

# Characterization of the Shapley-Shubik Power Index Without the Efficiency Axiom\*

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## Abstract

We show that the Shapley-Shubik power index on the domain of simple (voting) games can be uniquely characterized without the efficiency axiom. In our axiomatization, the efficiency is replaced by the following weaker requirement that we term the *gain-loss* axiom: any gain in power by a player implies a loss for someone else (the axiom does not specify the extent of the loss). The rest of our axioms are standard: transfer (which is the version of additivity adapted for simple games), symmetry or equal treatment, and dummy.

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## 1 Introduction

Shortly after the introduction of the Shapley (1953) value, Shapley and Shubik (1954) suggested to use its restriction to the domain of simple (voting) games in order to

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assess the *a priori voting power* of players. This restriction had since become known as the Shapley-Shubik power index (SSPI). The SSPI measures the chance each player has of being critical to the success of a winning coalition, if players join in a random order. In addition to being an attempt to quantify the elusive voting power, the SSPI can also be regarded as the utility of playing a simple game under a certain posture towards risk (see Roth (1977)).

The construction of the SSPI points the way to define other indices with broadly similar features. The Banzhaf (1965) index is a prime example, with a long history of successful applications and sustained academic interest. Like the SSPI, it evaluates players' probabilities of having a swing vote in the game, but under the assumption that each player joins a coalition or abstains from joining with equal probability and that the choices of different players are independent. Yet more general are the probabilistic assumptions on coalition formation behind the family of *semivalues* of Dubey et al (1981), defined for all games, whose restriction to the domain of simple games produces a variety of power indices (studied in Einy (1987) and also termed semivalues). One way of delineating the differences between indices is the axiomatic approach – certain critical properties (axioms) of the given index are identified, and then are shown to uniquely characterize it, thereby setting it apart from other indices.

The Shapley (1953) value was the first solution concept of cooperative game theory to be axiomatized. However, the SSPI, its spin-off, received similar treatment much later (in Dubey (1975)). Dubey (1975) uses the axioms of Shapley (1953), with the exception of additivity, which has to take up a special form<sup>1</sup> due to the non-linear structure of the set of simple games. The axiom of efficiency, according to which the total power of the players must be equal to 1, the worth of the grand coalition in the game, is the primary distinguishing feature of the SSPI. It is, in fact, the only efficient semivalue on the domain of simple games.<sup>2</sup>

The normalization of the total voting power to 1, embodied in the efficiency axiom, is a natural requirement if the power index is used to assess the *relative* power of each player in the *same* game. However, it is not clear whether an efficient index can allow comparisons of individual voting power across games, since the uniform normalization to 1 of the total power may in principle rescale individual power differently in different

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<sup>1</sup>It later became known as the *transfer axiom*, due to Weber (1988).

<sup>2</sup>See Theorem 2.5 and Remark 2.7 in Einy (1987).

games. Thus, a-priori, the efficiency axiom may appear to be too strong in the context of power indices.

In this paper we propose a new axiomatization of the SSPI, that replaces efficiency (which has been central in most axiomatizations) by a weaker axiom. We make no direct assumption on the total power in the game, and concentrate on the individual voting power instead. Our new axiom, which we term the axiom of *gain-loss*<sup>3</sup>, makes the minimal, ordinal, requirement that is still consistent with the constant-sum nature of power as measured by the SSPI. According to the gain-loss axiom, if the power of *some* player increases as a result of changes in the game, the power cannot concomitantly increase for *all* players. That is, any gain in power by a player implies a loss for someone else – this expresses the intuitive idea that in the "strife for power", if there are winners then there must also be losers. The axiom is weak since it specifies neither the identity of players that lose power, nor the extent of their loss.

The rest of our axioms are standard. We adopt the *transfer* axiom of Dubey (1975), and the *symmetry* and the *null player* axioms of Shapley (1953). Our Theorem 1 shows that the four axioms characterize the SSPI up to rescaling. But if the null player axiom is replaced by the stronger *dummy* axiom (which is the second axiom in our set, after *gain-loss*, to contain a mild aspect of efficiency), the SSPI is characterized uniquely, as we state in Corollary 2. Moreover, in this characterization, the symmetry can be replaced by the weaker *equal treatment* axiom, as shown in Theorem 3.

In several other works, a characterization of the SSPI or the Shapley value was done with substitutes for the efficiency axiom. When a power index is viewed as a utility function that represents a prospective player's preference over the set of games, Roth (1977) considers the axiom of "strategic risk neutrality". This axiom requires that a player be indifferent between playing a unanimity game with carrier  $T$ , and participating in a lottery that assigns probability  $1/|T|$  to being a dictator and probability  $1 - 1/|T|$  to being a null player. It thus determines the index (up to an affine transformation) as the efficient SSPI on unanimity games. Also in the context of treating power indices as utility functions, Blair and McLean (1990) introduce

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<sup>3</sup>We borrow this term from Laruelle and Valenciano (2001), although they use it to refer to a rather different property, as we will mention below.

another substitute of efficiency that pinpoints the efficient Shapley value in the set of semivalues that they characterize. They require that a player be indifferent between all symmetric simple games in which the minimal winning coalitions have the same size. This axiom, when restated in our purely game-theoretic setting, is related to *gain-loss*. Specifically, it is weaker than the combination of *gain-loss* and *symmetry*, and can replace *gain-loss* in our axiomatization as we point out in Remark 1. We do not introduce it as an explicit axiom in our setting, however, believing that *gain-loss* has a stronger aesthetic and conceptual appeal in many contexts.

More recently, Laruelle and Valenciano (2001) introduced the “constant total gain-loss balance” axiom, which requires that if a minimal winning coalition  $S$  is deleted from a simple game, then the total loss in power of players in  $S$  be the same as the total gain in power of players in the complement of  $S$ . Unlike our *gain-loss* axiom, however, this axiom *by itself* is sufficiently close to efficiency. Indeed, it implies that the total power of the players is constant in all games, and hence is a fixed multiple of the worth of the grand coalition. This means that the power index is *efficient* up to rescaling, provided the total power is non-zero.<sup>4</sup>

The structure of our paper is as follows: Section 2 introduces simple games and the definition of the SSPI, and Section 3 contains the statements of our axioms, the characterization results, and two remarks.

## 2 Simple Games and the Shapley-Shubik Index of Power

Let  $N = \{1, 2, \dots, n\}$  be the set of *players*. Denote the collection of all *coalitions* (subsets of  $N$ ) by  $2^N$ , and the empty coalition by  $\emptyset$ . Then a *game* on  $N$  is given by a map  $v : 2^N \rightarrow R$  with  $v(\emptyset) = 0$ . The space of all games on  $N$  is denoted by  $\mathcal{G}$ . A coalition  $T \in 2^N$  is called a *carrier* of  $v$  if  $v(S) = v(S \cap T)$  for any  $S \in 2^N$ .

The domain  $\mathcal{SG} \subset \mathcal{G}$  of *simple games* on  $N$  consists of all  $v \in \mathcal{G}$  such that

- (i)  $v(S) \in \{0, 1\}$  for all  $S \in 2^N$ ;

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<sup>4</sup>Our gain-loss axiom also implies that the power (both individual and total) is constant, but only in conjunction with the equal treatment axiom, and only for the small class of *symmetric* games, as can be easily seen (similarly to (5) in Lemma 1).

(ii)  $v(N) = 1$ ;

(ii)  $v$  is *monotonic*, i.e., if  $S \subset T$  then  $v(S) \leq v(T)$ .

A coalition  $S$  is said to be *winning* in  $v \in \mathcal{SG}$  if  $v(S) = 1$ , and *losing* otherwise.

A *power index* is a mapping  $\varphi : \mathcal{SG} \rightarrow \mathbb{R}^n$ . For each  $i \in N$  and  $v \in \mathcal{SG}$ , the  $i^{\text{th}}$  coordinate of  $\varphi(v) \in \mathbb{R}^n$ ,  $\varphi(v)(i)$ , is interpreted as the voting power of player  $i$  in the game  $v$ . The *Shapley-Shubik power index (SSPI)*  $\varphi_{ss}$  is among the best known. It is given for each  $v \in \mathcal{SG}$  and  $i \in N$  by

$$\varphi_{ss}(v)(i) = \sum_{S \subset N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} [v(S \cup \{i\}) - v(S)]. \quad (1)$$

For each  $i \in N$ ,  $\varphi_{ss}(v)(i)$  is exactly the probability that player  $i$  is pivotal in a random ordering of  $N$  (with the uniform distribution of orderings), i.e., the probability that the coalition of players preceding  $i$  in a random ordering is losing, but becomes a winning coalition if joined by  $i$ .

### 3 The Axioms and the Results

We start by exploring the degree of similarity to  $\varphi_{ss}$  that a power index on  $\mathcal{SG}$  must have if it satisfies the four axioms stated below. As mentioned, one of the axioms is new, while the other three are standard. The new axiom is a relaxation of the usual efficiency requirement. The axiom captures what could be expected intuitively from a measure of power – while it might be the case that the power of *some* players increases as a result of changes in the game, power cannot concomitantly increase for *all* players. That is, any gain in power by a player must come at the expense of someone else.

**Axiom I: Gain-loss (GL).** If

$$\varphi(v)(i) > \varphi(w)(i) \quad (2)$$

for some  $v, w \in \mathcal{SG}$  and  $i \in N$ , then there exists  $j \in N$  such that

$$\varphi(v)(j) < \varphi(w)(j). \quad (3)$$

The standard efficiency axiom requires that the equality  $\sum_{i \in N} \varphi(v)(i) = 1$  hold for every  $v \in \mathcal{SG}$ . Axiom **GL** is weaker than efficiency and quantitatively less demanding. It specifies neither the identity of  $j$  that loses power on account of  $i$ 's gain, nor the extent of  $j$ 's loss.

The three axioms that follow and their variants have been used in other characterizations of the SSPI (see, e.g., Dubey (1975)). To state our next axiom, we introduce the following notation. For  $v, w \in \mathcal{SG}$  define  $v \vee w, v \wedge w \in \mathcal{SG}$  by:

$$(v \vee w)(S) = \max \{v(S), w(S)\},$$

$$(v \wedge w)(S) = \min \{v(S), w(S)\}$$

for all  $S \in 2^N$ . (It is evident that  $\mathcal{SG}$  is closed under operations  $\vee, \wedge$ .) Thus a coalition is winning in  $v \vee w$  if, and only if, it is winning in at least one of  $v$  or  $w$ , and it is winning in  $v \wedge w$  if, and only if, it is winning in both  $v$  and  $w$ .

**Axiom II: Transfer (T).**  $\varphi(v \vee w) + \varphi(v \wedge w) = \varphi(v) + \varphi(w)$  for all  $v, w \in \mathcal{SG}$ .

As remarked in Dubey et al (2005), **T** can be restated in the following equivalent form. Consider two pairs of games  $v, v'$  and  $w, w'$  in  $\mathcal{SG}$ , and suppose that the transitions from  $v'$  to  $v$  and  $w'$  to  $w$  entail adding the *same* set of winning coalitions (i.e.,  $v \geq v', w \geq w'$ , and  $v - v' = w - w'$ ). An equivalent axiom would require that

$$\varphi(v) - \varphi(v') = \varphi(w) - \varphi(w'),$$

i.e., that the change in power depends only on the change in the voting game.

Next, denote by  $\Pi(N)$  the set of all *permutations* of  $N$  (i.e., bijections  $\pi : N \rightarrow N$ ). For  $\pi \in \Pi(N)$  and a game  $v \in \mathcal{SG}$ , define  $\pi v \in \mathcal{SG}$  by

$$(\pi v)(S) = v(\pi(S))$$

for all  $S \in 2^N$ . The game  $\pi v$  is the same as  $v$  except that players are relabeled according to  $\pi$ .

**Axiom III: Symmetry (Sym).**  $\varphi(\pi v)(i) = \varphi(v)(\pi(i))$  for every  $v \in \mathcal{SG}$ , every  $i \in N$ , and every  $\pi \in \Pi(N)$ .

According to **Sym**, if players are relabeled in a game, their power indices will be relabeled accordingly. Thus, irrelevant characteristics of the players, outside of their role in the game  $v$ , have no influence on the power index.

**Axiom IV: Null player (NP).** If  $v \in \mathcal{SG}$ , and  $i$  is a null player in  $v$ , i.e.,  $v(S \cup \{i\}) = v(S)$  for every  $S \subset N \setminus \{i\}$ , then  $\varphi(v)(i) = 0$ .

**Theorem 1.** A power index  $\varphi$  satisfies **GL**, **T**, **Sym**, and **NP** if and only if  $\varphi = a\varphi_{ss}$  for some  $a \in \mathbb{R}$ .

**Proof.** It is well known that  $\varphi_{ss}$  satisfies **T**, **Sym**, and **NP**. Axiom **GL** is satisfied since  $\varphi_{ss}$  is efficient, which is a stronger requirement. It is also obvious that the axioms are invariant under any rescaling of  $\varphi_{ss}$ .

We now show that the axioms uniquely determine  $\varphi_{ss}$  up to rescaling. To this end, fix a power index  $\varphi$  that satisfies **GL**, **T**, **Sym**, and **NP**.

**Lemma 1.** For each  $s = 0, 1, \dots, n - 1$ , consider the game  $w_s \in \mathcal{SG}$  given by

$$w_s(S) = \begin{cases} 1, & \text{if } |S| > s, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Then

$$\varphi(w_0)(i) = \varphi(w_1)(i) = \dots = \varphi(w_{n-1})(i) \quad (5)$$

for every  $i \in N$ .

**Proof of Lemma 1.** By **Sym**, for every  $i, j \in N$  and  $s = 0, 1, \dots, n - 1$ ,

$$\varphi(w_s)(i) = \varphi(w_s)(j). \quad (6)$$

If there were  $0 \leq s', s'' \leq n - 1$  such that (w.l.o.g.)  $\varphi(w_{s'})(i) > \varphi(w_{s''})(i)$  for some  $i \in N$ , there would exist  $j \in N$  with  $\varphi(w_{s'})(j) < \varphi(w_{s''})(j)$  by **GL**, contradicting (6) taken for  $s = s', s''$ . Hence (5) follows.  $\square$

Following the proof of Lemma in Dubey et al (1981),<sup>5</sup> consider the set  $F$  of all power indices on  $\mathcal{SG}$  that satisfy **T**, **Sym**, and **NP**. Clearly,  $F$  is a linear subspace of the space of all mappings  $\mathcal{SG} \rightarrow \mathbb{R}^n$ . Now consider, for each  $T \subset N$ , the *unanimity game*  $u_T$  given by

$$u_T(S) = \begin{cases} 1, & \text{if } T \subset S, \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

Any  $v \in \mathcal{SG}$  can be written as a maximum of a finite number of unanimity games:

$$v = u_{T_1} \vee u_{T_2} \vee \dots \vee u_{T_k},$$

where  $T_1, \dots, T_k$  are the minimal winning coalitions in  $v$ . Since any  $f \in F$  satisfies **T**, Lemma 2.3 of Einy (1987) can be applied to any such  $f$  to obtain:

$$f(v) = \sum_{I \subset \{1, \dots, k\}, I \neq \emptyset} (-1)^{|I|+1} f(u_{\cup_{m \in I} T_m}). \quad (8)$$

It follows from (8) that the values of a power index  $f \in F$  on unanimity games uniquely determine the index on the entire  $\mathcal{SG}$ . Moreover, by **Sym** and **NP**,  $f$  is in fact fully determined by the following  $n$  values:  $f(u_{\{1\}})(1)$ ,  $f(u_{\{1,2\}})(1)$ ,  $\dots$ ,  $f(u_{\{1,2,\dots,n\}})(1)$ . Thus, the dimension of  $F$  is at most  $n$ . It is moreover easy to see that the  $n$  power indices  $f_0, \dots, f_{n-1} \in F$ , where

$$f_s(v)(i) = \sum_{S \subset N \setminus \{i\}, |S|=s} [v(S \cup \{i\}) - v(S)]$$

for each  $s = 0, 1, \dots, n-1$  and  $v \in \mathcal{SG}$ ,  $i \in N$ , are linearly independent. Thus, in fact,  $\dim F = n$ , and  $f_0, \dots, f_{n-1}$  form a linear basis for  $F$ . Consequently, there exists a collection  $(p_s)_{s=0}^{n-1}$  of coefficients such that  $\varphi = \sum_{s=0}^{n-1} p_s f_s$ , or, put alternatively,

$$\varphi(v)(i) = \sum_{S \subset N \setminus \{i\}} p_{|S|} [v(S \cup \{i\}) - v(S)] \quad (9)$$

for every  $v \in \mathcal{SG}$  and  $i \in N$ .

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<sup>5</sup>Although our aim is to obtain equality (9), which is precisely what is claimed in the Lemma of Dubey et al (1981), we cannot apply the lemma directly. Indeed, the lemma is stated for semivalues defined on the space  $\mathcal{G}$  of *all* games on  $N$ , and not just simple games. Moreover, the axioms of Dubey et al (1981) are more demanding. Our proof introduces an adjustment (the use of (8)) that needs to be made in the context of power indices for simple games.



Using (9) for  $v = w_s$ ,

$$\varphi(w_s)(i) = \binom{n-1}{s} p_s \quad (10)$$

for every  $s = 0, 1, \dots, n-1$ , and  $i \in N$ . Denote

$$a \equiv \sum_{s=0}^{n-1} \binom{n-1}{s} p_s$$

and observe that (5) in Lemma 1 and (10) yield

$$p_s = \frac{a}{n} \binom{n-1}{s}^{-1} = a \cdot \frac{s!(n-s-1)!}{n!} \quad (11)$$

for every  $s = 0, 1, \dots, n-1$ . Substituting (11) into (9), and comparing the resulting equality with (1), yields  $\varphi = a\varphi_{ss}$ .  $\square$

Now consider the following well-known stronger version of the **NP** axiom:

**Axiom V: Dummy (D).** If  $v \in \mathcal{SG}$ , and  $i$  is a dummy player in  $v$ , i.e.,  $v(S \cup \{i\}) = v(S) + v(\{i\})$  for every  $S \subset N \setminus \{i\}$ , then  $\varphi(v)(i) = v(\{i\})$ .

Note that this is the only axiom in our set that contains a mild *quantitative* aspect of efficiency: **D** implies that  $\sum_{i \in N} \varphi(v)(i) = 1$  in every game  $v \in \mathcal{SG}$  where all players are dummies. However, it suffices to uniquely characterize the SSPI along with **GL**, **T**, and **Sym**:

**Corollary 2.** There exists one, and only one, power index satisfying **GL**, **T**, **Sym** and **D**, and it is the SSPI  $\varphi_{ss}$ .

**Proof.** It is well-known that  $\varphi_{ss}$  satisfies **D**, and the rest of our axioms are also satisfied, by Theorem 1. Furthermore, if a power index  $\varphi$  satisfies the axioms, by Theorem 1  $\varphi = a\varphi_{ss}$  for some  $a \in \mathbb{R}$ . By **D**,  $\varphi(u_{\{1\}})(1) = 1$  (see (7) for the definition of  $u_{\{1\}}$ ) while  $a\varphi_{ss}(u_{\{1\}})(1) = a$ , which implies that  $a = 1$ .  $\square$

**Remark 1 (Weakening of GL).** Note that in the proof of Theorem 1, the *only* use of **GL** was to derive, in conjunction with **Sym**, the equality (5) in Lemma 1, i.e., that

$$\varphi(w_0)(i) = \varphi(w_1)(i) = \dots = \varphi(w_{n-1})(i) \quad (12)$$

for every  $i \in N$ , where  $w_0, \dots, w_{n-1}$  are the games defined by (4).<sup>6</sup> Thus, the *ad hoc* requirement on  $\varphi$  that (12) holds for every  $i \in N$ , which is weaker than the combination of **GL** and **Sym**, can be used as a substitute of **GL** in Theorem 1 and Corollary 2.

This requirement was used by Blair and McLean (1990) in the context of characterising "subjective valuations" of playing a game, as the axiom that pinpoints the efficient Shapley value in the set of (not necessarily efficient) semivalues. We do not introduce (12) as an explicit axiom, however, as we believe that **GL** is a natural and desirable property in most contexts, and that it has a stronger aesthetic appeal.

**Remark 2 (Using GL to characterize the Shapley value on the entire  $\mathcal{G}$ ).** Formula (1), applied to every  $v \in \mathcal{G}$ , defines the *Shapley value*  $\varphi_{ss} : \mathcal{G} \rightarrow \mathbb{R}^n$  on the entire  $\mathcal{G}$ . However, in the context of mappings  $\varphi : \mathcal{G} \rightarrow \mathbb{R}^n$  the straightforward extension of axiom **GL** is of little interest. Indeed, if it is assumed that (2)  $\implies$  (3) for *any*  $v, u \in \mathcal{G}$ , then **GL** does not follow from efficiency unlike in simple games, and in fact  $\varphi_{ss}$  does not satisfy it. The "right" version of **GL** for general games would assume (2)  $\implies$  (3) *provided*  $v(N) = u(N)$ . This version would characterize the Shapley value on  $\mathcal{G}$  together with other, standard axioms (linearity, equal treatment, and dummy).<sup>7</sup> However, the explicit conditioning on the worth of the grand coalition in the axiom significantly limits its appeal as a weaker substitute for efficiency.

Our last result strengthens Corollary 2 by replacing **Sym** in the characterization of the SSPI with the following well-known weaker version:

**Axiom VI: Equal Treatment (ET).** If  $i, j \in N$  are substitute players in the game  $v \in \mathcal{SG}$ , i.e., for every  $S \subset N \setminus \{i, j\}$   $v(S \cup \{i\}) = v(S \cup \{j\})$ , then  $\varphi(v)(i) = \varphi(v)(j)$ .

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<sup>6</sup>Note that the same argument as in the proof of Lemma 1 shows that  $\varphi(v)(i)$  is the same for all *symmetric* games  $v \in \mathcal{SG}$ , and not only for  $v = w_0, \dots, w_{n-1}$ .

<sup>7</sup>This follows from Corollary 1, since the axioms uniquely determine the restriction of the value to  $\mathcal{SG}$ , and in particular for unanimity games. Since these games are a linear basis for  $\mathcal{G}$ , the value is in fact uniquely determined on  $\mathcal{G}$ .

While **Sym** postulates that irrelevant characteristics of the players, outside of their role in the game  $v$ , have no influence on a power index, the weaker **ET** merely forbids discrimination between substitute players (with the same role in the game). In most axiomatizations **ET** suffices for the uniqueness of the index on  $\mathcal{SG}$  (or the value on  $\mathcal{G}$ ) when the efficiency is included in the set of axioms. The stronger **Sym** is necessary primarily in the context of infinite number of players (see, e.g., Aumann and Shapley (1974), Dubey et al (1981)). Our next theorem shows that, even with **GL** instead of efficiency, **ET** can replace **Sym** in the characterization of SSPI.

**Theorem 3.** There exists one, and only one, power index satisfying **GL**, **T**, **ET**, and **D**, and it is the SSPI  $\varphi_{ss}$ .

**Proof.** By Corollary 2,  $\varphi_{ss}$  satisfies all the axioms. It remains to show that the axioms uniquely determine  $\varphi_{ss}$ . Fix any power index  $\varphi$  that satisfies **GL**, **T**, **ET**, and **D**.

**Lemma 2.** For every  $T \subset N$

$$\varphi(u_T) = \varphi_{ss}(u_T), \quad (13)$$

where  $u_T$  is the unanimity game on  $T$  defined in (7).

**Proof of Lemma 2.** Consider the following two cases:

*Case 1:*  $T = N$ . Let  $\bar{\varphi}$  be the power index given by

$$\bar{\varphi}(v)(i) \equiv \frac{1}{n!} \sum_{\pi \in \Pi(N)} \varphi(\pi v)(\pi^{-1}(i))$$

for every  $v \in \mathcal{SG}$  and  $i \in N$ . It is easy to check that  $\bar{\varphi}$  satisfies **T**, **Sym** (and not just **ET**), and **D**.

For every  $i, j \in N$  and  $s = 0, 1, \dots, n-1$ ,

$$\varphi(w_s)(i) = \varphi(w_s)(j) \quad (14)$$

by **ET**, where  $w_s$  is the game defined in (4). Since  $\pi w_s = w_s$  for every  $\pi \in \Pi(N)$ , it follows from the definition of  $\bar{\varphi}$  and (14) that

$$\bar{\varphi}(w_s)(i) = \varphi(w_s)(i) \quad (15)$$

for every  $s = 0, 1, \dots, n-1$  and  $i \in N$ . Since Lemma 1 holds with **ET** instead of **Sym** as is easy to check, for every  $i \in N$

$$\varphi(w_1)(i) = \varphi(w_2)(i) = \dots = \varphi(w_n)(i),$$

and thus, using (15), also

$$\bar{\varphi}(w_1)(i) = \bar{\varphi}(w_2)(i) = \dots = \bar{\varphi}(w_n)(i). \quad (16)$$

This shows that  $\bar{\varphi}$  satisfies (12) in Remark 1 (in addition to **T**, **Sym**, and **D**), and hence by Remark 1 and Corollary 2,  $\bar{\varphi} = \varphi_{ss}$ . Since  $\bar{\varphi}(w_{n-1}) = \varphi(w_{n-1})$  by (15), in fact

$$\varphi(w_{n-1}) = \varphi_{ss}(w_{n-1}).$$

But  $w_{n-1} = u_N$ , and thus (13) is established for  $T = N$ .

*Case 2:  $T \subsetneq N$ .* Denote by  $\mathcal{SG}(T)$  the set of simple games on the set of players  $T$ , that can at the same time be viewed as the games in  $\mathcal{SG}$  whose carrier is a subset of  $T$ . Consider the *restricted* power index  $\varphi|_T: \mathcal{SG}(T) \rightarrow \mathbb{R}^T$ , given by  $(\varphi|_T)(v)(i) \equiv \varphi(v)(i)$  for every  $v \in \mathcal{SG}(T)$  and  $i \in T$ . Since by **D**  $\varphi(v)(i) = 0$  for every  $v \in \mathcal{SG}(T)$  and  $i \notin T$ , the knowledge of  $\varphi|_T$  completely determines  $\varphi$  on  $\mathcal{SG}(T)$  when viewed as a subset of  $\mathcal{SG}$ .

It is easy to see that  $\varphi|_T$  satisfies axioms **T**, **ET**, and **D** on  $\mathcal{SG}(T)$ . It also satisfies **GL**. To check this, assume that (2) holds for some  $i \in T$  and  $v, u \in \mathcal{SG}(T)$ . By the **GL** property of  $\varphi$ , there exists  $j \in N$  satisfying (3). It cannot be that  $j \in N \setminus T$  since then the inequality in (3) would not be strict by the **D** property of  $\varphi$ , and we conclude that  $j \in T$ .<sup>8</sup>

Now, just as in Case 1 (mimic the proof by taking  $N = T$  and  $n \equiv |T|$ ),

$$(\varphi|_T)(u_T) = (\varphi_{ss}|_T)(u_T),$$

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<sup>8</sup>This is the only place in our proofs where the assumption of *strict* inequality in (3) is used. For our previous results, Theorem 1 and Corollary 2, a weak inequality in (3) would have sufficed.

and thus also

$$\varphi(u_T) = \varphi_{ss}(u_T).$$

Consequently, (13) holds for every  $T \subset N$ .  $\square$

By Lemma 2,  $\varphi$  and  $\varphi_{ss}$  coincide on all unanimity games. Using equality (8) in the proof of Theorem 1 (which holds for any power index  $f$  that satisfies **T**, by Lemma 2.3 of Einy (1987), and in particular for  $\varphi$  and  $\varphi_{ss}$ ) finally implies that  $\varphi = \varphi_{ss}$  on the entire  $\mathcal{SG}$ .  $\square$

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