COUNTABLE SPACES AND COMMON PRIORS

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Abstract. We show that the no betting characterisation of the existence of common priors over finite type spaces extends only partially to improper priors in the countably infinite state space context: the existence of a common prior implies the absence of a bounded agreeable bet, and the absence of a common improper prior implies the existence of a bounded agreeable bet. However, a type space that lacks a common prior but has a common improper prior may or may not have a bounded agreeable bet. The iterated expectations characterisation of the existence of common priors extends almost as is, as a sufficient and necessary condition, from finite spaces to countable spaces, but fails to serve as a characterisation of common improper priors. As a side-benefit of the proofs here, we also obtain a constructive proof of the no betting characterisation in finite spaces.

1. Introduction

The common prior assumption (as first introduced in Harsányi (1967-1968)) is taken as an integral assumption in the vast majority of models of incomplete information. It asserts that the beliefs of individuals in different states of the world are the posteriors that they form, after each is given private information, from a prior that is common to them all.

Despite its pervasiveness, the common prior assumption was, and still is, debated and challenged (see Gul (1998) and Aumann (1998)). It has been noted that, in many cases of interest, all that observers have are profiles of posteriors, not priors, and that there are examples of posteriors that could not possibly have been derived from common priors.

Given the importance of the common prior assumption, intense interest has been focussed on fully characterising the existence of a common prior in terms of the posterior profiles — since we are interested in the players at the present time, it is desirable to express the assumption of a common prior in present-time terms only. Aumann (1976) in his

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agreement theorem, gave a necessary condition for the existence of a common prior in terms of present beliefs: if there is a common prior, then it is impossible to agree to disagree, i.e., to have common knowledge of differences in the beliefs of any given event. By extending the notion of disagreement to differences in the expectation of a general random variable, several researchers (Morris (1995a); Feinberg (2000); Samet (1998b)) were able to show that the impossibility of there being common knowledge of disagreement is not only a necessary, but also a sufficient condition for the existence of a common prior. Since this characterisation is based on the criterion of whether or not there exists a bet such that the players take opposite sides of the bet, yet each player ascribes positive expected value to the bet at every state of the world (we will henceforth term such a bet an *agreeable bet*), it is often termed the 'no betting' characterisation. It has also been proved that this characterisation obtains for type spaces over compact, continuous state spaces (see Feinberg (2000) and Heifetz (2006)).

That left open the question of characterising the existence of common priors in type spaces over countable state spaces, a major lacuna given the many models of incomplete information in the game theory and economics literature that involve countable state spaces. That the no betting characterisation cannot be extended 'as is' to countable spaces was shown in Feinberg (2000), which presents an example of a type space over a countable state space that has no common prior, yet also admits no bounded agreeable bet (in fact, even no agreeable bet bounded from only above or from below).

Several researchers, however, noted that the counter-example in Feinberg (2000), and several other counter-examples (see, for example, Simon (2000) and Lehrer and Samet (2011)) admit no common prior, but satisfy the property of having a common *improper* prior. An improper prior for a player is a measure over the state space that may not be normalisable, i.e., the measure of the entire space may be infinite. On encountering this idea for the first time, it may seem strange to consider measures that are not probability measures in the context of then deriving posterior probabilities, but there is a sense in which an improper prior is an entirely natural construction. Consider, for example, the standard uniform probability distribution in finite spaces, which represent the intuitive idea that 'any state is equally likely'. Clearly, there is no equivalent probability distribution over a countable space. The closest thing would be a non-normalisable measure μ that assigns equal weight to each state ω, say $\mu(\omega) = 1$. If now E is a finite partition element and one derives a posterior probability for an element

$\omega \in E$ by defining it to be $\mu(\omega)/\mu(E)$, the result is indistinguishable from the posterior probability that would result if E were a subset of a finite space with the uniform distribution. Extending this idea to arbitrary measures gives the intuition behind improper priors.

Thus, an improper prior does not enable one to consider the probabilities that a player assigns to events at the *ex ante* stage, but it still enables discussion of the relative likelihood that he ascribes to pairs of events, as well as the interim probability assessments of the player — i.e. his types — constitute a disintegration of the improper prior. If one is interested in interim probability assessments and how they may be derived from *ex ante* considerations, without necessarily demanding that a player have a full-fledged *ex ante* probability measure, an improper prior can serve much the same purpose as a proper prior. An improper prior common to all the players is then a common improper prior.

There has for several years been an open conjecture that the no betting characterisation, or a close variant of it (see, for example, Heifetz (2006)) might obtain with respect to common improper priors over countable state spaces. In this paper, we directly address this conjecture, and prove that the absence of a common improper prior is a sufficient condition for the existence of a bounded agreeable bet among players. It is not, however, a necessary condition; we exhibit a simple example of a type spaces over a countable state spaces that has both a common improper prior and a bounded agreeable bet. We also show that the existence of a (proper) common prior is a sufficient condition for the absence of a bounded agreeable bet. These results, along with the example in Feinberg (2000), indicate that the no betting criterion is rather weak in the countable state space case. In particular, the 'intermediate' case of a type space with no common prior but a common improper prior is consistent both with the existence of a bounded agreeable bet and the absence of a bounded agreeable bet, and it is unclear what extra criteria can be used to distinguish between these two sorts of type spaces.

One reason that the study of no betting in countable state spaces remained an open problem for several years was due to the fact that the known proofs of the no betting characterisation, in both the finite case and the compact, continuous case, were all based on one or another variant of the Separation Theorem for convex sets. As Heifetz (2006) notes, this theorem is inapplicable in non-compact type spaces. This dictated seeking a different approach in this paper, and the proof of

Theorem 1(b), which takes up most of Section 4, is entirely combinatorial and constructive. Since the same construction applies in the finite state space case, we have as a side-effect the first constructive proof of the no betting characterisation for finite spaces. In fact, putting together the elements of the proofs in Section 4 essentially yields an algorithm which, given a finite type space, determines whether or not it has a common prior; if it does have a common prior, the algorithm then constructs the common prior; if it does not have a common prior, the algorithm constructs an agreeable bet, thus indicating a random variables about whose expected values the players 'agree to disagree'.

In the last section, Section 5, we turn our attention to studying, in the countable state space context, a second characterisation of the existence of a common prior in finite state spaces, the iterated expectations characterisation introduced in Samet (1998a) (extended to the case of compact, continuous spaces in Hellman (2010)). The main result in Samet (1998a) is that each iterated expectation of a random variable converges, and the value of its limit is common knowledge. Moreover, there exists a common prior if and only if for each random variable it is common knowledge that all its iterated expectations converge to the same value. In that paper, it is pointed out that 'the stochastic analysis of type spaces is finer than the convex analysis used for the nonagreement condition'. It turns out that this statement holds true even more strongly in the countable space context. Whereas, as pointed out above, the no betting criterion fails to extend as a characterisation even of common improper priors in countable type spaces, it is proved in Section 5 that the iterated expectations criterion extends, almost 'as is', to a full characterisation of the existence of a common prior. But the iterated expectations criterion *does* fail as a characterisation of common improper priors, and cannot be used for the goal of identifying common improper priors.

The study of common priors in the context of countable spaces involves a much richer set of concepts than the finite space context. Unlike the finite case, in the infinite one, the non existence of a common prior can have different characteristics that have bearing on the question of consistency. Mapping the relationships between these concepts is an on-going effort, as of this writing. Research studies in this field complementary to this paper include Lehrer and Samet (2010).

2. Preliminaries

2.1. **Knowledge and Belief.**

A *knowledge space* for a nonempty set of *players* I, is a pair $(\Omega, \mathbf{\Pi})$. In this context, Ω is a nonempty set called a *state space*, and $\mathbf{\Pi} = (\Pi_i)_{i \in I}$ is a *partition profile*, where for each $i \in I$, Π_i is a partition of Ω into measurable sets with positive measure. We will assume throughout this paper that every state space Ω is either finite, or countably infinite, and that $|I| = m$, where $m \geq 2$ is a finite integer. Denote by Δ^Ω the set of probability distributions over Ω.

When working with a knowledge space $(\Omega, \mathbf{\Pi})$, an element $\omega \in \Omega$ is typically termed a *state*. For each $\omega \in \Omega$, we denote by $\Pi_i(\omega)$ the element of Π_i containing ω. Π_i is interpreted as the information available to player i; $\Pi_i(\omega)$ is the set of all states that are indistinguishable to i when ω occurs. Player i is said to *know* an event E at ω if $\Pi_i(\omega) \subseteq E$. We define for each i a *knowledge operator* $K_i \colon 2^\Omega \to 2^\Omega$, by $K_i(E) = \{\omega \mid \Pi_i(\omega) \subseteq E\}$. Thus, $K_i(E)$ is the event that i knows E.

A partition Π' is a *refinement* of Π if every element of Π' is a subset of an element of Π. Refinement intuitively describes an increase of knowledge. The *meet* of $\mathbf{\Pi}$, denoted $\wedge\mathbf{\Pi}$, is the partition that is the finest among the partitions that are simultaneously coarser than all the partitions Π_i. $\mathbf{\Pi}$ is called *connected* when $\wedge\mathbf{\Pi} = \{\Omega\}$. (By abuse of notation, when $\mathbf{\Pi}$ is clear from context, we will sometimes say that Ω is connected when the intention is to say that $\mathbf{\Pi}$ is connected).

A *type function* for Π_i is a function $t_i : \Omega \to \Delta^\Omega$ that associates with each state ω a distribution in Δ^Ω, in which case the latter is termed the *type* of i at ω. Each type function t_i further satisfies the following two conditions:

(a) $t_i(\omega)(\Pi_i(\omega)) = 1$, for each $\omega \in \Omega$;
(b) t_i is constant over each element of Π_i.

A *type profile* for $\mathbf{\Pi}$ is a vector of type functions, $\tau = (t_i)_{i \in I}$, where for each i, t_i is a type function for Π_i, which intuitively represents the player's beliefs. A type profile τ is *positive* if $t_i(\omega)(\omega) > 0$ for each i, and each state ω.

By definition of a type function, abusing notation, we may write $t_i(\omega)$ as short-hand for $t_i(\omega)(\omega)$, with the distinction between the intended interpretation of $t_i(\omega)$ as an element of Δ^Ω or as an element of $\mathbb{R}$ clear from context.

A *random variable* f over Ω is any element of $\mathbb{R}^\Omega$. Given a probability measure $\mu \in \Delta^\Omega$ and a random variable f, the *expected value of* f with respect to μ, is

$$E_\mu f := \sum_{\omega \in S} f(\omega)\mu(\omega). \tag{1}$$

For a random variable f, denote by $E_i f$ the element of $\mathbb{R}^\Omega$ defined by

$$E_i f(\omega) := \sum_{\omega' \in \Pi_i(\omega)} t_i(\omega') f(\omega'). \tag{2}$$

We will alternatively also sometimes write $E_i(f|\omega)$ in place of $E_i f(\omega)$, and call this the *interim expected value* player i ascribes to f at ω.

2.2. **Priors.**

If Ω is a countable state spaces, an *improper prior* for a type function t_i is a non-negative and non-zero function $p \colon \Omega \to \mathbb{R}$ such that for each $\pi \in \Pi_i$, $p(\pi) < \infty$ and $p(\pi)t_i(\omega) = p(\omega)$ for all $\omega \in \pi$. Note that although for any $\pi \in \Pi_i$, $p(\pi) < \infty$, the possibility that $p(\Omega) = \infty$ is not ruled out, so that p may not be normalisable.

A *prior* for a type function t_i is a probability distribution $p \in \Delta^\Omega$ such that for each $\pi \in \Pi_i$, and $\omega \in \pi$, the equation $p(\pi)t_i(\omega) = p(\omega)$ is satisfied. Obviously, a prior is in particular an improper prior (normalising if necessary in the finite case), so that when we use the term 'improper prior' we will mean both concepts, but the term 'prior' alone will mean a normalisable prior.

An improper prior does not allow us to talk about the 'probabilities' that player i assigns to events at the *ex ante* stage, but it still allows us to discuss the relative likelihood that he ascribes to pairs of events; and the interim probability assessments of the player i.e., his types constitute a disintegration of the improper prior.

A *common improper prior* for a type profile τ is a vector $p \in \mathbb{R}^\Omega$ which is an improper prior for each player i.[1] A type profile τ is called *consistent* (the term is due to Harsányi) when it has a common prior.

Note also that if p is a common improper prior, then for any constant $\gamma > 0$, γp is also a common improper prior. In particular, if p is a common improper prior and $p(\Omega) < \infty$ then $[p(\Omega)]^{-1} p$ is a common prior. Thus, for a finite space, a type profile has a common prior if and only if it has a common improper prior. In light of this, we extend the definition of consistency to countable spaces: a type profile τ is *consistent* when it has a common improper prior and *inconsistent* otherwise.

[1] Contrasting a prior for t_i with the types $t_i(\omega, \cdot)$, the latter are referred to as the posterior probabilities of i.

2.3. **Type Matrices and Common Priors.**

Samet (1998a) introduced a matrix-based approach to the analysis of common priors that is convenient for the results of this paper. Although that paper restricts attention to finite state spaces, the matrix definitions extend readily to countable state spaces.

We will denote by $\mathbf{I}$ the square identity matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

trusting that the size of this matrix in each case will be clear by context (including cases in which the matrix is countably infinite).

For a type space τ, over either finite or countable Ω, define for each player i a *type matrix* M_i in $\mathbb{R}^{\Omega^2}$, by $M_i(\omega, \omega') = t_i(\omega)(\{\omega'\})$. M_i is a Markov matrix representing the transition function t_i. Each type matrix M_i satisfies the following properties:

- M_i is idempotent, i.e., $M_i M_i = M_i$. This follows from the fact that the multiplication of the j-th row of M_i with the k-th column of M_i is equal to the multiplication of the j-th row by the constant $t_i(\omega_j)(\{\omega_k\})$. But the j-th row sums to 1.
- A vector p such that for each $\pi \in \Pi_i$, $p(\pi) < \infty$, is an improper prior for i if and only if $pM_i = p$. This follows from the fact that the condition that $p(\omega_j)$ equals p times the j-th column of M_i is equivalent to the condition $p(\pi) t_i(\omega_j)(\pi) = p(\omega_j)$.
- For each random variable f, $M_i f = E_i f$. We will therefore use the notation $E_i f$ and $M_i f$ interchangeably.

Furthermore, for any $q \in \mathbb{R}^{\Omega}$, if $p = qM_i$ is a well-defined element $p \in \mathbb{R}^{\Omega}$, then p is an improper prior for i. To see this, using the idempotence of M_i, note that $pM_i = qM_iM_i = qM_i = p$.

Given a permutation σ of I, denote

$$M_\sigma := M_{\sigma_1} \dots M_{\sigma_n} = t_{\sigma_1} \dots t_{\sigma_n}$$

and term this a *permutation matrix*.

For every $P \in \wedge \mathbf{\Pi}$, and every i, denote by M_i^P the matrix M_i restricted to P, and similarly denote by M_σ^P the matrix M_σ restricted to P

Given the identification of $E_i f$ with $M_i f$, we can write, given a permutation σ of I,

$$E_\sigma := E_{\sigma_1} \dots E_{\sigma_n} = M_\sigma.$$

The *iterated expectation of f with respect to σ* is the sequence $\{E^k_\sigma f\}_{k=1}^\infty$. Samet (1998a) proves that in the finite state space case, p is a common prior if and only if p is an invariant probability measure of the Markov matrix M_σ for each permutation σ. Furthermore, if the knowledge space is connected, there exists a common prior if and only if for each random variable f, the iterated expectations of f, with respect to all permutations σ, converge to the same limit, and if p is the common prior, then this limit is pf. The analogues of these results are explored in this paper in Section 5.

2.4. **Common Knowledge.**

An event $E \subseteq \Omega$ is *self-evident* if for all $\omega \in E$ and each $i \in I$

$$\Pi_i(\omega) \subseteq E. \tag{3}$$

In particular, every element of the meet, $M \in \wedge \mathbf{\Pi}$, is self-evident.

An event E is *common knowledge at* $\omega \in \Omega$ if and only if there exists a self-evident event $F \ni \omega$ such that for all $i \in I$

$$F \subseteq K_i(E). \tag{4}$$

In fact, the element of the meet containing ω is also known as the *common knowledge component* of ω, because it is the smallest self-evident set containing ω.

Working with a connected space is thus particularly convenient for theorems involving common knowledge, because if $\mathbf{\Pi}$ is connected, then an event E is common knowledge at ω if and only if $E = \Omega$.

2.5. **Characterisation of the Existence of Common Priors.**

We adopt the notation that for vectors $x_1, x_2 \in \mathbb{R}^\Omega$, $x_1 > x_2$ means that $x_1(\omega) > x_2(\omega)$ for all $\omega \in \Omega$, and $x_1 > 0$ means that $x_1(\omega) > 0$ for all ω. $x_1 \geq x_2$ will be interpreted to mean that $x_1(\omega) \geq x_2(\omega)$ for all $\omega \in \Omega$, and there is at least one ω^* such that $x_1(\omega^*) = x_2(\omega^*)$.

Definition 1. Given a two-player type space τ, a random variable $f \in \mathbb{R}^\Omega$ is an *agreeable bet* if $E_1 f > 0 > E_2 f$.

An m-tuple of random variables $\{f_1, \dots, f_m\}$ is a *bet* if $\sum_{i=1}^m f_i = 0$. Given an m-player type space τ, a bet f is an *agreeable bet* if $E_i f_i > 0$ for all i. ♦

In the two-player case, $f \in \mathbb{R}^\Omega$, it is an agreeable bet by the first definition if and only if $\{f, -f\}$ is an agreeable bet by the second definition.

We can also slightly tweak the definition, replacing strict inequality with weak inequality, to obtain:

Definition 2. Given an m-player type space τ, an m-tuple of random variables $\{f_1, \ldots, f_m\}$ is a *weakly agreeable bet* if $\sum_{i=1}^m f_i = 0$, and $E_i f_i \geq 0$ for all i, with $E_j f_j > 0$ for at least one $j \in I$. ♦

We will say that an agreeable bet $\{f_1, \ldots, f_m\}$ is bounded if $|f_i|$ is bounded for all $i \in I$. In addition, given two sequences of r.v. $f = \{f_1, \ldots, f_m\}$ and $g = \{g_1, \ldots, g_m\}$, we define $f + g := \{f_1 + g_1, \ldots, g_1 + g_m\}$

The characterisation of the existence of common priors in finite spaces is accomplished by:

> A finite type space τ has a common prior if and only if there is no agreeable bet.

The functions f_i, which sum to zero, can be interpreted as a bet between the players. The condition $E_i(f_i|\omega) > 0$, for each state ω, amounts to saying that the positivity of $E_i f_i$ is always common knowledge amongst the players. Thus, a necessary and sufficient condition for the existence of a common prior is that there is no bet for which it is always common knowledge that all players expect a positive gain. This establishes a fundamental, and remarkable, two-way connection between posteriors and priors.

The most accessible proof of this result is in Samet (1998b). It was proved by Morris (1995a) for finite type spaces and independently by Feinberg (2000) for compact type spaces. Bonanno and Nehring (1999) proved it for finite type spaces with two agents.

3. Main Results

3.1. Agreeable Betting.

Theorem 1. *Let τ be a type space over $\{\Omega, \mathbf{\Pi}\}$, where Ω is countable.*

(a) *If τ has a common prior, then there is no bounded weakly agreeable bet relative to τ.*
(b) *If τ has no common improper prior, then there exists a bounded agreeable bet relative to τ.*

Claim 1. *Let τ be a type space over $\{\Omega, \mathbf{\Pi}\}$, where Ω is countable. If the set of priors of at least one player is compact, then τ has a common prior if and only if there is no bounded agreeable bet relative to τ.*

Corollary 1. *Let τ be a type space over $\{\Omega, \mathbf{\Pi}\}$, where Ω is countable. If the partition Π_j of at least one player j is finite, then τ has a common prior if and only if there is no bounded agreeable bet relative to τ.*

3.2. **Counterexamples.**

We show here by counter-examples that the converses to the statements in Theorem 1 do not obtain. We also include an example showing that the statement in Theorem 1(a) does not hold for unbounded bets.

The first example shows that the converse to Theorem 1(b) does not obtain, by exhibiting a type space with both a common improper prior and a bounded agreeable bet.

Example 1. The state space is $\Omega = \{\omega_0, \omega_1, \ldots\}$. There are two players, Anne and Ben. Anne's knowledge partition, Π_A, is given by

$$\{\{0\}, \{1,2\}, \{3,4\}, \{5,6\}, \ldots\}.$$

Ben's partition, Π_B, is given by

$$\{\{0,1\}, \{2,3\}, \{4,5\}, \ldots\}.$$

Anne's type function, t_A, is given by $t_A(\omega_0, \omega_0) = 1$, and $t_A(\omega_n, \omega_n) = 0.5$ for all $n \geq 1$. Ben's type function, t_B, is given by $t_B(\omega_n, \omega_n) = 0.5$ for all $n \geq 0$. See Figure 1.

Anne	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\ldots$
Ben	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\ldots$

Figure 1: The partition profile of Example 1.

Let $p(\omega) = 1$ for all $\omega \in \Omega$. Then p is a common improper prior.

Define a bounded random variable f as follows:

$$f(\omega_n) = \begin{cases} 1 & \text{if } n = 0 \\ 1 + \sum_{i=1}^{n} \frac{1}{2^i} & \text{if } n \text{ is even} \\ -(1 + \sum_{i=1}^{n} \frac{1}{2^i}) & \text{if } n \text{ is odd} \end{cases}$$

Then $E_A(f|\omega) > 0 > E_B(f|\omega)$ for all $\omega \in \Omega$, hence $\{f, -f\}$ is a bounded agreeable bet. ♦

Example 1 exhibits a type space with a common improper prior and no common prior, but it cannot serve as a counterexample to Theorem 1(a) because of the existence of a bounded agreeable bet. The next example however, exhibits a type space with neither a common prior nor a bounded agreeable bet. The example is taken from Feinberg (2000), and is included here for completeness.

Example 2. The state space is $\Omega = \{\omega_1, \omega_2, \ldots\}$. There are two players, Alice and Bob. Alice's knowledge partition, Π_A, is given by

$$\{\{1\}, \{2,3\}, \{4,5\}, \{6,7\}, \ldots\}.$$

Bob's partition, Π_B, is given by

$$\{\{1,2\}, \{3,4\}, \{5,6\}, \ldots\}.$$

Bob's type function, t_B, is given by $t_B(\omega_n, \omega_n) = 0.5$ for all $n \geq 1$.

Anne's type function, t_A, is given by

$$t_A(\omega_n, \omega_n) = \begin{cases} 1 & \text{if } n = 1 \\ 2/3 & \text{if } n = 2^{k+1} - 2 \text{ for } k > 0 \\ 1/3 & \text{if } n = 2^{k+1} - 1 \text{ for } k > 0 \\ 1/2 & \text{otherwise} \end{cases}$$

See Figure 2.

Anne	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	...
Ben	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	...

Figure 2: The partition profile of Example 2.

Let p be a candidate for being a common improper prior; by the structure of the partition profile, the value of $p(\omega_1)$ determines the value of $p(\omega_n)$ for all n. It cannot be the case that $p(\omega_1) = 0$, because then $p(\omega_n) = 0$ for all n. There must therefore be a real number $\alpha > 0$ such that $p(\omega_1) = \alpha$. This then implies that $p(\omega_2) = \alpha$, $p(\omega_3) = p(\omega_4) = p(\omega_5) = p(\omega_6) = \alpha/2$, $p(\omega_7) = p(\omega_8) = \ldots = p(14) = \alpha/2$, and so on. It follows that for any positive α, setting $p(\omega_1) := \alpha$ determines a common improper prior p; it cannot, however, be a common prior, because $\sum_{n=1}^{\infty} p(\omega_n) = \infty$.

There is no agreeable bet in this example. Let f be a candidate for a random variable satisfying $E_A(f|\omega) > 0 > E_B(f|\omega)$ for all $\omega \in \Omega$. Then $f(\omega_1) = E_A(f|\omega_1) > 0$, $f(\omega_2) = 2E_B(f|\omega_1) - f(\omega_1) < -f(\omega_1)$, $f(\omega_3) = 3E_A(f|\omega_2) - 2f(\omega_2) > -2f(2)$. It emerges that $f(n)$ is an alternating sequence whose absolute value tends to infinity. ♦

Examples 1 and 2 show that the no betting criterion is insufficiently subtle to be used as a tool for determining when a type space has a common improper prior but no common prior, as both examples satisfy that property, but one has a bounded agreeable bet and the other does not.

Finally, here is an example showing that the statement in Theorem 1(a) does not hold for unbounded bets.

Example 3. The state space is $\Omega = \{1, 2, \ldots\}$. There are two players, Anne and Ben. Anne's knowledge partition, Π_A, is given by

$$\{\{1\}, \{2,3\}, \{4,5\}, \{6,7\}, \ldots\}.$$

Ben's partition, Π_B, is given by

$$\{\{1,2\}, \{3,4\}, \{5,6\}, \ldots\}.$$

Anne's type function, t_A, is given by

$$t_A(n,n) = \begin{cases} 1 & \text{if } n = 1 \\ \frac{2}{3} & \text{if } n \text{ is even} \\ \frac{1}{3} & \text{if } n \text{ is odd, } n > 1 \end{cases}$$

Ben's type function, t_B, is given by

$$t_B(n,n) = \begin{cases} \frac{2}{3} & \text{if } n \text{ is odd} \\ \frac{1}{3} & \text{if } n \text{ is even} \end{cases}$$

Anne	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	...
Ben	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	...

Figure 3: The partition profile of Example 3.

Let $p(n) = 2^{-n}$ for all n. Then p is a common prior.

Fix $\varepsilon > 0$. Define an unbounded random variable f as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 1 \\ -(2f(n-1) + \varepsilon) & \text{if } n \text{ is even} \\ 2f(n-1) + \varepsilon & \text{if } n \text{ is odd, } n > 1 \end{cases}$$

Then $E_A f > 0 > E_B f$, hence $\{f, -f\}$ is an unbounded agreeable bet. ⧫

3.3. **The Iterated Expectations Criterion.**

In contrast to the no betting criterion, which fails to characterise either common priors or common improper priors in countable spaces, the iterated expectations criterion of Samet (1998a) does extend to a full characterisation of the existence of common priors.

Theorem 2. *Let τ be a positive type space, and suppose that $\wedge\mathbf{\Pi} = \{\Omega\}$. Then τ has a common prior if and only if for each bounded non-zero random variable $f \geq 0$, the iterated expectations of f, with respect to all permutations σ, converge to the same non-zero limit. Moreover, if p is the common prior, then this limit is pf.*

4. Acceptable Bets and Countable Spaces

First note that for the proof of Theorem 1, it suffices to prove the two statements of when the knowledge space $\mathbf{\Pi}$ is connected. For part (a), if there exists a common prior p over a non-connected $\mathbf{\Pi}$, we can decompose Ω into disjoint connected components, $\Omega = \sum_j S_j$, and then p restricted to each S_j is a common prior; if there is no acceptable bet over S_j, then there can be no acceptable bet over Ω. For part (b), again decompose Ω disjointly as $\Omega = \sum_j S_j$. If there is no common improper prior over Ω, then there can be no *cip* over each S_j (otherwise that *cip* could serve as a *cip* for all of Ω), and if we show the existence of a bounded agreeable bet f_{S_j} over each S_j, then the function $f(\omega) := f_{S_j}(\omega)$ for $\omega \in S_j$ is a bounded agreeable bet over Ω.

We will therefore assume that for all type spaces τ over $\{\Omega, \mathbf{\Pi}\}$ in this section, $\mathbf{\Pi}$ is connected.

The proof of Theorem 1(a) is a straightforward extension of the proof of the same statement in the finite state space case.

Proof of Theorem 1(a). Suppose by contradiction that there exists a bounded weakly agreeable bet f. First note that for any $j \in I$,

$$E_p(f_j) = \sum_{\omega' \in \Omega} f_j(\omega')p(\omega')$$

This quantity is well-defined given the assumptions that f is bounded, and that p is a (proper) common prior.

Next, as p is a prior for each $j \in I$, it follows that $E_p(f_j) = E_p(E_j f_j)$, where $E_j f_j$ is regarded as a function from Ω to $\mathbb{R}$.

As f is a bet, $\sum_{i \in I} f_i = 0$. Hence

$$0 = E_p\left(\sum_i f_i\right) = \sum_i E_p f_i = \sum_i E_p(E_i f_i). \tag{5}$$

But by the assumption that f is a weakly agreeable bet, there is at least one player i such that $E_i f_i > 0$, in which case $E_p E_i f_i > 0$, hence $\sum_i E_p(E_i f_i) > 0$, contradicting Equation 5. ∎

Proof of Claim 1. For each $i \in I$, denote the set of all priors of player i by P_i. Further denote $P := \times_{i\in I} P_i$, and let D denote the diagonal, i.e., the set of all vectors $(p, \ldots, p)$ of length $m = |I|$ such that $p \in \Delta(\Omega)$. Clearly, τ has a common prior if and only if $P \cap D \neq \emptyset$. Suppose that P_i is compact, for some $i \in I$.

Suppose next that $P \cap D = \emptyset$, but that the distance between P and D (in any appropriate metric) is 0, i.e., P and D are not strictly separated. Then there exists a sequence $\overline{p}_1, \overline{p}_2, \ldots$, where for each t, $\overline{p}_t = (p_t^1, \ldots, p_t^m) \in P$, and there exists a sequence of probabilities $q_1, q_2, \ldots$ such that $\|\overline{p}_n - d_n\| \to 0$, where $d_n = (q_n, \ldots q_n) \in D$.

Then w.l.o.g. there exists a sequence $q_1^i, q_2^i \ldots$ of elements of P_i such that $\|q_n^i - q_n\| \to 0$. By the assumption that P_i is compact, there is a point $q \in P_i$ such that both q_n^i and q_n converge to q. We can also show that $q \in P_j$ for all j: for each player j, there is a sequence $q_1^j, q_2^j \ldots$ of elements of P_j such that $\|q_n^j - q_n\| \to 0$. Since $q_n \to q$, it follows that $q_n^j \to q$. As P_j is closed, $q \in P_j$.

We therefore deduce that there does not exist common prior if and only if P and D can be strictly separated. The rest of the proof then proceeds as in the analogous case in the finite state space setting, as presented in Samet (1998b). ∎

4.1. **Chains.**

Following a concept introduced in Hellman and Samet (2010), we have the following definition:

Definition 3. A *chain* of length $n \geq 0$, for a partition profile $\mathbf{\Pi}$, from one state to another, is defined by induction on n. A state ω_0 is a chain of length 0 from ω_0 to ω_0. A chain of length $n + 1$, from ω_0 to ω, is a sequence $c \xrightarrow{i} \omega$, where c is a chain of length n from ω_0 to ω', and $\omega \in \Pi_i(\omega')$. Thus, a chain of positive length n is a sequence $c = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n$, such that $\omega_{s+1} \in \Pi_{i_s}(\omega_s)$ for $s = 0, \ldots, n-1$. ♦

We write $\omega \to \omega'$ when there is a chain from ω to ω', in which case we say that ω and ω' are connected by a chain. The binary relation $\to$ is the transitive closure of the union of the relations $\xrightarrow{i}$, and it is an equivalence relation.

Definition 4. Given a chain $c = \omega_0 \overset{i_0}{\to} \omega_1 \overset{i_1}{\to} \cdots \overset{i_{n-1}}{\to} \omega_n$, its *reverse chain* c^{-1} is defined as

$$c^{-1} := \omega_n \overset{i_{n-1}}{\to} \omega_{n-1} \overset{i_{n-2}}{\to} \cdots \overset{i_0}{\to} \omega_0.$$

A chain c is *alternating* if no two consecutive states ω_s and ω_{s+1} in c are the same, and no two consecutive agents i_s and i_{s+1} in c are the same.

♦

Hellman and Samet (2010) prove that a partition profile $\mathbf{\Pi}$ is connected if and only if every two states are connected by at least one chain.

4.2. **Positive, Zero, and Singular States.**

Towards the aim of proving Theorem 1(b), we introduce here a categorisation of states in Ω, relative to a type profile τ, which will be needed for the proofs.

Definition 5. Given a type profile τ, a state $\omega \in \Omega$ is:

- *positive* if $t_i(\omega) > 0$ for all $i \in I$;
- *zero* if $t_i(\omega) = 0$ for all $i \in I$;
- *singular* if it is neither positive nor zero. ♦

Based on the categorisation of states in Definition 5, define the following:

Definition 6. Given a type profile τ,

- A subset $S \subseteq \Omega$ is *i-positive* if $t_i(\omega) > 0$ for all $\omega \in S$.
- A subset $S \subseteq \Omega$ is *positive* if it is i-positive for all i (equivalently, if every $\omega \in S$ is a positive state). A chain c satisfying the condition that every element $\omega \in c$ is a positive state is a *positive chain.*
- A subset $S \subseteq \Omega$ *i-non-singular* if $t_i(\omega) = 0$ for every singular $\omega \in S$.
- A subset $S \subseteq \Omega$ is *non-singularly positive* if it is positive, and every maximal chain c entirely contained in S satisfies the property that for every $\omega \in c$ and every $i \in I$, $\pi_i(\omega)$ is i-non-singular. ♦

A subset S of Ω is thus non-singular positive if it is positive, and for every $\omega \in S$, and every $i \in I$, every $\omega' \in \Pi_i(\omega)$ satisfies the condition that either

- $\omega' \in S$, or
- ω' is a zero state, or
- ω' is a singular state such that $t_i(\omega') = 0$.

Note that it is immediate by definition that if Ω is a positive state space, then trivially the entire space Ω is non-singularly positive.

The distinctions made in Definition 6 are illustrated in the following examples.

Example 4. Let $\Omega = \{\omega_1, \omega_2\}$, $\Pi_1 = \Pi_2 = \Omega$, and let τ_1 be $\{\{1, 0\}\}$ and τ_2 be $\{\{0, 1\}\}$.

Ω has no positive subset, because all the states in Ω are singular. Clearly, this partition profile also can have no common prior. ♦

Example 5. $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$, and let τ_1 be given by

$$\tau_1 = \{\{1\}, \{1/2, 1/2, \}, \{1/2, 1/2\}, \{1\}\},$$

and τ_2 be given by

$$\tau_2 = \{\{1/2, 1/2\}, \{1/2, 1/2\}, \{0, 1\}\}$$

(with the structures of Π_1 and Π_2 clear from the structures of τ_1 and τ_2). See Figure 4.

$$\begin{array}{c|c|cc|cc|c|}
\tau_1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\
\hline
\tau_2 & \frac{1}{2} & \frac{1}{2} \,| & \frac{1}{2} & \frac{1}{2} \,| & 0 & 1
\end{array}$$

Figure 4: The partition profile of Example 5.

The subset $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ is positive, but it is not non-singularly positive, because $\omega_5 \in \Pi_1(\omega_4)$, but ω_5 is a singular state, and $\tau_1(\omega_5) = \frac{1}{2} > 0$. Ω, however, does have a non-singularly positive subset: the subset $\{\omega_6\}$ meets the conditions listed in Definition 6 for a positive subset. This partition profile has a common prior: $\{0, 0, 0, 0, 0, 1\}$ ♦

Example 6. $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7\}$, and let τ_1 be given as

$$\tau_1 = \{\{0, 1/2, 1/2\}, \{1/2, 1/2\}, \{0, 1\}\},$$

and τ_2 be given as

$$\tau_2 = \{\{1, 0\}, \{1/2, 1/2\}, \{1/2, 1/2, 0\}\}$$

(with the structures of Π_1 and Π_2 clear from the structures of τ_1 and τ_2). See Figure 5.

$$\begin{array}{c|ccc|cc|cc|}
\tau_1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 1 \\
\hline
\tau_2 & 1 & 0 \,| & \frac{1}{2} & \frac{1}{2} \,| & \frac{1}{2} & \frac{1}{2} & 0 \,|
\end{array}$$

Figure 5: The partition profile of Example 6.

The subset $S = \{\omega_3, \omega_4, \omega_5\}$ is positive (as is any subset of it), but Ω has no non-singularly positive subset. For example, S is not non-singularly positive, because $\omega_2 \in \Pi_1(\omega_3)$, but ω_2 is a singular state, and $t_1(\omega_2) = \frac{1}{2} > 0$. A similar analysis can be conducted on each subset of Ω to show that it is not non-singularly positive. This partition profile has no common prior. ♦

Lemma 1. *If S is a non-singularly positive subset of Ω, then for any $\omega \in S$, every $\omega' \in \Omega$ that is connected to ω via a positive chain is also an element of S. It follows that every non-singularly positive subset S can be decomposed as $S = \cup T_j$, where each T_j is a non-singularly positive subset such that all of the members of T_j are connected to each other by positive chains.*

The proof is in the appendix.

4.3. **Type Ratios.**

The proofs of the propositions in this subsection, which are mainly technical, are located in the appendix.

Definition 7. Let τ be a type profile and (ω_1, ω_2) an ordered pair of positive states in $\pi \in \Pi_i$. The *type ratio* of (ω_1, ω_2) relative to i is[2] $\mathrm{tr}^i_\tau(\omega_1, \omega_2) = t_i(\pi, \omega_1)/t_i(\pi, \omega_2)$. If a chain $c = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n$ of length $n > 0$ is a positive chain, the *type ratio* of c is $\mathrm{tr}_\tau(c) = \times_{k=0}^{n-1} \mathrm{tr}^{i_k}_\tau(\omega_k, \omega_{k+1})$. For a positive chain c of length 0, $\mathrm{tr}_\tau(c) = 1$. Thus, if $c = c' \xrightarrow{i} \omega$ where c' is a positive chain from ω_0 to ω' and ω' is a positive state, $\mathrm{tr}_\tau(c) = \mathrm{tr}_\tau(c')\mathrm{tr}^i_\tau(\omega', \omega)$.[3] ♦

We note here for later use two equalities involving type ratios that follow immediately from the definitions:

- For any chain c,

$$\mathrm{tr}(c^{-1}) = [\mathrm{tr}(c)]^{-1}, \tag{6}$$

- If $c = \omega_1 \xrightarrow{i} \omega_2 \xrightarrow{i} \omega_3$ (i.e., $\omega_2, \omega_3 \in \Pi_i(\omega_1)$), then

$$\mathrm{tr}(c) = \mathrm{tr}^i(\omega_1, \omega_3). \tag{7}$$

---

[2] The type ratio defined in Hellman and Samet (2010) is the inverse of the one defined here, i.e., there $\mathrm{tr}^i_t(\omega_1, \omega_2) = t_i(\pi, \omega_2)/t_i(\pi, \omega_1)$. Which definition is used is immaterial, as long as one keeps to it consistently in an exposition. The definition chosen here is more convenient for the equations developed in this paper.

[3] When we discuss only one type profile we omit the subscript τ in tr_τ.

The following proposition extends the results of a proposition appearing in Hellman and Samet (2010), from positive type spaces to general type spaces.

Proposition 1. *Let τ be a type profile over a connected knowledge space. Then there exists a common improper prior for τ if and only if Ω has a non-singularly positive subspace S with respect to τ, and for each ω_0 and ω in S, and chains c and c' entirely contained in S from ω_0 to ω,* $\mathrm{tr}_\tau(c) = \mathrm{tr}_\tau(c')$.

Definition 8. A chain $c = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n$, where $\omega_{s+1} \in \Pi_{i_s}(\omega_s)$ for $s = 0, \ldots, n-1$, is a *cycle*[4] if $\omega_n = \omega_0$. If with respect to a cycle c of length n there is a pair $s, s' \in \{0, \ldots, n-1\}$ such that $s' > s+1$ and $\omega_{s'} \in \Pi_{i_s}(\omega_s)$, then we say that c has a *self-crossing point* at $\omega_{s'}$. A cycle c is a *non-crossing* cycle if it is alternating, and has no self-crossing points, i.e., for every pair $s, s' \in \{0, \ldots, n-1\}$ such that $s' > s + 1$, $\omega_{s'} \notin \Pi_{i_s}(\omega_s)$. ♦

Definition 8 leads to an immediate corollary of Proposition 1:

Corollary 2. *Let τ be a type profile over a connected knowledge space. Then there exists a common improper prior for τ if and only if Ω has a non-singularly positive subspace S with respect to τ, and for each ω in S, every cycle $\overline{c} = \omega \to \omega$ that is entirely contained in S satisfies* $\mathrm{tr}(\overline{c}) = 1$.

In fact, we can do even better, and show that it suffices to check the type ratios only of non-crossing cycles, instead of all cycles:

Proposition 2. *Let τ be a type profile over a connected knowledge space. Then there exists a common improper prior for τ if and only if Ω has a non-singularly positive subspace S with respect to τ, and every non-crossing cycle $\overline{c}$ that is entirely contained in S satisfies* $\mathrm{tr}(\overline{c}) = 1$.

4.4. **No Common Improper Prior Implies Existence of a Bounded Agreeable Bet.**

Definition 9. Let τ be a type space over $(\Omega, \mathbf{\Pi})$, and let $X \subseteq \Omega$ be a subset of Ω. Define $\mathbf{\Pi}$ *restricted to* X, denoted $\mathbf{\Pi}^X$, to be the partition profile over X given by $\Pi_i^X(\omega) := \Pi_i(\omega) \cap X$ for any state ω. Further, let τ^X, a type function τ *restricted to* X, to be any type function over $(X, \mathbf{\Pi}^X)$ that satisfies the property that for any $\omega \in \Omega$, $t_i(\omega)(\Pi_i^X) t_i^X(\omega) = t_i(\omega)$.

[4] Cf. a similar definition in Rodrigues-Neto (2009).

In the special case in which X is a positive subset of Ω, τ^X is explicitly given by:

$$\tau_i^X(\omega) := \frac{\tau_i(\omega)}{\tau_i(\Pi_i^X(\omega))}$$

for any $\omega \in X$. ♦

Intuitively, Π_i^X is the partition of X derived from the partition Π_i of Ω by 'ignoring all states outside of X'. Then, $\tau_i^X(\omega)$, for each state $\omega \in X$, is $\tau_i(\omega)$ scaled relative to the other states in $\Pi_i^X(\omega)$ so that $\sum_{\omega \in X} \tau_i^X(\omega) = 1$.

For a random variable f, denote

$$E_i^X(f|\omega) := \sum_{\omega' \in \Pi_i^X(\omega)} t_i^X(\omega') f(\omega').$$

A set of r.v. $f = \{f_1, \ldots, f_m\}$ is an agreeable bet relative to τ^X if for all $\omega \in X$, $\sum_i f_i(\omega) = 0$, and $E_i^X(f|\omega) > 0$ for all $i \in I$.

Note that it follows from the definitions that if (ω_1, ω_2) is an ordered pair of positive states in $\pi \in \Pi_i^X$ that

$$\text{(8)} \qquad \operatorname{tr}^i_{\tau^X}(\omega_1, \omega_2) = \frac{t_i^X(\omega_1)}{t_i^X(\omega_2)} = \frac{t_i(\omega_1)}{\tau_i(\pi)} \frac{\tau_i(\pi)}{t_i(\omega_2)} = \frac{t_i(\omega_1)}{t_i(\omega_2)} = \operatorname{tr}^i_\tau(\omega_1, \omega_2),$$

from which it further immediately follows that for any chain c of τ whose elements are entirely contained in X,

$$\text{(9)} \qquad \operatorname{tr}_{\tau^X}(c) = \operatorname{tr}_\tau(c).$$

We need one more definition.

Definition 10. Let τ be a type space over $(\Omega, \mathbf{\Pi})$, let $X \subseteq \Omega$ be a positive subset of Ω, and let f be a random variable. A state $\omega \in X$ is a *surplus state for player* i relative to f and τ^X if $E_i^X(f|\omega) > 0$. In the context of a sequence $f = \{f_1, \ldots, f_m\}$ of r.v., we will say that ω is an *i-surplus state* if f_i is a surplus state for player i.

Proposition 3. *Let τ be a type space over $(\Omega, \mathbf{\Pi})$, let S be a finite connected subset of positive states in Ω, and let $X \subseteq S$. Suppose that there exists an agreeable bet relative to τ^X. Then there exists an agreeable bet relative to τ^S.*

Proof. Let f be an agreeable bet relative to τ^X. If $X = S$, there is nothing to prove. If $X \subset S$, then by the assumption of the connectedness of S, we can find at least one player i and a point $\omega' \notin X$ such that $\Pi_i(\omega') \cap X \neq \emptyset$. By the assumption of positivity, $t_i(\omega') > 0$, and by the assumption that f is an agreeable bet, every state $\omega \in \Pi_i$ is an i-surplus state relative to f.

Denote $Y := X \cup \omega'$, and let ε be the (by the i-surplus state assumption) positive value

$$\varepsilon := \sum_{\omega'' \in \Pi_i(\omega') \cap X} f_i(\omega'') t_i^X(\omega''). \tag{10}$$

Next, let $\overline{f}_i(\omega')$ be a negative real number satisfying

$$0 > \overline{f}_i(\omega') > \frac{-(1 - t_i^Y(\omega'))}{t_i^Y(\omega')} \varepsilon \ , \tag{11}$$

and for $j \neq i$, set $\overline{f}_j(\omega') := -\overline{f}_i(\omega')/(m-1) > 0$, where $m = |I|$. Clearly, by construction, $\sum_{j \in I} \overline{f}_j(\omega') = 0$. Complete the definition of $\overline{f}$ by letting $\overline{f}(\omega'') := f(\omega'')$ for all $\omega'' \in X$.

It is straightforward to check that $\overline{f}$ is an agreeable bet relative to τ^Y. Now simply repeat this procedure as often as necessary to extend the agreeable bet to every state in the finite set S. ∎

Lemma 2. *Let τ be a type space over $(\Omega, \mathbf{\Pi})$, and let X be a non-crossing cycle such that* $\mathrm{tr}(X) \neq 1$. *Then there exists a random variable f that is an agreeable bet relative to τ^X.*

Proof. Write the non-crossing cycle as $X = \omega_1 \xrightarrow{i_1} \omega_2 \xrightarrow{i_2} \dots \omega_n \xrightarrow{i_n} \omega_{n+1} = \omega_0$, where $\omega_{s+1} \in \Pi_{i_s}(\omega_s)$ for $s = 1, \dots, n$. Assume without loss of generality that $\mathrm{tr}(X) < 1$ (otherwise simply reverse the ordering of states in the cycle). To cut down on notational clutter, write $r_s := \mathrm{tr}^{i_s}(\omega_s, \omega_{s+1})$, hence $\mathrm{tr}(X) = r_1 r_2 \cdots r_n$. Furthermore, denote by P the set such that $i \in P$ if and only if i is one of $i_1, i_2, \dots i_n$ used in the presentation above of the cycle X.

We will several times make use of the following simple technical observation: suppose that i is a player, π is a partition element of Π_1, and $\omega', \omega'' \in \pi$. Furthermore, suppose that g is a random variable satisfying the property that $g(\omega) = 0$ for all $\omega \in \pi$ such that $\omega \neq \omega', \omega''$. Then:

$$\left\{ \begin{array}{ll} E_i(g|\omega') = 0 & \text{if } g(\omega'') = -\mathrm{tr}^i(\omega', \omega'') g(\omega'), \\ E_i(g|\omega') > 0 & \text{if } g(\omega'') > -\mathrm{tr}^i(\omega', \omega'') g(\omega'), \\ E_i(g|\omega') < 0 & \text{if } g(\omega'') < -\mathrm{tr}^i(\omega', \omega'') g(\omega'). \end{array} \right\} \tag{12}$$

We now proceed to the construction of an agreeable bet, in stages.

First note that by the assumption that $r_1 r_2 \cdots r_n < 1$, we may choose a $\delta_n > 1$ such that $r_1 r_2 \cdots r_n \delta_n < 1$. We may then further define δ_{n-1}

such that $\delta_n > \delta_{n-1} > 1$ and so on, to yield a sequence $\delta_n > \delta_{n-1} > \ldots > \delta_2 > 1$. Use this to define $\overline{f}$ by:

$$\begin{array}{ll} \overline{f}_{i_1}(\omega_1) = -1 & \\ \overline{f}_{i_n}(\omega_1 = \omega_{n+1}) = 1 & \\ \overline{f}_{i_{s-1}}(\omega_s) = \delta_s r_1 r_2 \cdots r_{s-1} & \text{for } 2 \leq s \leq n \\ \overline{f}_{i_s}(\omega_s) = -\delta_s r_1 r_2 \cdots r_{s-1} & \text{for } 2 \leq s \leq n \\ \overline{f}_j(\omega) = 0 & \text{for all other } \omega \text{ and } j. \end{array}$$

Using Equation (12) repeatedly (here is also where we use the assumption that X is non-crossing, which ensures that in each partition element of every player i there are at most two states at which $\overline{f}_i$ takes on non-zero values) we deduce that $E_{i_s}(\overline{f}_{i_s}|\omega_s) > 0$ for $1 \leq s \leq n+1$.

We still need to ensure that the players who are not in P have positive expectations at the states participating in X. To do so, note the following: since $r_1 r_2 \cdots r_n \delta_n < 1$, we can choose an ε_{n+1} such that $1 - \varepsilon_{n+1} > r_1 r_2 \cdots r_n \delta_n$. Similarly, since for any $2 \leq s \leq n$, $\delta_s r_1 r_2 \ldots r_{s-1} > \delta_{s-1} r_1 r_2 \ldots r_{s-1}$, we can choose ε_s such that

$$\delta_s r_1 r_2 \ldots r_{s-1} - \varepsilon_s > \delta_{s-1} r_1 r_2 \ldots r_{s-1}.$$

At each state in $\omega_s \in X$, therefore, we intuitively can take away a positive part of the positive value of $\overline{f}_{i_{s-1}}(\omega_s)$ and 'redistribute' it among the other players. This enables the following construction:

$$\begin{array}{ll} f_{i_s}(\omega_s) = \overline{f}_{i_s}(\omega_s) & \text{for } 1 \leq s \leq n \\ f_{i_{s-1}}(\omega_s) = \overline{f}_{i_{s-1}}(\omega_s) - \varepsilon_s & \text{for } 1 \leq s \leq n \\ f_j(\omega_s) = \varepsilon_s/(m-2) & \text{for } 1 \leq s \leq n, \text{ for } j \in I, j \neq i_s, i_{s-1} \\ f_j(\omega) = 0 & \text{for all other } \omega \text{ and } j. \end{array}$$

By construction, $f = \{f_1, \ldots, f_m\}$ is an agreeable bet relative to τ^X, which is what we needed to show. ∎

The following proposition, one half of the no betting characterisation for finite type spaces, has several proofs in the literature, all of which ultimately rely on convex separation theorem, or equivalents thereof. The proof presented here (building on the previous lemmas) is, in contrast, constructive and entirely combinatorial.

Proposition 4. *Let τ be a type space over $(\Omega, \mathbf{\Pi})$, where Ω is a finite state space, and suppose that τ has no common prior. Then there exists an f that is an agreeable bet relative to τ.*

Proof. Recalling that in a finite state space every improper prior can be normalised, and is therefore a proper prior, we may refer to all the previous lemmas and apply them restricting attention to the special case of common priors, rather than the more general common improper prior.

Let $S \subseteq \Omega$ be the set of non-singularly positive states in Ω, and (using Lemma 1) decompose S disjointly as $S = \cup_{j=1}^{k} T_j$, where each T_j is non-singularly positive and connected. By Proposition 2, for each $1 \leq j \leq k$, there is a non-crossing cycle c_{T_j} contained in T_j such that $\mathrm{tr}(c_{T_j}) \neq 1$. By Proposition 3, there exists $f^{T_j} = \{f_1^{T_j}, \ldots, f_k^{T_j}\}$ such that $f_i^{T_j}(\omega) = 0$ for all $\omega \notin T_j$ and $i \in I$, and f^{T_j} is an agreeable bet relative to τ^{T_j}.

Let Z be the set of zero states, and denote by Q the complement of $S \cup Z$. Define f^Q in stages as follows.

In stage 0, let W_0 be the set of singular states in Q. For every $\omega \in W_0$ decompose I as $I = J \cup K$, where $\tau_j(\omega) > 0$ for all $j \in J$ and $\tau_i(\omega) = 0$ for $i \in K$, and choose an arbitrary $i' \in K$. Then define $g_j^0(\omega) = 1/|J|$ for every $j \in J$, with $g_{i'}^0(\omega) = -1$, and $g_i^0(\omega) = 0$ for all other $i \in K$, $i \neq i'$. Set $g_k^0(\omega) = 0$ for all states ω not in W_0 and all players $k \in I$.

Note, for the rest of this proof, that by definition every state in $Q \backslash W_0$ is a positive state.

In stage 1, let W_1 be the set of all states $\omega \in Q \setminus W_0$ satisfying the property that there is at least one player i, and a state $\omega' \in W_0$, such that $\omega \in \pi_i(\omega')$. For each $\omega \in W_1$, choose ω' as just described. If $t_i(\omega') > 0$, then by construction ω' is an i-surplus state relative to g^0. Hence we can apply a process similar to that described in the paragraph preceding and following Equation 11 to extend g^0 to a function g^1 such that $g_i^1(\omega) < 0$, but for all $j \neq i$, $g_j^1(\omega) > 0$, $\sum_{j \in I} g_j^1(\omega) = 0$, and is a surplus state for all players.

If $t_i(\omega') = 0$, then note that since ω is positive but not contained in any non-singularly positive set, there must be a chain $c = \omega \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n$ entirely contained in $Q \setminus W_0$ such that there is a player i and a state $\omega' \in X_0$ satisfying $\omega_n \in \pi_i(\omega')$ and $t_i(\omega') > 0$. But then we can apply the same argument as in the previous paragraph to define extend g^0 to g^1 by induction over all the states $\omega_n, \omega_{n-1}, \ldots, \omega$, yielding a function such that $\sum_{j \in I} g_j^1(\omega) = 0$, and is a surplus state for all players.

In all stages $l > 1$, in stage $l-1$, a function g^{l-1} has been defined such that each state $\omega \in W_{l-1}$ is an i-surplus state for every player i relative to g^{l-1}. Denote by W_l the set containing every state $\omega \in Q \setminus W_{l-1}$ satisfying the property that there is at least one $i \in I$ and $\omega \in W_{l-1}$ such that $\omega' \in \pi_i(\omega)$. Since ω is an i-surplus state relative to g^{l-1}, we can again apply the same technique as in the previous paragraphs to extend g^{l-1} to g^l.

By the finiteness of Ω, this iterative process ends after a finite number of stages r. Finally, set $f^Q := g^r$, and define

$$f := f^{T_1} + f^{T_2} + \ldots f^{T_k} + f^Q.$$

It is straightfoward to check that f, by construction, is an agreeable bet. ∎

Proof of Theorem 1(b). Let τ be a type space over $\{\Omega, \mathbf{\Pi}\}$. If Ω is finite, then Proposition 4.4 suffices.

Suppose therefore that Ω is countably infinite. We apply a theorem appearing in Lehrer and Samet (2010), stating that there exists a *cip* for τ iff there exists an increasing infinite sequence $\bigcup_n \Omega_n$ of finite subsets of Ω such that $\bigcup \Omega_n = \Omega$ and for each n there is a common prior for τ^{Ω_n}. Let Ω_n be an arbitrary such increasing sequence of finite subsets such that $\bigcup_n \Omega_n = \Omega$. Since we are assuming that τ has no *cip*, there must be infinitely many n's such that there is no common prior for Ω_n. Thus we may assume without loss of generality that for each n there is no common prior for τ^{Ω_n}. By Proposition 4.4, for each n there is an agreeable bet f^n on Ω_n bounded by 1. Let $\hat{f}^n$ be the extension of f^n to all of Ω defined to be 0 at all points not in f^n. Then $f = \sum_n 2^{-n} \hat{f}^n$ is a bounded agreeable bet over Ω. ∎

As mentioned in the introduction, putting together the elements of the proofs in this section yields an algorithm that can be applied in finite type spaces. The algorithm determines whether the space has a common prior, by listing the connected non-singularly positive subspaces and checking whether there is such a subspace such that every non-crossing cycle contained in it satisfies $\mathrm{tr}(c) = 1$ (see Proposition 2). If it does have a common prior, the algorithm then constructs the common prior, by the method in the proof of Proposition 1. If the space does not have a common prior, the algorithm constructs an agreeable bet, by the method in the proof of Proposition , thus finding a sequence of random variables about whose expected values the players 'agree to disagree'.

5. Iterated Expectations, Countable Spaces, and Common Priors

5.1. **Markov Theory Preliminaries.**

We recall some basic concepts and results from Markov theory.

Let M be a Markov matrix over a state space Ω that is either finite or countable. Fix an ordering of the elements of Ω as $\{\omega_1, \omega_2, \ldots\}$.

States ω_i and ω_j *communicate* if there are $k, m \geq 0$ such that $M^k(\omega_i, \omega_j) > 0$ and $M^m(\omega_j, \omega_i) > 0$. This relation partitions the state space into equivalence classes. M is *irreducible* if all the states communicate. A state ω_i is said to have *period* d if $M^k(\omega_i, \omega_i) = 0$ whenever k is not divisible by d, and d is the largest integer with this property. A state with period 1 is *aperiodic*. Every pair of communicating states have the same period, hence an irreducible Markov matrix has a well-defined period.

A state ω_i is called *recurrent* if

$$\sum_{k=0}^{\infty} M^k(\omega_i, \omega_i) = \infty.$$

Otherwise the state is called transient. Every pair of communicating states are either both recurrent or both transient, have we can categorise irreducible Markov matrices into recurrent and transient matrices.

An irreducible aperiodic Markov matrix is called *null recurrent* if it is recurrent and for every pair of states ω_i, ω_j

$$\lim_{k \to \infty} M^k(\omega_i, \omega_j) = 0.$$

If M is recurrent but not null recurrent then it is called *positive recurrent.* Denoting

$$\gamma_i := \lim_{k \to \infty} M^k(\omega_i, \omega_i), \tag{13}$$

M is positive recurrent if and only if $\gamma_i > 0$ for all i. If M is irreducible, aperiodic, and positive recurrent it is termed *ergodic.*

A vector $\mu \in \Omega^{\mathbb{R}}$ satisfying $\mu \geq 0$ is an *invariant measure* of M if it is a left eigenvector of M, i.e. $\mu M = \mu$, or equivalently for each state ω':

$$\mu(\omega') = \sum_{\omega \in \Omega} \mu(\omega) M(\omega, \omega').$$

If μ is an invariant measure and also satisfies the property that it is normalisable, i.e. $\sum_{\omega \in \Omega} \mu(\omega) < \infty$, then it is termed an *invariant probability measure.*

The following are well-known results in Markov theory:

(1) If M is recurrent then it has an invariant measure that is unique up to scalar multiplication.

(2) M has a unique invariant probability measure if and only if M is ergodic.
(3) If M is ergodic, then the vector π given by $\pi(i) = \gamma_i > 0$ (where γ_i is defined in Equation 13) is the unique invariant probability measure.
(4) If M is irreducible, aperiodic, and recurrent, then $\lim_{k\to\infty} M^k$ exists. If M is ergodic, then

$$\lim_{k\to\infty} M^k = \mathbf{I}\pi.$$

If M is null recurrent, then $\lim_{k\to\infty} M^k = 0$.

5.2. **Iterated Expectations and Common Priors over Countable State Spaces.**

This section clearly owes a large debt to Samet (1998a), and most of the proof ideas are taken from that paper. The need to distinguish between null recurrent and positive recurrent classes, however, adds subtleties that are not present in the finite state space.

Proposition 5. *Let σ be a permutation of I, and let $P \in \wedge\mathbf{\Pi}$ be an element of the meet. Then P is an irreducible, aperiodic, and recurrent class of M_σ. Thus, M_σ^P has a unique (up to scalar multiplication) invariant measure μ_σ^P on P.*

Proof. Let $i \in I$, and let $\omega, \omega' \in P \in \Pi_{\sigma(i)}$. Then

$$\begin{aligned} M_\sigma(\omega,\omega') \geq M_{\sigma(1)}(\omega,\omega)\cdots M_{\sigma(i-1)}(\omega,\omega)M_{\sigma(i)}(\omega,\omega') \\ \times M_{\sigma(i+1)}(\omega',\omega')\cdots M_{\sigma(n)}(\omega',\omega'). \end{aligned}$$

Therefore, any two states in the same element of a partition of an arbitrary player communicate. Hence, if ω is in an equivalence class of states, then $\Pi_i(\omega)$, for each i, is a subset of this class. This means that each class is a union of elements of $\wedge\mathbf{\Pi}$. Also, for each $P \in \wedge\mathbf{\Pi}$, the probability of $\omega \in P$ staying in P under M_σ is 1, and therefore P is an irreducible equivalence class. That the Markov matrix M_σ is aperiodic and recurrent follows from the fact that $M_\sigma(\omega,\omega) > 0$ for every state. ∎

Corollary 3. *Let σ be a permutation of I, and let $P \in \wedge\mathbf{\Pi}$ be an element of the meet. If P is a positive recurrent class of M_σ, then M_σ^P has a unique invariant probability measure μ_σ^P.* ∎

Proposition 6. *The following conditions are equivalent for each $P \in \wedge\mathbf{\Pi}$:*

(1) *p is the common prior on P.*

(2) *p is the invariant probability measure of M_i^P for every $i \in I$.*
(3) *The permutation matrix M_σ^P is ergodic for every permutation σ, and p is the invariant probability measure of M_σ^P.*

Proof. With slight abuse of notation, write M_i in place of M_i^P throughout this proof, for ease of exposition.

That (1) and (2) are equivalent is straightforward. To see that (2) implies (3), note that if p is an invariant probability measure of M_i for every i, then it must also be an invariant probability measure of M_σ for every permutation σ. But M_σ has an invariant probability measure if and only if it is ergodic.

Next, suppose that (3) is true and let p be the invariant probability measure of $M_{\sigma_1} := M_1 M_2 \cdots M_n$ that exists by (3). Then

$$pM_1 M_2 \cdots M_n = p.$$

Multiplying M_{σ_1} from the right by M_1 yields

$$pM_1 M_2 \cdots M_n M_1 = pM_1.$$

It follows that pM_1 is an invariant probability measure of $M_{\sigma_2} := M_2 \cdots M_n M_1$. However, by (3), p is also an invariant probability measure of M_{σ_2}, and by Corollary 3, each permutation matrix over restricted to P has a unique invariant probability measure. Hence $pM_1 = p$, and similarly, $pM_i = p$ for each $i \in I$. ∎

Proof of Theorem 2. Since $E_\sigma^k f = M_\sigma^k f$ for each f and k,

$$\lim_{k\to\infty} E_\sigma^k f = \lim_{k\to\infty} M_\sigma^k f.$$

As M_σ is recurrent for each σ by Proposition 5, $\lim_{k\to\infty} M_\sigma^k$ exists.

If τ has a common prior, then by Proposition 6, M_σ is ergodic for each σ, and therefore $\lim_{k\to\infty} M_\sigma^k = \mathbf{I}p_\sigma$, where p_σ is the unique invariant probability measure of M_σ. But by the common prior assumption, $p_\sigma = \varphi$ for all σ. Furthermore, $\varphi > 0$, hence the iterated expectations of f with respect to all permutations converge to the same non-zero limit pf.

In the other direction, suppose that for each bounded non-zero random variable $f \geq 0$, the iterated expectations of f, with respect to all permutations σ, converge to the same non-zero limit. Then it cannot be the case that $\lim_{k\to\infty} M_\sigma^k = 0$, hence M_σ is ergodic. It follows that $\lim_{k\to\infty} M_\sigma^k = \mathbf{I}p_\sigma$, where p_σ is the unique invariant probability measure of M_σ, and therefore the iterated expectation of every f with respect to σ converges to $p_\sigma f$. By the assumption that $p_\sigma f$ is the same for all permutations σ, there is a single p, $p = p_\sigma$ for all σ, such that p

is the unique invariant probability measure of M_σ. We conclude from Proposition 6 that p is the common prior. ∎

Theorem 2 provides a characterisation of common priors in countable spaces using the iterated expectations criterion. That criterion cannot be used for identifying common improper priors in cases in which a common prior does not exist; by Proposition 6, if there is no common prior, then for at least one permutation σ, the permutation matrix M_σ^P is not ergodic. This in turn means that M_σ^P must be null recurrent, hence $\lim_{k\to\infty} M_\sigma^k = 0$.

6. Appendix

Proof of Lemma 1. Suppose that $S \subseteq \Omega$ is non-singularly positive, and let $\omega_0 \in S$ be chosen arbitrarily. Suppose that ω_0 is connected to ω_n by a positive chain $c = \omega_0 \overset{i_0}{\to} \omega_1 \overset{i_1}{\to} \cdots \overset{i_{n-1}}{\to} \omega_n$. Since ω_0 is in S and ω_1 is in the same partition element of player i_0 as ω_0, by Definition 6, ω_1 must also be in S, and continuing the same argument by induction, we conclude that all the elements in c are members of S. ∎

Proof of Proposition 1. Suppose that there exists a common improper prior p for τ. Let $S = \{\omega \in \Omega \mid p(\omega) > 0\}$. S is guaranteed to be positive, because $p \neq 0$. We next show that S is non-singularly positive: Suppose that for arbitrary $i \in I$ and $\omega \in S$, $\omega' \in \Pi_i(\omega)$. Furthermore, suppose that $\omega' \notin S$. Then $p(\omega') = 0$, while $p(\omega) > 0$. Hence $p(\Pi_i(\omega)) > 0$, and by the definition of an improper prior,

$$t_i(\omega') = \frac{p(\omega')}{p(\Pi_i(\omega))} = 0.$$

It follows from Definition 6 that S is non-singularly positive.

To complete this part of the proof, note that for any pair of states $\omega_1, \omega_2 \in S$ such that ω_1 and ω_2 are in the same element of Π_i for some $i \in I$, $\mathrm{tr}_t^i(\omega_1, \omega_2) = p(\omega_1)/p(\omega_2)$. It then easily follows from the definition of the type ratio of a chain that for any chain c entirely contained in S and connecting ω_0 and ω, one has $\mathrm{tr}_\tau(c) = p(\omega_0)/p(\omega)$.

Conversely, suppose that Ω has a non-singularly positive subspace S with respect to τ, and that for each ω_0 and ω in S, any pair of chains c and c' entirely contained in S connecting ω_0 to ω satisfy $\mathrm{tr}_\tau(c) = \mathrm{tr}_\tau(c')$. Using Lemma 1, we may assume that S is connected (replacing S by a connected subset of itself if necessary).

We will construct a *cip* p. For $\omega \notin S$, set $p(\omega) = 0$. Otherwise, fix $\omega_0 \in S$ and for each $\omega \in S$, let $p(\omega) = \mathrm{tr}(c)$ for some chain c from ω_0 to ω contained in S.

To see that p is a *cip* consider $\pi \in \Pi_i$. Suppose first that $\pi \cap S = \emptyset$. Then for all $\omega \in \pi$, $p(\omega) = 0$, hence $p(\pi) = 0$, and $p(\pi)t_i(\omega) = p(\omega)$ is satisfied.

Suppose instead that $\pi \cap S \neq \emptyset$, and that $\omega \in \pi \cap S$. Let c be a chain from ω_0 to ω entirely contained in S. For $\omega' \in \pi \cap S$, consider the chain $c' = c \xrightarrow{i} \omega'$. Then, by the definitions of tr and p, $p(\omega') = \mathrm{tr}(c') = \mathrm{tr}(c)\mathrm{tr}^i(\omega, \omega') = p(\omega)t_i(\pi, \omega')/t_i(\pi, \omega)$. Thus, $p(\pi) = \sum_{\omega' \in \pi \cap S} p(\omega') = [p(\omega)/t_i(\pi, \omega)] \sum_{\omega' \in \pi \cap S} t_i(\pi, \omega') = p(\omega)/t_i(\pi, \omega) < \infty$, and $p(\omega) = p(\pi)t_i(\pi, \omega)$.

Finally, suppose that $\pi \cap S \neq \emptyset$, and that $\omega \in \pi$ is such that $\omega \notin S$. By construction, $p(\omega) = 0$, and by the assumption that S is non-singularly positive, it must be the case that $t_i(\omega) = 0$. We have already shown that $p(\pi) < \infty$, hence $p(\pi)t_i(\omega) = p(\omega)$ is satisfied. ∎

Proof of Corollary 2. It suffices to note the following: suppose that c_1 and c_2 are two distinct chains entirely contained in S connecting a pair of states ω and ω'. Then $\overline{c} := c_1 c_2^{-1}$ is a cycle connecting ω to itself. By Equation (6), $\mathrm{tr}(c_1) = \mathrm{tr}(c_2)$ if and only if $\mathrm{tr}(\overline{c}) = 1$. ∎

Proof of Proposition 2. If there exists a *cip*, then by Corollary 2, there is a non-singularly positive $S \subseteq \Omega$ such that every cycle contained in S has type ratio equal to 1, hence in particular every non-crossing cycle satisfies the same property.

In the other direction, if there does not exist a *cip*, then for every non-singularly positive $S \subseteq \Omega$, there is at least one cycle $\overline{c}$ entirely contained in S such that $\mathrm{tr}(\overline{c}) \neq 1$. Suppose that $\overline{c}$ is not non-crossing.

If $\overline{c}$ fails to be non-crossing because it is not alternating, this 'flaw' can easily be corrected: if two consecutive states ω_s and ω_{s+1} in $\overline{c}$ are identical, since that implies that $\mathrm{tr}^i(\omega_s, \omega_{s+1}) = 1$, the state ω_{s+1} is redundant and can be removed from $\overline{c}$ without changing the type ratio. Similarly, if ω_s, ω_{s+1}, ω_{s+2} and ω_{s+3} are consecutive states that are all members of the same partition element of player i, then $\mathrm{tr}^i(\omega_s, \omega_{s+1})\mathrm{tr}^i(\omega_{s+2}, \omega_{s+3}) = \mathrm{tr}^i(\omega_s, \omega_{s+3})$, hence we may remove ω_{s+1} and ω_{s+2} from $\overline{c}$ without changing the type ratio.

We will therefore assume that $\overline{c}$ is alternating but not non-crossing, and that we can then write $\overline{c} = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n = \omega_0$, where $\omega_{s+1} \in \Pi_{i_s}(\omega_s)$ for $s = 0, \ldots, n-1$, where there exists at least one pair r, k such that $k > r + 1$, and $\omega_k \in \Pi_{i_r}(i_r)$.

We can 'shorten' $\overline{c}$ into another cycle:

$$\widehat{c} = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \ldots \omega_r \xrightarrow{i_r} \omega_k \xrightarrow{i_k} \cdots \xrightarrow{i_{n-1}} \omega_n = \omega_0.$$

If $\mathrm{tr}(\widehat{c}) \neq 1$, then we have a cycle of type ratio not equal to 1, with a number of self-crossing points that is strictly less than the number of self-crossing points in $\overline{c}$, and we can continue by induction to apply the same process to $\mathrm{tr}(\widehat{c})$.

Suppose, therefore, that $\mathrm{tr}(\widehat{c}) = 1$. Denote:

$$c_0 = \omega_0 \stackrel{i_0}{\to} \omega_1 \stackrel{i_1}{\to} \cdots \stackrel{i_{r-1}}{\to} \omega_r,$$

$$c_k = \omega_k \stackrel{i_k}{\to} \omega_{k+1} \stackrel{i_{k+1}}{\to} \cdots \stackrel{i_{n-1}}{\to} \omega_n,$$

and

$$c_l = \omega_r \stackrel{i_r}{\to} \omega_{r+1} \stackrel{i_{r+1}}{\to} \cdots \stackrel{i_{k-1}}{\to} \omega_k.$$

Then $\overline{c} = c_0 c_l c_k$, and $\widehat{c} = c_0(\omega_r, \omega_k)c_k$. By assumption, $1 = \mathrm{tr}(\widehat{c}) = \mathrm{tr}(c_0)\mathrm{tr}^{i_r}(\omega_r, \omega_k)\mathrm{tr}(c_k)$. It follows that $[\mathrm{tr}^{i_r}(\omega_r, \omega_k)]^{-1} = \mathrm{tr}^{i_r}(\omega_k, \omega_r) = \mathrm{tr}(c_0)\mathrm{tr}(c_k)$. We also assumed that $\mathrm{tr}(\overline{c}) \neq 1$, so $1 \neq \mathrm{tr}(c_0 c_l c_k) = \mathrm{tr}(c_0)\mathrm{tr}(c_k)\mathrm{tr}(c_l) = \mathrm{tr}^{i_r}(\omega_k, \omega_r)\mathrm{tr}(c_l)$.

Writing out the last inequality in full yields

$$\mathrm{tr}(\omega_k \stackrel{i_r}{\to} \omega_r \stackrel{i_r}{\to} \omega_{r+1} \stackrel{i_{r+1}}{\to} \cdots \stackrel{i_{k-1}}{\to} \omega_k) \neq 1.$$

But by Equation 7, $\mathrm{tr}(\omega_k \stackrel{i_r}{\to} \omega_r \stackrel{i_r}{\to} \omega_{r+1}) = \mathrm{tr}(\omega_k \stackrel{i_r}{\to} \omega_{r+1})$, hence

$$\mathrm{tr}(\omega_k \stackrel{i_r}{\to} \omega_{r+1} \stackrel{i_{r+1}}{\to} \cdots \stackrel{i_{k-1}}{\to} \omega_k) \neq 1.$$

We deduce then that the cycle $\widetilde{c} = \omega_k \stackrel{i_r}{\to} \omega_{r+1} \stackrel{i_{r+1}}{\to} \cdots \stackrel{i_{k-1}}{\to} \omega_k$ satisfies both that $\mathrm{tr}(\widetilde{c}) \neq 1$, and that it has a number of self-crossing points that is strictly less than the number of self-crossing points in $\overline{c}$. We can continue by induction to apply the same process to $\mathrm{tr}(\widetilde{c})$.

After applying this reasoning as often as necessary, we arrive at the existence of a cycle entirely contained in S with no self-crossing points, i.e., a non-crossing cycle, whose type ratio is not equal to 1, which is what we needed to show. ∎

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