_p$. Thus, passing to the limit in (19) one obtains:

$$0 \le H(p,q) + \phi'(t).$$

■

The comparison principle in this case is given by the next result.

**Lemma 17** *Let $W_1$ and $W_2$ in $\mathcal{T}$ satisfying* **PS1**, **PS2** *and*

- ***PS3:*** *$W_1(1,p,q) \le W_2(1,p,q)$ for any $(p,q) \in \Delta(K)\times\Delta(L)$.*

*Then $W_1 \le W_2$ on $[0,1]\times\Delta(K)\times\Delta(L)$.*

**Proof.** The proof is very similar to the proof of Lemma 9.
*We argue by contradiction, assuming that*

$$\max_{t\in[0,1],p\in P,q\in Q}[W_1(t,p,q) - W_2(t,p,q)] = \delta > 0 .$$

*Then, for $\varepsilon > 0$ sufficiently small,*

$$\delta(\varepsilon) := \max_{t\in[0,1],s\in[0,1],p\in P,q\in Q}[W_1(t,p,q) - W_2(s,p,q) - \frac{(t-s)^2}{2\varepsilon} + \varepsilon s] > 0 . \tag{20}$$

*Moreover $\delta(\varepsilon)\to\delta$ as $\varepsilon\to 0$.*
*We claim that there is $(t_\varepsilon,s_\varepsilon,p_\varepsilon,q_\varepsilon)$, point of maximum in (13), such that $\mathbf{P}(t_\varepsilon,p_\varepsilon,q_\varepsilon,W_1)$ is non-revealing for player 1 and $\mathbf{Q}(s_\varepsilon,p_\varepsilon,q_\varepsilon,W_2)$ is non-revealing for player 2. Let $(t_\varepsilon,s_\varepsilon,p'_\varepsilon,q'_\varepsilon)$ be a maximum point of (13) and $C(\varepsilon)$ be the set of maximum points in $P\times Q$ of the map $(p,q)\mapsto W_1(t_\varepsilon,p,q) - W_2(s_\varepsilon,p,q)$. This is a compact set. Let $(p_\varepsilon,q_\varepsilon)$ be an extreme point of the convex hull of $C(\varepsilon)$. By Caratheodory's theorem, this is also an element of $C(\varepsilon)$. Let $\mu_\varepsilon\in\mathbf{P}(t_\varepsilon,p_\varepsilon,q_\varepsilon,W_1)$ and $\nu_\varepsilon\in\mathbf{Q}(s_\varepsilon,p_\varepsilon,q_\varepsilon,W_2)$. Since $W_1$ and $W_2$ are in $\mathcal{T}$, we have:*

$$W_1(t_\varepsilon,p_\varepsilon,q_\varepsilon) - W_2(s_\varepsilon,p_\varepsilon,q_\varepsilon) \le \int_{P\times Q}\left[W_1(t_\varepsilon,p',q') - W_2(s_\varepsilon,p',q')\right]\mu_\varepsilon(dp')\nu_\varepsilon(dq')$$

*By extremality of $(p_\varepsilon,q_\varepsilon)$, one deduces that $\mu_\varepsilon = \delta_{p_\varepsilon}$ and $\mu_\varepsilon = \delta_{p_\varepsilon}$. Therefore we have constructed $(t_\varepsilon,s_\varepsilon,p_\varepsilon,q_\varepsilon)$ as claimed.*
*Finally we note that $t_\varepsilon < 1$ and $s_\varepsilon < 1$ for $\varepsilon$ sufficiently small, because $\delta(\varepsilon) > 0$ and $W_1(1,p,q) \le W_2(1,p,q)$ for any $p,q$ by* **P3**.

*Since the map* $t \mapsto W_1(t, p_\varepsilon, q_\varepsilon) - \frac{(t-s_\varepsilon)^2}{2\varepsilon}$ *has a global maximum at* $t_\varepsilon$ *and since* $\mathbf{P}(t_\varepsilon, p_\varepsilon, q_\varepsilon, W_1)$ *is non-revealing for player I, condition* **PS1** *implies that*

$$u(p_\varepsilon, q_\varepsilon) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} \geq 0 \ . \tag{21}$$

*In the same way, since the map* $s \mapsto W_2(s, p_\varepsilon, q_\varepsilon) + \frac{(t_\varepsilon - s)^2}{2\varepsilon} - \varepsilon s$ *has a global minimum at* $s_\varepsilon$ *and since* $\mathbf{Q}(s_\varepsilon, p_\varepsilon, q_\varepsilon, W_2)$ *is non-revealing for player J, we have by condition* **PS2** *that*

$$u(p_\varepsilon, q_\varepsilon) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \varepsilon \leq 0 \ .$$

*This latter inequality contradicts (21).* ■

*I suggest to drop the above part*

We are now ready to prove the convergence result for $\lim_{n\to\infty} v_n$:

**Proposition 18** $W_n$ *converges uniformly to the unique point* $W \in \mathcal{T}$ *that satisfies the variational inequalities* **PS1** *and* **PS2** *and the terminal condition* $W(1, p, q) = 0$.
*Consequently,* $v_n(p, q)$ *converges uniformly to* $v(p, q) = W(0, p, q)$ *and* $W(t, p, q) = (1 - t)v(p, q)$.
*Moreover* $v = \mathbf{MZ}(H)$.

**Proof.** Let $W$ be any limit point of the relatively compact family $W_n$. Then, from Lemma 16, $W \in \mathcal{T}_0$ satisfies the variational inequalities **PS1** and **PS2**. Moreover, $W(1, p, q) = 0$. Since, from Lemma 17, there is at most one map fulfilling these conditions, we obtain convergence.
Consequently, $v_n(p, q)$ converges uniformly to $V(p, q) = W(0, p, q)$ and $W(t, p, q) = (1-t)V(p, q)$. In particular if one choose as test function $\phi(t) = W(t, p, q)$, then $\phi'(t) = -V(p, q)$, so that **PS1** and **PS2** reduce **PP1** and **PP2**. On concludes by using the variational characterization of $\mathbf{MZ}(u)$ in Proposition 14. ■

## 3.3 General evaluation

The same results extend to the general evaluation case defined by a partition $\Pi$ with $\mu_n$ decreasing. The existence of $V_\Pi$ is obtained in two steps. We first let $V_\Pi^n$ to be 0 on $[t_n, 1]$ and define inductively $V_\Pi^n(t_m, ., .)$ for $m < n$ by

$$V_\Pi^n\,(t_m, p, q) = \mathtt{val}_{M_p^P \times M_q^Q} \int_{P\times Q} [\mu_{m+1} H(p', q') + V_\Pi^n(t_{m+1}, p', q')]\mu(dp')\nu(dq'). \tag{22}$$

It follows that $V_\Pi^n \in \mathcal{G}$ by Lemma 12 and converges uniformly to $V_\Pi$. Then the proof follows exactly the same steps than in Part 2.

## 3.4 Time dependent case

We consider here the case where the function $H$ may depend on the stage.
To be able to study the asymptotic behavior one has to define $H$ directly in the limit game: the map $H$ is a continuous real function on $[0, 1] \times P \times Q$.

For each integer $n$, let $Z_n(1, p, q) := 0$ and for $m = 0, ..., n - 1$ define $Z_n(\frac{m}{n}, p, q)$ inductively as follows:

$$Z_n\left(\frac{m}{n}, p, q\right) = \mathtt{val}_{M_p^P \times M_q^Q} \int_{P\times Q} [\frac{1}{n} H(\frac{m}{n}, p', q') + Z_n(\frac{m+1}{n}, p', q')]\mu(dp')\nu(dq'). \tag{23}$$

By induction each function $Z_n(\frac{m}{n},.,.)$ is in $\mathcal{G}$ and one can show as in Lemma 15 that $Z_n$ is uniformly Lipschitz continuous on $\{\frac{m}{n},\ m \in \{0,\dots,n\}\} \times P \times Q$.

**Remark :** An alternative choice is to replace $\frac{1}{n}H(\frac{m}{n},p',q')$ by $\int_{\frac{m}{n}}^{\frac{m+1}{n}} H(t,p',q')dt$.

Note that the projective operator is the same than in the autonomous case. Let $\mathcal{T}$ be the set of real functions $Z$ on $[0,1]\times P\times Q$ such that for all $t\in[0,1]$, $Z(t,.,.)\in\mathcal{S}$. We define $\mathbf{P}(t,p,q,Z)$ and $\mathbf{Q}(t,p,q,Z)$ as before and denote by $\mathcal{Z}_0$ the set of accumulation points of the family $Z_n$. We note that $\mathcal{Z}_0\subset\mathcal{T}$.

We define two properties for a function $Z\in\mathcal{T}$ and all $C^1$ test function $\phi:[0,1]\to I\!R$.

- **PST1:** If, for some $t\in[0,1)$, $\mathbf{P}(t,p,q,Z)$ is non-revealing and $Z(\cdot,p,q)-\phi(\cdot)$ has a global maximum at $t$, then $H(t,p,q)+\phi'(t)\geq 0$.
- **PST2:** If, for some $t\in[0,1)$, $\mathbf{Q}(t,p,q,Z)$ is non-revealing and $Z(\cdot,p,q)-\phi(\cdot)$ has a global minimum at $t$ then $H(t,p,q)+\phi'(t)\leq 0$.

**Lemma 19** *Any* $Z\in\mathcal{Z}_0$ *satisfies* **PST1** *and* **PST2**.

**Proof.** Let $t\in[0,1)$, $p$ and $q$ be such that $\mathbf{P}(t,p,q,Z)$ is non-revealing and $Z(\cdot,p,q)-\phi(\cdot)$ admits a global maximum at $t$. Adding $(\cdot-t)^2$ to $\phi$ if necessary, we can assume that this global maximum is strict.
Let $Z_{\varphi(n)}$ be a sequence converging uniformly to $Z$. Define $\theta(n)\in\{0,\dots,\varphi(n)-1\}$ such that $\frac{\theta(n)}{\varphi(n)}$ is a global maximum of $Z_{\varphi(n)}(\cdot,p,q)-\phi(\cdot)$ on $\{0,\dots,\varphi(n)-1\}$. $t$ being a strict maximum $\frac{\theta(n)}{\varphi(n)}\to t$. By (23) we have that:

$$Z_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)},p,q\right)=\sup_{\mu\in M_p^P}\inf_{\nu\in M_q^Q}\int_{P\times Q}[\frac{1}{\varphi(n)}H(\frac{\theta(n)}{\varphi(n)},p',q')+Z_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)},p',q')]\mu(dp')\nu(dq').$$

Let $\mu_n$ be optimal for player I in the above formula and let $\nu=\delta_q$ be the Dirac mass at $q$. Then:

$$Z_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)},p,q\right)\leq\int_P\frac{1}{\varphi(n)}H(\frac{\theta(n)}{\varphi(n)},p',q')\mu_n(dp')+\int_P Z_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)},p',q)\mu_n(dp').$$

By concavity of $Z_{\varphi(n)}$ with respect to $p$, we have

$$\int_P Z_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)},p',q)\mu_n(dp')\leq Z_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)},p,q)$$

Hence:

$$0\leq\int_P H(\frac{\theta(n)}{\varphi(n)},p',q')\mu_n(dp')+\varphi(n)\left[Z_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)},p,q\right)-Z_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)},p,q\right)\right].$$

Since $\frac{\theta(n)}{\varphi(n)}$ is a global maximum of $Z_{\varphi(n)}(\cdot,p,q)-\phi(\cdot)$ on $\{0,\dots,\varphi(n)-1\}$ one has:

$$Z_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)},p,q\right)-Z_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)},p,q\right)\ \leq\phi\left(\frac{\theta(n)+1}{\varphi(n)}\right)-\phi\left(\frac{\theta(n)}{\varphi(n)}\right)$$

Since $M_p^P$ is compact, one can assume without loss of generality that $\{\mu_n\}$ converges to some $\mu$. Note that $\mu$ belongs to $\mathbf{P}(t,p,q,Z)$ by upper semicontinuity and uniform convergence of $Z_{\varphi(n)}$ to $Z$. Hence $\mu=\delta_p$ is non-revealing. Thus, passing to the limit one obtains:

$$0\leq H(t,p,q)+\phi'(t).$$

■

The comparison principle in this case is given by the next result.

**Lemma 20** *Let $Z_1$ and $Z_2$ in $\mathcal{T}$ satisfying* **PS1**, **PS2** *and*

- ***PS3:*** $Z_1(1,p,q) \leq Z_2(1,p,q)$ *for any* $(p,q) \in \Delta(K) \times \Delta(L)$.

*Then* $Z_1 \leq Z_2$ *on* $[0,1] \times \Delta(K) \times \Delta(L)$.

**Proof.** We argue by contradiction, assuming that

$$\max_{t\in[0,1],p\in P,q\in Q} [Z_1(t,p,q) - Z_2(t,p,q)] = \delta > 0 \ .$$

Then, for $\varepsilon > 0$ sufficiently small,

$$\delta(\varepsilon) := \max_{t\in[0,1],s\in[0,1],p\in P,q\in Q} [Z_1(t,p,q) - Z_2(s,p,q) - \frac{(t-s)^2}{2\varepsilon} + \varepsilon s] \ > 0 \ . \tag{24}$$

Moreover $\delta(\varepsilon) \to \delta$ as $\varepsilon \to 0$.
Hence as before there is $(t_\varepsilon, s_\varepsilon, p_\varepsilon, q_\varepsilon)$, point of maximum in (13), such that $\mathbf{P}(t_\varepsilon, p_\varepsilon, q_\varepsilon, W_1)$ is non-revealing for player I and $\mathbf{Q}(s_\varepsilon, p_\varepsilon, q_\varepsilon, W_2)$ is non-revealing for player J.
Finally we note that $t_\varepsilon < 1$ and $s_\varepsilon < 1$ for $\varepsilon$ sufficiently small, because $\delta(\varepsilon) > 0$ and $Z_1(1,p,q) \leq Z_2(1,p,q)$ for any $p,q$ by **P3**.
Since the map $t \mapsto Z_1(t,p_\varepsilon,q_\varepsilon) - \frac{(t-s_\varepsilon)^2}{2\varepsilon}$ has a global maximum at $t_\varepsilon$ and since $\mathbf{P}(t_\varepsilon, p_\varepsilon, q_\varepsilon, W_1)$ is non-revealing for player I, condition **PST1** implies that

$$H(t_\varepsilon, p_\varepsilon, q_\varepsilon) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} \geq 0 \ . \tag{25}$$

In the same way, since the map $s \mapsto W_2(s,p_\varepsilon,q_\varepsilon) + \frac{(t_\varepsilon - s)^2}{2\varepsilon} - \varepsilon s$ has a global minimum at $s_\varepsilon$ and since $\mathbf{Q}(s_\varepsilon, p_\varepsilon, q_\varepsilon, W_2)$ is non-revealing for player J, we have by condition **PST2** that

$$H(s_\varepsilon, p_\varepsilon, q_\varepsilon) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \varepsilon \leq 0 \ .$$

This latter inequality contradicts (25), since $H(.,p,q)$ is Lipschitz and $|t_\varepsilon - s_\varepsilon| = o(\varepsilon)$.

■

We are now ready to prove the convergence result for $Z_n$:

**Proposition 21** *$Z_n$ converges uniformly to the unique point $Z \in \mathcal{T}$ that satisfies the variational inequalities* **PST1** *and* **PST2** *and the terminal condition $Z(1,p,q) = 0$.*

**Remark :** the same result obviously holds for any sequence of decreasing evaluation.

**Proof.** Let $Z$ be any limit point of the relatively compact family $Z_n$. Then, from Lemma 19, $W \in \mathcal{T}_0$ satisfies the variational inequalities **PST1** and **PST2**. Moreover, $Z((1,p,q) = 0$. Since, from Lemma 20, there is at most one map fulfilling these conditions, we obtain convergence. ■

*On peut ajouter l'extension des resultats d'A.S. On definit*

$$A_n\left(\frac{m}{n},p,q\right) = \mathbf{MZ}[\frac{1}{n}H(\frac{m}{n},p,q) + A_n(\frac{m+1}{n},p,q)]. \tag{26}$$

*ou plus generalement pour toute evaluation $\Pi$, alors $A_n$ converge la solution de la Proposition 21.*

# 4 Absorbing games

An absorbing game is a stochastic game where only one state is non absorbing. In the other states one can assume that the payoff is constant (equal to the value) thus the game is defined by the following elements: two finite sets $I$ and $J$, two (payoff) functions $f$, $g$ from $I \times J$ to $[-1,1]$ and a function $p$ from $I \times J$ to $[0,1]$.
The repeated game with absorbing states is played in discrete time as follows. At stage $m = 1, 2, ...$ (if absorbtion has not yet occurred) player 1 chooses $i_m \in I$ and, simultaneously, player 2 chooses $j_m \in J$:
(i) the payoff at stage $m$ is $f(i_m, j_m)$;
(ii) with probability $1 - p(i_m, j_m)$ absorbtion is reached and the payoff in all future stages $n > m$ is $g(i_m, j_m)$ and
(iii) with probability $p(i_m, j_m)$ the situation is repeated at stage $m+1$.

Recall that the asymptotic analysis for these games is due to Kohlberg (1974) [5] who also proved the existence of a uniform value in case of standard signalling.

## 4.1 The discounted game

The $\lambda$ discounted game has a value, $v_\lambda$. Using the Shapley operator, $v_\lambda$ is the unique real number in $[-1,1]$ satisfying

$$v_\lambda = \max_{x \in \Delta(I)} \min_{j \in J} \left[\lambda f(x,j) + (1-\lambda) p(x,j) v_\lambda + (1-\lambda) f^*(x,j)\right]. \tag{27}$$

where $p^*(i,j) = 1 - p(i,j)$ and $f^*(i,j) = p^*(i,j)g(i,j)$ and any map $\varphi : I \times J \to I\!R$ is extended linearly to $I\!R^I \times I\!R^J$ as follows: $\varphi(\alpha,\beta) = \sum_{i \in I,\ j \in J} \alpha^i \beta^j \varphi(i,j)$.
A simple computation implies that the payoff $r(\lambda,x,y)$ induced by the stationary strategies $x \in \Delta(I)$ and $y \in \Delta(J)$ is

$$r(\lambda,x,y) = \frac{\lambda f(x,y) + (1-\lambda) f^*(x,y)}{\lambda p(x,y) + p^*(x,y)} \tag{28}$$

so that

$$v_\lambda = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} r(\lambda,x,j) \tag{29}$$

The next result by Laraki (2010) [10] identifies the limit as the value of a one shot game on $(\Delta(I) \times I\!R^I_+) \times (\Delta(J) \times I\!R^J_+)$ with payoff

$$A(x,\alpha,y,\beta) = \frac{f^*(x,y)}{p^*(x,y)} \mathbf{1}_{\{p^*(x,y)>0\}} + \frac{f(x,y) + f^*(\alpha,y) + f^*(x,\beta)}{1 + p^*(\alpha,y) + p^*(x,\beta)} \mathbf{1}_{\{p^*(x,y)=0\}}.$$

**Proposition 22 ([10])** *$v_\lambda$ converges, as $\lambda$ goes to zero, to*

$$v := \mathtt{val}_{(\Delta(I)\times I\!R^I_+)\times(\Delta(J)\times I\!R^J_+)} A(x,\alpha,y,\beta) \tag{30}$$

**Proof.** Let $w = \lim_{n\to\infty} v_{\lambda_n}$ be an accumulation point of $\{v_\lambda\}$ and consider an optimal stationary strategy $x(\lambda_n)$ of player 1 for $v_{\lambda_n}$ in (29). Thus, for every $y \in \Delta(J)$ and $\beta \in I\!R^J_+$ one has, using homogeneity:

$$v_{\lambda_n} \leq \frac{\lambda_n f(x(\lambda_n), y + \lambda_n \beta) + (1-\lambda_n) f^*(x(\lambda_n), y + \lambda_n \beta)}{\lambda_n p(x(\lambda_n), y + \lambda_n \beta) + p^*(x(\lambda_n), y + \lambda_n \beta)}. \tag{31}$$

By compactness of $\Delta(I)$, we can assume that $x(\lambda_n) \to x$.

**Case 1:** $p^*(x,y) > 0$. Letting $\lambda_n$ go to zero in (31) implies

$$w \le \frac{f^*(x,y)}{p^*(x,y)}.$$

**Case 2:** $p^*(x,y) = 0$. Let $\alpha(\lambda_n) = \left(\frac{x^i(\lambda_n)}{\lambda_n}\right)_{i\in I} \in I\!R^I_+$. Hence, from equation (31), and because $p(x,y) = 1$,

$$w \le \lim_{n\to\infty} \inf \frac{f(x,y) + f^*(x,\beta) + (1-\lambda_n)\, f^*(\alpha(\lambda_n), y)}{1 + p^*(x,\beta) + p^*(\alpha(\lambda_n), y)}. \tag{32}$$

Since $f^*$ and $p^*$ are linear in $y$ and $J$ is finite, for any $\varepsilon > 0$, there is $N(\varepsilon)$ such that, for every $y \in \Delta(J)$,

$$w \le \frac{f(x,y) + f^*(x,\beta) + f^*(\alpha(\lambda_{N(\varepsilon)}), y)}{1 + p^*(x,\beta) + p^*(\alpha(\lambda_{N(\varepsilon)}), y)} + \varepsilon.$$

Hence there exists $(x,\alpha)$ such that for any $(y,\beta)$

$$w \le A(x,\alpha,y,\beta).$$

Consequently, $w \le \max_{(\Delta(I)\times I\!R^I_+)} \min_{(\Delta(J)\times I\!R^J_+)} A(x,\alpha,y,\beta)$ and the result follows by symmetry.

■

## 4.2 The finitely repeated game

The values $\{v_n\}_{n=1,...}$ of the finitely repeated games satisfy:

$$v_n = \max_{x\in\Delta(I)} \min_{y\in\Delta(J)} \left[\frac{1}{n} f(x,y) + \frac{n-1}{n} p(x,y) v_{n-1} + \frac{n-1}{n} f^*(x,y)\right],$$

with $v_0 = 0$.
For each integer $n$, define a function $W_n$ on $[0,1]$ as follows: $W_n(1) = 0$ and for $m = 0, ..., n-1$, $W_n(\frac{m}{n})$ is specified inductively by:

$$W_n(\frac{m}{n}) = \max_{x\in\Delta(I)} \min_{y\in\Delta(J)} \left[\frac{1}{n} f(x,y) + p(x,y) W_n(\frac{m+1}{n}) + \frac{n-m-1}{n} f^*(x,y)\right].$$

By induction, $W_n(\frac{m}{n}) = \left(1 - \frac{m}{n}\right) v_{n-m}$. Extend $W_n(\cdot,p,q)$ to $[0,1]$ by linear interpolation. Consequently: $W_n(\cdot)$ is a $C$ Lipschitz function and if $W_n$ converges uniformly to some function $W$, $v_n$ converges to $W(0)$ and $W(t) = (1-t)\, W(0)$.
The projective operator is $\Phi(v) = \max_{x\in\Delta(I)} \min_{y\in\Delta(J)} (p(x,y)v + f^*(x,y))$ and $\mathcal{S}$ is his set of fixed points. As usual the set $\mathcal{S}_0$ of accumulation points of $\{v_n\}$ is included in $\mathcal{S}$.
Define the Hamiltonian $H$ from $[0,1] \times I\!R \times I\!R \to I\!R$ as follows. It is the value of a zero-sum game where the strategies of Player 1 are of the form $(x,\alpha) \in \Delta(I) \times I\!R^I_+$ while strategies for Player 2 are $(y,\beta) \in \Delta(J) \times I\!R^J_+$ and the payoff is given by:

$$\begin{aligned} h(t,a,b,x,\alpha,y,\beta) &= \Big(\frac{(1-t) f^*(x,y)}{p^*(x,y)} - a\Big)\mathbf{1}_{\{p^*(x,y)>0\}} \\ &+ \frac{f(x,y) + (1-t) f^*(\alpha,y) + (1-t) f^*(x,\beta) - [p^*(\alpha,y) + p^*(x,\beta)]\, a + b}{1 + p^*(\alpha,y) + p^*(x,\beta)} \mathbf{1}_{\{p^*(x,y)=0\}}. \end{aligned}$$

According to Proposition 22, this game has a value (just replace in the absorbing game the function $f$ with $f+b$ and the function $g$ with $g-a$):

$$H(t,a,b) = \mathtt{val}_{(\Delta(I)\times I\!R^I_+)\times(\Delta(J)\times I\!R^J_+)}\, h(t,a,b,x,\alpha,y,\beta)$$

The variational characterization for this class uses the following properties: for all $t \in [0,1)$ and any $C^1$ function $\phi : [0,1] \to I\!R$ :

- **R1:** If $U(\cdot)-\phi(\cdot)$ admits a global maximum at $t$ then $H(t,U(t),\phi'(t))\geq 0$.
- **R2:** If $U(\cdot)-\phi(\cdot)$ admits a global minimum at $t$ then $H(t,U(t),\phi'(t))\leq 0$.

**Lemma 23** *Any accumulation point $U(\cdot)$ of $W_n(\cdot)$satisfies **R1** and **R2**.*

**Proof.** Let us prove the first variational inequality, the second being obtained by symmetry. Let $t$ be such that $U(\cdot)-\phi(\cdot)$ admits a global maximum at $t$. Adding $(\cdot - t)^2$ to $\phi$ if necessary, we can assume that this global maximum is strict.
Let $W_{\varphi(n)}$ converge to $U$ and let $\frac{\theta(n)}{\varphi(n)}$ be a global maximum of $W_{\varphi(n)}(\cdot)-\phi(\cdot)$ over the set $\{\frac{m}{\varphi(n)}; m=0,\cdots,\varphi(n)-1\}$. Then, $\frac{\theta(n)}{\varphi(n)}\to t$. Recall that, by definition:

$$W_{\varphi(n)}(\frac{\theta(n)}{\varphi(n)})=\max_{x\in\Delta(I)}\min_{y\in\Delta J)}\left[\frac{1}{\varphi(n)}f(x,y)+p(x,y)W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)})+\frac{\varphi(n)-\theta(n)-1}{\varphi(n)}f^*(x,y)\right].$$

Let $x_n$ be optimal for player 1 in the above formula and let $y\in\Delta(J)$. Thus:

$$W_{\varphi(n)}(\frac{\theta(n)}{\varphi(n)})\leq\frac{1}{\varphi(n)}f(x_n,y)+p(x_n,y)W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)})+\frac{\varphi(n)-\theta(n)-1}{\varphi(n)}f^*(x_n,y)$$

By compactness one can assume that $x_n$ converges to some $x$.

**Case 1:** $p^*(x,y)>0$. Letting $n\to\infty$ implies:

$$U(t)\leq p(x,y)U(t)+(1-t)f^*(x,y)$$

hence

$$0\leq\frac{(1-t)f^*(x,j)}{p^*(x,j)}-U(t)$$

**Case 2:** $p^*(x,y)=0$.
Since $p(x_n,y)=1-p^*(x_n,y)$, we deduce that:

$$\begin{aligned}&W_{\varphi(n)}(\frac{\theta(n)}{\varphi(n)})-W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)})\\ \leq\ &\frac{1}{\varphi(n)}f(x_n,y)+\frac{\varphi(n)-\theta(n)-1}{\varphi(n)}f^*(x_n,y)-p^*(x_n,y)W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)}).\end{aligned}$$

Since $\frac{\theta(n)}{\varphi(n)}$ is a global maximum of $W_{\varphi(n)}(\cdot)-\phi(\cdot)$ one has:

$$\phi(\frac{\theta(n)}{\varphi(n)})-\phi(\frac{\theta(n)+1}{\varphi(n)})\leq W_{\varphi(n)}(\frac{\theta(n)}{\varphi(n)})-W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)}).$$

Consequently:

$$0\leq\frac{1}{\varphi(n)}f(x_n,y)+\frac{\varphi(n)-\theta(n)-1}{\varphi(n)}f^*(x_n,y)-p^*(x_n,y)W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)})+\phi(\frac{\theta(n)+1}{\varphi(n)})-\phi(\frac{\theta(n)}{\varphi(n)})$$

Let $\alpha_n=\varphi(n)x_n\in M_+(I)$. Thus,

$$0\leq\frac{f(x_n,y)+\frac{\varphi(n)-\theta(n)-1}{\varphi(n)}f^*(\alpha_n,y)-p^*(\alpha_n,y)W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)})+\varphi(n)\left[\phi(\frac{\theta(n)+1}{\varphi(n)})-\phi(\frac{\theta(n)}{\varphi(n)})\right]}{p(x_n,y)+p^*(\alpha_n,y)}$$

Up to a subsequence, by linearity using the fact that $J$ is finite, one may suppose that the right hand term converges uniformly in $y$. Thus, for any $\varepsilon>0$, there exist $(x,\alpha)$ such that for any $y\in\Delta(J)$:

$$-\varepsilon\leq\frac{f(x,y)+(1-t)f^*(\alpha,y)-p^*(\alpha,y)U(t)+\phi'(t)}{1+p^*(\alpha,y)}$$

Consequently for any $\varepsilon > 0$, $H(t, U(t), \phi'(t)) \geq -\varepsilon$ and so $H(t, U(t), \phi'(t)) \geq 0$. ■

The comparison principle for this class is the next result.

**Lemma 24** *Let $U_1$ and $U_2$ be two Lipschitz functions with $U_i(t) \in \mathcal{S}$ for all $t \in [0,1]$ satisfying* ***R1****,* ***R2****, and*

- ***R3:*** $U_1(1) \leq U_2(1)$.

*Then $U_1 \leq U_2$ on $[0,1] \times \Delta(K) \times \Delta(L)$.*

**Proof.** By contradiction, suppose

$$\max_{t \in [0,1]} [U_1(t) - U_2(t)] = \delta > 0 \ .$$

Let $\varepsilon > 0$ and set

$$\delta(\varepsilon) = \max_{(t,s) \in [0,1] \times [0,1]} [U_1(t) - U_2(s) - \frac{(t-s)^2}{2\varepsilon} + \varepsilon s] \ .$$

Then, $\delta(\varepsilon) \to \delta$ as $\delta \to 0$. Let

$$(t_\varepsilon, s_\varepsilon) \in \arg\max_{t,s,p,q} U_1(t) - U_2(s) - \frac{(t-s)^2}{2\varepsilon} + \varepsilon s$$

Note that, for $\varepsilon$ sufficiently small, $t_\varepsilon < 1$ and $s_\varepsilon < 1$ because $U_1(1) \leq U_2(1)$. Moreover, from standard arguments, $t_\varepsilon - s_\varepsilon \to 0$ as $\varepsilon \to 0$.
Since the map $U_1(t) - \frac{(t-s_\varepsilon)^2}{2\varepsilon}$ has a global maximum at $t_\varepsilon \in [0,1)$, we have by condition **R1** that

$$H\left(t_\varepsilon, U_1(t_\varepsilon), \frac{t_\varepsilon - s_\varepsilon}{\varepsilon}\right) \geq 0 \ . \tag{33}$$

In the same way, since the map $s \to U_2(s) + \frac{(t_\varepsilon - s)^2}{2\varepsilon} - \varepsilon s$ has a global minimum at $s_\varepsilon$, we have by condition **R2** that

$$H\left(s_\varepsilon, U_2(s_\varepsilon), \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \varepsilon\right) \leq 0 \ . \tag{34}$$

To simplify the expressions, let us set $w_1^\varepsilon = U_1(t_\varepsilon)$, $w_2^\varepsilon = U_2(s_\varepsilon)$ and $b_\varepsilon = \frac{t_\varepsilon - s_\varepsilon}{\varepsilon}$. Let $(x_\varepsilon, \alpha_\varepsilon) \in \Delta(I) \times I\!R_+^I$ be such that:

$$H\left(t_\varepsilon, w_1^\varepsilon, b_\varepsilon\right) \leq \varepsilon^2 + \inf_{(y,\beta)} h\left(t_\varepsilon, w_1^\varepsilon, b_\varepsilon, x_\varepsilon, \alpha_\varepsilon, y, \beta\right)$$

and $(y_\varepsilon, \beta_\varepsilon) \in \Delta(J) \times I\!R_+^J$ be such that

$$H\left(s_\varepsilon, w_2^\varepsilon, b_\varepsilon + \varepsilon\right) \geq -\varepsilon^2 + \sup_{(x,\alpha)} h\left(s_\varepsilon, w_2^\varepsilon, b_\varepsilon + \varepsilon, x, \alpha, y_\varepsilon, \beta_\varepsilon\right) \ .$$

We now discuss two cases:

**Case 1:** There is a subsequence $\varepsilon \to 0$ such that $p^*(x_\varepsilon, y_\varepsilon) > 0$.
Then, from (33),

$$(1 - t_\varepsilon) f^*(x_\varepsilon, y_\varepsilon) - p^*(x_\varepsilon, y_\varepsilon) w_1^\varepsilon \geq -\varepsilon^2 p^*(x_\varepsilon, y_\varepsilon)$$

while, from (34),

$$(1 - s_\varepsilon) f^*(x_\varepsilon, y_\varepsilon) - p^*(x_\varepsilon, y_\varepsilon) w_2^\varepsilon \leq \varepsilon^2 p^*(x_\varepsilon, y_\varepsilon) \ .$$

Computing the difference between the two last inequalities, we get

$$(t_\varepsilon - s_\varepsilon) f^*(x_\varepsilon, y_\varepsilon) + (w_1^\varepsilon - w_2^\varepsilon) p^*(x_\varepsilon, y_\varepsilon) \leq 2\varepsilon^2 p^*(x_\varepsilon, y_\varepsilon) \ ,$$

Since $f^*(x_\varepsilon, y_\varepsilon) = \sum_{i,j} x_\varepsilon(i) y_\varepsilon(j) p^*(i,j) g(i,j)$ and $|g(i,j)| \leq C$ one has $|f^*(x_\varepsilon, y_\varepsilon)| \leq C p^*(x_\varepsilon, y_\varepsilon)$, hence

$$-C\,|t_\varepsilon - s_\varepsilon| + (w_1^\varepsilon - w_2^\varepsilon) \leq 2\varepsilon^2.$$

This leads to a contradiction as $\varepsilon \to 0$ because $\delta(\varepsilon) \to \delta > 0$ and $t_\varepsilon - s_\varepsilon \to 0$.

**Case 2:** $p^*(x_\varepsilon, y_\varepsilon) = 0$ for any $\varepsilon > 0$ sufficiently small.
Then, letting in order to simplify the expressions: $f_\varepsilon = f(x_\varepsilon, y_\varepsilon)$, $p_\varepsilon^* = p^*(\alpha_\varepsilon, y_\varepsilon) + p^*(x_\varepsilon, \beta_\varepsilon)$, and $f_\varepsilon^* = f^*(\alpha_\varepsilon, y_\varepsilon) + f^*(x_\varepsilon, \beta_\varepsilon)$, we obtain:

$$f_\varepsilon + (1 - t_\varepsilon) f_\varepsilon^* - p_\varepsilon^* w_1^\varepsilon + b_\varepsilon \geq -\varepsilon^2 \left(1 + p_\varepsilon^*\right), \tag{35}$$

and

$$f_\varepsilon + (1 - s_\varepsilon) f_\varepsilon^* - p_\varepsilon^* w_2^\varepsilon + b_\varepsilon + \varepsilon \leq \varepsilon^2 \left(1 + p_\varepsilon^*\right). \tag{36}$$

Computing and simplifying the difference between (36) and (35) we get

$$-C p_\varepsilon^*\,|t_\varepsilon - s_\varepsilon| + \varepsilon + p_\varepsilon^*(w_1^\varepsilon - w_2^\varepsilon) \leq 2\varepsilon^2(1 + p_\varepsilon^*).$$

So, dividing the above inequality by $(1 + p_\varepsilon^*)$ implies that

$$\min\{\varepsilon, w_1^\varepsilon - w_2^\varepsilon - C\,|t_\varepsilon - s_\varepsilon|\} \leq 2\varepsilon^2,$$

which is impossible because $w_1^\varepsilon - w_2^\varepsilon \to \delta > 0$ and $t_\varepsilon - s_\varepsilon \to 0$ as $\varepsilon \to 0$. ■

*On n'a pas utilisé Lipschitz dans la preuve, seulement continu*

Consequently, we obtain the uniform convergence of $v_n$ to $v$:

**Theorem 25** *$v_n$ converges to $v = \lim v_\lambda$ and $W_n(t)$ converges uniformly to $(1-t)v$.*

**Proof.** The last two lemmas implies that $W_n$ converges uniformly to the unique C-Lipschitz function satisfying **R1-R3**. On the other hand, the function $V(t) = (1-t)v$ is $C$-Lipschitz and trivially satisfies the inequalities. Consequently, $V(t)$ is the limit of $W_n$ and so $v$ is the limit of $v_n$ ■

## 4.3 General evaluation

The proof and the result extend in a straightforward way to any sequence of decreasing evaluation of the payoffs.

## 4.4 Remark

Lemma 24 does not use the fact that $U_i(t) \in \mathcal{S}$. In fact this property implies that for any couple of optimal strategies $x$ in the projective game at $U_1$ and $y$ in the projective game at $U_2$, one has $p^*(x,y) = 0$. This leads to a shorter proof and simpler variational characterization.

For $w \in \mathcal{S}$, let $X(w)$ be the set of optimal strategies for player 1 and $Y(w)$ be the set of optimal strategies for player 2 in the limiting game at $w$. $X(w)$ is called non absorbing if for every $x \in X(w)$, there exists $y \in \Delta(J)$ such that $p^*(x,y) = 0$ and similarly for $Y(w)$.

We introduce now two properties:

**Definition 26** *Let $w \in \mathcal{S}$.*

- ***R1:*** *if $X(w)$ is non absorbing then for any $\varepsilon > 0$, there exists $(x, \alpha) \in X(w) \times I\!R_+^I$ such that for every $(y, \beta) \in \Delta(J) \times I\!R_+^J$ with $p^*(x, y) = 0$, one has $v \leq \frac{f(x,y)+f^*(\alpha,y)+f^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)} + \varepsilon$.*

- ***R2:*** *if $Y(w)$ is non absorbing then for any $\varepsilon > 0$, there exists $(y, \beta) \in Y(w) \times I\!R_+^J$ such that for every $(x, \alpha) \in \Delta(I) \times rit_+^I$ with $p^*(x, y) = 0$, one has $v \geq \frac{f(x,y)+f^*(\alpha,y)+f^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)} - \varepsilon$.*

**Lemma 27** *If $w \in \mathcal{S}_0$ then $w$ satifies **R1** and **R2**.*

**Proof.** Follows from the proof of Proposition 22. ■

**Lemma 28** *If $w_1 \in \mathcal{S}$ satisfies **R1** and $w_2 \in \mathcal{S}$ satisfies **R2** then $w_1 \leq w_2$.*

**Proof.** Suppose that $w_1 - w_2 = \delta > 0$. Let $x_1 \in X(w_1)$ and $x_2 \in Y(w_2)$. Then:

$$\delta = w_1 - w_2 \leq p(x, y)\,(w_1 - w_2) = p(x, y)\delta$$

Thus, $p^*(x, y) = 1 - p(x, y) = 0$. Consequently, $X(w_1)$ and $Y(w_2)$ are non absorbing. Hence, for any $\varepsilon > 0$ there exists $(x, y, \alpha, \beta)$ such that:

$$w_1 \leq \frac{f(x, y) + f^*(\alpha, y) + f^*(x, \beta)}{1 + p^*(\alpha, y) + p^*(x, \beta)} + \varepsilon,$$

and

$$w_2 \geq \frac{f(x, y) + f^*(\alpha, y) + f^*(x, \beta)}{1 + p^*(\alpha, y) + p^*(x, \beta)} + \varepsilon,$$

implying that $\delta \leq 2\varepsilon$, a contradiction. ■

**Corollary 29** *$v$ converges to the unique point in $\mathcal{S}$ satisfying **R1** and **R2**.*

## 5 Concluding comments

The main contribution of this approach is to provide a unified treatment of the asymptotic analysis of the value of repeated games:
- it applies to all decreasing evaluations and shows the interest of the limiting game played on $[0, 1]$. Further research will be devoted to a formal construction and to the analysis of optimal strategies.
- it allows to treat incomplete infomation games as well as absorbing games. We strongly believe that similar tools will allow to analyze more general classes,
- it shows that technics introduced in diferential games where the dynamics on the state are smooth can be used in a repeated game framework. On the other the stationary aspect of the payoff in repeated games is no longer necessary to obtain asymptotic properties.

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