A Folk theorem for repeated games played on a network

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1 Introduction

We consider repeated games on a social network: each player has a set of neighbors with whom he interacts and communicates. The payoff of a player depends only on the actions chosen by himself and his neighbors, and at each stage, a player can send different messages to his neighbors. Players observe their stage payoff but not the actions chosen by their neighbors. We establish a necessary and sufficient condition on the network for the existence of a protocol which identifies a deviating player in finite time. We derive a Folk theorem for a relevant class of payoff functions.

2 The model

2.1 Preliminaries

We consider a repeated game on a network where the players interact and communicate with their neighbors. This is described by the following data.

- A finite set $N = \{1, \ldots, n\}$ of players, $n \geq 2$.

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- For each player $i \in N$, a non-empty finite set A^i of his actions (with $\sharp A^i \geq 2$). We denote $A = \prod_{i \in N} A^i$.
- An undirected graph $G = (N, E)$ where the vertices are the players N and $E \subseteq N \times N$ is a set of links. Let $\mathcal{N}(i) = \{j \neq i \ : \ ij \in E\}$ be the set of neighbors of player i. Since G is undirected, we have that $i \in \mathcal{N}(j) \Leftrightarrow j \in \mathcal{N}(i)$.
- For each player $i \in N$, a payoff function $g^i \ : \ \prod_{j \in \mathcal{N}(i) \cup \{i\}} A^j \to \mathbb{R}$, *i.e.* the stage payoff of player i in N only depends on the actions chosen by him and his neighbors.

For each player i in N, we denote by $G - i$ the graph obtained from G by removing i and its links. More precisely, $G - i = (N \setminus \{i\}, E')$ where $E' = \{jk \in (N \setminus \{i\}) \times (N \setminus \{i\}) : j \in \mathcal{N}(k) \text{ and } k \in \mathcal{N}(j)\}$.

The graph represents both the interactions and the communication possibilities. At each stage, players choose an action and send messages to their neighbors. We use the following notations: $A^{\mathcal{N}(i) \cup \{i\}} = \prod_{j \in \mathcal{N}(i) \cup \{i\}} A^j$, $a^{\mathcal{N}(i)} = (a^j)_{j \in \mathcal{N}(i)}$ and $g = (g^1, \ldots, g^n)$ denotes the vector of payoffs. The repeated game unfolds as follows. At any stage $t \in \mathbb{N}^*$:

- the players choose actions simultaneously in their action sets. Let $a_t : (a_t^i)$ be the profile of actions taken at stage t.
- Then, each player $i \in N$ observes his stage payoff $g^i(a_t^i, a_t^{\mathcal{N}(i)})$. A player cannot observe the actions chosen by others, even by his neighbors.
- Finally, messages are sent by each player to his neighbors. Communication is *unicast* in that each player can send different messages to his neighbors. Communication between a pair of neighbors is private, *i.e.* no other player can learn the message or change it. Let M^i be a non-empty finite set of messages of player i and denote by $m_t^i(j)$ the message sent by player i to his neighbor $j \in \mathcal{N}(i)$ at stage t. The specification of the set M^i is part of the solution concept and is described in section 3.

We assume players have perfect recall (*i.e.* they never forget what they have learnt) and that the whole description of the game is common knowledge. For each stage t, denote by H^i_t the set of histories of player i up to stage t, that is $(A^i \times (M^i)^{\mathcal{N}(i)} \times (M^j)_{j\in\mathcal{N}(i)} \times \{g^i\})^t$, where $\{g^i\}$ is the range of g^i (H^i_0 is a singleton). An element of H^i_t is called an i-history of length t. A *behavior strategy* of a player i is a pair (σ^i, ϕ^i) where $\sigma^i = (\sigma^i_t)_{t\geq 1}$, $\phi^i = (\phi^i_t)_{t\geq 1}$, and for each stage t, σ^i_t is a mapping from H^i_{t-1} to $\Delta(A^i)$, with $\Delta(A^i)$ the set of probabilities distributions over A^i, and ϕ^i_t is a mapping from $H^i_{t-1} \times \{a^i_t\} \times \{g^i_t\}$ to $\Delta((M^i)^{\mathcal{N}(i)})$. We call σ^i the *action strategy* of player i and ϕ^i his *communication strategy.* Let Σ^i be the set of action strategies of player i and Φ^i his set of communication strategies. We denote by $\sigma = (\sigma^i)_{i\in N} \in \prod_{i\in N} \Sigma^i$ the joint action strategy of the players and by $\phi = (\phi^i)_{i\in N} \in \prod_{i\in N} \Phi^i$ their joint communication strategy. Let H_t be the set of histories of length t, that is the sequences of actions, payoffs and messages for t stages. H_∞ is the set of all possible infinite histories. A profile (σ, ϕ) defines a probability distribution, $\mathbb{P}_{\sigma,\phi}$, over the set of plays H_∞, and we denote $\mathbb{E}_{\sigma,\phi}$ the corresponding expectation operator. We consider the infinitely discounted repeated game, where the overall payoff function of any player i in N is the expected normalized sum of discounted payoffs. That is, for each player i in N:

$$\gamma^i_\delta(\sigma, \phi) = \mathbb{E}_{\sigma,\phi}\left[(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} g^i_t(a^i_t, a^{\mathcal{N}(i)}_t)\right],$$

where $\delta \in [0, 1)$ is a common discount factor. A strategy profile (σ, ϕ) is a Nash equilibrium if no player can increase his discounted payoff by switching unilaterally to an alternative strategy (τ^i, ψ^i). We study equilibrium outcomes of repeated games with high discount factor.

Definition 2.1 *A payoff vector $x = (x^i)_{i\in N} \in \mathbb{R}^N$ is an equilibrium payoff of the repeated game if there exists a discount factor $\bar{\delta} \in (0, 1)$ such that, for any $\delta \in (\bar{\delta}, 1)$, x is induced by a Nash equilibrium of the δ-discounted game.*

Let $\Gamma_\delta(G, g)$ be the δ-discounted game, and let $E_\delta(G, g)$ be its associated set of equilibrium payoffs. For any action profile $a \in A$, let $g(a) = (g^1(a^1, a^{\mathcal{N}(1)}), \ldots, g^n(a^n, a^{\mathcal{N}(n)}))$ and $g(A) = \{g(a) : a \in A\}$. We denote by $\operatorname{co} g(A)$ the convex hull of $g(A)$: $\operatorname{co} g(A)$ is the set of feasible payoffs. It is straightforward that $E_\delta(G, g) \subseteq \operatorname{co} g(A)$. The (independent) minmax level of player i is defined by:

$$v^i = \min_{x^{\mathcal{N}(i)} \in \prod_{j \in \mathcal{N}(i)} \Delta(A^j)} \max_{x^i \in \Delta(A^i)} g^i(x^i, x^{\mathcal{N}(i)}).$$

We denote by $SIR(G, g) = \{x = (x^1, \ldots, x^n) \in \mathbb{R}^N \,:\, x^i > v^i \, \forall i \ \in N\}$ the set of strictly individually rational payoffs. The aim of this paper is to characterize the networks G for which a Folk theorem holds, *i.e.* each feasible and strictly individually rational payoff is an equilibrium payoff of the repeated game for large enough discount factors. The next section presents our conditions on payoff functions and on networks.

We recall now some usual definitions of graph theory (the reader is referred to [Die00]).

Definition 2.2

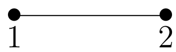
- *A* graph *is a pair* $G = (V, E)$ *where* V *is a set of nodes and* $E \subseteq V \times V$ *is a set of links (or edges).*
- *A* subgraph G' *of* G*, written as* $G' \subseteq G$*, is a pair* $G' = (V', E')$ *where* $V' \subseteq V$ *and* $E' \subseteq E$*.*
- *A* walk *in* $G = (V, E)$ *is a sequence of links* $e_1, \ldots, e_K$ *such that* $e_k e_{k+1} \in E$ *for each* $k \in \{1, \ldots, K-1\}$*. A* path *from node* i *to node* j *is a walk such that* $e_1 = i$*,* $e_K = j$ *and each node in the sequence* $e_1, \ldots, e_K$ *is distinct. The number of links of a path is referred as its* length*.*
- *The* distance $d_G(i, j)$ *in* G *of two nodes* i, j *is the length of a shortest path from* i *to* j *in* G*; if no such path exists, we set* $d_G(i, j) := +\infty$*.*
- *Two paths from* i *to* j *are* independent *if there are no common links except* i *and* j*.*
- G *is called* k-connected *(for* $k \in \mathbb{N}$*) if* $\sharp V \geq k$ *and* $G - X$ *is connected for every set* $X \subseteq V$ *with* $\sharp X < k$*, where* $G - X$ *represents the graph where all nodes in* X *have been removed (and the corresponding links). In other words, a graph is* k*-connected if any two of its nodes can be joined by* k *independent paths.*
- *A* connected component *of a graph* G *is a subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices.*

2.2 Structures of payoffs and networks

2.2.1 Payoff functions

Payoff functions need to be sufficiently rich for the players to be able to detect a deviation. The next example shows that a Folk theorem may be impossible for some particular payoff functions.

Example 2.3 *Consider the 2-player game played on the following network:*



and with payoff matrix (player 1 chooses the row, player 2 the column):

	L	M	R
U	$0,0$	$1,0$	$0,0$
D	$3,4$	$3,3$	$1,0$

The payoff vector $(3,3)$ is feasible and strictly individually rational, but is not an equilibrium payoff. Indeed, for any discount factor $\delta \in [0,1)$, player 2 has an incentive to deviate by playing L at each stage. When player 1 chooses D, he does not know whether player 2 plays L or R, and thus does not know if player 2 deviates.

On another hand, suppose that the payoff matrix is the following:

	L	M	R
U	$0,0$	$1,0$	$0,0$
D	$3,4$	$3+\epsilon,3$	$1,0$

with $\epsilon > 0$. Then, player 1 detects player 2's deviation and starts punishing by playing U. For a large enough discount factor δ, (D,M) is an equilibrium of the discounted game.

The previous example shows that if a deviation of a player i does not change the payoffs of his neighbors, then they do not detect the deviation and as a consequence, it may possible for some feasible and strictly individually rational payoffs are not equilibrium payoffs of the repeated game. For a similar phenomenon, see [Leh89]. So it is not possible to have a Folk theorem for all payoff functions g. We introduce the following assumption.

Assumption 2.4 *For any player $i \in N$, for any neighbor $j \in \mathcal{N}(i)$, for any pair of actions b^j and c^j in A^j:*

$$g^i(a^i, a^{\mathcal{N}(i)\setminus\{j\}}, b^j) \neq g^i(a^i, a^{\mathcal{N}(i)\setminus\{j\}}, c^j).$$

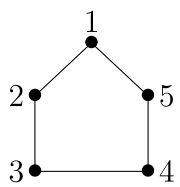
Example 2.5 *The following payoff functions satisfy Assumption 2.4:*

- *For each player i in N, let $A^i \subset \mathbb{N}$ and $g^i(a^i, a\mathcal{N}(i)) = \sum_{j\in\mathcal{N}(i)\cup\{i\}} a^j$.*
- *For each player i in N, let $A^i \subset \mathbb{R}$ and g^i is strictly monotone with respect to any coordinate.*
- *For each player i in N, let $A^i = A \subset \mathbb{R}$ and g^i is strictly monotone with respect to any coordinate and invariant by permutation of the actions chosen by i's neighbors.*

2.2.2 Condition on networks

The next example puts forwards the importance of the connectivity of the graph.

Example 2.6 *Consider the 5-player game played on the following network:*



where for each player i in N, $\mathcal{N}(i) \equiv \{i-1, i+1\} \mod 5$. Suppose that each player chooses between two actions, a and b. For simplicity, we focus only on payoff functions of players 1, 2 and 3 given by the following matrix (player 1 chooses the row, player 2 the column, player 3 the matrix):

	a	b
a	$1,1,1$	$0,1,3$
b	$3,0,0$	$0,0,4$

a

	a	b
a	$3,0,3$	$2,0,4$
b	$4,1,0$	$2,1,2$

b

In addition, we assume that g^4 and g^5 are such that, if player 4 (respectively 5) plays b, each of his neighbors gets a bonus of ϵ and if player 1 (respectively 3) chooses a, his *neighbor* 5 *(respectively* 4*) receives a bonus of* ϵ*. It is easy to check that Assumption 2.4 is satisfied.*

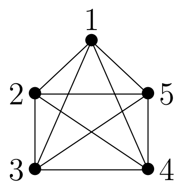
The minmax levels of players 1, 2 and 3 are zero, so the payoff $(1, 1, 1)$*, only obtained by the profile of actions* (a, a, a)*, is feasible and strictly individually rational. However, players 1 and 3 have both an incentive to deviate by playing b in order to get a payoff of 3. Player 1 then gets a payoff of 0 no matter who is the deviator. Player 5 (respectively 4) can differentiate between the two deviations: if player 1 deviates (respectively player 3), then player 5 (respectively player 4) has a decrease of* ϵ *in his payoff. But, both players 4 and 5 cannot communicate truthfully this information to player 1, since all paths between them and player 1 go through players 1 or 3, both of them being suspected*[1]*. So, player 1 cannot differentiate between a deviation of player 1 and a deviation of player 3.*

Moreover, player 1 cannot punish both players 1 and 3: player 2 has to play b to minmax player 2 and player 3 then gets at least 3 by playing a; and he has to play a to punish player 3, player 1 getting then at least 3 by playing b. So, the element $(1, 1, 1)$ *is not an equilibrium payoff and with this network, the Folk theorem does not hold.*

In the network with five players presented in the previous example, a Folk theorem does not hold as there exists a particular payoff function (which satisfies Assumption 2.4) for which a feasible and strictly individually rational payoff is not an equilibrium payoff. The situation would be different if players 4 and 5 could have transmit directly (by a link) to player 1 their information concerning an eventual deviation of player 1 or 3. This puts forwards the importance of the connectivity of a network: the more connected the network is, the less manipulated the communication between two players is. The example below shows that too much connectivity is also an obstacle in order to have a Folk theorem.

Example 2.7 *Consider the 5-player game played on the following network:*
where for each i in N, $\mathcal{N}(i) = N \setminus \{i\}$*. Assume that, for each player i in N,* $A^i = \{0, 1\}$ *and* $g^i(a^i, a^{\mathcal{N}(i)}) = g^i(a^1, \ldots, a^5) = g^i(a) = \sum_{i \in N} a^i$*. For any player i in N, it is not possible to identify the deviator when there is an action deviation. For instance, it is not possible to differentiate a deviation of player 1 from a deviation of player 2. Indeed, if*

[1] A formal proof of this idea is given in a general case in section 4.



player 1 deviates in action at some stage t, everybody detects the deviation because of the change in their stage payoffs. However, as the graph is complete, each player suspects every other player and nobody can differentiate between players 1 and 2 in particular. In this example, an action deviation of any player is detected, but the deviator is not identified.

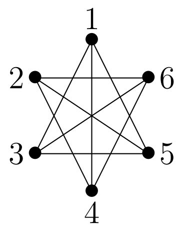
The previous example shows that the networks must not be complete and players should have different neighbors in order to have a Folk theorem. We now explicit the sufficient and necessary condition on networks to have a Folk theorem.

Condition 2.8 ***Necessary and sufficient condition***
For any $i \in N$, for any $j, k \in \mathcal{N}(i)$ such that $j \neq k$, there exists $l \in \mathcal{N}(j) \triangle \mathcal{N}(k)$ such that there is a path from l to i which goes neither through j nor k, where:

$$\mathcal{N}(j) \triangle \mathcal{N}(k) = (\mathcal{N}(j) \setminus \mathcal{N}(k)) \bigcup (\mathcal{N}(k) \setminus \mathcal{N}(j)) .$$

The basic idea of Condition 2.8 is the following. Take any player i in N. If at some stage there is an action deviation of one of his neighbors, he needs to identify him in order to punish him. So, for any pair of neighbors j, k in $\mathcal{N}(i)$, he needs to differentiate between a deviation of player j and a deviation of player k. If the network satisfies Condition 2.8, then there exists some player l who is a neighbor of only one of players j and k, say j. On one hand, if player j is the deviator, then player l observes a change in his payoff whenever Assumption 2.4 is satisfied and concludes that k is innocent (because k is not his neighbor). On the other hand, if k is the deviator, player l concludes that j is innocent because he does not observe a change in his payoff. In both situations, player l can transmit this information to player i as there is path from l to i which goes not through the suspects j nor k. Moreover, when a player, say m tells to player i that some other player, say j, is innocent, one concludes that player j is indeed innocent. Actually, either player m is not telling the truth, thus deviates, and player j is innocent as we focus on unilateral deviations; or, player m is innocent, thus telling the truth and player j also is innocent. This idea is resumed in the example below.

Example 2.9 *Consider the 6-player game played on the following network:*



Suppose that the payoff functions are such that Assumption 2.4 is satisfied. Notice that Condition 2.8 is satisfied in this network. Consider without loss of generality player 1's point of view (the network is symmetric) and suppose that player 3 deviates in action at some stage $t > 0$. Then player 1 observes a change in his payoff due to Assumption 2.4 and suspects his neighbors 3, 4 and 5. On the other hand, player 2 is not a neighbor of player 3 and does not observe a change in his payoff, so he concludes that his neighbors players 4 and 5 (and 6) are innocent. Player 2 transmits this information to players 4 and 5. Then, player 4 is able to tell to player 1 that 5 is innocent and player 5 that 4 is innocent. Player 1 then knows that player 1 is guilty since he is the only suspect left.

The important fact here is that player 2 can clear players 4 and 5 and that there exists a path from player 2 *to player* 1 *which goes neither trough 3 nor 4 and this path is used to transmit the information that 4 is innocent. In the same way, there is a path from 2 to 1 which goes neither trough 3 nor 5.*

We now exhibit some basic properties of the networks which satisfy Condition 2.8.

Proposition 2.10 *If the graph G satisfies Condition 2.8, then for each pair of players i and j in the same connected component of at least 3 players, $\mathcal{N}(i) \setminus \{j\} \neq \mathcal{N}(j) \setminus \{i\}$.*

Proof. Take a graph G which satisfies Condition 2.8 and a connected component $C \subseteq G$ with at least three nodes. Assume that there exists a pair of players j and k in C such that

$\mathcal{N}(j) \setminus \{k\} = \mathcal{N}(k) \setminus \{j\}$. Since C contains at least three nodes, there exists $i \neq j, k$ such that $i \in \mathcal{N}(j) \cap \mathcal{N}(k)$ (otherwise j and k would not be in the same connected component). So j and k are in $\mathcal{N}(i)$, but then there exists no $l \in \mathcal{N}(j) \triangle \mathcal{N}(k)$ such that $l \neq j$, k, which contradicts Condition 2.8. ■

Proposition 2.11 *If the network G satisfies Condition 2.8, then no player can have exactly two neighbors: $\forall i \in N$, $\sharp\mathcal{N}(i) \neq 2$.*

Proof. Take a graph G which satisfies Condition 2.8. Suppose that there exists a player i in N who has exactly two neighbors, say j and k, that is: $\mathcal{N}(i) = \{j, k\}$. Then, because of Condition 2.8, there must exist a player l in N such that $l \neq j$, k and such that there is a path from l to i going neither trough j nor k. This implies that i must have another neighbor different from j and k, which is impossible by assumption. ■

We can deduce from the two previous propositions a family of graphs that satisfy Condition 2.8.

Corollary 2.12 *Let $n \geq 3$. If a network G is 3-connected and if, for any pair $(i, j) \in N^2$, $\mathcal{N}(i) \setminus \{j\} \neq \mathcal{N}(j) \setminus \{i\}$, then G satisfies Condition 2.8.*

Remark 2.13 *Except the 2-player case, one can see easily that there is no connected graph satisfying Condition 2.8 with strictly less than six players.*

Before presenting the main result, we exhibit some graphs which satisfy Condition 2.8 (Figure 1) and some which do not (Figure 2). Regarding the first ones, notice that G_1 is 3-connected, whereas G_2, G_3 and G_4 are not even 2-connected.

2.3 Main result

The main result of the paper is the following.

Theorem 2.14 *The following statements are equivalent.*

1. *The network G satisfies Condition 2.8.*

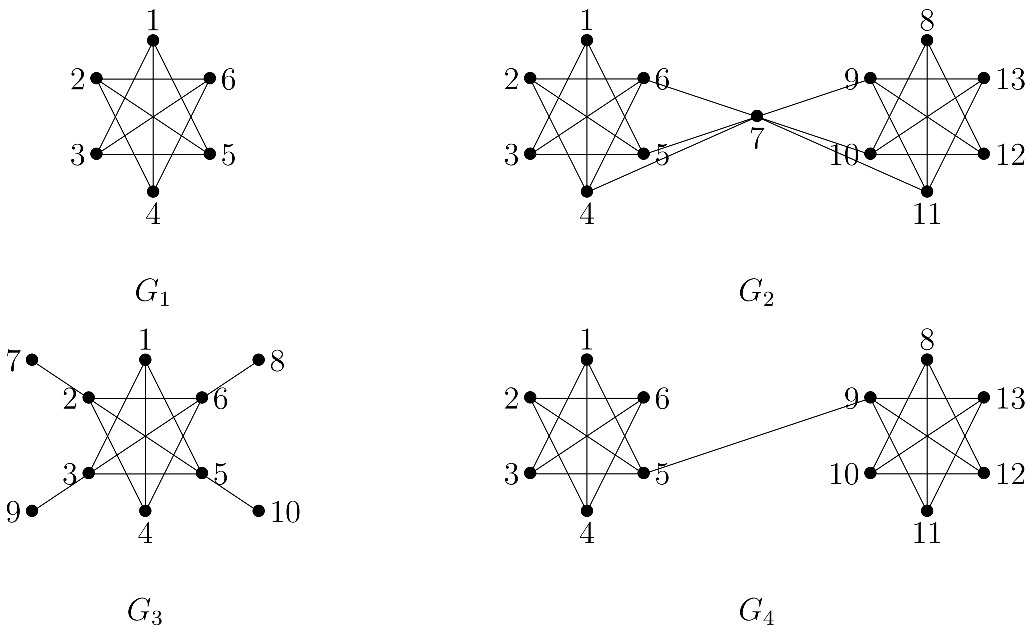


Figure 1: Networks satisfying Condition 2.8

Figure 2: Networks not satisfying Condition 2.8

2. *For any payoff function which satisfies Assumption 2.4, any feasible and strictly individually rational payoff is an equilibrium payoff of the discounted game, that is: there exists* $\bar{\delta} \in (0,1)$ *such that for any* $\delta \in (\bar{\delta}, 1)$,

$$\operatorname{co} g(A) \cap SIR(G,g) \subseteq E_\delta(G,g).$$

The second assertion of the above theorem can be seen as a Folk theorem. Section 3 is devoted to prove that Condition 2.8 is sufficient for this Folk theorem to hold. In section 4, we prove that this is also a necessary condition, *i.e.* for the family of networks which do not satisfy Condition 2.8, we find a particular payoff function g such that there exists a feasible and strictly individually rational payoff which is not an equilibrium payoff of the discounted game.

Remark 2.15 *Henceforth, we only focus on connected networks. Notice that when a network G has several connected components, the payoff function of each player only depends on the actions chosen by himself and his neighbors and no player in a connected component can communicate with any player of another component. Therefore, each connected component can be seen as a game in itself.*

3 Construction of the equilibrium strategy

In this section, we assume that the network G satisfies Condition 2.8 and show the inclusion $\operatorname{co} g(A) \cap SIR(G, g) \subseteq E_\delta(G, g)$ for any large enough discount factor and any payoff function that satisfies Assumption 2.4. From now on, suppose that g satisfies Assumption 2.4. We take a point $\bar{\gamma} = (\bar{\gamma}^1, \ldots, \bar{\gamma}^n)$ in $\operatorname{co} g(A) \cap SIR(G, g)$ and construct an equilibrium of the discounted game $(\bar{\sigma}, \bar{\phi})$ with payoff $\bar{\gamma}$ for a large enough discount factor.

The strategy is made of three parts: a stream of pure actions leading to the payoff $\bar{\gamma}$ and to be played in case of non deviation, periods of communication allowing the neighbors of the deviator to identify him exactly if there is an action deviation, and punishments phases. We need first to precise these three phases, then describe the communication part in order to identify the deviator when there is a deviation (section 3.2). In section 3.3, we prove the sufficient part of theorem 2.14. Finally, in section 3.4, we study the particular case of 2-connected networks which have some additional properties.

3.1 Description of the equilibrium strategy

The equilibrium path. For any player i in N and any stage $t > 0$, choose $\bar{a}_t^i \in A^i$ such that

$$(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} g_t^i(\bar{a}_t^i, \bar{a}_t^{\mathcal{N}(i)}) = \bar{\gamma}^i$$

In this phase, player i should play action $\bar{a}_t^i$ at stage t. There is a blank message $\varnothing$ that should be sent by player i to his neighbors at each stage t: it corresponds to the case where player i does not transmit any information to his neighbors.

Punishment phase. For any player i in N and any neighbor $k \in \mathcal{N}(i)$, fix $\bar{\sigma}^{i,k} \in \Sigma^i$ such that for any ϕ^i and for any (σ^k, ψ^k),

$$\gamma_\delta^k(\sigma^k, (\bar{\sigma}^{i,k})_{i\in\mathcal{N}(k)}, \psi^k, (\phi^i)_{i\in\mathcal{N}(k)}) \leq \sum_{t=1}^{\infty}(1-\delta)\delta^{t-1}v^k.$$

During this phase, player i plays a minmax strategy against his neighbor k. Notice that only the neighbors of player k are able to minmax him, so the strategies of other players are arbitrary.

Communication phase. Notice that there are two kinds of deviations, action and communication ones: a faulty player k may stop playing $\bar{a}_t^k$ at some stage t and/or send spurious messages. The deviator is supposed to be Byzantine: he is allowed to choose any possible strategy (τ^k, ψ^k). In particular, this deviator may stop playing $\bar{a}_t^k$ and try to avoid other players to identify him by sending false messages. He may also take action $\bar{a}_t^k$ at any stage t but send spurious messages to induce wrong conclusions on behalf of the others.

By assumption, communication is non-costly, so we only need to identify the deviator when there is an action deviation. In this case, we only need the neighbors of the deviator to identify him as only they can minmax him. However, all the neighbors of the deviator have to identify him because for some payoff functions, it may not be possible to punish several neighbors and it may also be needed that all the neighbors of the deviator start playing the punishment strategy in order to force him to his minmax. During this phase, each player i in N should play action $\bar{a}_t^i$ at stage t.

In the next section, we construct the communication part of the strategy $(\bar{\sigma}, \bar{\phi})$ in order to identify the deviator when there is an action deviation.

3.2 Communication protocol

For the communication phase of the equilibrium strategy, we use a communication protocol, *i.e.* the specification of how players choose their messages, the number of communication rounds and an output rule for each player. In our context, we need the communication protocol to start when there is a deviation and if it is an action deviation, that the neighbors of the deviator output the identity of the faulty player. In particular, the protocol has to start even if there is only a communication deviation, as it may not be possible for some players to differentiate between action and communication deviations. We first introduce some definitions, then construct a communication protocol which has the desired properties.

3.2.1 Definitions

A communication protocol specifies: a finite set of messages, a strategy for each player, a number of rounds of communication and an output rule for each player. We need to construct a communication protocol such that:

- if, for any stage $t > 0$ and any player i in N, $g_t^i = g^i(\bar{a}_t^i, \bar{a}_t^{\mathcal{N}(i)})$ and for any neighbor $j \in \mathcal{N}(i)$, player i received the blank message from j, that is $m_t^j(i) = \varnothing$, then the protocol does not start;
- if there is a stage $t > 0$ such that:
  - for any stage $s < t$ and any player i in N, $g_s^i = g^i(\bar{a}_t^i, \bar{a}_t^{\mathcal{N}(i)})$ and $m_t^j(i) = \varnothing$ for any neighbor $j \in \mathcal{N}(i)$;
  - and there is a single player k who starts deviating at stage t, *i.e.* either $a_t^k \neq \bar{a}_t^k$ or there exists a player $j \in \mathcal{N}(j)$ such that $m_t^k(j) \neq \varnothing$;

  then, the protocol starts. In addition, if there is an action deviation at any stage $t' \geq t$, *i.e.* $a_{t'}^k \neq \bar{a}_{t'}^k$, and if all the players, except possibly player k, perform the protocol, then after the specified number of rounds, each neighbor of player k outputs the name of k.

In this case, we say that **deviator identification by neighbors** is possible for the network G.

Remark 3.1 *An action deviation of player k at some stage $t > 0$ is not directly observable, since players do not observe actions. However, since the payoffs satisfy Assumption 2.4, an action deviation leads to a change in payoff for each player $j \in \mathcal{N}(k)$, who then should start the protocol.*

More precisely, we denote by θ the first stage at which some player starts deviating, *ie* θ is the stopping time $\theta = \inf\{t \geq 1 : \exists k \in N \text{ s.t. } \big(a_t^k \neq \bar{a}_t^k \text{ or } \exists j \in \mathcal{N}(k) \text{ s.t. } m_t^k(j) \neq \varnothing\big)\}$. Also denote by θ_A the first stage at which some player starts deviating in action, *i.e.* θ_A is the stopping time $\theta_A = \inf\{t \geq 1 \,:\, \exists k \in N \text{ s.t. } a_t^k \neq \bar{a}_t^k\}$.

Definition 3.2 ***Communication protocol***
A communication protocol is a tuple $(T, (M^i)_{i\in N}, \sigma, \phi, (\mathbf{x}^i)_{i\in N})$ with:

- *an integer T,*
- *an action strategy $\sigma = (\sigma^i)_{i\in N}$ and a communication strategy $\phi = (\phi^i)_{i\in N}$,*
- *a family of random variables $(\mathbf{x}^i)_{i\in N}$ where $\mathbf{x}^i$ is $\mathcal{H}^i_{\theta_A+T}$-mesurable with values in $N \cup \{OK\}$.*

Definition 3.3 ***Deviator identification by neighbors***
Deviator identification by neighbors is possible for the network G if there exists a communication protocol $(T, (M^i)_{i\in N}, \sigma, \phi, (\mathbf{x}^i)_{i\in N})$ such that, for any player $k \in N$, behavior strategy (τ^k, ψ^k) and integer t such that $\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}(\theta_A = t) > 0$, then:

$$\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}\big(\forall i \in \mathcal{N}(k),\, \mathbf{x}^i = k \mid \theta_A = t\big) = 1.$$

The interpretation is the following. The action strategy σ^k prescribes player k to choose $a_t^k = \bar{a}_t^k$ at each stage $t > 0$. The communication strategy ϕ^k prescribes player k to send to any neighbor $j \in \mathcal{N}(k)$ the blank message $m_t^k(j) = \varnothing$ at stage t as far as for any stage $s < t$, $g_s^k = g_s^k(\bar{a}^k, \bar{a}^{\mathcal{N}(k)})$ and $m_s^j(k) = \varnothing$ for any neighbor $j \in \mathcal{N}(k)$. An alternative strategy (τ^k, ψ^k) for player k such that $\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}(\theta_A < +\infty) > 0$ is henceforth called an action deviation. The definition requires that, under an action deviation, the protocol ends in finite time for the deviator's neighbors. That is, T stages of communication after the action deviation, each player $i \in \mathcal{N}(k)$ comes up with a value for $\mathbf{x}^i$. Deviator identification by neighbors is possible if, whenever there is an action deviation of some player $k \in N$, any of his neighbors $i \in \mathcal{N}(k)$ finds out the name of the deviating player with probability one (other players output OK). If no player ever deviates (neither in action nor in communication), each player concludes OK. Recall that the players may not be able to distinguish between action and communication deviations, that is why we need the protocol to start in both cases. In the next section, we construct a communication protocol to prove that under Assumption 2.4, deviator identification by neighbors is possible for any network which satisfies Condition 2.8.

3.2.2 Deviator identification by neighbors

We now prove the following result.

Proposition 3.4 *Suppose that Assumption 2.4 is satisfied. Then, deviator identification by neighbors is possible for the network G if G satisfies Condition 2.8.*

The formal proof is given in section 6. Given a network which satisfies Condition 2.8, we construct a protocol which satisfies the requirements of Definition 3.3 for deviator identification by neighbors. This protocol has the additional property of being deterministic: players who perform the protocol do not use random strategies, although the deviator might. The main ideas are as follows. When a player i in N detects a deviation, that is either he observes a change in his payoff or he receives a message different from the blank one, then he starts broadcasting to all of his neighbors sets of innocent players computed as follows. If the deviation detected is in action, *i.e.* he observed a change in his payoff, then he clears all the players that are not his neighbors since the deviator must be one of his neighbors. On the other hand, if he does not observe a change in his payoff, he clears all his neighbors. Player i also updates his set of innocent players with the sets received by his neighbors: for instance, if one of his neighbors $j \in \mathcal{N}(i)$ sends the set of innocents $\{j, l, m\}$, he adds players l and m to his own set of innocents. Actually, either j is the deviator then l and m are cleared (recall that we only focus on unilateral deviations);

or j is performing the protocol then l and m are truthfully innocent. However, player i cannot clear player j. Besides, there may be several deviations at different stages, either in communication and/or in action, so each player i sends in fact a list of sets, each set being linked with a delay corresponding to the stage of the eventual deviation considered. At the end of the protocol, for any player $i \in \mathcal{N}(k)$, if only one of player i's neighbors, say player $k \in \mathcal{N}(i)$, is not cleared, then player i outputs the name of k.

Formally, take any player i in N. Let t^i be the first stage at which player i detects a deviation, that is $t^i = \inf\{t \geq 1 \,:\, \exists k \in \mathcal{N}(i),\, a_t^k \neq \bar{a}_t^k \text{ or } m_t^k(i) \neq \varnothing\}$. We denote also by $t_A^i = \{t \geq 1 \,:\, \exists k \in \mathcal{N}(i),\, a_t^k \neq \bar{a}_t^k\}$ the first stage at which player i detects an action deviation. Equivalently, since Assumption 2.4 is satisfied, t_A^i represents the first stage at which player i observes a change in his payoff: $t_A^i = \inf\{t \geq 1 \,:\, g_t^i \neq g_t^i(\bar{a}_t^i, \bar{a}_t^{\mathcal{N}(i)})\}$. In the same way, let $t_C^i = \inf\{t \geq 1 \,:\, \exists k \in \mathcal{N}(i),\, m_t^k(i) \neq \varnothing\}$. Obviously, $t^i = \inf\{t_A^i, t_C^i\}$. Let also $\theta = \inf_{i \in N} t^i$. We then have $\theta = \inf\{t \geq 1 \,:\, \exists k \in N,\, a_t^k \neq \bar{a}_t^k \text{ or } \exists j \in \mathcal{N}(k) \text{ s.t. } m_t^k(j) \neq \varnothing\}$. In the same way, let $\theta_A : \inf_{i \in N} t_A^i$ and $\theta_C = \inf_{i \in N} t_C^i$. One have $\theta = \inf\{\theta_A, \theta_C\}$. The communication protocol is as follows.

PROTOCOL FOR DEVIATOR IDENTIFICATION BY NEIGHBORS

The message space. All players communicate using the same finite set of messages M with:

$$M = \{(s, I_s)_{s \in S} \mid S \subseteq \{0, \ldots, t\},\ \text{with } t = n \text{ and } \forall s \in S,\ I_s \subseteq N\}$$

where n is the number of players in the network. A message is a list of couples, each couple being composed of an integer and a subset of players. The interpretation is as follows. In each couple, the integer s represents a delay and refers to a stage. The subset of players I_s is a set of innocent players corresponding to a deviation that occurred s stages before. The maximal delay is restricted to the number of players n.

The strategy of player i. Player i always takes action $\bar{a}^i_t$ when he performs the protocol and the message sent by him is a function of his observations. At the end of stage t^i, player i starts the protocol. For each stage $t \geq t^i$, let $S^i(t) \subseteq \{0, \ldots, n\}$ be the set of delays used in the message sent by player i at stage t. We denote also by $I^i_s(t)$ the set of innocent players according to player i at stage t concerning the deviation that occurred s stages before, with $s \in S^i(t)$: $I^i_s(t)$ thus represents player i's set of innocent players for a deviation that happened at stage $t - s$. At each stage $t \geq t^i$, player i broadcasts to his neighbors $j \in \mathcal{N}(i)$ the message $m^i_j(t) = (s, I^i_s(t))_{s \in S^i(t)}$ computed as follows.

(i) <u>Delay 0</u>: if player i detects an action deviation at stage t, that is $g^i_t \neq g^i_t(\bar{a}^i_t, \bar{a}^{\mathcal{N}(i)}_t)$, then, at the end of stage t, player i broadcasts to all of his neighbors the couple $(0, I^i_0(t))$ where his set of innocent is $I^i_0(t) = N \setminus \mathcal{N}(i)$. This means that for any neighbor $j \in \mathcal{N}(i)$,

$$(0, N \setminus \mathcal{N}(i)) \in m^i_t(j).$$

(ii) <u>Delays $1, \ldots, n$:</u> suppose that player i receives at stage t the messages $(m^j_t(i))_{j \in \mathcal{N}(i)}$ from his neighbors, where for any player $j \in \mathcal{N}(i)$, $m^j_t(i) = (s, I^j_s(t))_{s \in S^j(t)}$ (all other messages are disregarded). Then, the message sent by player i at stage $t+1$, $m^i_{t+1}(j) = (s, I^i_s(t+1))_{s \in S^i(t+1)}$, satisfies the following rule.

1. Each delay increases of one stage, that is: if $s \in \bigcup_{j \in \mathcal{N}(i)} S^j(t) \cup S^i(t)$ and $s \neq n$, then $s + 1 \in S^i(t+1)$.
2. Then, player i updates his sets of innocents linked to each delay $s \in S^i(t+1) \setminus \{0\}$ as follows.
   - If $s - 1 \in S^i(t)$, then:

$$I^i_s(t+1) = \bigcup_{j \in \mathcal{N}(i)} \left(I^j_{s-1}(t) \setminus \{j\}\right) \cup I^i_{s-1}(t).$$

     The new set of innocents of player i at stage $t+1$ for the deviation at stage $t + 1 - s$ contains player i's set of innocents at stage t and all the players cleared by each neighbor $j \in \mathcal{N}(i)$ at stage t (except player j himself).

- Otherwise, if $s-1 \notin S^i(t)$, then:

$$I_s^i(t+1) = \bigcup_{j \in \mathcal{N}(i)} \left(I_{s-1}^j(t) \setminus \{j\}\right) \cup \mathcal{N}(i).$$

  For deviations undetected by player i at stage t, his set of innocents at stage $t+1$ contains all the players cleared by his neighbors at stage t (where the sender is removed from his set of innocents as previously) as well as his neighbors $j \in \mathcal{N}(i)$.

For all other histories, the message is arbitrary (histories which are not consistent with unilateral deviations are disregarded). This ends the definition of the strategies. Denote by $(\tilde{\sigma}, \tilde{\phi})$ this strategy profile.

The output rule. If $t_A^i < +\infty$, then for any $t \geq t_A^i$, let $X^i(t)$ be the set of suspects of player i regarding the deviation at stage t_A^i, that is, for any $t \geq t_A^i$:

$$X^i(t) = \mathcal{N}(i) \cap \left(N \setminus I_{t-t_A^i}^i(t)\right).$$

This set of suspects contains the neighbors of player i that have not been cleared up to stage t. The output rule $\mathbf{x}^i$ of player i is defined as follows. Consider the first stage T^i at which player i identifies the faulty player in case of an action deviation of a neighbor:

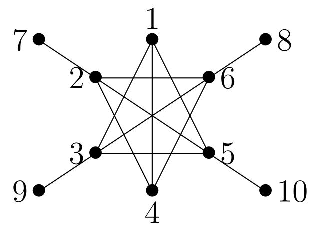
$$T^i = \inf\{t \geq t_A^i \ : \ \sharp X^i(t) = 1\}.$$

If $T^i = +\infty$, we set $\mathbf{x}^i = OK$. Otherwise, there exists x such that $X^i(T^i) = \{x\}$ and we define $\mathbf{x}^i = x$. In other words, when player i's set of suspects concerning an action deviation of one of his neighbors is reduced to x, player i concludes that x is faulty.

The number of rounds. We let the number of rounds of communication (after an action deviation) be $T = n$ where n is the number of players in the game. Notice that if $\theta_A = +\infty$, the protocol may never stop if the deviator keeps sending spurious messages infinitely. However, this is not an issue as communication is non costly.

Let us now present an example to illustrate how this protocol works.

Example 3.5 *Consider the game played on the following network:*



and assume that the payoff function g satisfies Assumption 2.4. Suppose now that player 2 deviates from the equilibrium path $\bar{a}_t^2$ at some stage t. Players 4, 5, 6 and 9 start the protocol at the end of stage t; players 1, 3, 7 and 9 at stage $t+1$ and player 10 at stage $t+2$. The evolution of the sets of innocent players is described in the following table:

Player	t	$t+1$	$t+2$	$t+3$	*t+10*
1		$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
3		$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
4	$\{3,4,5,7,8,9,10\}$	$N \setminus \{2,6\}$	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
5	$\{4,5,6,7,9,10\}$	$N \setminus \{1,2,3,8\}$	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
6	$\{1,5,6,8,9,10\}$	$N \setminus \{2,4\}$	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
7		$N \setminus \{2,3,4,6\}$	$N \setminus \{2,4,6\}$	$N \setminus \{2,6\}$	$N \setminus \{2,6\}$
8		$N \setminus \{1,2,3,5\}$	$N \setminus \{1,2,3,5\}$	$N \setminus \{2,5\}$	$N \setminus \{2,5\}$
9	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$	$N \setminus \{2\}$
10			$N \setminus \{2,3\}$	$N \setminus \{2,3\}$	$N \setminus \{2,3\}$

At the end of stage $t+2$, all the neighbors of player 2 output the name of player 2 and they start to minmax him at stage $t+3$.

Notice however that players 7, 8 and 10 never know who the deviator is since all the information they have comes from their unique neighbor. In particular, when player 5 starts punishing player 2 at stage $t+3$, player 8 keeps performing the protocol and outputs the name of player 5. Then, player 8 starts punishing player 5 at stage $t+4$. This is *due to the fact that player 8 cannot differentiate between the two following histories: a communication deviation followed by an action deviation of player 5 on one hand, and an action deviation of player 2 on the other hand. There are too many punished players, however this is not an issue in order to have a Folk theorem.*

In the next section, we prove that the strategy thus constructed is an equilibrium of the discounted game for any large enough discount factor.

3.3 The equilibrium property

In this section, we prove the sufficient part of Theorem 2.14, that is: *if the network G satisfies Condition 2.8, then for any payoff function which satisfies Assumption 2.4, any feasible and strictly individually rational payoff is an equilibrium payoff of the discounted game.*

Proof. Take a network G which satisfies Condition 2.8 and assume that the payoff function g satisfies Assumption 2.4. We prove that the strategy $(\bar{\sigma}, \bar{\phi})$ constructed before and which consists in what follows:

- each player i in N takes action $\bar{a}_t^i$ and sends $m_t^i(j) = \varnothing$ for each $j \in \mathcal{N}(i)$ in case of no deviation (equilibrium path);
- when there is a deviation of a player k, each player i, except possibly k, starts the protocol for deviator identification by neighbors (Section 3.2.2) and plays $(\tilde{\sigma}, \tilde{\phi})$, with $\tilde{\sigma}_t^i = \bar{a}_t^i$ for each stage t during the protocol. Since some players may output the name of the deviator k before the end of the protocol, each player should not stop communicating before T rounds (recall that $T = n$ with n the number of players in the game).
- Finally, if there is an action deviation of player k, then each neighbor of player k outputs the name of k before stage $\theta_A + T$ and starts minmaxing him at stage $\theta_A + T + 1$. If a player i outputs OK then he comes back to the equilibrium path by playing $\bar{a}_t^i$ and sending $m_t^i(j) = \varnothing$ (if for some player i in N, the protocol never ends, then he takes action $\bar{a}_t^i$ and sends messages according to the protocol). Notice

that there may be several players minmaxed, as in example 3.5: each player i who outputs the name of a player starts to punish him.

Now, we check that $\bar{\gamma} \in \operatorname{co} g(A) \cap SIR(G, g)$ is an equilibrium payoff. If there is no action deviation, the induced payoff vector is indeed $\bar{\gamma}$: if there are only communication deviations, each player i who performs the protocol either outputs OK since he does not observe a change in his payoff, or keeps following the protocol forever. Suppose that player k stops playing action $\bar{a}_t^k$ at some stage t; without loss of generality, we let $t = 1$. The protocol ends before stage $T = n + 1$. The discounted payoff of player k is no more than:

$$\sum_{t=1}^{T}(1-\delta)\delta^{t-1}B + \sum_{t\geq T}(1-\delta)\delta^{t-1}v^k = (1-\delta^T)B + \delta^T v^k$$

where B is an upper bound on the payoffs in the stage game, *i.e* for each player i in N, for any $a^i \in A^i$ and any $a^{\mathcal{N}(i)} \in A^{\mathcal{N}(i)}$, $g^i(a^i, a^{\mathcal{N}(i)}) \leq B$. Since $\bar{\gamma}^k > v^k$, the expected discounted payoff of player k is less than $\bar{\gamma}^k$ for δ close to one. ■

3.4 2-connected networks

In this section, we present some additional properties for 2-connected networks (notice that Condition 2.8 does not ensure 2-connectedness, see for instance the networks G_2, G_3 and G_4 in Figure 1). We show first that it is then possible for each player to differentiate between action and communication deviations. Then in Section 3.4.2, we prove that there exists a protocol such that each player i in N who performs it outputs the name of the deviating player when there is an action deviation (and not only the neighbors of the deviator anymore).

3.4.1 Differentiation between action and communication deviations

In this section, we construct a communication protocol such that:

- if, for any stage $t > 0$ and any player i in N, $g_t^i = g^i(\bar{a}_t^i, \bar{a}_t^{\mathcal{N}(i)})$ and for any neighbor $j \in \mathcal{N}(i)$, player i receives the blank message from j, $m_t^j(i) = \varnothing$, then the protocol does not start;

- if there is a stage $t > 0$ such that:
  - for any stage $s < t$ and any player i in N, $g^i_s = g^i(\bar{a}^i_t, \bar{a}^{\mathcal{N}(i)}_t)$ and $m^j_t(i) = \varnothing$ for any neighbor $j \in \mathcal{N}(i)$;
  - and there is a single player k who starts deviating at stage t, *i.e.* either $a^k_t \neq \bar{a}^k_t$ or there exists a player $j \in \mathcal{N}(j)$ such that $m^k_t(j) \neq \varnothing$;

  then, the protocol starts. In addition, for any stage $t' \geq t$, if there is an action deviation at stage t', *i.e.* $a^k_{t'} \neq \bar{a}^k_{t'}$, and if all the players, except possibly player k, perform the protocol, then after the specified number of rounds, each player i outputs A regarding the deviation at stage t'. On the other hand, for any stage $t' \geq t$, if there was no action deviation at stage t', each player i in N outputs NA.

In other words, for each stage $t' > t$, each player who performs the protocol is able to tell if there was an action deviation or not at stage t' after a finite number of rounds of communication. In this case, we say that **deviation differentiation** is possible for the network G. More precisely, we modify the definition of a communication protocol, which is now a tuple $(T, (M^i)_{i\in N}, \sigma, \phi, (\mathbf{x}^i(t))_{i\in N})$ with:

- an integer T,
- an action strategy $\sigma = (\sigma^i)_{i\in N}$ and a communication strategy $\phi = (\phi^i)_{i\in N}$,
- a family of random variables $(\mathbf{x}^i(t))_{i\in N}$ where $\mathbf{x}^i(t)$ is $\mathcal{H}^i_{t+T}$-mesurable with values in $A \cup NA$.

Such a protocol can be seen as a multi-protocol: the output function now depends on the stage of the deviation considered. Formally, let θ and θ_A be defined as in Section 3.2.1.

Definition 3.6 ***Deviation differentiation***
Deviation differentiation is possible for the network G if there exists a communication protocol $(T, (M^i)_{i\in N}, \sigma, \phi, (\mathbf{x}^i(t))_{i\in N})$ such that, for any player $k \in N$, behavior strategy

(τ^k, ψ^k) *and integer* t *such that* $\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}(\theta = t) > 0$, *then, for each* $t' \geq t$*:*

$$\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}\left(\forall i \in N,\ \mathbf{x}^i(t') = A \mid a^k_{t'} \neq \bar{a}^k_{t'}\right) = 1$$
$$\text{and} \quad \mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}\left(\forall i \in N,\ \mathbf{x}^i(t') = NA \mid a^k_{t'} = \bar{a}^k_{t'}\right) = 1.$$

The interpretation is the following. The action strategy σ^k prescribes player k to choose $a^k_t = \bar{a}^k_t$ at each stage $t > 0$. The communication strategy ϕ^k prescribes player k to send to any neighbor $j \in \mathcal{N}(k)$ the blank message $m^k_t(j) = \varnothing$ at stage t as far as for any stage $s < t$, $g^k_s = g^k_s(\bar{a}^k, \bar{a}^{\mathcal{N}(k)})$ and $m^j_s(k) = \varnothing$ for any $j \in \mathcal{N}(k)$. An alternative strategy (τ^k, ψ^k) for player k such that $\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}(\theta = t) > 0$ is henceforth called a deviation. The definition requires that, for each stage $t' \geq t$ and after T stages of communication, each player $i \in N$ comes up with a value for $\mathbf{x}^i(t')$. Deviation differentiation is possible if, whenever there is an action deviation at stage t', any player $i \in N$ outputs A with probability one. On the contrary, whenever there is no action deviation at stage t', each player outputs NA. Now, we show that 2-connected networks have the following property.

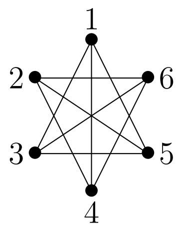
Proposition 3.7 *Suppose that* g *satisfies Assumption 2.4. Then, deviation differentiation is possible for the network* G *if* G *is 2-connected and satisfies Condition 2.8.*

The formal proof is given in section 7. Given a 2-connected network which satisfies Condition 2.8, we construct a protocol which satisfies the requirements of Definition 3.6 for deviation differentiation. This protocol has the additional property of being deterministic: players who perform the protocol to not use random strategies, although the deviator might. The main ideas are as follows. Let t^i be defined as in Section 3.2.2. When a player i in N detects a deviation at stage t^i, that is either he observes a change in his payoff or he receives a message different from the blank one, then he starts broadcasting to all of his neighbors ordered sequences of players linked with a date. If player i observes a change in his payoff at stage t^i, then he sends $(0, (i))$ to his neighbors at the end of stage t^i, which means "player i detects an action deviation 0 stages before". If player i receives $m^i_{t^i}(j) \neq \varnothing$ from a neighbor $j \in \mathcal{N}(i)$, then i transmits this message adding first his name at the end and $+1$ to the delay. For instance, if player i receives $(s, (m, l))$ from neighbor $l \in \mathcal{N}(i)$, then he transmits to all his neighbors at stage $t^i + 1$ the new message $(s + 1, (m, l, i))$ which means "i says that l says that m says that he detected a deviation $s + 1$ stages before". Other messages are disregarded. The delay represents the date of the eventual action deviation, and the sequence of players the path of communication. After T rounds of communication, each player who performs the protocol analyzes all the sequences of players received for each delay. We claim the following points (see Section 7 for a formal proof):

- If there is at least one player in all the sequences concerning stage t, then there was no action deviation at stage t and each player i outputs NA.
- If there is no single player who appears in all the sequences regarding stage t, then there was an action deviation at stage t and each player i outputs A.

The idea is that if there is no action deviation at stage t, then each message received comes from the deviating player and his name thus appears in all the paths of communication. On the other hand, if there is an action deviation at stage t, there are at least two neighbors of the deviator who start sending messages since G is 2-connected, and each player receives this information from at least two independent paths. The protocol is presented formally in Appendix 7. Let us now present a simple example to illustrate how this protocol works.

Example 3.8 *Consider a 6-player game played on the following network:*



and suppose that Assumption 2.4 is satisfied. Suppose also that at some stage $t > 0$*, player 1 deviates in communication and sends to players 4 and 5 the message* $m_t^1(j) = (1, (3, 1))$*,* $j \in \{4, 5\}$*, which means: "player 3 tells me that he detected an action deviation 1 stage before (so at stage t-1)". Player 4 then starts the protocol for deviation differentiation* *and, for any $j \in \{1,2,6\}$, $m^4_{t+1}(j) = (2,(3,1,4))$. In the same way, for any $j \in \{1,2,3\}$, $m^5_{t+1}(j) = (2,(3,1,5))$. Each message after stage $t+1$ starts with the same sequence $(s,(3,1))$, $s > 0$, except if player 1 sends new spurious messages. But, in each case, player 1 appears in each sequence of players. And at stage $t+n+1$, each player, except possibly player 1, outputs NA. However, notice that player 3 may also appear in all the sequences, so it may be impossible to distinguish between a communication deviation at stage $t-1$ of player 3 on one hand, and a communication deviation of player 1 at stage t on the other hand.*

In the next section, we use the protocol for deviation differentiation in order to show that 2-connected networks enable all the players to identify the deviator when there is an action deviation.

3.4.2 Deviator identification

In this section, we introduce the notion of deviator identification by all players, not only by neighbors.

Definition 3.9 ***Deviator identification***

Deviator identification is possible for the network G if there exists a communication protocol $(T,(M^i)_{i\in N},\sigma,\phi,(\mathbf{x}^i)_{i\in N})$ such that, for any player $k \in N$, behavior strategy (τ^k,ψ^k) and integer t such that $\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}(\theta_A = t) > 0$, then:

$$\mathbb{P}_{\tau^k,\sigma^{-k},\psi^k,\phi^{-k}}\left(\forall i \in N,\ \mathbf{x}^i = k \mid \theta_A = t\right) = 1.$$

The difference between deviator identification and deviator identification by neighbors is that in the first notion, every player outputs the name of the deviator when there is an action deviation, while the second notion only implies that the neighbors of the deviator output his name. We then have the following result:

Proposition 3.10 *Suppose that Assumption 2.4 is satisfied. Then, deviator identification is possible for the network G if G is 2-connected and satisfies Condition 2.8.*

The formal proof is given in Appendix 8. The idea is to combine both protocols constructed before for deviator identification by neighbors and for deviation differentiation. As before, each neighbor of the deviator identifies him in finite time. Moreover, when there is an action deviation at some stage t, each player i in N detects that it is an action deviation (Proposition 3.7). Then the results follows from the fact that each player who is not neighbor of the deviator does not suspect any communication deviation of any neighbor anymore. Notice that if deviator identification is possible for a network G, then, when strategy $(\bar{\sigma}, \bar{\phi})$ (as described in Section 3.1) is played and when there is an action deviation at some stage t, thereafter only the deviator is minmaxed since all the players in the game identify him.

In the next section, we prove that Condition 2.8 is necessary to derive a the Folk theorem.

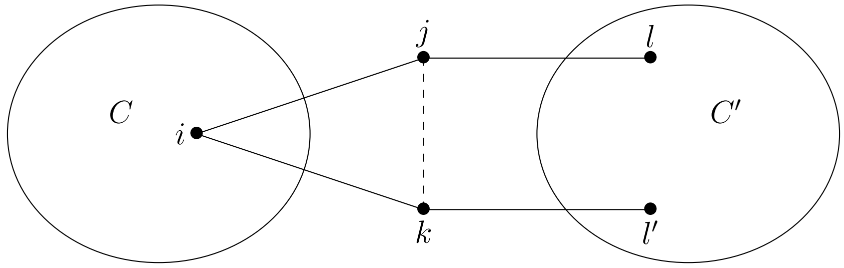
4 Necessary condition

In this section, we prove that: *if for any payoff function which satisfies Assumption 2.4, any feasible and strictly individually rational payoff is an equilibrium payoff of the discounted game, then the network G satisfies Condition 2.8.* To do so, we construct a payoff function g such that, whenever Condition 2.8 is not satisfied, then there exists a payoff $u \in \operatorname{co} g(A) \cap SIR(G,g)$ such that u is not an equilibrium payoff of the discounted game.

Assume hereafter that Condition 2.8 is not satisfied. We then must have that:

$$\exists i \in N,\ \exists j,k \in \mathcal{N}(i),\ j \neq k, \text{ s.t. } \forall l \in \mathcal{N}(j) \triangle \mathcal{N}(k),$$
$$\text{every path from } l \text{ to } i \text{ goes either through } j \text{ or } k.$$

This means that, in the graph $G - \{jk\}$ where j and k have been removed, i is not in the same connected component as l for any $l \in \mathcal{N}(j) \triangle \mathcal{N}(k)$. The network G is thus such as below:



where any player $l \in \mathcal{N}(j) \triangle \mathcal{N}(k)$ is in the subgraph C'; each player i in the subgraph C is either a neighbor of both players j and k, *i.e.* $i \in \mathcal{N}(j) \cap \mathcal{N}(k)$, or of none of them, *i.e.* $i \in N \setminus (\mathcal{N}(j) \cup \mathcal{N}(k))$; moreover, C and C' are not connected in $G - \{jk\}$; finally, the dashed line between j and k means that players j and k may be neighbors or not, and we study both cases.

First case: j and k are not neighbors.

Consider the payoff function for players i, j and k represented by the following matrix (where player i chooses the row, player j the column and player k the matrix):

i \ j	C	D
U	1,1,1	0,3,1
M	1,0,3	0,0,3
D	1,4,0	0,4,0

i \ j	C	D
U	0,1,4	1,3,4
M	0,0,3	1,0,3
D	0,4,0	0,4,0

To complete the description of g, let also assume that each player $m \neq i, j, k$ has two actions C and D such that:

- if $m \in \mathcal{N}(i)$, then at each stage $t > 0$:

$$g_t^m(a_t^m, a_t^{\mathcal{N}(m)}) = \begin{cases} l\frac{\epsilon}{n} + \frac{\epsilon}{n} & \text{if } a_t^i = U, \\ l\frac{\epsilon}{n} & \text{otherwise} \end{cases}$$

  where $l = \{\sharp l \ : \ l \in \mathcal{N}(m) \cup \{m\} \setminus \{i\} \text{ and } a_t^l = C\}$.

- If $m \notin \mathcal{N}(i)$, then at each stage $t > 0$:

$$g_t^m(a_t^m, a_t^{\mathcal{N}(m)}) = l\frac{\epsilon}{n} + \frac{\epsilon}{n}$$

with $l = \{\sharp l \ : \ l \in \mathcal{N}(m) \cup \{m\} \text{ and } a_t^l = C\}$.

- Finally, for each $t > 0$:

$$\begin{aligned} g_t^i(a_t^i, a_t^{\mathcal{N}(i)}) &= g_t^i(a_t^i, a_t^j, a_t^k) + l^i \frac{\epsilon}{n} \\ g_t^j(a_t^j, a_t^{\mathcal{N}(j)}) &= g_t^j(a_t^j, a_t^i) + l^j \frac{\epsilon}{n} \\ g_t^k(a_t^k, a_t^{\mathcal{N}(k)}) &= g_t^k(a_t^k, a_t^i) + l^k \frac{\epsilon}{n} \end{aligned}$$

  where $l^i = \{\sharp l : l \in \mathcal{N}(i) \backslash \{j,k\} \text{ and } a_t^{l^i} = C\}$, $l^j = \{\sharp l : l \in \mathcal{N}(j) \backslash \{i\} \text{ and } a_t^{l^j} = C\}$, $l^k = \{\sharp l \ : \ l \in \mathcal{N}(k) \setminus \{i\} \text{ and } a_t^{l^k} = C\}$ and $g_t^i(a_t^i, a_t^j, a_t^k)$, $g_t^j(a_t^j, a_t^i)$ and $g_t^k(a_t^k, a_t^i)$ are defined by the matrix above.

If player i has more than three actions and if the other players have more than two actions, we shall duplicate rows, columns, matrix...etc, after decreasing each payoff by $\frac{1}{k} \times \frac{\epsilon}{n}$ where each new action is numbered by $k > 1$. The payoff function g thus defined has the following properties:

- g satisfies Assumption 2.4;
- $v^i = 0$, $v^j = 0$ and $v^k = 0$;
- C is a dominant strategy for each player $l \neq i, j, k$;
- the payoff $(1,1,1)$ (representing the payoffs of players i, j and k) is in $\operatorname{co} g(A) \cap SIR(G, g)$;
- for any $a^i \in \{U, M, D\}$ and any a^j, a^k in $\{C, D\}$:

$$g^j(a^j, a^i) + g^k(a^k, a^i) \geq 3,$$

  to the only way to get the payoff of $(1,1,1)$ is that player i chooses action U and all other players take action C;

- player i cannot punish both players j and k: player i has to play M in order to minmax player j and player k thus gets a payoff of 3, and has to choose action D in order to minmax player k and player j thus gets a payoff of 4.

Assume now that $(1,1,1)$ is in $E_\delta(G,g)$ and let $\sigma = (\sigma^i, \sigma^j, \sigma^k, (\sigma^m)_{m\neq i,j,k})$ and $\phi = (\phi^i, \phi^j, \phi^k, (\phi^m)_{m\neq i,j,k})$ be an equilibrium of the discounted game with payoff $\gamma_\delta = (1,1,1)$ for players i, j and k. We define deviations for players j and k as follows. We construct (τ^j, ψ^j) and (τ^j, ψ^j) such that $\mathbb{P}_{\tau^j,\sigma^{-j},\psi^j,\phi^{-j}}()$.

5 Extensions

In this section, we present some extensions of our model and open problems.

Partially known networks. One can see easily that Theorem 2.14 is still valid if the network G is partially known from the players, that is players only know their neighbors and the number of players in the game. However, a deviator needs to know the graph in order to know if he has an incitation to deviate.

Uniform equilibrium. Condition 2.8 is also a necessary and sufficient condition to have a Folk theorem for the undiscounted repeated game. In this case, we consider the notion of uniform equilibrium (see [Sor92] and [FL91]).

Networks of interaction and communication. We modify our model as follows. Suppose that we have two networks G_1 and G_2, where G_1 is the graph of interaction which gives the payoff of the players (depending on the actions of their neighbors) and G_2 is the graph of communication (each player can communicate with his neighbors in G_2). Denote by $\mathcal{N}_1(i)$ the set of neighbors of player i in G_1. We can have a similar result as Theorem 2.14 which is:

Corollary 5.1 *The following statements are equivalent.*

1. *For any payoff function which satisfies Assumption 2.4, any feasible and strictly individually rational payoff is an equilibrium payoff of the discounted game, that is:*

there exists $\bar{\delta} \in (0,1)$ *such that for any* $\delta \in (\bar{\delta}, 1)$,

$$\operatorname{co} g(A) \cap SIR(G,g) \subseteq E_\delta(G,g).$$

2. *For each player* i *in* N, *for each neighbors* j,k *in* $\mathcal{N}_1(i)$, *there exists a player* l *in* $\mathcal{N}_1(j) \triangle \mathcal{N}_1(k)$ *such that there is a path from player* l *to* i *in* G_2 *which goes neither through* j *nor* k.

The proof is let to the reader.

Sequential equilibrium. Since a player does not observe the actions of other players, the notion of subgame perfection does not capture the fact that each player optimizes after every history. Therefore we would like to extend our result with the notion of sequential equilibrium introduced by Kreps and Wilson ([KW82]). This notion requires the equilibrium strategies to be optimal off the equilibrium path besides being a Nash equilibrium (that is there exists no profitable deviation from the equilibrium strategies). However, it remains an open problem in our model.

Broadcast communication. If we modify our model such that communication is multicast (or broadcast), that is each player is restricted to send the same message to all his neighbors at each stage, the necessary and sufficient condition to have a Folk theorem remains an open problem. We believe that this condition may be less strong than Condition 2.8 since in this case, some coding can be used to transmit information even if the graph does not satisfy Condition 2.8 but this condition may not be intuitive and one can refer to [RT08] to see that.

6 Appendix A: Proof of Proposition 3.4

We take up the protocol presented in Section 3.2.2 and prove the following lemma.

Lemma 6.1 *Let $T = n$. Consider a player k, a strategy (τ^k, ψ^k) and an integer t such that $\mathbb{P}_{\tau^k,\tilde{\sigma}^{-k},\psi^k,\tilde{\phi}^{-k}}(\theta_A = t) > 0$. Then, for each player $i \neq k$,*

$$\mathbb{P}_{\tau^k,\tilde{\sigma}^{-k},\psi^k,\tilde{\phi}^{-k}} \big(\forall i \in \mathcal{N}(k),\ X^i(t+n) = \{k\} \mid \theta_A = t\big) = 1.$$

Note that Proposition 3.4 directly follows.

Proof. Suppose Assumption 2.4 is satisfied. Take a network G that satisfies Condition 2.8 and is connected (see Remark 2.15). Since the 2-player case is trivial, we assume that $n \geq 6$ (see Remark 2.13). Fix a player k and assume that player k stops playing action $\bar{a}^k_t$ at stage t, while each player i in N chooses action $\bar{a}^i_s$ for each stage $s \leq t$. We prove that, the protocol defined in Section 3.2.2 is such that:

- for each player $i \neq \mathcal{N}(k)$, for each player $j \in \mathcal{N}(i)$ such that $j \neq k$, $j \notin X^i(t+T)$;
- $k \in X^i(t+T)$ for any $i \in \mathcal{N}(k)$;

with $T = n$.

First, each neighbor of player k, $i \in \mathcal{N}(k)$, observes a change in his payoff at stage t (because of Assumption 2.4) and thus starts the protocol at the end of stage t by sending the message $m^i_t(j) = (0, N \setminus \mathcal{N}(i))$ to his neighbors $j \in \mathcal{N}(i)$[2]. At the end of stage t, $X^i(t) = \mathcal{N}(i)$. Since Condition 2.8 is satisfied, for each player $j \in \mathcal{N}(i)$, $j \neq k$, there exists a player $l \in \mathcal{N}(j) \bigtriangleup \mathcal{N}(k)$ such that there is a path from l to i which goes neither trough j nor k. We consider the two following cases:

- if $l \in \mathcal{N}(k) \setminus \mathcal{N}(j)$, then player l starts the protocol at the end of stage t by sending the message $(0, I^l_0(t))$ with $j \in I^l_0(t)$. Since G is connected, the distance between l and j is at most $n-2$ (recall that there is a path which goes not trough neither j nor k). Then, at some stage $s \leq t+n-3$, there exists a player $m \in \mathcal{N}(i)$ such that $m \neq j$, k, and $m^m_t(i) = (s-t, I^m_{s-t}(s))$ with $j \in I^m_{s-t}(s)$ since along the path which

[2] One notice that the protocol could have start before, if there were communication deviations only. However, it does not change the proof to omit them since it only adds some pairs of delays and sets in the messages that are not taken into account to analyze the deviation that occurred at stage t. For simplicity we omit all the deviations that happens before stage t.

goes through neither j nor k, all the players are following the protocol. Finally, at stage $s+1 \leq t+n-2$, we have $j \in I^i_{s+1-t}(s)$, so $j \notin X^i(t+n-1)$.

- On the other hand, if $l \in \mathcal{N}(j) \setminus \mathcal{N}(k)$, then the distance between l and k is at most 3, $d_G(l,k) \leq 3$, as the path k, i, j, l exists. So, player l starts the protocol at stage $s \leq t+3$. Again, there is a path from player l to i which goes through neither k nor j, its maximal length is $n-3$ and all the players along this path follow the protocol. So, as before, at some stage $s \leq t+3+n-4 = t+n-1$, there exists a player $m \in \mathcal{N}(i)$ such that $m \neq j$, k, and $m^m_t(i) = (s-t, I^m_{s-t}(s))$ with $j \in I^m_{s-t}(s)$. Finally, at stage $s+1 \leq t+n$, we have $j \in I^i_{s+1-t}(s)$, so $j \notin X^i(t+n)$

We now prove the second point. Each player $i \in \mathcal{N}(k)$ starts the protocol at the end of stage t because of Assumption 2.4. So, at the end of stage t and for each player $i \in \mathcal{N}(k)$, $I^i_0(t) = N \setminus \mathcal{N}(i)$ and $k \notin I^i_0(t)$. On the other hand, any player $j \notin \mathcal{N}(k)$ starts the protocol at stage $s \leq t+n$ as the graph is connected. Since all the players except k perform the protocol, there exists no player $l \neq k$ such that $l \in \mathcal{N}(j)$ and $k \in I^l_{s+n-1}(s-1)$ (recall that if player k sends a set of innocents in which he is, a player who follows the protocol do not clear player k). Adding the fact that player j did not observe a change in his payoff at stage t, we conclude that $k \notin I^j_{s-t}(s)$. Then, the only player that may transmit the name of player k in his set of innocents regarding the deviation at stage t is player k himself. So, for each player $i \in \mathcal{N}(k)$, $k \in X^i(t+n)$.

Finally, we conclude that for each player $i \in \mathcal{N}(k)$, $X^i(t+n) = \{k\}$, which proves Lemma 6.1. ■

7 Appendix B: Proof of Proposition 3.7

Before proving Proposition 3.7, we introduce the following communication protocol.

PROTOCOL FOR DEVIATION DIFFERENTIATION

The message space. All players communicate using the same finite set of messages M with:

$$M = \left\{(s, (x_1, \ldots, x_{s+1})_{x_1,\ldots,x_{s+1} \in N})_{s \in S, x_1,\ldots,x_{s+1} \in N} \mid S \subseteq \{0, \ldots, t\} \text{ with } t = n\right\}$$

where n is the number of players in the network. Now $(x_1, \ldots, x_{s+1})$ is an ordered sequence of players and s represents a delay which is restricted to the number of players n.

The strategy of player i. Player i always takes his action $\bar{a}_t^i$ when he performs the protocol and the message sent by him is a function of his observations. Let t^i be defined as in Section 3.2.2. At the end of stage t^i, player i starts the protocol. For each stage $t \geq t^i$ and each player i in N, let $S^i(t) \subseteq \{0, \ldots, n\}$ be the set of delays used in the message player i sends at stage t. For each stage $t \geq t^i$, player i broadcasts to his neighbors $j \in \mathcal{N}(i)$ the message $m_j^i(t) = (s, (x_1, \ldots, x_{s+1})_{x_1,\ldots,x_{s+1} \in N})_{s \in S^i(t)}$ computed as follows.

(i) Delay 0: if player i detects an action deviation at stage t, that is $g_t^i \neq g_t^i(\bar{a}_t^i, \bar{a}_t^{\mathcal{N}(i)})$, then, at the end of stage t, player i broadcasts to all of his neighbors the pair $(0, (x_1 = i))$.

(ii) Delays $1, \ldots, n$: for each player $i \in N$, let $(x_1^{is}(t), \ldots, x_{s+1}^{is}(t))_{s \in S^i(t)}$ be the ordered sequences of players used in the message player i sends at stage t. Suppose that player i receives at stage t the messages $(m_t^j(i))_{j \in \mathcal{N}(i)}$ from his neighbors, where for any player $j \in \mathcal{N}(i)$, $m_t^j(i) = (s, (x_1^{js}(t), \ldots, x_{s+1}^{js}(t) = j))_{s \in S^j(t)}$ (all other messages are disregarded). Then, the message sent by player i at stage $t+1$, $m_{t+1}^i(j)$, satisfies the following rule:

$$m_{t+1}^i(j) = \Big((s+1, (x_1^{i(s+1)}(t+1), \ldots, j, x_{s+2}^{i(s+1)}(t+1) = i))_{s+1 \in S^j(t) \setminus \{n\}}\Big)_{j \in \mathcal{N}(i)}.$$

For all other histories, the message is arbitrary (histories which are not consistent with unilateral deviations are disregarded). This ends the definition of the strategies. Denote by (σ^*, ϕ^*) this strategy profile.

The output rule. For any $i \in N$, for each $t' \geq \theta$ (where θ is defined as in Section 3.2.1), if the two following points are satisfied:

1. there exists a player $l \in N$ such that, $\exists l' \in \mathcal{N}(l)$ such that $\exists t^l \in [t', t'+T]$ such that there exists a message of the form $m^l_{t^l}(l') = (s, (x^{ls}_1(t^l), \ldots, x^{ls}_{s+1}(t^l) = l))_{s \in S^l(t^l)}$ such that $\exists s^l \in S^l(t^l)$ such that $t^l - s^l = t'$ *(there exists at least one player who sends before stage $t'+T$ a message containing a sequence of players linked to an eventual deviation at stage t', which implies that stage t' is a potential stage of deviation)*;

2. there exists no player $k \in N$ such that:

$$\forall j \in \mathcal{N}(i),\ \forall t \in [t', t'+T],\ \forall m^j_t(i) = (s, (x^{js}_1(t), \ldots, x^{js}_{s+1}(t) = j))_{s \in S^j(t)},$$
$$\forall s \in S^j(t) \text{ s.t. } t - s = t',\ k \in (x^{js}_1(t), \ldots, x^{js}_{s+1}(t) = j)$$

   (concerning the eventual deviation at stage t', no player appears in all the sequences of players transmitted before stage $t'+T$);

then $\mathbf{x}^i(t') = A$. Otherwise, $\mathbf{x}^i(t') = C$.

The number of rounds. We let the number of rounds of communication after each $t \geq \theta$ such that there exists $k \in N$ such that $a^k_t \neq \bar{a}^k_t$ or $\exists j \in \mathcal{N}(k)$ s.t. $m^k_t(j) \neq \varnothing$ be $T = n$ where n is the number of players in the game. Notice that, as before, the protocol may never stop if the deviator keeps sending spurious messages infinitely. However, this is not an issue as communication is non costly.

We now prove Proposition 3.7.

Proof. Suppose that Assumption 2.4 is satisfied. Take a network G that satisfies Condition 2.8 and such that G is 2-connected. Since the 2-player case is trivial, we assume that $n \geq 6$ (see Remark 2.13). Fix a player k and assume that player k deviates at some stage t, that is either $a^k_t \neq \bar{a}^k_t$ or there exists $j \in \mathcal{N}(k)$ such that $m^k_t(j) \neq \varnothing$, although for each stage $s \leq t$ and each player $i \in N$, $a^i_s = \bar{a}^i_s$ and $m^i_s(j) = \varnothing$ for any

$j \in \mathcal{N}(i)$. We prove that the protocol for deviation differentiation defined before is such that, for any $t' \geq t$:

- if $a^k_{t'} \neq \bar{a}^k_{t'}$, for each player i in N, $\mathbf{x}^i(t') = A$ at stage $t' + T$;
- otherwise, $\mathbf{x}^i(t') = C$ at stage $t' + T$;

with $T = n$.

First, take $t' \geq t$ such that $a^k_{t'} \neq \bar{a}^k_{t'}$. Since G is connected, each player i in N, except possibly player k, has started the protocol with a delay referring to stage t' before stage $t' + n$. Take a neighbor j of player k, $j \in \mathcal{N}(k)$. Since G is 2-connected, for each player i in N, there are at least two independent paths between player j and player i. So, if all the players except possibly k perform the protocol for deviation differentiation, then for each player i in N, we have that:

1. player j performs the protocol so there exists a player $j' \in \mathcal{N}(j)$ such that $m^j_{t'}(j') = (s, (x^{js}_1(t'), \ldots, x^{js}_{s+1}(t') = j))_{s \in S^j(t')}$ is such that $0 \in S^l(t')$ and $t' - 0 = t'$;

2. and:

$$
\begin{gathered}
\exists j^1, j^2 \in \mathcal{N}(i) \text{ s.t. } \exists t^1, t^2 \in [t', T + t'] \text{ s.t.} \\
\exists m^{j^1}_{t^1}(i) = (s, (x^{j^1 s}_1(t^1), \ldots, x^{j^1 s}_{s+1}(t^1) = j^1))_{s \in S^{j^1}(t^1)} \\
\text{and } \exists m^{j^2}_{t^2}(i) = (s, (x^{j^2 s}_1(t^2), \ldots, x^{j^2 s}_{s+1}(2^1) = j^2))_{s \in S^{j^2}(t^2)} \text{ s.t.} \\
\exists (s^1, s^2) \in S^{j^1}(t^1) \times S^{j^2}(t^2) \text{ s.t. } t^1 - s^1 = t^2 - s - 2 = t' \text{ s.t.} \\
\forall (p, q) \in [1, \ldots, s^1 + 1] \times [1, \ldots, s^2 + 1],\ \ x^{j^1 s^1}_p(t^1) \neq x^{j^2 s^2}_r(t^2).
\end{gathered}
$$

Notice that this results still holds if player i is in $\mathcal{N}(k)$, because $\sharp\mathcal{N}(k) \geq 2$ since G is 2-connected. We deduce then easily that there exists no player $l \in N$ such that for any player $i \in N$:

$$
\begin{gathered}
\forall j \in \mathcal{N}(i),\ \forall t \in [t', t' + T],\ \forall m^j_t(i) = (s, (x^{js}_1(t), \ldots, x^{js}_{s+1}(t) = j))_{s \in S^j(t)}, \\
\forall s \in S^j(t) \text{ s.t. } t - s = t',\ l \in (x^{js}_1(t), \ldots, x^{js}_{s+1}(t) = j)
\end{gathered}
$$

and so for each player i in N, $\mathbf{x}^i(t') = A$ at stage $t' + T$ with $T = n$.

For the second point, suppose that $a^k_{t'} = \bar{a}^k_{t'}$. Then, either there exists no player $l \in N$ such that, $\exists l' \in \mathcal{N}(l)$ such that $\exists t^l \in [t', t'+T]$ such that there exists a message of the form $m^l_{t^l}(l') = (s, (x^{ls}_1(t^l), \ldots, x^{ls}_{s+1}(t^l) = l))_{s \in S^l(t^l)}$ such that $\exists s^l \in S^l(t^l)$ such that $t^l - s^l = t'$ *(there exists no player who sends before stage $t' + T$ a message containing a sequence of players linked to an eventual deviation at stage t', which implies that stage t' is not a potential stage of deviation).*

Or, there exists a player $l \in N$ such that, $\exists l' \in \mathcal{N}(l)$ such that $\exists t^l \in [t', t' + T]$ such that there exists a message of the form $m^l_{t^l}(l') = (s, (x^{ls}_1(t^l), \ldots, x^{ls}_{s+1}(t^l) = l))_{s \in S^l(t^l)}$ such that $\exists s^l \in S^l(t^l)$ such that $t^l - s^l = t'$. In that latter case, as all the players except possibly k performs the protocol, we must have that $l = k$, and moreover, for any $i \in N$:

$$\forall j \in \mathcal{N}(i),\ \forall t \in [t', t' + T],\ \forall m^j_t(i) = (s, (x^{js}_1(t), \ldots, x^{js}_{s+1}(t) = j))_{s \in S^j(t)},$$
$$\forall s \in S^j(t) \text{ s.t. } t - s = t',\ k \in (x^{js}_1(t), \ldots, x^{js}_{s+1}(t) = j)$$

since player k is the only player who starts to transmit messages concerning stage t' (and because no player $i \neq k$ observes a change in his payoff at stage t').

So, if $a^k_{t'} = \bar{a}^k_{t'}$, then each player $i \in N$, except possibly player k, outputs NA at stage $t' + T$ with $T = n$. This ends the proof of Proposition 3.7. ∎

8 Appendix C: Proof of Proposition 3.10

Proof. Suppose Assumption 2.4 is satisfied. Take a network G that satisfies Condition 2.8 and such that G is 2-connected. Since the 2-player case is trivial, we assume that $n \geq 6$ (see Remark 2.13). Let define the following protocol constructed with the two previous protocols. For each player $i \in N$, player i starts at stage t^i (where t^i is defined as in Section 3.2.2) the protocol for deviation differentiation (see Section 7). Then, if $\theta_A = +\infty$, player i keeps performing the protocol for deviation differentiation. On the other hand, if $\theta_A < +\infty$, then player i starts the protocol for deviator identification by neighbors at stage $\theta_A + n + 1$. The output rule of player i only needs to be adapted from the protocol for deviator identification by neighbors. Now, the output rule $\mathbf{x}^i$ of player i is defined as follows. For any player i and any $t \geq \theta_A + n + 1$, let $X^i(t)$ be the set of suspects of player i at stage t, that is, for any $t \geq \theta_A + n + 1$:

$$X^i(t) = N \setminus I^i_{t-\theta_A}(t).$$

The output rule $\mathbf{x}^i$ of player i. Consider the first stage T^i at which player i identifies the faulty player at stage θ_A:

$$T^i = \inf\{t \geq \theta_A + n + 1 \; : \; \sharp X^i(t) = 1\}.$$

If $T^i = +\infty$, we set $\mathbf{x}^i = OK$. Otherwise, there exists x such that $X^i(T^i) = \{x\}$ and we define $\mathbf{x}^i = x$. In other words, when player i's set of suspects is reduced to x, player i concludes that x is faulty.

We now prove that this new protocol, with $T = 3n - 1$ the number of communication rounds, satisfies the requirements of Definition 3.9. Fix a player k and assume that player k stops playing action $\bar{a}^k_t$ at stage t, while each player i in N chooses action $\bar{a}^i_s$ for each stage $s \leq t$.

First, since G is 2-connected, Proposition 3.7 is satisfied, and then, at stage $t+n$, each player i in N outputs A and thus knows that there was a deviation at stage θ_A. Then, each player $i \in N$, except possibly player k, starts the protocol for deviator identification by neighbors at stage $t + n + 1$. For each player $j \neq k$, consider the network $G - \{jk\}$ where players j and k have been removed. We only need to prove the following claim:

Claim 8.1 *Fix a player $j \neq k$. In each connected component C of $G - \{jk\}$, there exists a player $l \in C$ such that $l \in \mathcal{N}(j)$.*

Indeed, take a player $j \neq k$. If this claim is true, take any component C of $G - \{jk\}$, then there exists $l \in C$ such that $l \in \mathcal{N}(j)$. Then, the following cases are possible:

- either $l \in \mathcal{N}(j) \cap \mathcal{N}(k)$, then $j \in I^l_{2n+1}(t+2n+1)$ (where $I^l_{2n+1}(t+2n+1)$ is defined as in Section 3.2.2) because of Proposition 3.4. We concludes that for each player $i \in C$, $j \in I^i_{3n-1}(t+3n-1)$, since the distance between l and i is at most $n-2$.

- or $l \in \mathcal{N}(j) \setminus \mathcal{N}(k)$, then because player l knows that there was an action deviation at stage t (Proposition 3.7) and since player l did not observe a change in his payoff at stage t (2.4), then player l knows that the deviator is not one of his neighbors (and player l does not suspect any of his neighbors to deviate in communication as we only consider unilateral deviations). So again $j \in I^l_{n+1}(t+n+1)$ at stage $t+n+1$, and for each player $i \in C$, $j \in I^i_{2n+1}(t+2n-1)$ as before.

We conclude that if Claim 8.1 is true, then for any player $j \neq k$ and any player $i \in N$, except possibly player k, $j \in I^i_{3n-1}(t+3n-1)$, so each player i outputs the name of k.

We now prove that Claim 8.1 is true. Take a player $j \in N$, such that $j \neq k$ and consider the graph $G - \{jk\}$. Fix a connected component C of $G - \{jk\}$. Take any player $i \in C$. Since the original graph G is 2-connected, there exists a path in G between player i and j which does not go through k. Then there exists a player $l \in N$ such that this path can be written as $i, \ldots, l, j$. In that case, $l \in C$ and $l \in \mathcal{N}(j)$, which proves Claim 8.1 and concludes the proof of Proposition 3.10. ■

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