Coalition Formation in Simple Games

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April 14, 2011

Abstract The characterization of core stable partitions in a simple game is a difficult task. We consider a property called *absence of the strong paradox of smaller coalitions* which is a generalization of Shenoy's well-known condition for a nonempty core and give complete characterizations of strict and semistrict core.
Since this condition is not necessary we consider the Shapley value as allocation rule on the class of generalized apex games. We characterize those games which have a nonempty core and compare core stability with Nash stability.

1 Introduction

Talking about the core of a simple game usually means talking about the core of a TU game. This is not what we consider in this paper. We are interested in coalitional games which are induced by simple games.
In the next section we introduce proper monotonic simple games as well as hedonic games. We define a class of solutions which will play the key role throughout this paper and show how we can derive hedonic games with their help.
The third section is devoted to characterizations of the core of hedonic games which are induced by simple games and solutions. We are considering a property which we call *absence of the strong paradox of smaller coalitions*. It has been introduced by Dimitrov and Haake in [4] and is a generalization of Shenoy's well-known sufficient condition for the nonemptiness of the core. Particularly, we show how strict core and semistrict core are associated in this situation.
Unfortunately the absence of the strong paradox it not necessary for the existence of a core stable partition. In the last section we restrict our attention to the Shapley value and analyze generalized apex games. We a give necessary and sufficient condition for the nonemptiness of its core and strict core.

2 Preliminaries

In this section we will give a brief introduction into the theory of simple games and hedonic games. Throughout this paper let N be a set of players and v be a transferable utility (TU) game, i.e. a function $v : \mathfrak{P}(N) \to \mathbb{R}$ with $v(\emptyset) = 0$. If $S \subset N$ we define the subgame v_S by $v_S(T) = v(S \cap T)$ for all $T \subset N$.

2.1 Simple Games

Simple games are a special class of TU games.

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Definition 2.1 *Let N be a set of players and v be a TU game.*

1. *v is a simple game if $v(S) \in \{0,1\}$ for all $S \subset N$ and $v(\emptyset) = 0$.*
2. *v is a proper simple game if $v(S) = 1$ implies $v(N \setminus S) = 0$ for all $S \subset N$.*
3. *v is a monotonic simple game if $v(S) = 1$ implies $v(T) = 1$ for all $S, T \subset N$ with $S \subset T$.*

Let (N, v) be a proper monotonic simple game. A winning coalition in N is a subset $S \subset N$ such that $v(S) = 1$. If $S \subset N$ such that $v(S) = 0$ then $v(T) = 0$ for all $T \subset S$.
In the following let $\mathcal{V}$ be the set of all proper monotonic simple games. We are interested in the power of different players within a subset S of N. Let $i \in S$. Then we define $\delta_i(S) = v(S) - v(S \setminus \{i\})$. In our context of simple games we say i is a vetoer if $\delta_i(N) = 1$. The concept of vetoers is not only interesting for N but also for subsets of N. For this purpose we extend it. Let therefore $S \subset N$. Note that if $v \in \mathcal{V}$ then also $v_S \in \mathcal{V}$. For $i \in S$ we define

$$\delta_i^S(T) = v_S(T) - v_S(T \setminus \{i\}).$$

Note that $\delta_i^S(S) = \delta_i(S)$. We call i a vetoer in S if $\delta_i^S(S) = 1$. We say S is a minimal winning coalition if all players $i \in S$ are vetoers in S. On the other hand, if $k \in N$ is such that $v_S(T \cup \{k\}) - v_S(T) = 0$ for all $T \subset N \setminus \{k\}$ then k is called null player in S. Note that this is the case if and only if k is not contained is any minimal winning coalition.

Remark 2.2 *Let $S \subset N$ be a winning coalition and $i \subset S$.*

1. *If i is vetoer, he is also vetoer in S; but a vetoer in S does not need to be a vetoer in N.*
2. *If i is null player, he is also null player in S; but a null player in S does not need to be a null player in N.*

Since we want to describe a coalitional game in which a player's preferences over coalitions depend on his payoffs, we need to introduce a system to distribute a coalition's value $v(S)$ among the players it contains.

Definition 2.3 *An efficient nonnegative solution is a mapping $\varphi : \mathcal{V} \to \mathbb{R}^N_{\geq 0}$ such that*

$$\sum_{i \in S} (\varphi(v_S))_i = v(S)$$

for all $v \in \mathcal{V}$ and for all $S \subset N$.

In the following we use $\varphi_i(v)$ instead of $(\varphi(v))_i$. Since we are talking about nonnegative efficient solutions throughout this paper, we will just call them solutions. We denote the set of all efficient nonnegative solutions by $\mathcal{S}$.
We can extend the solution concept to subgames since these are also proper monotonic simple games. Therefore we will write $\varphi(S)$ for $\varphi(v_S)$.
Before we start with the main results we introduce some properties of solutions.

Definition 2.4 *Let $\varphi \in \mathcal{S}$.*

1. *φ satisfies the equal treatment property if for all $v \in \mathcal{V}$ and all players $i, j \in N$ which satisfy $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$*

$$\varphi_i(v) = \varphi_j(v).$$

2. φ *is strongly monotonic iff for all* $u, v \in \mathcal{V}$ *and* $i \in N$ *with*

$$u(S \cup \{i\}) - u(S) \geq v(S \cup \{i\}) - v(S)$$

for all $S \subset N$ *we have* $\varphi_i(u) \geq \varphi_i(v)$.

Although these definitions are quite abstract we can use them to derive some more intuitive properties.

First, if a nonnegative efficient solution satisfies equal treatment on a proper monotonic simple game (N, v) and $S \subset N$ is a minimal winning coalition then

$$\varphi_i(S) = \frac{1}{|S|} \tag{1}$$

A solution which satisfies (1) for each minimal winning coalition in a proper monotonic simple game satisfies equal treatment of minimal winning coalitions. The second property we derive closes this section.

Lemma 2.5 *Let* (N, v) *be a proper monotonic simple game and* φ *a strongly monotonic solution. Let further* $S \subset N$, $T \supset S$ *and* $i \in S$ *be a vetoer in* T*. Then*

$$\varphi_i(T) \geq \varphi_i(S).$$

Proof: Let $S \subset N$, $T \supset S$ and $i \in S$ be a vetoer in T. Since v is monotonic i is a vetoer in S. We have to show that

$$\delta_i^T(U) \geq \delta_i^S(U) \tag{2}$$

for all $U \subset N$. So, let $U \subset N$ be arbitrary and assume without loss of generality that $\delta_i^S(U) = 1$. Then $(U \cup \{i\}) \cap S$ is winning (in v) and hence $v_T(U \cup \{i\}) = 1$. As i is vetoer in T we see $v_T(U \setminus \{i\}) = 0$ and hence, $\delta_i^T(U) = 1 = \delta_i^S(U)$. Since equation (2) is satisfied for all $U \subset N$ and since φ is strongly monotonic we conclude $\varphi_i(T) \geq \varphi_i(S)$.

□

2.2 Hedonic Games

In the last section we introduced simple games. In this section we will describe the relation between simple games and and hedonic games. The idea is simple: Given a solution, each player has preferences over the potential partitions, more precisely over the coalitions he might belong to.
We still work with proper monotonic simple games $v \in \mathcal{V}$ and nonnegative efficient solutions $\varphi \in \mathcal{S}$ without stating it all the time.

Definition 2.6 *A hedonic game is a pair* $(N, \succeq)$*, where* N *is a set of agents and* $\succeq = (\succeq_i)_{i \in N}$ *is a profile of reflexive, transitive and complete binary relations* $\succeq_i$ *on* $\mathfrak{P}_i(N)$*, the collection of all subsets of* N *containing* i.

Using the properties of $\succeq$ we are able to find functions $u_i : \mathfrak{P}_i(N) \to \mathbb{R}$ which describes i's preferences. We can also turn it around: If we have a set of utility functions, we can describe a hedonic game: Let (N, v) be a proper monotonic simple game and $\varphi \in \mathcal{S}$. The binary relations $\succeq_i$, defined as

$$S \succeq_i T \qquad \Leftrightarrow \qquad \varphi_i(S) \geq \varphi_i(T)$$

induce a hedonic game.
We are interested in how coalitions will form. We therefore need the concept of a

partition. A partition Γ of N is a collection $(S_1, \ldots, S_m)$ of pairwise disjoint subsets of N, such that $\bigcup_{i=1}^{m} S_i = N$. We write $C^\Gamma(i)$ to denote the unique coalition in Γ which contains i.
In case of proper monotonic simple games, we are not interested in all partitions: If S is a winning coalition, we represent all partitions containing S by $(S, N \setminus S)$, since in all those partitions $\varphi_j(T) = 0$ for all $j \in N \setminus S$ and all $T \subset N \setminus S$.
We therefore also write $C^S(i)$ instead of $C^\Gamma(i)$ if S is the winning coalition in Γ. When we talk about a non trivial partition $(S, N \setminus S)$ we always mean that S is the winning coalition.
The preferences of different players over coalitions will in general not coincide. However, there might be partitions such that no group of players would leave their coalition to form a new one. The following stability concept formalizes this idea..

Definition 2.7 *Let $(N, \succeq)$ be a hedonic game and Γ be a partition of N.*

1. *A deviation of Γ is a coalition $S \subset N$ such that*
$$S \succ_i C^\Gamma(i)$$
for all $i \in S$.

2. *A weak deviation of Γ is a coalition $S \subset N$ such that*
$$S \succeq_i C^\Gamma(i)$$
for all $i \in S$ and
$$S \succ_i C^\Gamma(i)$$
for some $i \in S$.

3. *Γ is called core stable (lies in the core) if it has no deviations.*

4. *Γ is called strictly core stable (lies in the strict core) if it has no weak deviations.*

5. *Γ is called semistrictly core stable (lies in the semistrict core) if it has no weak deviation D such that for all $S \in \Gamma$ with $S \cap D \neq \emptyset$ there is $i \in D \cap S$ with $D \succ_i S$.*

Whereas for core and strict core we need only to consider the winning coalition of a partition, semistrict core stability might depend on all elements of a partition. But at least we can say something.

Proposition 2.8 *Let v be a proper monotonic simple game and $\varphi \in \mathcal{S}$. Let $\Gamma = (S_1, \ldots, S_m)$ be a partition in the induced hedonic game such that S_1 is winning.*

1. *If $(S_1, N \setminus S_1)$ is semistrictly core stable then Γ is, too.*

2. *If $\left(S_1, \{i_1\}, \ldots, \{i_{|N \setminus S|}\}\right)$ is not semistrictly core stable then Γ is neither.*

Proof:

1. Has been shown by Dimitrov and Haake in [3].

2. Let $\left(S_1, \{i_1\}, \ldots, \{i_{|N \setminus S|}\}\right)$ not be semistrictly core stable. Then there is a weak deviation D of $\left(S_1, \{i_1\}, \ldots, \{i_{|N \setminus S|}\}\right)$ such that there is $i \in D \cap S_1$ with $\varphi_i(D) > \varphi_i(S_1)$ and
$$\varphi_i(D) > \varphi_i(\{i\}) = 0 = \varphi_i\left(C^\Gamma(i)\right)$$
for all $i \in D \setminus S_1$. Hence, Γ cannot be semistrictly core stable.

□

3 Core Partitions of Simple Games

It is difficult to decide whether or not a hedonic game has a nontempty core, even when they are induced by simple games. However, several authors have been able to derive suffient conditions. In this section we present some known results and develop a language which allows us to recognize, why they cannot be necessary. Particularly, we give new characterizations of the cores given these conditions.

3.1 A Characterization of the Core

Let (N, v) be a proper monotonic simple game and $S \subset N$ be a winning coalition. A solution tells us how to allocate the payoff among the players in S. Let $T \subsetneq S$ be a winning coalition and a proper subset of S. We might expect that the payoff allocated to the players of T might increase (or should not decrease) if we distribute it only under them. Unfortunately, this is a quite strong expectation. The following definition was given by Shenoy in [8].

Definition 3.1 *Let $v \in \mathcal{V}$ and $\varphi \in \mathcal{S}$. φ does not exhibit the paradox of smaller coalitions on v if for all winning coalitions $S, T \subset N$ with $T \subset S$ we have*

$$\varphi_i(T) \geq \varphi_i(S)$$

for all $i \in T$.

Shenoy showed in [8] that the absence of the paradox of smaller coalitions is a sufficient condition to ensure the existence of a core stable partition. Dimitrov and Haake gave in [4] weaker conditions which are still sufficient.

Definition 3.2 *Let $v \in \mathcal{V}$ and $\varphi \in \mathcal{S}$. φ does not exhibit the strong paradox of smaller coalitions on v if for all winning coalitions $W \subset N$ there is a minimal winning coalition $T \subset W$ such that*

$$\varphi_i(T) \geq \varphi_i(W)$$

for all $i \in T$.

Proposition 3.3 (Dimitrov, Haake, 2007) *Let $v \in \mathcal{V}$ and $\varphi \in \mathcal{S}$ such that φ satisfies equal treatment of minimal winning coalitions and does not exhibit the strong paradox of smaller coalitions on v. Then there is a core stable partition.*

The idea of this proposition becomes clear with its proof. Because in this case minimal winning coalitions with minimal cardinality are core stable. We say that a minimal winning coalition S has minimal cardinality if $|S| \leq |T|$ for all minimal winning coalitions $T \subset N$. We can extend this idea to subgames as well by considering only sets which are minimal winning in the respective subgame. However, if we are talking about minimal winning coalitions of minimal cardinality we always mean minimal cardinality with respect to N if not specified elsewise.
The next lemma gives a characterization of coalitions which lie in the core without using the absence of the paradox.

Lemma 3.4 *Let $v \in \mathcal{V}$ and φ be a solution.*

1. *A partition $(S, N \setminus S)$ is core stable if and only if for all winning coalitions $T \subset N$ with $\varphi_j(T) > 0$ for all $j \in T$ there is $i \in S \cap T$ such that*

$$\varphi_i(S) \geq \varphi_i(T). \tag{3}$$

2. A partition $(S, N \setminus S)$ is strictly core stable if and only if there is no winning coalition $T \subset N$ such that

$$\varphi_i(S) \leq \varphi_i(T). \tag{4}$$

for all $i \in S \cap T$ and

$$\varphi_j(S) > 0 \tag{5}$$

for some $j \in S \setminus T$

Proof:

1. First let $S \subset N$ be such that (3) is satisfied for all winning coalitions T. Assume that $(S, N \setminus S)$ is no core partition. Then there is a winning coalition T, such that $\varphi^i(T) > \varphi^i\left(C^S(i)\right)$ for all $i \in T$. By properness $T \cap S$ cannot be empty. Hence, equation (3) cannot be satisfied.
Let now S be a winning coalition such that $(S, N \setminus S)$ is core stable. Assume that there is a winning coalition T such that equation (3) is not satisfied for any $i \in S \cap T$. In this case T is a deviation for all $i \in S \cap T$, since $\varphi_i(S) < \varphi_i(T)$ for all $i \in T \cap S$. Further, T is a deviation for all $i \in T \setminus S$ since $\varphi_i(T) > 0 = \varphi_i\left(C^S(i)\right)$ for all $i \in T \setminus S$. Hence, T is a deviation. This is a contradiction, since S is core stable.

2. First let $S \subset N$ and assume that T is a weak deviation of S which does not satisfy both equation (4) and (5). Let first T be a coalition such that equation (4) is unsatisfied. Then T cannot be a weak deviation per definition. If (4) is satisfied but (5) is not then there cannot be any $k \in T \setminus S$ such that $\varphi_k(T) > 0$. If there was $i \in S \cap T$ such that $\varphi_i(T) > \varphi_i(S)$ then there must be $j \in S \cap T$ such that $\varphi_i(T) < \varphi_i(S)$. Then T cannot be a weak deviation. In both cases we ended up in a contradiction, hence T cannot be a weak deviation.
Let now S be a winning coalition such that $(S, N \setminus S)$ is strictly core stable. Assume that there is $T \subset N$ such that equations (4) and (5) are satisfied. If there is $i \in S \cap T$ such that $\varphi_i(S) < \varphi_i(T)$ then T is a weak deviation of S. This is a contradiction since S is strictly core stable. Hence, $\varphi_i(S) = \varphi_i(T)$ for all $i \in S \cap T$. Since

$$\sum_{i \in S} \varphi_i(S) = \sum_{i \in T} \varphi_i(T) = 1$$

and because of equation (5) there must be $k \in T \setminus S$ such that $\varphi_k(T) > 0$. Hence, T is a weak deviation of S. This is a contradiction since S is strictly core stable. Thus, there cannot be $T \subset N$ such that both equation (4) and (5) are satisfied.

□

Note that if $(S, N \setminus S)$ lies in the (strict) core then every partition which contains S does so. We therefore say that S lies in the (strict) core. However, this shows, that it is not so easy to derive a similarly brief characterization of the semistrict core since it depends on the partition of the players which are not included in the winning coalition.

3.2 The Paradox of Smaller Coalitions

We can now interpret the role of Dimitrov and Haake's condition from a new point of view and give a stricter version.

Theorem 3.5 *Let $v \in \mathcal{V}$ and φ be a solution which satisfies equal treatment of minimal winning coalitions and which does not exhibit the strong paradox of smaller coalitions on v.*

1. *A partition $(S, N \setminus S)$ is core stable if and only if for all minimal winning coalitions $T \subset N$ there is $i \in S \cap T$ such that*
$$\varphi_i(S) \geq \varphi_i(T). \tag{6}$$

2. *A partition $(S, N \setminus S)$ is strictly core stable if and only if there is a unique minimal winning coalition $S' \subset S$ of minimal cardinality (with respect to N) such that $\varphi_i(S') = \varphi_i(S)$ for all $i \in S'$.*

3. *A partition $(S, N \setminus S)$ is semistrictly core stable if and only if there is a minimal winning coalition $S' \subset S$ of minimal cardinality (with respect to N) such that $\varphi_i(S') = \varphi_i(S)$ for all $i \in S'$.*

Proof:

1. Let $(S, N \setminus S)$ be core stable. By lemma 3.4 for all winning coalitions $T \subset N$ there is $i \in T$ such that equation (6) is satisfied. Particularly, it is for all minimal winning coalitions.
Let equation (6) be satisfied for all minimal winning coalitions and assume that there is a winning coalition W such that for all $i \in W \cap S$ $\varphi_i(W) > \varphi_i(S)$. W cannot be minimal, but because of the absence of the strong paradox of smaller coalitions we can find a minimal winning coalition $T \subset W$ such that $\varphi_i(T) \geq \varphi_i(W)$ for all $i \in T$. Since T is minimal there is $i \in S \cap T$ such that $\varphi_i(S) \geq \varphi_i(T)$. Hence, we have $\varphi_i(S) \geq \varphi_i(T) \geq \varphi_i(W)$ what is a contradiction.

2. Let S be strictly core stable. If S is the only minimal winning coalition with minimal cardinality then there is nothing to show. So, let S not be unique or be minimal winning without minimal cardinality. Then there is a minimal winning coalition of minimal cardinality T which is a (weak) deviation of S. So, let S not be minimal winning. Then there is a minimal winning coalition $T \subset S$ such that $\varphi_i(T) \geq \varphi_i(S)$ for all $i \in T = T \cap S$. There cannot be $i \in T$ such that $\varphi_i(T) > \varphi_i(S)$, else T would be a weak deviation of S. Hence, $\varphi_i(T) = \varphi_i(S)$. This implies that $\varphi_k(S) = 0$ for all $k \in S \setminus T$. If T is not the only minimal winning coalition with minimal cardinality then there is again a (weak) deviation of T which is also a deviation of S. Hence, S must contain the unique minimal winning coalition S' of minimal cardinality and $\varphi^i(S') = \varphi^i(S)$ for all $i \in S'$.
Now, let S be a winning coalition, $S' \subset$ be the unique minimal winning coalition and of minimal cardinality and $\varphi_i(S') = \varphi_i(S)$ for all $i \in S'$. Assume that there is a deviation W of S. W is also weak deviation of S', so W cannot be minimal. Hence, there is $T \subset W$ such that $\varphi_i(T) \geq \varphi_i(W)$ for all $i \in T$. From properness follows that $S' \cap T \neq \emptyset$. Consequently there is $i \in S' \cap T$ such that
$$\frac{1}{|T|} = \varphi_i(T) \geq \varphi_i(W) > \varphi_i(S') = \frac{1}{|S'|} \geq \frac{1}{|T|}$$
what is a contradiction.

3. Note that a minimal winning coalition with minimal cardinality cannot be deviated by another minimal winning coalition with minimal cardinality in sense of semistrictly core stability. With this knowledge in mind we can proceed the same steps as we did in 2 for each coalition S which contains a minimal winning coalition S' of minimal cardinality such that $\varphi_i(S') = \varphi_i(S)$ for all $i \in S'$.

□

An interesting point is, that in this case the semistrict core stability of a partition depends only on its winning coalition.

Corollary 3.6 *If* $(S, N \setminus S)$ *is (strictly, semistrictly) core stable, then each partition* Γ *which contains* S *is also (strictly, semistrictly) core stable.*

4 Apex Games

In the last chapter we considered a sufficient condition for the existence of core stable partitions in simple games. However, this property is not necessary. In this section we investigate a class of games which does not satisfy the absence of the strong paradox with respect to the Shapley value. We show which elements of the class have a nonempty core and which have not. Before we start we briefly introduce the Shapley value and give some useful properties.

4.1 The Shapley Value

The most popular solution for TU games is the Shapley Value. For a proper monotonic simple game (N, v) we define

$$Sh_i(N, v) = \sum_{S \subset N} \frac{(|N| - |S|)!\,(|S| - 1)!}{|N|!} \delta_i(S).$$

Particularly, we can define the Shapley Value on reduced games. In the following we write $Sh_i(S) = Sh_i(S, v_S)$.

Lemma 4.1 *Let* (N, v) *be a proper monotonic simple game,* $S \subset N$*,* $i, k \in S$ *and* i *be a vetoer in* S*. Then*

$$Sh_i(S) \geq Sh_k(S) + \frac{1}{|S|}.$$

Proof: Let $T \subsetneq S$. Then

$$v_T(T) - v_T(T \setminus \{k\}) \leq v_T(T) - v_T(T \setminus \{i\}),$$

since whenever the left side is equal to 1 we have $v(T) = 1$. But in this case the right also equals 1 since i is a vetoer in T.
Consequently, we have

$$\begin{aligned}
Sh_i(S) &= \sum_{T \subset S} \frac{(|S| - |T|)!\,(|T| - 1)!}{|S|!} \delta_i^S(T) \\
&= \frac{1}{|S|} + \sum_{T \subsetneq S} \frac{(|S| - |T|)!\,(|T| - 1)!}{|S|!} \delta_i^S(T) \\
&\geq \frac{1}{|S|} + \sum_{T \subsetneq S} \frac{(|S| - |T|)!\,(|T| - 1)!}{|S|!} \delta_i^S(T) \\
&= \frac{1}{|S|} + Sh_k(S).
\end{aligned}$$

The last equality follows from the fact that $v_S(S \setminus \{k\}) = 1$ since k is per definition no vetoer in S.

□

A last lemma in this section shall focus on the influence of null players on the Shapley value.

Lemma 4.2 *Let N, M be sets of players and v be a proper monotonic game on $N \cup M$ such that all players in M are null players and all players in N are non null players. Then for all $S \subset N$, $T \subset M$ and for all $i \in S$*

$$Sh_i(S) = Sh_i(S \cup T).$$

Proof: Can be found in the appendix.

□

4.2 The Core of Generalized Apex Games

A certain subclass of simple games are so called apex games. Hart and Kurz gave an analysis of these games in [5]. An axiomatization of the Shapley value in apex games can be found in van den Brink's article [2]. First, we want to give the formal definition of those games.

Definition 4.3 *Let N be a set of agents, $J \subsetneq N$ and $i \in N \setminus J$. An apex-(i, J)-game is a proper monotonic simple game (N, a_{iJ}), such that*

$$a_{iJ}(S) = \begin{cases} 1, & \textit{if } (i \in S \textit{ and } S \cap J \neq \emptyset) \textit{ or } J \subset S \\ 0, & \textit{else.} \end{cases}$$

If $|J| = 1$ then we call $j \in J$ a dictator. This case is not very interesting since $Sh_j(S) = 1$ for all $S \subset N$ with $j \in S$ and all other players are null players. Hence, $|J| \geq 2$ from here.

One result of van den Brink is the following:

Lemma 4.4 *Let a_{iJ} be an apex game. Then $k \in N$ is a null player if and only if $k \in N \setminus (J \cup \{i\})$.*

Our aim is to investigate the core of those games. Particularly, we will prove the following lemma.

Lemma 4.5 *The core of an apex-(i, J)-game (N, a_{iJ}) with $|J| \geq 3$ with respect to the Shapley value is empty.*

The proof is not difficult. Since we want to give a more general version later, we just give a short sketch of the proof here.
We can assume without loss of generality that $J = N \setminus \{i\}$. Further, J is dominated by $\{i, j\}$ for some $j \in J$. Taking an arbitrary set S of the form $S = \{i\} \cup J'$ where $J' \subset J$ and $|J'| < \frac{1}{2}|J|$ we can find the deviation $T = \{i\} \cup J \setminus J'$. Finally, if $S = \{i\} \cup J'$ where $J' \subset J$ and $|J'| \geq \frac{1}{2}|J|$, we see that S is dominated by J.
The same idea with some more calculations will lead to a more general result. But first, we give a new definition.

Definition 4.6 *Let N be a set of agents, $I \subsetneq N$ and $J \in N \setminus I$. An apex-(I, J)-game is a proper monotonic simple game (N, a_{IJ}), such that*

$$a_{IJ}(S) = \begin{cases} 1, & \textit{if } (I \subset S \textit{ and } S \cap J \neq \emptyset) \textit{ or } J \subset S \\ 0, & \textit{else.} \end{cases}$$

First, we have a closer look on players outside of I and J. Particularly, we are interested in an analogon of van den Brink's lemma 4.4.

Lemma 4.7 *Let N be a set of players, $I, J \subset N$ such that a_{IJ} is an apex game on N. Then all players $k \in N \setminus (I \cup J)$ are null players in N.*

Proof: Let $k \in N \setminus (I \cup J)$ and $T \subset N$ such that $k \in T$ and $a_{IJ}(T) = 1$. Then either $J \subset T$ or $I \subset T$ and $T \cap I \neq \emptyset$. Assume that $a_{IJ}(T \setminus \{k\}) = 0$ and consider the first case. Since $k \notin J$, $J \subset T \setminus \{k\}$ and this coalition is winning, what is a contradiction to the assumption. Hence, consider the second case. Since neither $k \in I$ nor $k \in J$, we still have $I \subset T$ and $T \cap I \neq \emptyset$. Hence, $a_{IJ}(T \setminus \{k\}) = 1$, again a contradiction. Our assumption must have been false, i.e. there is no such coalition $T \subset N$ in which k is a veto player. We can conclude, that k is null player.

□

The next lemma investigates the Shapley value of apex players. Its proof is quite technical and can be found in the appendix.

Lemma 4.8 *Let a_{IJ} be an apex game and $S', T' \subset J$ such that $|S'| > |T'|$. Let further $S = S' \cup I$ and $T = T' \cup I$. Then for all $i \in I$*

$$Sh_i(S) > Sh_i(T). \tag{7}$$

Proof: See appendix.

□

We can now proof our main theorem about generalized apex games.

Theorem 4.9 *Let (N, a_{IJ}) be an apex-(I, J)-game.*

1. *If $|J| > |I| + 1$ then the core of (N, a_{IJ}) with respect to the Shapley value is empty.*
2. *If $|J| \leq |I| + 1$ then S is contained in the core of (N, a_{IJ}) with respect to the Shapley if and only if $J \subset S$ and $S \neq N$. In this case semistrict core and strict core are empty.*
3. *If $|J| < |I| + 1$ then the core of (N, a_{IJ}) with respect to the Shapley value conincides with the strong core.*

Proof:

1. First we show, that we can assume without loss of generality that $J = N \setminus \{I\}$. Let $(S, N \setminus S)$ be a core stable partition. Define N' as the subset of N which does not contain any null players. Assume that $S' = N' \cap S$ is not core stable in N'. Then there is a winning coalition $T \subset N$ such that for all $k \in S' \cap T$

$$Sh_k(T) > Sh_k(S')$$

and $Sh_k(T) > 0$ for all $k \in T \setminus S$. Let now $k \in S \cap T$. Then either $k \in S' \cap T$ and consequently $Sh_k(T) > Sh_k(S)$ or $k \in (S \setminus S') \cap T$ leading to $Sh_k(S) = 0 < Sh_k(T)$. This means that T is a deviation of S, a contradiction to S being core stable. Thus, the core of (N', a_{IJ}) is nonempty. So, it is sufficient to show that the core of (N', a_{IJ}) is empty. But we know that $N' = J \cup I$.
Let $J = N \setminus I$. We have to show that there is a deviation for each winning coalition S. The case $S = N$ is not very interesting since $J \subset N$ would be a deviation. So, let $J \subset S \neq N$. All $i \in I \cup S$ are null players in S since they are not contained in any minimal winning coalition in S. Hence,

$Sh_j(S) = Sh_j(J)$ by lemma 4.2. Since this is a minimal winning coalition we have

$$Sh_j(S) = \frac{1}{|J|} < \frac{1}{|I|+1} = Sh_j(I \cup \{j\})$$

for $j \in J$ and $Sh_i(I \cup \{j\}) = \frac{1}{|I|+1} > 0$ for all $i \in I$. Hence, $I \cup \{j\}$ is a deviation.

Let now $S \subset J$. In this case we know due to lemma 4.8 that $Sh_i(S \cup I)$ is strictly increasing for all $i \in I$ with increasing cardinality of $|S|$.

First let $|S| < \frac{1}{2}|J|$. Then $I \cup (J \setminus S)$ is a deviation since $|J \setminus S| > |S|$.

So, let $|S| \geq \frac{1}{2}|J|$. We see that

$$\begin{aligned}
\sum_{j \in S} Sh_j(S \cup I) &= 1 - \sum_{i \in I} Sh_i(S \cup I) = 1 - |I| \left(\frac{1}{|I|} - \frac{|S|!\,(|I|-1)!}{(|S|+|I|)!} \right) \\
&= \binom{|S|+|I|}{|S|}^{-1}
\end{aligned}$$

By the equal treatment property of the Shapley value (see for instance [7]) we see that for all $j \in S$

$$Sh_j(S \cup I) = \frac{1}{|S|} \binom{|S|+|I|}{|I|}^{-1}$$

Since $|I|, |S| \geq 1$ we get

$$\binom{|S|+|I|}{|I|} \geq |S| + |I|$$

Finally we see

$$Sh_j(S \cup I) \leq \frac{1}{|S|} \frac{1}{|S|+|I|} < \frac{4}{|J|^2} < \frac{1}{|J|}$$

since $|S| \geq \frac{1}{2}|J|$, $|I| > 0$ and $|J| > |I| + 1 \geq 2$. Hence, J is a deviation since by the equal treatment property $Sh_j(J) = \frac{1}{|J|}$ for all $j \in J$. At this point we considered all winning coalitions and the proof is complete.

2. We see that the last two cases of the previous proof remain the same as before, hence the only candidate for the core are coalitions which contain J. N is not core stable as before since it can be deviated by J. So, let $J \subset S \neq N$. Again all $i \in I \cup S$ are null players in S. Hence, $Sh_j(S) = \frac{1}{|J|}$. Since $|J| \leq |I| + 1$, for each $j \in J$

$$\max_{T \subset N, I \subset T, j \in T} Sh_j(T) \leq Sh_j(I \cup \{j\}) = \frac{1}{|I|+1} \leq \frac{1}{|J|} = Sh_j(S),$$

thus, S cannot be deviated. However, this equation shows that $S \cup \{j\}$ is a weak deviation of S, hence S cannot be part of the semistrict core.

3. Let $J \subset S \neq N$. We see for the same reason as before

$$\max_{T \subset N, I \subset T, j \in T} Sh_j(T) \leq Sh_j(I \cup \{j\}) = \frac{1}{|I|+1} < \frac{1}{|J|} = Sh_j(J)$$

Hence, S is contained in the strong core.

□

Consider the last two cases of theorem 4.9. In both cases the core is nonempty. However, if $S \subset N$ is a winning coalition which is not minimal and $J \not\subseteq S$ then $Sh_i(S) > \frac{1}{|I|+1} = Sh_i(I \cup \{j\})$ for each $i \in I$ and $j \in S \cap J$. This means that the Shapley value exhibits the strong paradox of smaller coalitions on a_{IJ}, since each minimal winning subset $W \subset S$ must contain I.

4.3 Nash Stability

After the analysis of the core of generalized apex games, it is interesting to investigate their properties regarding noncooperative aspects. Therefore we introduce Nash stability for these games

Definition 4.10 *Let (N, v) be a proper monotonic game and φ be a solution. A partition $\Gamma = (S_1, \ldots, S_k)$ is Nash stable with regard to φ if for all $i \in N$*

$$\varphi_i\left(C^{\Gamma}(i)\right) \geq \varphi_i\left(S_j \cup \{i\}\right)$$

for all $j = 1, \ldots, k$.

We call a partition Γ trivial if $v(S) = 0$ for all coalitions $S \in \Gamma$. Note that in an apex game the grand coalition N is always Nash stable.

Theorem 4.11 *Let a_{IJ} be an apex game on N.*

1. *Let $|J| > |I| + 1$. Then a nontrivial partition Γ is Nash stable if and only if*
   (a) *J is contained in the winning coalition, and*
   (b) *I is not contained in a nonwinning coalition in Γ, and*
   (c) *there are either zero or at least two players $i, k \in I$ which are not contained in the winning coalition.*
2. *Let $|J| \leq |I| + 1$. Then a nontrivial partition Γ is Nash stable if and only if*
   (a) *J is contained in the winning coalition, and*
   (b) *there are either zero or at least two players $i, k \in I$ which are not contained in the winning coalition.*

Proof:

1. Let Γ be a partition with winning coalition S, which satisfies the conditions. First, let $I \subset S$. Each player inside of S would deviate from a winning into a nonwinning coalition and not improve. Each player $k \in N \setminus S$ is null player in N and remains null player in $S \subset \{k\}$. Consequently, he cannot improve. Hence, Γ is Nash stable.
   Let now $I \not\subseteq S$. If $j \in J$ leaves S there is no winning coalition any more since $I \not\subseteq S$. For all $i \in S \setminus J$ the coalition $S \setminus \{i\}$ is still winning. Hence, there is no $i \in S$ which could improve by deviating. Let $k \in N \setminus S$. Either $k \in I$ or k is a null player in N. If k is null player, he cannot improve, since he remains null player in $S \cup \{k\}$. So, let $k \in I$. Then, the only minimal winning coalition of $S \cup \{k\}$ is J since there is $i \in I$ such that $i \in N \setminus (S \cup \{k\})$. Thus, k is null player in the winning coalition, i.e. he could not improve by deviating. Hence, a partition which satisfies these conditions is Nash stable.
   Let now Γ be a Nash stable partition with winning coalition S. Assume that $J \not\subseteq S$. Then $I \subset S$ and for all $j \in J$

$$Sh_j\left(S \cup \{j\}\right) > 0 = Sh_j\left(C^{\Gamma}(j)\right).$$

   This means that the partition cannot be Nash stable, what is a contradiction. Hence, $J \subset S$.
   Assume now that there is T in Γ such that $I \subset T$ and $v(T) = 0$. Then all $k \in T \setminus I$ are null players in N according to lemma 4.7. Hence, for all $j \in J$

$$Sh_j\left(T \cup \{j\}\right) = Sh_j\left(I \cup \{j\}\right) = \frac{1}{|I| + 1} > \frac{1}{|J|} = Sh_j(J).$$

Consequently, Γ cannot be Nash stable. This is a contradiction and we conclude that $I \not\subseteq T$ for all $T \in \Gamma$.
Assume now, that $I \setminus S = \{i\}$. Then, $I \subset S \cup \{i\}$ and for $j \in J$ the coalition $I \cup \{j\} \subset S$ is minimal winning and contains i. Thus, $Sh_i(S \cup \{i\}) > 0 = Sh_i(C^\Gamma(i))$. Since this is a contradiction of Γ being Nash stable we conclude that the third condition must be satisfied as well.

2. Let Γ be a Nash stable partition with winning coalition S. Assume that $J \not\subseteq S$. Then $S = I \cup T$ where all $k \in T$ are null players in N. In this case $Sh_j(S \cup \{j\}) > 0$ for $j \in J$. Hence, Γ cannot be Nash stable, what is a contradiction.
Let now Γ be a partition with winning coalition S and $J \subset S$ which satisfies both conditions. Let $j \in S$. Without loss of generality let there be a coalition T in Γ such that $T \cup \{j\}$ is winning. Then $I \subset T$ and $j \in J$. Particularly, $I \cup \{j\}$ is the only minimal winning coalition in T. Hence,
$$Sh_j(T \cup \{j\}) = \frac{1}{|I|+1} \leq \frac{1}{|J|} = Sh_j(S).$$
Hence, there is no $j \in S$ which would deviate. Let now $k \in N \setminus S$. Without loss of generality we can assume that k is not a null player in N. J is the only minimal winning coalition in $S \cup \{k\}$ since $I \not\subseteq S \cup \{k\}$. This implies that k is null player in $S \cup \{k\}$. Thus, k cannot improve his situation. We conclude that Γ is Nash stable.

$\square$

Although we can give this characterizations of Nash stable partitions it is not fully satisfactory. We can neither say that a nontrivial Nash stable partition is core stable nor the other way around. However, if the core is nonempty there is only one class of core stable partitions which is not Nash stable: The winning coalition must contain J and $I \setminus \{i\}$ for some $i \in I$.

5 Conclusion

We have characterized strict and semistrict core in case of the absence of the strong paradox of smaller coalitions. Unfortunately this condition is not necessary for their existence. However, we characterized a class of games with nonempty core on which the Shapley value exhibits the strong paradox. There remain two tasks: First, a further relaxation of the condition, such that it is a necessary and sufficient condition for the existence of core stable partitions. Second, a characterization of core, strict and semistrict core in this case.

A Proofs

Proof of lemma 4.2 : It is sufficient to show that for all $S \subset N \cup M$, for all $i \in S$ and for all $k \in M$

$$Sh_i(S) = Sh_i(S \cup \{k\}).$$

So, let S, i, and k as described. Since k is a null player we have $\delta_i^S(T) = \delta_i^S(T \cup \{k\})$ for all $T \subset N$ and $i \subset S$. Then

$$\begin{aligned}
Sh_i(S \cup \{k\}) &= \sum_{T \subset S} \frac{(|S| + 1 - |T|)!\,(|T| - 1)!}{(|S| + 1)!} \delta_i^S(T) \\
&\quad + \sum_{T \subset S} \frac{(|S| - |T|)!\,(|T|)!}{(|S| + 1)!} \delta_i^S(T) \\
&= \sum_{T \subset S} \frac{(|S| + 1 - |T|)!\,(|T| - 1)! + (|S| - |T|)!\,(|T|)!}{(|S| + 1)!} \\
&= \sum_{T \subset S} \frac{((|S| + 1 - |T|) + (|T|))\,(|S| - |T|)!\,(|T| - 1)!}{(|S| + 1)!} \delta_i^S(T) \\
&= \sum_{T \subset S} \frac{((|S| + 1))\,(|S| - |T|)!\,(|T| - 1)!}{(|S| + 1)\,|S|!} \delta_i^S(T) \\
&= Sh_i(S)
\end{aligned}$$

□

Proof of lemma 4.8 : We start with the proof of the following lemma.

Lemma A.1 *Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that $m \geq -1$. Then*

$$\sum_{k=1}^{n} \frac{(k + m)!}{(k - 1)!} = \frac{1}{m + 2} \prod_{i=0}^{m+1} (n + i). \tag{8}$$

Proof of lemma A.1 : Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, $m \geq -1$. We prove the lemma via complete induction over n. First, let $n = 0$ obviously both side of equation (8) are 0. So, we need to show the induction step. Let $n \geq 0$ and equation (8) be true for n. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{(k + m)!}{(k - 1)!} &= \sum_{k=1}^{n} \frac{(k + m)!}{(k - 1)!} + \frac{(m + n + 1)!}{n!} \\
&= \frac{1}{m + 2} \prod_{i=0}^{m+1} (n + i) + \frac{(m + n + 1)!}{n!} \\
&= \frac{1}{m + 2} \left(\prod_{i=0}^{m+1} (n + i + 1) - (m + 2) \prod_{i=0}^{m+1} (n + i) \right) + \frac{(m + n + 1)!}{n!} \\
&= \frac{1}{m + 2} \prod_{i=0}^{m+1} (n + i + 1).
\end{aligned}$$

Thus, the lemma holds.

□

We now come to the main part. Let first S be a winning coalition containing I. We start with a calculation of the Shapley value.

$$
\begin{aligned}
Sh_i(S \cup I) &= \sum_{T \subset S \cup I} \frac{(|S|+|I|-|T|)!\,(|T|-1)!}{(|S|+|I|)!}\left(a_{IJ}(T) - a_{IJ}(T \setminus \{i\})\right) \\
&= \sum_{k=|I|+1}^{|S|+|I|} \frac{(|S|+|I|-k)!\,(k-1)!}{(|S|+|I|)!}\binom{|S|}{k-|I|} \\
&= \sum_{k=|I|+1}^{|S|+|I|} \frac{(|S|+|I|-k)!\,(k-1)!}{(|S|+|I|)!}\,\frac{|S|!}{(k-|I|)!\,(|S|-(k-|I|))!} \\
&= \sum_{k=|I|+1}^{|S|+|I|} \frac{|S|!}{(|S|+|I|)!}\,\frac{(k-1)!}{(k-|I|)!} \\
&= \frac{|S|!}{(|S|+|I|)!}\sum_{k=1}^{|S|}\frac{(k+|I|-1)!}{k!}
\end{aligned}
$$

This formula is quite similar to the one of lemma A.1:

$$
\begin{aligned}
\sum_{k=1}^{|S|}\frac{(k+|I|-1)!}{k!} &= \sum_{k=1}^{|S|+1}\frac{(k+|I|-2)!}{(k-1)!} - (|I|-1)! \\
&= \frac{1}{|I|}\prod_{i=0}^{|I|-1}(|S|+1+i) - (|I|-1)! \\
&= \frac{1}{|I|}\frac{(|I|+|S|)!}{|S!|} - (|I|-1)!
\end{aligned}
$$

and this leads to

$$
\begin{aligned}
Sh_i(S \cup I) &= \frac{|S|!}{(|S|+|I|)!}\left(\frac{1}{|I|}\frac{(|I|+|S|)!}{|S!|} - (|I|-1)!\right) \\
&= \frac{1}{|I|} - \frac{|S|!\,(|I|-1)!}{(|S|+|I|)!}
\end{aligned}
$$

The last part of the right side is decreasing with an increasing $|S|$. So, the Shapley value is increasing with the cardinality of $|S|$. This completes the proof since $|I|$ is constant.

□

References

[1] L. Billera, *Some Theorems on the Core of an n-person Game without Side Payments*, SIAM Journal of Applied Mathematics, 18, pp.567-579, 1970

[2] R. van den Brink, *The Apex Power Measure for Directed Networks*, Soc. Choice Welfare, 19, pp. 845-867, 2002

[3] D. Dimitrov, C.-J. Haake, *Government versus Opposition: Who Should be Who in the 16th German Bundestag*, Journal of Economics, 89(2), pp. 115-128, 2006

[4] D. Dimitrov, C.-J. Haake, *A Note on the Paradox of Smaller Coalitions*, Soc. Choice Welfare, 30, pp. 571-579, 2008

[5] S. Hart, M. Kurz, *Stable Coalition Structures*, Coalitions and Collective Actions, Manfred J. Holler (Editor), Physica-Verlag, pp. 235-258, 1984

[6] A. Bogomolnaia, M. Jackson, *The Stability of Hedonic Coalition Structures*, Games and Economic Behaviour, 38(2), pp. 201-230, 2002

[7] B. Peleg, P. Sudhölter, *Introduction of the Theory of Cooperative Games*, Second Edition, Springer, 2007

[8] P. Shenoy, *On Coalition Formation: A Game-theoretical Approach*, Int. J. Game Theory, 8, pp.133-164, 1979