An Algorithmic Approach to Find Iterated Nash Equilibria in Extended Cournot and Bertrand Games with Potential Entrants

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Abstract

In this paper, in order to come up with a simple entry-exit model, we extend Cournot and Bertrand models by considering the fact that some firms might choose to remain as potential entrants in equilibrium. This might be related to less brand name recognition and consumer loyalty, cost disadvantages, or incapability of differentiating their products from others. In that regard, we study firms under Cournot and Bertrand game settings with heterogenous production costs in differentiated product markets and propose several iteration algorithms to find which potential players produce positive quantities in equilibrium. Our results show that there is a unique iterated Cournot-Nash equilibrium. Additionally, we study Bertrand models and present a new approach for understanding why an established firm can decrease its price in equilibrium when it is faced with a low threat potential entrant firm. Further, we show several examples in which pure strategies lead to multiple undominated iterated Bertrand-Nash equilibria. This result is very different from the existing literature on Bertrand models, where uniqueness usually holds under a linear market demand assumption. Next, we characterize the set of undominated equilibria for the Bertrand game. Our results provide additional evidence for why the Bertrand game is more competitive than the Cournot game. As an application of the model, we show that mergers increase incentives for market entry, which contradicts to the conventional wisdom.

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INTRODUCTION

Barriers to entry has been widely discussed in the literature. It is argued that established firms have comparative advantages over potential entrant firms through various channels. Those include brand name recognition and consumer loyalty via the use of advertising strategies, absolute cost advantages due to early entry, sunk costs, inability of differentiating their products, high capital requirements, international trade restrictions, and so on (McAfee et. al, 2004). We distinguish three types of potential entrants: 1-) High threat ones, 2-) Low threat ones, and 3-) No-threat ones. In an incumbent-entrant game, if the monopoly firm faces with a high threat potential entrant, a duopoly market formation is possible. On the other hand, low threat and no-threat potential entrant firms cannot enter into the market due to entry barriers, yet the former has an influence on the pricing decision of the monopoly firm. The literature is mainly concentrated on the interactions between established firms and high threat potential entrants. It is suggested that predatory pricing, which means the practice of a dominant firm selling its product at a very low price to make competition more difficult for new firms in expectation of getting more profits in the future, might create a barrier for entry. In many countries predatory pricing is considered anti-competitive and is illegal under competition laws. Moreover, it is argued that there are not so much real-life examples of predatory pricing because after the decline in prices, the surviving firms will not be in an industry with high barriers to entry. Hence, it is difficult to maintain high prices after some firms are driven out from the business.

We argue that the threat of some potential entrants might be low or none because of the discussed advantages of the incumbent firms. In that regard, we mainly focus on these types of potential entrant firms. We present a new approach for understanding why an established firm can decrease its price when it is faced with a low threat potential entrant firm. We work in a one-shot game setting and therefore incumbent firms cannot decrease their prices because of high profit expectations in the future. Nevertheless, they decrease their prices in total because it is each firm's optimal strategy to do so given other firms' strategies in the current period, a notion known as equilibrium.

Although Nash (1951) introduces the concept of equilibrium for any non-cooperative game, Cournot (1838) and Bertrand (1883) are the first to come with non-cooperative firm models based on quantity and price competitions respectively. Friedman (1983) later provides sufficiency conditions for the existence and uniqueness of equilibrium for these models. His sufficiency condition for uniqueness requires best response functions to be contractions. Shapiro (1989) argues that it is very difficult to have a unique equilibrium simply because best response functions of firms might intersect more than once. However, both firm models assume that all potential firms produce in equilibrium. In this study, in order to present our simple entry-exit model, we extend Cournot and Bertrand models by considering the fact that some firms might choose to remain as potential entrants in equilibrium even in the absence of any fixed costs. This might be related to bad market conditions, cost disadvantages or incapability of differentiating their products from others. It is well known that when two firms with different marginal cost levels play the Cournot game, it might be optimum for the cost-inefficient firm to not produce, which makes the cost-efficient firm monopoly in the market. Theoretically, the best response functions of the firms intersect at the boundary. To see this possibility, consider the following simple example:

Example 1: Let the set of firms be $N = \{1, 2\}$ and the marginal cost vector be $c = (8, 9.5)$. The linear market demand is specified as $p = 10 - q$. Under Cournot competition, the best response functions of the firms are $BR_1(q_2) = \frac{2-q_2}{2}$ and $BR_2(q_1) = \frac{0.5-q_1}{2}$. The related graphs of these functions are for each $i \in \{1, 2\}$, $Gr(BR_i) = \{\mathbf{q} \in \mathbb{R}^2 : q_i = BR_i(q_{-i})\}$. However, these unrefined best response graphs intersect in the negative region (see Figure 1) and the coordinates of this intersection, i.e. point N', are $q_1^* = 7/6$ and $q_2^* = -1/3$. However, negative production is not feasible by definition. Therefore, if firm one produces more than 0.5, it is optimal for the inefficient firm to not produce. Similarly, if firm two produces more than 2, firm one does not produce. Under this refinement, it is easy to see that the best response graphs intersect at point N characterized by $q_1^* = 1$ and $q_2^* = 0$ as shown in Figure 2. Thus, in the unique Nash equilibrium of this game, firm one is a monopoly and firm two is the no-threat potential entrant firm, which does not have any effect on the equilibrium strategy of the monopoly firm, in our terminology.

In this paper, we study firms under Cournot and Bertrand game settings with complete information of heterogenous production costs in differentiated product markets. To generalize the above two-firm example to an n-firm setting, we propose several iteration algorithms to find which potential players produce positive quantities in equilibrium. These iteration algorithms divide our initial potential finite number of firms into three: 1-) Incumbent firms 2-) Low threat potential entrant firms 3-) No-threat potential entrant firms, where low-threat entrants are only specific to the Bertrand game. Note that Friedman's (1983) uniqueness of equilibrium proof for the Cournot game is also valid in our extended set-up. We first provide an alternative proof for the uniqueness and propose a simple algorithm to find the closed form solution of the extended Cournot game. Our results show that there is a unique iterated Cournot-Nash equilibrium. Next, we consider a Bertrand game setting with imperfectly substitutable goods and impose nonnegativity of output constraints. We present theoretically possible examples in which multiple undominated Bertrand-Nash equilibria exist in pure strategies. Note that low threat of potential entrant firms force incumbent firms to decrease their price levels together in equilibrium in our one-shot game setting. Otherwise, if the potential entrant firm is not taken into account by the incumbent firms and it is efficient enough, then there might be some demand left to it, which is greater than its marginal cost of production, and it deviates to produce accordingly. Moreover, we come up with examples of games such that given any incumbent firm, there are multiple equilibria at which it has different pricing strategies. However, these examples involve at least three firms, where the most cost efficient two firms are incumbent and the least efficient one is potential entrant. This result is very different from the existing literature on Bertrand models, where uniqueness usually holds under a linear market demand assumption. In addition, we characterize the set of multiple undominated iterated Bertrand-Nash equilibria. In each equilibrium, the most cost efficient n^* number of firms actively produce, where n^* is the maximum number that leads to positive production for the least efficient one(s) among these n^* firms.

It is argued that Bertrand game is more competitive than the Cournot game (See Vives (1985), Cumbul (2011)). In this paper, we provide additional evidence for this claim. To see that, consider a two firm setup, where unrevised best response graphs of Cournot and Bertrand games lead to negative quantities for firm two, which contradicts with feasibility. We observe that in the Bertrand game, the efficient firm can sometimes force the inefficient firm to exit the market only by charging a price lower than the monopoly price in equilibrium. However, in the Cournot equilibrium, the efficient firm can always produce the monopoly quantity in order to ensure that the inefficient firm stays as a potential entrant. This simple argument partially validates the initial claim.

Next, we apply the presented entry-exit model to a merger setting and investigate the effects of mergers on incentives for market entry. We show that as opposed to the conventional wisdom, mergers increase incentives for market entry. (to be added..)

We then incorporate the axiomatic theory into the presented IO model and show that whereas the Cournot rule satisfies population monotonicity, consistency, and converse consistency, it is not replication invariant. However, Bertrand rule is not consistent. In the last section, we discuss possible generalizations on the cost and demand structures of the models.

The article is organized as follows. In Section 1, we state our formal models and get equilibrium prices, quantities and profits of firms under the assumption that there is enough demand for all potential entrant firms so that they all produce actively. In Section 2, we study the extended versions of the Cournot and Bertrand games respectively by considering possible demand insufficiencies, cost efficiency gaps, and product differentiation among firms. In that regard, we provide characterizations of undominated equilibrium(a) in both types of game settings. In Section 3, we apply the presented entry-exit model to a merger setting and investigate the effects of mergers on incentives for market entry. In Section 4, we define Cournot and Bertrand rules and study their axiomatic properties. In Section 5, we discuss possible generalizations of the models. The Appendix shows the related proofs.

1 MODEL

1.1 Notations and Assumptions:

Let $n \in \mathbb{N}$ and $N = \{1, 2..., n\}$ be a finite set of firms. The constant marginal cost levels of firms are specified as $c_1, c_2, ..., c_n$. Assume W.O.L.G. that the firms are ordered from the most efficient to the least efficient: $c_1 \leq c_2 \leq ... \leq c_{n-1} \leq c_n$. Each firm i is producing a differentiated product, q_i, and the assumed total consumer quadratic utility function over these products is given by:

$$U(q_1, q_2, ..., q_n) \quad = \quad \sum_{i=1}^{n} A_i q_i - \frac{1}{2} \sum_{i=1}^{n} \delta_i q_i^2 - \theta \sum_{i=1}^{n-1} \sum_{j>i}^{n} q_i q_j + G \tag{1}$$

where $G = M - \sum_{i=1}^{n} p_i q_i$ is the private income spent on other goods, for each $i \in N$, $\delta_i \geq \theta$, and the ratio $\theta / \delta_i \in [0, 1]$ shows how much product i is differentiated from others, and A_i is the demand parameter, which can capture brand name recognition and consumer loyalty. For each $i \in N$, taking derivative with respect to each differentiated product q_i gives us the demand function as[1]:

$$p_i \quad = \quad A_i - \delta_i q_i - \theta \sum_{j \neq i}^{n} q_j \tag{2}$$

Note the closer θ is to δ_i, the higher the degree of substitutability and the level of competition between firms. For example, as an extreme case, when $\theta = \delta_i$ for each $i \in N$, the products are perfect substitutes and no longer differentiable. On the other hand, when $\theta = 0$ for each $i \in N$, each firm is a monopoly for the good it produces.

Next, we make the following assumptions:

Assumption 1: Each firm i has a constant marginal cost c_i. Firms have full information.

Assumption 2: For each firm $i \in N$, $A_i > c_i$.

Assumption 3: There are no fixed costs and no capacity constraints.

In the next step, we calculate the price, quantity, and profit levels of firms under Cournot and Bertrand competitions.

[1] This demand function is also studied by Shapiro (1989).

1.2 COURNOT COMPETITION:

In the Cournot model, firms maximize their profits by taking other firms' optimum quantity levels as given. Hence, each firm faces the following objective function:

$$\max_{q_i} \pi_i = (A_i - \delta_i q_i - \theta \sum_{j \neq i}^{n} q_j - c_i) q_i \tag{3}$$

For each $j \in N$, let $\lambda_j = A_j - c_j$. For each nonempty subset $S \subseteq N$, each $j \in N$, let $\gamma_{-j}(S) = \prod_{l \neq j}^{S} (2\delta_l - \theta)$ and $\gamma(S) = \prod_{l \in S} (2\delta_l - \theta)$. If the induced quantities from the above optimization problem turn out to be positive, then equilibrium price, quantity and profit levels of firm $i \in N$ are respectively given by

$$q_i^C(S) = \frac{\lambda_i(\gamma(S) + \theta \sum_{j \in S} \gamma_{-j}(S)) - \theta \sum_{j \in S} \lambda_j \gamma_{-j}(S)}{(2\delta_i - \theta)(\gamma(S) + \theta \sum_{j \in S} \gamma_{-j}(S))} \tag{4}$$

$$p_i^C(S) = q_i^C(S) + c_i \tag{5}$$

$$\pi_i^C(S) = (q_i^C(S))^2 \tag{6}$$

where $S = N$ and the capital C denotes the Cournot competition.

1.3 BERTRAND COMPETITION:

For each $j \in N$, let $\delta_j = 1$, and $A_j = A$. In the Bertrand model, firms compete through prices rather than quantities. Hence, they take the optimum price levels of other firms as given and maximize their profits. To create our optimization problem, we first solve (2) for quantities and get the unrevised demand functions as

$$q_i = a_n - b_n p_i + d_n \sum_{j \neq i}^{n} p_j \tag{7}$$

where $a_n = \frac{A}{1+\theta(n-1)}$, $b_n = \frac{1+\theta(n-2)}{[1-\theta][1+\theta(n-1)]}$, and $d_n = \frac{\theta}{[1-\theta][1+\theta(n-1)]}$.

Note that eq.(7) may result in some negative quantities for some firms which is an infeasibility. Accordingly, we first propose the following iteration algorithm to get the revised demand functional forms.

Iteration Algorithm 0: Let $X_i = \{1, 2, ..., i\}$. Take a price vector $p = (p_1, p_2, ..., p_n)$. WLOG, assume that $p_1 \leq p_2 \leq ... \leq p_n$.

STEP 0: For all $i \in X_n$, if $p_i \leq \frac{a_n + d_n \sum_{j \in X_n} p_j}{b_n}$ then $q_i = a_n - b_n p_i + d_n \sum_{j \in X_n} p_j$. Otherwise, proceed to the next step [2] .

STEP 1: For all $i \in X_{n-1}$, if $p_i \leq \frac{a_{n-1} + d_{n-1} \sum_{j \in X_{n-1}} p_j}{b_{n-1}}$, then $q_i = a_{n-1} - b_{n-1} p_i + d_{n-1} \sum_{j \in X_{n-1}} p_j$ and $q_n = 0$. Otherwise, proceed to the next step.

STEP 2: For all $i \in X_{n-2}$, if $p_i \leq \frac{a_{n-2} + d_{n-2} \sum_{j \in X_{n-2}} p_j}{b_{n-2}}$, then $q_i = a_{n-2} - b_{n-2} p_i + d_{n-2} \sum_{j \in X_{n-2}} p_j$ and for all $l \in N \setminus X_{n-2}$, $q_l = 0$. Otherwise, proceed to the next step.

Since the number of firms is finite, this algorithm stops at a step k. The demand function that each firm receives can be stated as:

$$\begin{array}{ll} \forall i \in X_{n-k}, & q_i = a_{n-k} - b_{n-k} p_i + d_{n-k} \sum_{j \in X_{n-k}} p_j \\ \forall l \in N \setminus X_{n-k}, & q_l = 0 \end{array} \tag{8}$$

Next, we write down the objective function that a firm faces as

$$\max_{p_i} \pi_i = \max_{p_i} (a_n - b_n p_i + d_n \sum_{j \neq i} p_j)(p_i - c_i) \tag{9}$$

By taking the derivative with respect to p_i and equating it to zero, we get the best response function of firm i as:

$$BR_i(\mathbf{p}_{-i}) = p_i^* = \frac{a_n + d_n \sum_{X \setminus i} p_j + b_n c_i}{2b_n} \tag{10}$$

where $\mathbf{p}_{-i} = (p_1, p_2, ..., p_{i-1}, p_{i-1}, ..., p_n)$ is the price vector that does not contain the i^{th} dimension. If the induced quantities from the above optimization problem turn out to be positive, then equilibrium price, quantity and profit levels are respectively given by:

$$p_i^B = \frac{a_n(2b_n + d_n) + b_n(2b_n - d_n(n-1))c_i + b_n d_n c_t}{(2b_n + d_n)(2b_n - d_n(n-1))} \tag{11}$$

[2] Using eq.(7), for all $i \in X_n$, $q_i \geq 0$ if and only if $p_i \leq \frac{a_n + d_n \sum_{j \in X_n} p_j}{b_n}$. The inequalities in the other steps are found similarly by playing around the set X.

$$q_i^B = b_n(p_i^B - c_i) \tag{12}$$

$$\pi_i^B = \frac{(q_i^B)^2}{b_n} = b_n(p_i^B - c_i)^2 \tag{13}$$

2 EXTENDED MODELS

2.1 Extended Cournot Game with Potential Entrants

In this section, we study quantity setting games such that the induced quantities from the optimization problem given in eq(9) is negative for some firms. This might be partially related to low market demand conditions, cost disadvantages, or incapability of differentiating their products from others. Therefore, we have to impose non-negativity of production constraints. We first propose the following iteration algorithm in order to determine the critical set N^* to state our characterization theorem.

Iteration Algorithm 1

STEP 1: Let all firms play the Cournot game. If all equilibrium quantities are positive, then $N^* = N$. If not, denote the firms producing positive quantities by the N_1 and proceed to the next step.

STEP 2: Let firms in N_1 play the Cournot game. If all equilibrium quantities are positive, then $N^* = N_1$. If not, denote the firms producing positive quantities by the set N_2 and proceed to the next step.

STEP 3: Let firms in N_2 play the Cournot game. If all equilibrium quantities are positive, then $N^* = N_2$. If not, denote the ones producing positive quantities by the set N_3 and proceed to the next step.

Since the number of firms is finite, the set N^* will be determined at most $n-1$ steps. Note that this iteration algorithm divide our initial finite number of firms into two: 1-) Incumbent firms, e.g, firms in N^* 2-) No-threat potential entrant firms, e.g, firms in $N \setminus N^*$. Let N^* be found by Iteration Algorithm 1. We next state our characterization theorem

Theorem 1 (Characterization): *In the unique equilibrium, for each $i \in N^*$, the production and price vectors of firm i are given by eqs.(4) and (5) respectively, which are calculated at $S = N^*$. The remaining firms produce zero.*

Proof: Please see a proof of Theorem 1 in the Appendix.

The bottom line is that a firm is likely to be eliminated through Iteration Algorithm 1 if it faces a low demand, is incapable of differentiating its products from others, and produces at high cost level. Therefore, these three conditions are the main determinants of market entry incentives of firms. Moreover, the existence of no threat potential entrant firms do not have an effect on the quantity decisions of the incumbent firms.

Applications of Theorem 1:

In this section, we study the applications of Theorem 1 by giving some examples.

Example 2: Let $N = \{1, 2, 3\}$, $(c_1, c_2, c_3) = (8, 9, 9.5)$, and $\theta = 1$. The market demand is given by $p_i = 10 - q_i - q_{-i}$. The best responses can be calculated as $BR_1(q_2, q_3) = \frac{2-q_2-q_3}{2}$, $BR_2(q_1, q_3) = \frac{1-q_1-q_3}{2}$, and $BR_3(q_1, q_2) = \frac{0.5-q_1-q_2}{2}$. For each $i \in \{1, 2, 3\}$, let the unrevised best response graphs be $GR(BR_i) = \{q \in \mathbb{R}^3 : q_i = BR_i(q_{-i})\}$. These unrefined best response graphs intersect at $q = (9/8, 1/8, -3/8)$. Iteration Algorithm 1 gives $N^* = \{1\}$. Note that monopoly output of firm 1 is one. Hence, Theorem 1 assures that $q^\star = (q_1^*, q_2^*, q_3^*) = (1, 0, 0)$ is the unique Nash equilibrium of this game. Indeed, in Figure 2, we refine the best response graphs as follows: When firm 2 and firm 3 produce more than two in total, it is optimal for firm 1 to not produce. Therefore, firm 1's best response function becomes the $q_2 - q_3$ plane. Similarly, we refine the best response graphs of firms 2 and 3. Note that the refined best response functions graphs meet at a unique equilibrium point $N = (1, 0, 0)$, as desired.

Example 3: In the above example, we change the marginal cost vector from $(c_1, c_2, c_3) = (8, 9, 9.5)$ to $(c_1, c_2, c_3) = (8, 8.5, 9.2)$ keeping everything else the same. The reader can easily verify that the unrefined best response graphs intersect at $N' = (\frac{3.7}{4}, \frac{1.7}{4}, \frac{-1.1}{4})$ as seen in Figure 3. Iteration Algorithm 1 gives $N^* = \{1, 2\}$. When firm 1 and firm 2 play the game, their best response graphs intersect at the positive quadrant of the $q_1 - q_2$ plane, i.e. at $N = (\frac{2.5}{3}, \frac{1}{3})$. Observe that when we project point N' onto the $q_1 - q_2$ plane (not necessarily a perpendicular projection), we get point N. By Theorem 1, point N is the unique Nash equilibrium of this game. In addition, this point lies at the unique intersection of the refined best response planes of the firms, as desired.

2.2 Extended Bertrand Game with Potential Entrants

Similar to the Cournot analysis, we study the type of games in such when all firms play the Bertrand strategies, some of them are supposed to produce negative quantities, which is not feasible. Therefore non-negativity of production constraints should be imposed. We mainly focus on undominated equilibria. In each of the second type of equilibria, the most efficient n^* number of firms play the Bertrand game, where n^* is the maximum number that leads to a nonnegative production for the least efficient firm among these n^* firms. Unlike the Cournot case, if $n^* \geq 2$, there might be multiple equilibria among which an incumbent firm charge different prices.

Undominated Equilibria:

Let the most efficient n^* number of firms play the Bertrand game where n^* is the maximum number that leads to a nonnegative production for the least efficient firm among these n^* firms. We find this n^* through an iteration algorithm written for Bertrand game as follows:

Iteration Algorithm 2

STEP 1: Let all n firms play the Bertrand game. If all equilibrium prices lead to nonnegative production, then $n^* = n$. If not, proceed to the next step.

STEP 2: Let the most efficient $n-1$ firms play the Bertrand game. If all equilibrium prices of firms lead to nonnegative production, then $n^* = n-1$. If not, proceed to the next step.

STEP 3: Let the most efficient $n-2$ firms play the Bertrand game. If all equilibrium prices of firms lead to nonnegative production, then $n^* = n-2$. If not, proceed to the next step.

Since the number of firms is finite, n^* will be determined by at most $n-1$ steps. In particular, there exists a step k such that the most efficient $n-k+1$ firms play the Bertrand game and all equilibrium prices of firms lead to nonnegative productions, i.e, $n^* = n-k+1$. Interestingly, with this algorithm, we can identify three types of potential entrants: 1-) High threat ones, 2-) Low threat ones, and 3-) No-threat ones. Low threat potential entrant firms cannot enter into the market due to entry barriers, yet they have an influence on the pricing decision of the incumbent firms.

Next, consider a problem and let the above algorithm gives $n = n^*$. Let $X = \{1, 2, ..., n^*\}$. We define two critical marginal cost levels for firm n^*+1. Let $\bar{c}$ be the marginal cost level such that if firm n^*+1's marginal cost level were $\bar{c}$ (ceteris paribus), it would have produced exactly zero as a result of the Bertrand game played among the most efficient n^*+1 firms. On the other hand, R_{n^*+1} denotes the residual demand left to firm n^*+1 following the Bertrand game played among the most efficient n^* firms. Formally, these levels can be written as [3]

$$
\begin{aligned}
\bar{c} &= \tfrac{A(1-\theta)(2+\theta(2n^*-1))+\theta(1+\theta(n^*-1))c_T^*}{2+\theta(-3+\theta+n^*(3+\theta(n^*-3)))} \\
R_{n^*+1} &= \tfrac{A(1-\theta)(2+\theta(2n^*-3))+\theta(1+\theta(n^*-2))c_T^*}{(1+\theta(n^*-1))(2+\theta(n^*-3))}
\end{aligned}
\tag{14}
$$

where $c_T^* = \sum_{i \in X} c_i$.

Let $\mathbf{p}$ and $\mathbf{p'}$ denote any arbitrary n^* and n^*+1 dimensional price vectors respectively, i.e, $\mathbf{p} = (p_1, p_2, ..., p_{n^*})$ and $\mathbf{p'} = (p_1, p_2, ..., p_{n^*+1})$. Next, define simplexes as $\triangle^{n^*-1} = \{\mathbf{p} \in \mathbb{R}_+^{n^*} : \sum_{j \in X} p_j = \frac{b_{n^*+1}c_{n^*+1} - a_{n^*+1}}{d_{n^*+1}}\}$, where the superscript n^*-1 shows the dimension of the simplex. Take any $i \in X$. Let $BR_i^\star(.)$ denote the best response of firm i to other firms' prices in the $n^* - poly$ market formed by the most efficient n^* firms. Similarly, let $BR_i^{\star\star}(.)$ denote the best response of firm i to other firms' prices in the $(n^*+1) - poly$ market formed by the most efficient n^*+1 firms. Formally, using eq.(10), these best response functions of firm i are respectively given by

$$
\begin{aligned}
&BR_i^\star : \mathbb{R}_{++}^{n^*-1} \to \mathbb{R}_{++} \;\text{ s.t. }\; BR_i^\star(\mathbf{p}_{-i}) = \tfrac{a_{n^*} + d_{n^*} \sum_{X \setminus i} p_j + b_{n^*} c_i}{2b_{n^*}} \\
&BR_i^{\star\star} : \mathbb{R}_{++}^{n^*} \to \mathbb{R}_{++} \;\text{ s.t. }\; BR_i^{\star\star}(\mathbf{p'}_{-i}) = \tfrac{a_{n^*+1} + d_{n^*+1}(\sum_{X \setminus i} p_j + p_{n^*+1}) + b_{n^*+1} c_i}{2b_{n^*+1}}
\end{aligned}
$$

where $\mathbf{p}_{-i} = (p_1, ..., p_{i-1}, p_{i+1}, ..., p_{n^*})$ and $\mathbf{p'}_{-i} = (p_1, ..., p_{i-1}, p_{i+1}, ..., p_{n^*+1})$. Now define the unprojected and projected best response graphs as $Gr^u(BR_i^\star) = \{\mathbf{p} \in \mathbb{R}_+^{n^*} : p_i = BR_i^\star(\mathbf{p}_{-i})\}$ and $Gr^{proj}(BR_i^{\star\star}) = \{\mathbf{p} \in \mathbb{R}_+^{n^*} : p_i = BR_i^{\star\star}(\mathbf{p}_{-i}, p_{n^*+1} = c_{n^*+1})\}$, where in the latter we first constraint firm i's best response curve by letting $p_{n^*+1} = c_{n^*+1}$, then project the resulting constrained set into n^* dimensional space. Now, take any $\mathbf{x} \in \triangle^{n^*-1} \cap Gr^u(BR_i^\star)$ and $\mathbf{y} \in \triangle^{n^*-1} \cap Gr^{proj}(BR_i^{\star\star})$. The i^{th} dimension of the price vectors $\mathbf{x}$ and $\mathbf{y}$ are independent of the price vectors of firms in $X \setminus \{i\}$ and calculated as

$$
\begin{aligned}
x_i = p_i^\star &= \tfrac{(b_{n^*} + d_{n^*})c_{n^*+1} + b_{n^*} c_i}{2b_{n^*} + d_{n^*}} \\
y_i = p_i^{\star\star} &= \tfrac{(b_{n^*+1} + d_{n^*+1})c_{n^*+1} + b_{n^*+1} c_i}{2b_{n^*+1} + d_{n^*+1}}
\end{aligned}
\tag{15}
$$

In the Appendix, we prove that for each $i \in X$, $p_i^\star > p_i^{\star\star}$ [4].

Proposition 1: *Let $\theta \in (0,1)$ and $p_i^m = \frac{A+c_i}{2}$ be the monopoly price charged by firm i.*

[3] These equations are derived in the proof of Proposition 1.

[4] For a geometrical configuration of these points in two dimensions, please visit Figures 6 and 7, which are drawn for Example 6.

a- If $c_{n^*+1} \geq R_{n^*+1}$, then the following pricing strategies form iterated Bertrand-Nash equilibria:

$$\begin{array}{ll} \forall i \in X, & p_i^* = \frac{a_{n^*}(2b_{n^*}+d_{n^*})+b_{n^*}(2b_{n^*}-d_{n^*}(n^*-1))c_i+b_{n^*}d_{n^*}c_T^*}{(2b_{n^*}+d_{n^*})(2b_{n^*}-d_{n^*}(n^*-1))} \\ \forall j \in N \setminus X, & p_j^* \in [c_j, p_j^m] \end{array} \tag{16}$$

At each equilibrium, the equilibrium production levels of firms are given by

$$\begin{array}{ll} \forall i \in X, & q_i^* = b_{n^*}(p_i^* - c_i) \\ \forall j \in N \setminus X, & q_j^* = 0 \end{array} \tag{17}$$

b- If $c_{n^*+1} < R_{n^*+1}$, then firms in X charge prices of the form $E = \{\boldsymbol{p}^* \in \mathbb{R}_+^{n^*} : p_i^{\star\star} \leq p_i^* \leq p_i^\star$ and $\sum_{i \in X} p_i^* = \frac{b_{n^*+1}c_{n^*+1}-a_{n^*+1}}{d_{n^*+1}}\}$, where $p_i^\star$ and $p_i^{\star\star}$ are defined in eq(15). Along this equilibrium path, $p_{n^*+1}^* = c_{n^*+1}$, $q_{n^*+1}^* = 0$; and $\forall j \in N \setminus \{X \cup \{n^*+1\}\}$, $p_j^* \in [c_j, p_j^m]$ and $q_j^* = 0$.

Proof: Please see a proof of Proposition 1 in the Appendix.

Next, we state our characterization theorem for the Bertrand case.

Theorem 2 (Characterization): *The equilibria stated in Proposition 1 are the only undominated Bertrand-Nash equilibria of the game.*

Proof: Please see a proof of Theorem 2 in the Appendix.

Applications of Proposition 1 and Theorem 2

We discuss the results of Proposition 1 and Theorem 2 in three examples. We stress that we need at least three firms in order to come up with an example of game with multiple equilibria among which incumbent firms have different pricing strategies.

Example 4: Let $N = \{1, 2\}$ and $c = (9, 14.1)$. The market demand is given by $p_i = 16 - q_i - 0.5q_{-i}$. We let both firms play the Bertrand game and the interaction of the unrefined best response functions gives a price vector of $p^* = (12.01, 14.05)$. Note that the inefficient firm, i.e. firm 2, charges a price lower than his marginal cost. Given these prices, the production vector $q = (4.02, -0.06)$ is contradicting the nonnegativity constraint. Thus, p^* is not the equilibrium price vector. We let then firm 1 be the monopoly firm. It consequently charges the monopoly price of $p_1^m = 12.5$ and produces $q_1^m = 3.5$. Given this monopoly production, the demand left to firm 2 is $A - \theta q_1^m = 14.25$. However, firm 2 can make a profit by charging a price of $p_2 \in (14.1, 14.25)$. That is, firm 1 cannot be a monopoly firm in an equilibrium by charging the monopoly price of 12.5 either. Similarly, firm 2 cannot be a monopoly firm.

We claim that $p = (12.2, 14.1)$ constitutes the unique undominated equilibrium of this game[5].

i-)Existence: First note that when $q_2 = 0$, $p_1 = 12.2$ given by eq.(7). Firm 1 does not have an incentive to decrease its price level from 12.2, which would cause him to exercise less monopoly power trivially. Now consider the effect of increasing its price level slightly from $p_1 = 12.2$ to $\tilde{p}$. The potential entrant, i.e. firm 2, enters into the market and starts to produce. Given $p_2 = c_2 = 14.1$, the duopoly best response of firm 1 is a point given by $p_1 = 12.025$. However, the original price level 12.2 is closer to 12.025 than $\tilde{p}$ proving that firm 1 loses. Thus, $p = (12.2, 14.1)$ is an equilibrium point[6].

ii-)Uniqueness: In Figure 4, the unrevised best response functions of firms 1 and 2 are $B_1^U(p_2) = 8.5 + p_2/4$ and $B_2^U(p_1) = 11.05 + p_1/4$ respectively. The related graphs of these functions are $Gr(B_1^U) = \{\mathbf{p} \in \mathbb{R}^2 : p_1 = B_1^U(p_2)\}$ and $Gr(B_2^U) = \{\mathbf{p} \in \mathbb{R}^2 : p_2 = B_2^U(p_1)\}$, which are shown by rays $]CE[$ and $]EG[$ respectively. They intersect at E, which is associated with the infeasible production vector $q = (4.02, -0.06)$. Accordingly, we draw the revised best response graphs of firms one and two, i.e. $Gr(BR_1)$ and $Gr(BR_2)$, which are shown by blue and red lines respectively. W.L.O.G., consider firm two. $Gr(BR_2^U)$ intersect with c_2 at $N = (12.2, 14, 1)$. Thus, if $p_1 \leq 12.2$, it is optimal for firm two to not produce. Note that charging any price weakly above $seg[AN]$ results in zero production for firm two by iteration algorithm 0. Moreover, if firm one charges a price higher or equal the total demand, e.g, A, firm two will be the monopoly firm and her best response is to charge the monopoly price. Similarly, we revise the best response function of firm one. In sum, both best responses meet at a unique undominated Nash equilibrium outcome given by $N = (12.2, 14, 1)$. $Seg[DN)$ constitutes the weakly dominated equilibria of the game.

The bottom line is that there might be a threat of potential entrant firm and this threat changes the equilibrium pricing strategy of the incumbent firm. This observation gives additional evidence for why the Bertrand game is more competitive than the Cournot game. Firm 1 can only keep firm 2 out of the market by charging a smaller price than the monopoly price of 12.5. However, if we had considered a Cournot game set-up in which the inefficient firm is supposed to produce negatively, it would have always been optimal for this inefficient firm to be a potential entrant when the efficient firm produces the monopoly output as in Example 1.

To show the possibility of multiple undominated equilibria in a two-firm setup, we change c_2 from 14.1 to 15 in the above example. In Figure 5, we draw the revised best response functions of firms 1 and 2 in a similar fashion as above. Observe that we have multiple

[5]I am very grateful to Gábor Virág for pointing out this equilibrium in this game.

[6]Note that, this equilibrium point can itself be interpreted in two ways. First, firm 1 is a monopoly and charges a price of 12.2 and produces 3.8. Second, firm 1 and firm 2 is a duopoly and they charge $p = (12.2, 14.1)$. The related production vector is $q = (3.8, 0)$.

undominated equilibria in which firm 1 charges a monopoly price of $p_1^M = 12.5$ and firm 2 charges any price within $[c_2, p_2^M] = [15, 15.5]$. However, firm 2 does not produce in each equilibrium.

Example 5-Multiple Equilibria with Three Firms: Let $N = \{1, 2, 3\}$ and $c = (9, 14.8, 14.9)$. The market demand is given by $p_i = 17 - q_i - 0.5q_{-i}$. We let firms play the Bertrand game and find $q^* = (0.11, -0.09, -0.18)$, which is not feasible. Iteration algorithm 2 gives $n^* = 1$. We let then firm 1 be the monopoly firm. Accordingly, it charges the monopoly price of $p_1^m = 13$ and produces $q_1^m = 4$. However, $R_2 = R_3 = A - \theta q_1^m = 15 > \max\{c_2, c_3\}$. Therefore, they both have an incentive to deviate. In order to eliminate a possible deviation by firm 2, i.e. $q_2 = 0$ and $p_2 = 14.8$, firm 1 should decrease its unconstrained monopoly price to $p^* = \frac{b_2c_2 - a_2}{d_2} = 12.6$ coming from eq.(7). A similar argument shows that if $p_1 \leq 12.8$, firm 3 does not produce. Clearly, when $p_1 \leq 12.6$, firm 1 is still the monopoly firm and does not want decrease his production level because it wants to be as close as possible to the monopoly optimum price of 13. Now given $p_2 = 14.8$, consider firm 1 increasing his price level up to $\tilde{p}$. If $\tilde{p} \in (12.6, 12.8]$, then only firm 2 starts to produce. Given $p_2 = 14.8$, in a duopoly market formed by firms 1 and 2, firm 1 optimally charges a price of 12.45 found by using eq(10). Thus, since 12.6 is more closer to 12.45 than $\tilde{p}$, this deviation is not profitable for firm 1. If $\tilde{p} > 12.8$, then both firms 2 and 3 start to produce. Having in mind that $p_2 = 14.8$ and $p_2 = 14.9$, the optimal price of firm 1 is now 12.28 in the triopoly market. Hence, firm 1 is again worse off. Altogether, firm 1 charges $p_1 = 12.6$ and produces $q_1 = 3.6$ in the monopoly equilibrium.

Example 6-Multiple Equilibria with Three Firms: We change the marginal cost vector from $c = (9, 14.8, 14.9)$ to $c = (14, 16, 16.1)$ in the above example. We let firms play the Bertrand game and find $q^* = (1.78, 0.07, -0.02)$, which is not feasible. Iteration algorithm 2 gives $n^* = 2$. In the absence of firm 3, using eqs.(12), and (11), the duopoly equilibrium price and quantity levels are $p^d = (p_1^d, p_2^d) = (15.27, 16.07)$ (Point O in Figure 6) and $q^d = (q_1^d, q_2^d) = (1.69, 0.09)$ respectively. However, $R_3 = A - \theta(q_1^d + q_2^d) = 16.11 > c_3$. In order to prevent a possible deviation by firm 3, R_3 should be less than or equal to 16.1. Using eq.(7), $q_3 = 0$ simplifies to $p_1 + p_2 = \frac{b_3c_3 - a_3}{d_3} = 31.3$, which is denoted by the simplex $\triangle^1$ in Figure 6. Note that charging a total price lower or equal to 31.3 by firms 1 and 2 assures that firm 3 is not in the market. Given $p_3 = c_3 = 16$, for each $i \in \{1, 2\}$, we draw unprojected and projected best response graphs of firms 1 and 2, i.e. $Gr^U(BR_i^\star)$ and $Gr^{proj}(BR_i^{\star\star})$, in duopoly and triopoly markets denoted by straight and dotted lines respectively. It is important to note that the dotted and straight lines are respectively only valid on the above and below of the green line. The coordinates of the critical points are: $A = (14, 17.3)$, $B = (15.2, 16.1)$, $C = (15.24, 16.06)$, $D = (15.242, 16.05)$, $E = (15.26, 16.04)$, and $F = (15.3, 16)$. We also have the individual rationality constraints stating that $p_1^d \geq c_1$ and $p_2^d \geq c_2$ [7].

[7] A price lower than marginal cost can only be supported by a zero production level.

Claim: We claim that any price combination on the line segment $seg[CD]$ shown in Figure 6 are the only undominated equilibrium.

i-)Existence(Figure 6): Each price combination in $seg(AF)$ is a candidate for undominated equilibrium. $seg(AF)$ is a border for firm 3's production and if either firm 1 or 2 charges a slightly higher price on this segment, firm 3 starts to produce and projected best response graphs become valid. Whereas the vertical arrows represent the directions of the possible deviations by firm 2, horizontal arrows show firm 1's possible deviations. For example, on $seg(EF]$, given other firms' prices, firm 2 has an incentive to deviate to a slightly higher price from p_2 to $p_2+\varepsilon$ with $\varepsilon > 0$, because firm 3 starts to produce after this increase and the projected best response set of firm 2, i.e, $Gr^{proj}(BR_2^{\star\star})$, becomes valid. However, $p_2 + \varepsilon$ is more close to $Gr^{proj}(BR_2^{\star\star})$ than p_2 showing that firm 2 gains. Note that, there is no profitable deviation by firm 1 or firm 2 on line segment $seg[CD]$, where the feasibility (Green line) and firm rationality constraints (Yellow lines) are also satisfied. Hence all price combinations of firms 1 and 2 on the segment $seg[CD]$ constitute an undominated Bertrand-Nash equilibrium. As a check, for each $\hat{\mathbf{p}} \in seg[CD]$, for each $i \in \{1,2\}$, $p_i^{\star\star} \leq \hat{p}_i \leq p_i^{\star}$ and $\sum_{i\in 1,2} \hat{p}_i = 31.3$ as claimed in part b of Proposition 1. Note that along this equilibrium path, firm 3 charges his marginal cost level, i.e. $p_3 = 16.1$, and produces nothing.

ii-)Characterization(Figure 7): To see the equilibria stated above are the only undominated equilibria, consider a duopoly market formation by firms 1 and 3 playing the game. The graphs of the best response functions are drawn in a similar way to Figure 6. The critical points are $A = (14.95, 16.5)$,$B = (14.96, 16.04)$,$C = (15.14, 15.85)$, and $D = (15.2, 15.8)$. When firms one and three play the game, the duopoly price vector is at the intersection of unprojected two firm best response graphs, i.e. $GR^u(BR_i^{\star})'s$, and is given by O but the demand left to firm 2 is greater than c_2. In order to eliminate entry incentives of firm 2, incumbent firms decrease their total prices to 31. Let $\triangle^1 = \{\mathbf{p} \in \mathbb{R}^2 : p_1 + p_3 = 31\}$. But, arrows show that given other firms' prices, for every price combination on Δ^1, either firm 1 or firm 3 deviates to decrease or increase their price levels. Accordingly, their is not any equilibrium in a game played among the most and least cost efficient firms. As a check, there is no price vector $\hat{\mathbf{p}}$ on $\triangle^1$ such that for each $i \in \{1,3\}$, $p_i^{\star\star} \leq \hat{p}_i \leq p_i^{\star}$ and $\sum_{i\in 1,3} \hat{p}_i = 31$ as claimed in Theorem 2. Similarly, firm 2 and 3 cannot be duopoly firms in equilibrium because firm 1 does not produce only if $p_1+p_3 \leq 25$, which is not feasible. Moreover, neither firm 2 nor firm 3 can be monopoly firms, because that would require them to charge a price lower than their marginal costs, i.e $p = 11$, to guarantee zero production by firm 1. Lastly, consider the case where firm 1 is a monopoly. If $p_1 \leq 15$, firm 2 stays out as a potential entrant. Also, the optimal monopoly price of firm 1 is $p_1^m = 15.5$. Thus, a decrease from a price level of 15 is not beneficial for firm 1. However, as he increases his price level slightly, firm 2 enters into the market, and given $p_2 = 16$, firm 1 optimally sets $p^{\circ} = 15.25$. Thus, there is a room for deviation.

3 An Application to Mergers

In this subsection, we apply the presented entry-exit model to a merger setting. Our main motivation is identify the effects of mergers on the incentives of the potential entrant firms to enter into the market. In that regard, we adapt the heterogeneous cost model studied by Cumbul(2011).

3.1 Notations and Assumptions

Let $N = \{1, 2..., n\}$ be the finite set of firms and $k \in [1, n]$ of them exogenously merge. Let $N = I \cup O$, where I and O denote the sets of insiders and outsiders respectively. For each $j \in N$, let $\delta_j = 1$, and $A_j = A$. Each firm i produces a different product at a marginal cost of c_i. Let firm e be the most cost efficient insider with a marginal cost level of c_e. We still assume that there is a representative consumer, who gets the exact utility given in eq(1) and therefore each firm i faces the linear demand given in eq(2). We next make the following assumptions:

Assumption 1: Firms have full information and there are not any fixed costs and capacity constraints.

Assumption 2: A merger occurs if and only if it is profitable.

Assumption 3: There is rationalization of production and mergers bring full cost synergies. All insiders' post merger marginal cost level becomes c_e.

We next solve the Cournot and Bertrand models under these assumptions. Assume $A1$, $A2$, and $A3$ hold.

3.1.1 Cournot Model

Since each insider's marginal cost level becomes c_e following their merger, their optimization problem becomes symmetric. Therefore, for each insider firm $i \in I$, equilibrium quantity, price, and profit levels are

$$q = \frac{A(2-\theta) - c_e(2+\theta(n-k-1)) + \theta \sum_{O} c_i}{w} \tag{18}$$

$$p = (1+\theta(k-1))q + c_e \tag{19}$$

$$\pi = (1+\theta(k-1))q^2 \tag{20}$$

where $w = 4 + 2\theta(n + k - 3) + \theta^2(2 - k^2 + n(k - 2))$.

For each outsider firm $j \in O$ such that $O \neq \emptyset$, equilibrium quantity, price, and profits levels are

$$x_j = U - \frac{c_j}{2 - \theta} \tag{21}$$

$$r_j = x_j + c_j \tag{22}$$

$$\rho_j = {x_j}^2 \tag{23}$$

where $U = \frac{A(2-\theta)+\theta\sum_O c_i - \theta(2-\theta)kq}{(2-\theta)(2+\theta(n-k-1)))}$.

3.1.2 Bertrand Model

For each insider firm $i \in I$, equilibrium quantity, price, and profits levels are

$$\mathrm{p} = \frac{a(2b+d)+(b-d(k-1))(2b-d(n-k-1))c_e+bd\sum_O c_j}{2(b-d(k-1))(2b-d(n-k-1))-d^2(n-k)k} \tag{24}$$

$$\mathrm{q} = (b - d(k - 1))(\mathrm{p} - c_e) \tag{25}$$

$$\pi = (b - d(k - 1))(\mathrm{p} - c_e)^2 \tag{26}$$

For each outsider firm $j \in O$ such that $O \neq \emptyset$, equilibrium quantity, price, and profits levels are

$$\mathrm{r}_j = V + \frac{bc_j}{2b + d} \tag{27}$$

$$\mathrm{x}_j = b(\mathrm{r}_j - c_j) \tag{28}$$

$$\rho_j = b(\mathrm{r}_j - c_j)^2 \tag{29}$$

where $V = \frac{(2b+d)(a+dk\mathrm{p})+bd\sum_O c_j}{(2b+d)(2b-d(n-k-1))}$.

To be continued..

SECTION 4: AXIOMATIC APPROACH:

Converting the Model into the Axiomatic Framework:

We mainly follow the notations of Thomson (1998). There is an infinite set of "potential" firms indexed by the natural numbers $\mathbb{N}$. Each group of firms N is drawn from the family $\mathcal{N}$ of non-empty finite subsets of $\mathbb{N}$. Let the set of all possible demand functional forms be $\mathcal{P}$.

An economy is a five-tuple $\xi = (N, \mathcal{A}, \Theta, c, p) \in \mathcal{N} \times \mathbf{R}_+^N \times \mathbf{R}^{N(N-1)/2} \times \mathbf{R}_+^N \times \mathcal{P}^N$ where N is the number of firms, $\mathcal{A} = (\mathcal{A}_i)_{i \in N}$ is the demand parameter vector, $\Theta = (\Theta_{ij})_{i,j \in N, i<j}$ and Θ_{ij} is the degree of substitutability between firms i and j's products, $c = (c_i)_{i \in N}$ is the marginal cost vector; and $p = (p_i)_{i \in N}$ is the demand vector. Each firm in an economy is supposed to determine how much to produce[8]. Let ζ^N denotes the set of all possible problems. Let ζ_L^N be the domain of problems where each firm's demand is linear as stated in eq.(2); each has a constant marginal cost; and for each $i, j \in N$, we fix $\mathcal{A}_i = A$ and $\Theta_{i,j} = \theta$ where $\theta \in [0,1]$. A feasible allocation for $\xi = (N, \mathcal{A}, \Theta, c, p) \in \zeta_L^N$ is a vector $z = (z_i)_{i \in N} \in \mathbf{R}_+^N$ such that $0 \leq \sum_{i=1}^N z_i \leq \frac{A}{\theta}$ if $\theta \in (0,1]$ and $0 \leq z_i \leq A$ if $\theta = 0$. Let $Z(\xi)$ be the set of feasible allocations of ξ. A solution is a function $\varphi : \zeta_L^N \to \mathbf{R}_+^N$ which associates with each $N \in \mathcal{N}$ and for each $\xi \in \zeta_L^N$, $\varphi(\xi) \in Z(\xi)$. For each $i \in N$, let $\pi_i : \mathcal{N} \times \mathbf{R}_+ \times (0,1) \times \mathbf{R}_+^N \to \mathbf{R}_{++}$ such that $\pi_i = \pi_i(N, A, \theta, c)$ be the profit function that each firm receives. For all $i \in N$, let R_i be the following binary relation; $\forall \xi \in \zeta_L^N$, $\forall x, y \in Z(\xi)$, $x \, R_i \, y$ iff $\pi_i(x) \geq \pi_i(y)$.

Next, we define consistency, population monotonicity, and converse consistency to our model.

CONSISTENCY:

Consistency requires that each remaining firm get his original component of x as a solution to the reduced economy. Let $N \in \mathcal{N}$ and consider a problem ξ such that N could face. Let $x \in \varphi(\xi)$. Now, consider some of the firms leaving with their components of x. If N' is the subgroup of remaining firms, we denote $r_{N'}^x(\xi)$ as the set of alternatives where the firms who leave receive their components of x and refer to it as the reduced problem of ξ with respect to N' and x. Note that in the reduced economy, the residual demand that each remaining firm receives is given by $A - \theta \sum_{N \setminus N'} x_i$.

Consistency: For all groups $N \in \mathcal{N}$, all problems $\xi \in \zeta_L^N$, all subgroups $N' \subset N$, all $x \in \varphi(\xi)$, if the reduced problem of ξ with respect to N' and x, obtained from ξ by assigning to all agents in $N \setminus N'$ their components of x, belongs to $\zeta_L^{N'}$, then $x_{N'} \in \varphi(r_{N'}^x(\xi))$.

[8]Firms may compete on prices first and found optimal quantities accordingly as in the Bertrand game.

POPULATION MONOTONICITY:

Population monotonicity postulates that as the number of firms increases in an economy, the welfare of already existing firms decreases or vice versa. There are mainly two effects of increasing the number of firms in an economy. It might be the case the newcomers are so efficient that the number of firms that are actually participating in the game strictly decreases in the end. We call the change in the number of agents in the economy as the competition effect. Although a decrease in the number of agents is beneficial for the initial firms that are still in the game, it also has a cost. The overall efficiency in the market increases. We denote the second effect as the efficiency effect. The final effect depends on which effect dominates the other one.

Let $N \in \mathcal{N}$ and consider a problem ξ such that N could face. Let $N' \subseteq N$. The reduced economy ξ with respect to N' is defined as $r_{N'}(\xi) = (N', \mathcal{A} \mid_{N'}, \Theta \mid_{N'}, c \mid_{N'}, p \mid_{N'})$.

Population Monotonicity: A rule φ is population monotonic iff for each $\xi = (N, \mathcal{A}, \Theta, c, p) \in Z(\xi)$, each $N' \subseteq N$;

$$\forall i \in N' : \varphi_i(r_{N'}(\xi)) \; R_i \; \varphi_i(\xi)$$

CONVERSE CONSISTENCY:

This property deduces that an alternative x is chosen for some problem by the solution if its restriction to each two-firm subgroup is chosen for the reduced problem associated with the subgroup and x. Formally,

Converse Consistency: A rule φ is conversely consistent iff for each $\xi = (N, \mathcal{A}, \Theta, c, p) \in Z(\xi)$, each $x \in Z(\xi)$;

$$[\forall N' \in N, |N'| = 2, \forall i \in N' : \varphi_i(r^x_{N'}(\xi)) = x_i] \Rightarrow x = \varphi(\xi)$$

Let $\xi \in \zeta^N_L$. We define two rules for this problem. W.O.L.G., we order the firms from the most cost efficient to the least cost efficient as usual. Let $N \in \mathcal{N}$ and the critical values n^* and n^{**} is found by the iteration algorithms 1 and 3 respectively.

Cournot Rule($\varphi = CN$): This rule assigns production levels stated in eq.(4) to the most efficient n^* firms and 0 to the remaining firms. Iteration algorithm 1 guarantees the feasibility of this allocation. We stress that because the Cournot solution is single valued by

Theorem 1, it is indeed a rule.

Bertrand Rule($\varphi = BN$): Let $\theta \in (0,1)$. If $n^{**} \geq 1$ and $c_{n^{**}+1} \geq R_{n^{**}+1}$, then this rule assigns production levels stated in eq.(12) to the most efficient n^{**} firms and 0 to the remaining firms where $R_{n^{**}+1}$ is stated in eq.(39). Moreover, if $n^{**} = 1$ and $c_{n^{**}+1} \leq R_{n^{**}+1}$, firm 1's production is given by $q_1 = \frac{a_2(b_2+d_2)-c_2(b_2^2-d_2^2)}{d}$ whereas the remaining firms produce nothing. Iteration algorithm 2 guarantees the feasibility of this allocation [9].

PROPOSITION 2: *Let $\xi \in \zeta_L^N$. The Cournot rule is population monotonic, consistent, and conversely consistent.*

Proof: Please see a proof of Proposition 3 in the Appendix.

PROPOSITION 3: *The Bertrand rule is not consistent.*

Proof: We prove it by giving a counter example. In example 4, we have shown that the Bertrand rule assigns a quantity vector of $q = (3.8, 0)$, which is associated with the price vector of $p = (12.2, 14.1)$. However, if firm 2 goes away with his production, which is zero, the Bertrand rule assigns firm 1 to produce 3.5 and charge the monopoly price of 12.5. Observe that this production level and the initial level differ by an amount of 0.3, which creates inconsistency. ■

GENERALIZATIONS

We discuss possible generalizations on the cost and demand structures of the models. (To be added...)

[9] By Proposition 2, when $n^{**} \geq 2$ and $c_{n^{**}+1} < R_{n^{**}+1}$, the Bertrand Solution is multi-valued. In that regard, we restrict the domain of problems to define the Bertrand rule.

CONCLUSION:

In this paper, we developed a simple entry-exit model in which relative cost levels of the firms are the main determinants of who is producing in the market. In that regard, we studied the oligopoly models of Cournot and Bertrand in the case where some firms may have so high costs that they may stay out of the market. We considered a general model with product differentiation, and constant, but different marginal costs across firms. In the Cournot model, a unique equilibrium can be found by an iteration algorithm.

On the other hand, when firms engage in price competition, we developed a similar algorithm that stops at some critical level n^* like above. However, when $n^* \geq 2$ and we have at least n^*+1 firms, we showed that there may be multiple equilibria in which firms from 1 to n^* charges different prices at each equilibrium. These equilibria are sustained simply because if firm n^*+1 is just indifferent between entering or not, the profit functions of participating firms have kinks. Therefore, the first order conditions can be written as inequalities, which admit several solutions. In essence, these multiplicity of equilibria issues arise because price competition games are more competitive than quantity competition games. Hence, efficient incumbent firms may not simply ignore potential entrant firms and accordingly these firms altogether lower their prices just to keep the most efficient potential entrant firm out of the market. This result is very different from the existing literature on Bertrand models, where uniqueness usually holds under a linear market demand assumption.

In the last section, we converted our model into an axiomatic framework and showed that whereas Cournot rule is consistent, Bertrand rule is not. The main reason behind this result is that the existence of potential entrant firms might be effective in determining equilibrium outcomes as noted above. Moreover, we also see that Cournot rule satisfies population monotonicity and converse consistency.

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APPENDIX:

Proof of Theorem 1: Consider a problem $\xi = (N, \theta, (A_i)_{i\in N}, (\delta_i)_{i\in N}, (c_i)_{i\in N})$. Let $N = \{1, 2, ..., n\}$ and N^* be found by Iteration Algorithm 1. Let $U = N \setminus N^*$. For each nonempty $S \subseteq N$, let $\alpha(S) = \sum_{k\in S} \lambda_k \gamma_{-k}(S)$ and $\beta(S) = \gamma(S) + \sum_{k\in S} \theta\gamma_{-k}(S)$ where $\gamma(S) = \prod_{l\in S} (2\delta_l - \theta)$ and for each $j \in N$, $\gamma_{-j}(S) = \prod_{l\neq j}^{S} (2\delta_l - \theta)$.

a-) Existence: Consider any firm $k \in U$ with marginal cost $c_k = c_k^*$. Let $\bar{c}$ be the (hypothetical) marginal cost level such that $q_k(N^* \cup k) = 0$ if $c_k = \bar{c}$, ceteris paribus [10]. By Iteration Algorithm 1, $c_k^* > \bar{c}$. To calculate $\bar{c}$, letting $q_k(N^* \cup k) = 0$ and $c_k = \bar{c}$ in eq.(4) and simplifying the resulting equation yields

$$\bar{c} = \frac{\beta(N^*)A_k - \alpha(N^*)\theta}{\beta(N^*)} \tag{30}$$

Now consider the game played among firms in N^*. Their total production is found by summing across outputs given in eq(4) as

$$\sum_{i\in N^*} q_i(N^*) = \frac{\alpha(N^*)}{\beta(N^*)} \tag{31}$$

Finally, using eq(4), it can be shown that the residual demand left firm k, i.e. R_k, is equal to $\bar{c}$ as follows:

$$R_k = A_k - \theta \sum_{i\in N^*} q_i(N^*) = \bar{c} \tag{32}$$

Since $\bar{c} < c_k^*$, in order for firm k to produce a positive quantity, it needs to charge a price lower than his marginal cost level, which is an impossibility. Hence, equilibrium conditions are met. ■

b-) Uniqueness:

Take any two firms i and j with marginal costs c_i^* and c_j^* such that $i \in N^*$ but $j \in U$. By Iteration Algorithm 1, $q_j(N^* \cup j) \leq 0$ according to unrevised best response graphs. Using eq.(4), this condition simplifies to

$$\frac{\lambda_j}{\theta} \leq \frac{\alpha(N^*)}{\beta(N^*)} \tag{33}$$

[10] Depending on the parameters of the problem, $\bar{c}$ might be negative.

By Iteration Algorithm 1, $q_i(N^*) > 0$. Moreover, by using eq.(4) again, $q_i(N^* \cup j) = (2\delta_i - \theta)q_i(N^*) > 0$. Hence, $q_i(N^* \cup j) > q_j(N^* \cup j)$, which simplifies as

$$\frac{\lambda_i(2\delta_j - \theta) - \lambda_j(2\delta_i - \theta)}{2\theta(\delta_j - \delta_i)} > \frac{\alpha(N^*)}{\beta(N^*)} \tag{34}$$

Suppose to show a contradiction there exists an equilibrium in which firm j is in it but firm i is not. Denote the set of firms forming this equilibrium by Z. The same arguments in the existence proof shows that the demand left to firm i is

$$R_i = A_i - \theta\frac{\alpha(Z)}{\beta(Z)} \tag{35}$$

$q_j(Z) > 0$ by definition of equilibrium and it simplifies to $\frac{\lambda_j}{\theta} > \frac{\alpha(Z)}{\beta(Z)}$. Combining it with inequality given in (33) gives

$$\frac{\alpha(N^*)}{\beta(N^*)} > \frac{\alpha(Z)}{\beta(Z)} \tag{36}$$

Now let $Z' = Z \cup i$. Combining eq(36) with eq.(34) yields $q_i(Z') > q_j(Z')$. But, $q_j(Z') = (2\delta_j - \theta)q_j(Z) > 0$ and therefore $q_i(Z') > 0$ as well. Finally, let $\bar{c}$ be such that if $c_i = \bar{c}$ then $q_i(Z') = 0$. It becomes trivial that $\bar{c} > c_i^*$. Following the existence proof, $R_i = \bar{c} > c_i^*$. Hence firm i has an incentive to deviate to produce by charging a price in (c_i^*, R_i), as desired. ■

Proof of Proposition 1:

Consider a problem $\xi = (N, A, \theta, c)$. Iteration algorithm 2 gives $n = n^*$. Let $X = \{1, 2, ..., n^*\}$. If $q_{n^*} = 0$, then using eq.(11) and eq.(12)

$$c_{n^*} = \frac{a_{n^*}(2b_{n^*} + d_{n^*}) + b_{n^*}d_{n^*}c_T}{(b_{n^*} + d_{n^*})(2b_{n^*} - d_{n^*}(n^* - 1))} \tag{37}$$

where $c_T^* = \sum_{i \in X} c_i$. If $q_{n^*} > 0$, as done in the proof of Theorem 1, we find the critical value $\bar{c}$ such that if firm n^*+1's marginal cost level were $\bar{c}$ (ceteris paribus), it would have produced exactly zero as a result of the Bertrand game played among the most efficient $n^* + 1$ firms. Since $q_{n^*+1} = 0$, using eq.(12), we have $p_{n^*+1} = \bar{c}$. Therefore, substituting eq.(11) into this last equality gives:

$$\bar{c} = \frac{a_{n^*+1}(2b_{n^*+1} + d_{n^*+1}) + b_{n^*+1}d_{n^*+1}c_T^*}{(b_{n^*+1} + d_{n^*+1})(2b_{n^*+1} - d_{n^*+1}n^*) - b_{n^*+1}d_{n^*+1}} \tag{38}$$

In the next step, we calculate the residual demand left to firm $n^* + 1$, i.e, $R_{n^*+1} = A - \theta q_T$ where $q_T = \sum_{i \in X} q_i$ and for each $i \in X$, q_i is stated in eq.(12) and a, b,and d are calculated at $n = n^*$. Accordingly, it is simplified as

$$R_{n^*+1} = \frac{A(1-\theta)(2+\theta(2n^*-3)) + \theta(1+\theta(n^*-2))c_T^*}{(1+\theta(n^*-1))(2+\theta(n^*-3))} \tag{39}$$

In case, $R_{n^*+1} \leq c_{n^*+1}$, firm $n^* + 1$ does not have any incentive to deviate. Moreover, firms that are more inefficient than firm $n^* + 1$ cannot have profitable deviations. If $q_{n^*} = 0$, then comparing eq.(37) and eq.(39) gives $R_{n^*+1} = c_{n^*} < c_{n^*+1}$. Indeed, by iteration algorithm 0, for each $i \in N \setminus X$, $q_i = 0$.

Therefore assume that $q_{n^*} > 0$. Accordingly, $c_{n^*} < R_{n^*+1}$. By iteration algorithm 2, we have $c_{n^*+1} \geq \max\{\bar{c}, c_{n^*}\}$. Hence, we differentiate two cases.

CASE 1: $R_{n^*+1} \geq c_{n^*+1} \geq \bar{c} \geq c_{n^*}$

Subtracting eq.(38) from eq.(39) yields:

$$R_{n^*+1} - \bar{c} = \frac{\theta^3(1-\theta)(An^* - c_T^*)}{(1+\theta(n^*-1))M_1(\theta, n^*)} > 0 \tag{40}$$

which is positive as a result of assumption 2 whenever $\theta \in (0,1)$, where $M_1 : (0,1) \times \mathbb{N}_+ \to \mathbb{R}$ such that $M_1(\theta, n^*) = 2 + \theta(3(n^*-1) + \theta(1 + n^*(n^*-3)))$. [11] By lemma 1, when $n^* = 1$ and $\bar{c} < c_2 < R_2$, then the uniqueness of iterated Bertrand Nash equilibrium is assured. Consequently, to finish the proof of Case 1, we prove the following claim:

Claim 1: Let $n^* \geq 2$. If $\bar{c} < c_{n^*+1} < R_{n^*+1}$, then multiple equilibria exist.

Proof of Claim 1-(Figures 8-9): Let $n^* \geq 2$ and assume that $\bar{c} \leq c_{n^*+1} \leq R_{n^*+1}$. In order to prevent a possible deviation by firm $n^* + 1$, firms in X lower their price levels accordingly. Given that $q_{n^*+1} = 0$, eq.(7) yields $\sum_{i \in X} p_i = \frac{b_{n^*+1}c_{n^*+1} - a_{n^*+1}}{d_{n^*+1}}$.

Let $\mathbf{p}$ and $\mathbf{p'}$ denote any arbitrary n^* and $n^* + 1$ dimensional price vectors respectively, i.e, $\mathbf{p} = (p_1, p_2, ..., p_{n^*})$ and $\mathbf{p'} = (p_1, p_2, ..., p_{n^*+1})$. Next, for each $c_{n^*+1} \in [\bar{c}, R_{n^*+1}]$, define simplexes as $\triangle^{n^*-1}_{c_{n^*+1}} = \{\mathbf{p} \in \mathbb{R}^{n^*}_+ : \sum_{j \in X} p_j = \frac{b_{n^*+1}c_{n^*+1} - a_{n^*+1}}{d_{n^*+1}}\}$, where the superscript $n^* - 1$ shows the dimension of the simplex. Take any $i \in X$. Let $BR_i^{\star}(.)$ denote the best response of firm i to other firms' prices in the $n^* - poly$ market formed by the most efficient n^* firms. Similarly, let $BR_i^{\star\star}(.)$ denote the best response of firm i to other firms' prices in the

[11] Note that $\frac{\partial M_1(.)}{\partial n^*} = \theta(3 + \theta(2n^* - 3)) > 0$. Thus, M_1 is minimized at $n^* = 1$ and becomes $M_1(\theta, 1) = 2 - \theta^2 > 0$ as desired.

$(n^*+1)-poly$ market formed by the most efficient n^*+1 firms. Formally, using eq.(10), these best response functions of firm i are respectively given by

$$
\begin{aligned}
BR_i^\star : \mathbb{R}_{++}^{n^*-1} \to \mathbb{R}_{++} \text{ s.t. } BR_i^\star(\mathbf{p}_{-i}) &= \tfrac{a_{n^*}+d_{n^*}\sum_{X\setminus i} p_j + b_{n^*}c_i}{2b_{n^*}} \\
BR_i^{\star\star} : \mathbb{R}_{++}^{n^*} \to \mathbb{R}_{++} \text{ s.t. } BR_i^{\star\star}(\mathbf{p'}_{-i}) &= \tfrac{a_{n^*+1}+d_{n^*+1}(\sum_{X\setminus i} p_j + p_{n^*+1}) + b_{n^*+1}c_i}{2b_{n^*+1}}
\end{aligned}
$$

where $\mathbf{p}_{-i}=(p_1,...,p_{i-1},p_{i+1},...,p_{n^*})$ and $\mathbf{p}'_{-i}=(p_1,...,p_{i-1},p_{i+1},...,p_{n^*+1})$. Now define the unprojected and projected best response planes as $Gr^u(BR_i^\star)=\{\mathbf{p}\in\mathbb{R}_+^{n^*}: p_i = BR_i^\star(\mathbf{p}_{-i})\}$ and for each $c_{n^*+1}\in[\bar{c},R_{n^*+1}]$, $Gr^{proj}(BR_i^{\star\star}(c_{n^*+1}))=\{\mathbf{p}\in\mathbb{R}_+^{n^*}: p_i = BR_i^{\star\star}(\mathbf{p}_{-i}, p_{n^*+1}=c_{n^*+1})\}$, where in the latter we first constraint firm i's best response plane by letting $p_{n^*+1}=c_{n^*+1}$, then project the resulting constrained set into n^* dimensional space. For each $c_{n^*+1}\in[\bar{c},R_{n^*+1}]$, take any $\mathbf{x}\in\triangle_{c_{n^*+1}}^{n^*-1}\cap Gr^u(BR_i^\star)$ and $\mathbf{y}\in\triangle_{c_{n^*+1}}^{n^*-1}\cap Gr^{proj}(BR_i^{\star\star})$. The i^{th} dimension of the price vectors $\mathbf{x}$ and $\mathbf{y}$ are independent of the price vectors of firms in $X\setminus\{i\}$ and calculated as

$$
\begin{aligned}
x_i = p_i^\star(c_{n^*+1}) &= \tfrac{(b_{n^*}+d_{n^*})c_{n^*+1}+b_{n^*}c_i}{2b_{n^*}+d_{n^*}} \\
y_i = p_i^{\star\star}(c_{n^*+1}) &= \tfrac{(b_{n^*+1}+d_{n^*+1})c_{n^*+1}+b_{n^*+1}c_i}{2b_{n^*+1}+d_{n^*+1}}
\end{aligned} \tag{41}
$$

We now take the convex hull of all of the price vectors such that any distinct n^*-1 number of $Gr^u(BR_i^\star)$'s and $\triangle_{R_{n^*+1}}^{n^*-1}$ intersect. Accordingly, define $Z_1 = conhull(\bigcup_{i_1=1}^{n^*}\{\mathbf{p}\in\mathbb{R}_+^{n^*}: \mathbf{p}\in\bigcap_{i=i_1}^{i_2}(Gr^u(BR_i^\star)\cap\triangle_{R_{n^*+1}}^{n^*-1})\})$ where $i_2\equiv n^*-2+i_1 \pmod{n^*}$. (In Figures 8 and 9, $Z_1=seg[BC]$ and $Z_1=A[ABC]$ respectively.) Put it differently, we take the convex hull of $\binom{n^*}{n^*-1}$ critical intersection price vectors to form Z_1. Let the set of these price vectors be $U=\{A_1,A_2,...,A_{n^*}\}$, where for each $i\in X$, $A_i=(p_1^{\star\star}(R_{n^*+1}),p_2^{\star\star}(R_{n^*+1}),...,p_{i-1}^{\star\star}(R_{n^*+1}),$ $\sum_{i\in X}p_i^\star(R_{n^*+1})-\sum_{j\in X\setminus i}p_j^{\star\star}(R_{n^*+1}),p_{i+1}^{\star\star}(R_{n^*+1})...,p_{n^*}^{\star\star}(R_{n^*+1}))$ and $p_i^{\star\star}(.)$ is stated in eq.(41-b). In sum, $Z_1=conhull(A_i: i\in X)$. Let $Z_1'=strictconhull(A_i: i\in X)$.

Similarly, we take the convex hull of all of the price vectors such that any distinct n^*-1 number of Φ_i^{proj}'s and $\triangle_{\bar{c}}^{n^*-1}$ intersect. Formally, let $Z_2 = conhull(\bigcup_{i_1=1}^{n^*}\{\mathbf{p}\in\mathbb{R}_+^{n^*}: \mathbf{p}\in\bigcap_{i=i_1}^{i_2}(Gr^{proj}(BR_i^{\star\star}(\bar{c}))\bigcap\triangle_{\bar{c}}^{n^*-1})\})$ where $i_2\equiv n^*-2+i_1 \pmod{n^*}$. (In Figures 8 and 9, $Z_2=seg[GH]$ and $Z_2=A[DEF]$ respectively.). Note that Z_2 is formed by the convex hull of $\binom{n^*}{n^*-1}$ critical intersection points denoted in the set $V=\{B_1,B_2,...,B_{n^*}\}$, where for each $i\in X$, $B_i=(p_1^\star(\bar{c}),p_2^\star(\bar{c}),...,p_{i-1}^\star(\bar{c}),\sum_{l\in X}p_l^{\star\star}(\bar{c})-\sum_{j\in X\setminus i}p_j^\star(\bar{c}),p_{i+1}^\star(\bar{c})...,p_{n^*}^\star(\bar{c}))$ where for each $i\in X$, $p_i^\star(.)$ is stated in eq.(41-a). Hence, $Z_2=conhull\{B_1,B_2,...,B_{n^*}\}$. Lastly, define $Z_2'=strictconhull\{B_1,B_2,...,B_{n^*}\}$.

Subclaims 1 and 2 assures that feasibility and firm rationality constraints are satisfied.

Subclaim 1 (Feasibility Constraint): $\frac{b_{n^*+1}c_{n^*+1}-a_{n^*+1}}{d_{n^*+1}} > c_T^* = \sum_{i\in X} c_i$.

Proof Subclaim 1: The left hand side of the inequality is increasing in c_{n^*+1}, therefore it is enough to prove the inequality holds when c_{n^*+1} is substituted by its lower bound $\bar{c}$, which is stated in eq.(38). Hence, changing c_{n^*+1} by $\bar{c}$ and simplifying the result gives

$$\frac{\theta(1-\theta)(1+\theta n^*)(An^*-c_T^*)}{(1+\theta(n^*-1))M_1(\theta,n^*)} > 0 \tag{42}$$

where $M_1(.) > 0$ is stated in eq.(40), as desired.

Subclaim 2 (Firm Rationality Constraint): For each $i \in X$ and each $c_{n^*+1} \in \{\bar{c}, R_{n^*+1}\}$, $p_i^{\star}(c_{n^*+1}) > p_i^{\star\star}(c_{n^*+1}) > c_i$.

Proof Subclaim 2: Take any $i \in X$. Let $c_{n^*+1} \in \{\bar{c}, R_{n^*+1}\}$. Since $c_i < c_{n^*+1}$ by iteration algorithm 2, $p_i^{\star\star}(c_{n^*+1}) > c_i$ by eq.(41-b). Next, define $D : (0,1) \times \mathbb{N}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_{++}$ such that $D(\theta, n^*, c_i, c_{n^*+1}) = p_i^{\star}(c_{n^*+1}) - p_i^{\star\star}(c_{n^*+1})$. By using eq.(41), taking the derivative of the function $D(.)$ with respect to c_{n^*+1} and c_i respectively yields $\frac{\partial D(.)}{\partial c_{n^*+1}} = -\frac{\partial D(.)}{\partial c_i} = \frac{\theta^2}{(2+\theta(2n^*-1))(2+\theta(2n^*-3))} > 0$. Additionally, when $c_i = c_{n^*+1} = \bar{c}$, $D(.)$ becomes zero. However, Iteration Algorithm 2 provides $c_{n^*+1} > c_i$, which proves the claim.

Subclaim 3: For each $i \in X$, the price vectors A_i and B_i are both defined in the firm rational set $F = \{\mathbf{p} \in \mathbb{R}_{++}^{n^*} : \forall j \in X, p_j > c_j\}$.

Proof Subclaim 3: Take any $i \in X$. First, for each $j \in X \setminus \{i\}$, $p_j^{\star\star}(R_{n^*+1}) > c_i$ and $p_j^{\star}(\bar{c}) > c_i$ by subclaim 2. Therefore, it is enough to show that the i^{th} dimensions of price vectors A_i and B_i are both defined in F. To show the former, by subclaim 2, for each $i \in X$, $\frac{b_{n^*+1}R_{n^*+1}-a_{n^*+1}}{d_{n^*+1}} = \sum_{i\in X} p_i^{\star}(R_{n^*+1}) > \sum_{j\in X\setminus i} p_j^{\star\star}(R_{n^*+1}) + c_i$ proving that for each $i \in X$, the price vector A_i is defined on the firm rational region.

To show the i^{th} argument of B_i is positive, we claim that $\frac{b_{n^*+1}\bar{c}-a_{n^*+1}}{d_{n^*+1}} = \sum_{l\in X} p_l^{\star\star}(\bar{c}) > \sum_{j\in X\setminus i} p_j^{\star}(\bar{c})$. Now, define $E : (0,1) \times \mathbb{N}_+ \times \mathbb{R}_{++}^{n^*} \times \mathbb{R}_{++} \to \mathbb{R}_+$ such that $E(\theta, n^*, \mathbf{c}, \bar{c}) = \sum_{l\in X} p_l^{\star\star}(\bar{c}) - \sum_{j\in X\setminus i} p_j^{\star}(\bar{c})$ where $\mathbf{c} = (c_1, c_2, ..., c_{n^*})$. Substituting the related price vectors and the value of $\bar{c}$ respectively stated in eq.(41) and eq.(38) into $E(.)$ yields

$$E(\theta, n^*, c^*, \bar{c}) = \frac{A(1-\theta)(2+\theta(-3+\theta+2M_0)) + c_T^*\theta(1+\theta(-1+M_0)) + M_2 c_i}{(2+\theta(2n^*-3))M_1} \tag{43}$$

where $c_T^* = \sum_{i\in X} c_i$, $M_0 = n^*(2+\theta(n^*-2)) > 1$, $M_1 > 0$ is defined in eq.(42), and $M_2 = (1+\theta(n^*-2))M_1$. Since $\frac{\partial E(.)}{\partial c_i} = \frac{M_2}{(2+\theta(2n-3))M_1} > 0$, for a given value of n^* and θ, E(.) gets its minimum at $c_i = 0$. However, $E(c_i = 0) > 0$ by eq.(43) and consequently, $E(.) > 0$ verifying the claim.

In the next step, we show the uniqueness of the equilibrium at the extreme values that c_{n^*+1} can take.

Subclaim 4: The set $\bigcap\limits_{i\in X} Gr^{u}(BR_i^{\star})\cap \triangle_{R_{n^*+1}}^{n^*-1}$ is a singleton and denoted by $\{N_1\}$. Moreover, $\{N_1\} \in Z_1^{'}$.

Proof Subclaim 4: Let $c_{n^*+1} = R_{n^*+1}$. We first claim that for each $i \in X$, $Gr^{u}(BR_i^{\star})$ and $\triangle_{R_{n^*+1}}^{n^*-1}$ intersect at a fix point N_1[12]. To see that, using eq.(41-a) it is enough to prove that $\frac{b_{n^*+1}R_{n^*+1}-a_{n^*+1}}{d_{n^*+1}} = \sum_{i\in X} p_i(R_{n^*+1})$ as follows:

$$\frac{b_{n^*+1}R_{n^*+1} - a_{n^*+1}}{d_{n^*+1}} = \sum_{i\in X} \frac{(b_{n^*} + d_{n^*})R_{n^*+1} + b_{n^*}c_i}{2b_{n^*} + d_{n^*}} \tag{44}$$

Rewriting eq.(44) gives the residual demand equation, i.e, R_{n^*+1}, stated in eq.(39), as desired.

We claim that $N_1 \in Z_1^{'}$. Indeed, $N_1 = \sum_{i\in X} \alpha_i A_i$ where $\alpha_i = \frac{p_i^{\star}(R_{n^*+1})-p_i^{\star\star}(R_{n^*+1})}{p_T^{\star}(R_{n^*+1})-p_T^{\star\star}(R_{n^*+1})}$, where $p_T^{\star}(R_{n^*+1}) = \sum_{i\in X} p_i^{\star}(R_{n^*+1})$ and $p_T^{\star\star}(R_{n^*+1}) = \sum_{i\in X} p_i^{\star\star}(R_{n^*+1})$. Note that $\sum_{i\in X}\alpha_i = 1$ trivially. We claim that for each $i \in X$, $0 < \alpha_i < 1$. Take any $i \in X$. By subclaim 2, $p_i^{\star}(R_{n^*+1}) > p_i^{\star\star}(R_{n^*+1})$ and therefore $p_T^{\star}(R_{n^*+1}) > p_T^{\star\star}(R_{n^*+1})$ proving that $\alpha_i > 0$. Moreover, $p_T^{\star}(R_{n^*+1}) > p_T^{\star\star}(R_{n^*+1}) + p_i^{\star}(R_{n^*+1}) - p_i^{\star\star}(R_{n^*+1})$ by the same subclaim showing that $\alpha_i < 1$. Hence, $N_1 \in Z_1^{'}$.

Subclaim 5: The set $\bigcap\limits_{i\in X} Gr^{proj}(BR_i^{\star\star}(\bar{c}))\cap \triangle_{\bar{c}}^{n^*-1}$ is a singleton and denoted by $\{N_2\}$. Moreover, $\{N_2\} \in Z_2^{'}$.

Proof Subclaim 5: Assume that $c_{n^*+1} = \bar{c}$. We claim that for each $i \in X$, $Gr^{proj}(BR_i^{\star\star}(\bar{c}))$ and $\triangle_{\bar{c}}^{n^*-1}$ intersect at a fix point N_2. To see that, by using eq.(41-b), it is enough to prove that $\frac{b_{n^*+1}\bar{c}-a_{n^*+1}}{d_{n^*+1}} = \sum_{i\in X} p_i^{\star\star}(\bar{c})$ as follows:

$$\frac{b_{n^*+1}\bar{c} - a_{n^*+1}}{d_{n^*+1}} = \sum_{i\in X} \frac{(b_{n^*+1} + d_{n^*+1})\bar{c} + b_{n^*+1}c_i}{2b_{n^*+1} + d_{n^*+1}} \tag{45}$$

Rewriting eq.(45) gives exactly eq.(38), as desired. We now claim that $N_2 \in Z_2^{'}$. We note that $N_2 = \sum_{i\in X} \beta_i B_i$ where $\beta_i = \frac{p_i^{\star}(\bar{c})-p_i^{\star\star}(\bar{c})}{p_T(\bar{c})-p_T^{\star\star}(\bar{c})}$. Similar to Subclaim 3, using subclaim 2, we can show that $0 < \beta_i < 1$ and $\sum_{i\in X}\beta_i = 1$, which shows the claim.

Finally, define $W_1 = conhull(Z_1, N_2)$ and $W_2 = conhull(Z_2, N_1)$. By construction, $seg(N_1, N_2) \in W_1 \cap W_2$. By Subclaims 3,4, and 5, the convex hull Z_1 and the point N_2

[12] We sketched this intersection in markets formed by three and four firms, where $n^* = 2$ and $n^* = 3$ respectively, as drawn in Figures 13 and 14.

are both defined in the firm rational region. Accordingly, their convex hull, W_1 and therefore the grand intersection $W_1 \cap W_2$ are defined in the firm rational region of the problem.

Subclaim 6: For each $i \in X$ and each $c_{n^*+1} \in (\bar{c}, R_{n^*+1})$, every price vector in $W_1 \cap W_2 \cap \triangle_{c_{n^*+1}}^{n^*-1}$ constitutes an undominated Bertrand-Nash equilibrium[13].

Proof of Subclaim 6 : Consider the profit level of firm i, i.e, $\pi_i(.)$, is stated in eq.(9). We prove the claim under two sub-subclaims.

Sub-Subclaim 1: For each $i \in X$, for each price vector $\check{p} \in W_1$, $\frac{\partial \pi_i(A,\theta,n^*+1,c_i,p)}{\partial p_i} |_{\check{p}} \leq 0$.

Proof of Sub-Subclaim-1: Take any firm $i \in X$ and consider a price vector $\check{p} \in W_1$. Let $c_{n^*+1} = c^\star$ be such that the simplex $\triangle_{c^\star}^{n^*-1}$ passes through $\check{p}$. Note that, this simplex is uniquely defined by continuity. Let $\check{Z}_1 = W_1 \cap \triangle_{c^\star}^{n^*-1}$. Indeed, by linearity, eq.(41) assures that $\check{Z}_1$ is a monotonic transformation of Z_1. Noting that $\frac{\partial \pi_i(A,\theta,n^*+1,c_i,p)}{\partial p_i} |_{\check{p}} = a_{n^*+1} + d_{n^*+1}(\sum_{l \in X} \check{p}_l(c^\star) - \check{p}_i) + b_{n^*+1}c_i - 2b_{n^*+1}\check{p}_i$ and substituting $\sum_{l \in X} \check{p}_l(c^\star) = \frac{b_{n^*+1}c^\star - a_{n^*+1}}{d_{n^*+1}}$ into this last equality gives $\frac{\partial \pi_i(A,\theta,n^*+1,c_i,c^\star)}{\partial p_i} = (b_{n^*+1} + d_{n^*+1})c^\star + b_{n^*+1}c_i - (2b_{n^*+1} + d_{n^*+1})\check{p}_i$. Note that $\check{p} \in \check{Z}_1 = conhull(\check{A}_i : i \in X)$ where $\check{A}_i$ is equal to A_i stated in Case 1 but calculated at $c_{n^*+1} = c^\star$. Hence, for each $i \in X$, we get $\check{p}_i \geq p_i^{\star\star}(c^\star) = \frac{(b_{n^*+1}+d_{n^*+1})c^\star + b_{n^*+1}c_i}{2b_{n^*+1}+d_{n^*+1}}$ proving the claim. What is more, for every $\tilde{p} \in \triangle_{c^\star}^{n^*-1} \setminus W_1$, there exists a firm $j \in X$ such that $\tilde{p}_j < p_j^{\star\star}(c^\star)$ showing that he has a profitable deviation by increasing its price, i.e, $\frac{\partial \pi_j(A,\theta,n^*+1,c_j,p)}{\partial p_j} |_{\tilde{p}} > 0$.

Sub-Subclaim-2: For each $i \in X$, for each price vector $\hat{p} \in W_2$, we claim that $\frac{\partial \pi_i(A,\theta,n^*,c_i,p)}{\partial p_i} |_{\hat{p}} \geq 0$.

Proof of Sub-Subclaim-2: Take any price vector $\hat{p} \in W_2$. Let $c_{n^*+1} = c^{\star\star}$ be such that the simplex $\triangle_{c^{\star\star}}^{n^*-1}$ passes through $\hat{p}$. Let $\hat{Z}_2 = W_2 \cap \triangle_{c^{\star\star}}$. Indeed, by linearity, eq.(41) assures that $\hat{Z}_2$ is a monotonic transformation of Z_2. Take a firm $i \in X$. We claim that $\frac{\partial \pi_i(A,\theta,n^*,c_i,p)}{\partial p_i} |_{\hat{p}} \geq 0$. Noting that $\frac{\partial \pi_i(A,\theta,n^*,c_i,p)}{\partial p_i} |_{\hat{p}} = a_{n^*} + d_{n^*}(\sum_{l \in X} \hat{p}_l(c^{\star\star}) - \hat{p}_i) + b_{n^*}c_i - 2b_{n^*}\hat{p}_i$ and substituting $\sum_{l \in X} \hat{p}_l(c^{\star\star}) = \frac{b_{n^*+1}c^{\star\star} - a_{n^*+1}}{d_{n^*+1}}$ into this last equality gives $\frac{\partial \pi_i(A,\theta,n^*,c_i,p)}{\partial p_i} |_{\hat{p}} = \frac{b_{n^*+1}d_{n^*}c^{\star\star} + b_{n^*}d_{n^*+1}c_i}{d_{n^*+1}} - (2b_{n^*} + d_{n^*})\hat{p}_i$. Note that $\hat{p} \in \hat{Z}_2 = conhull(\hat{B}_i : i \in X)$ where $\hat{B}_i$ is equal to B_i stated in Case 2 but calculated at $c_{n^*+1} = c^{\star\star}$. Thus, for each $i \in X$, we have $\hat{p}_i \leq p_i^\star(c^{\star\star}) = \frac{d_{n^*}b_{n^*+1}c^{\star\star} + b_{n^*}d_{n^*+1}c_i}{d_{n^*+1}(2b_{n^*}+d_{n^*})}$ proving the claim. Moreover, for every $\tilde{p} \in \triangle_{c^{\star\star}} \setminus W_2$, there exists a firm $j \in X$ such that $\tilde{p}_j > p_j^\star(c^{\star\star})$ showing that he has a profitable deviation

[13] In Figure 9, $W_1 \cap W_2$ is shown by the green region.

by decreasing its price, i.e, $\frac{\partial \pi_j(A,\theta,n^*,c_j,p)}{\partial p_j} \mid_{\tilde{p}} < 0$.

For each $i \in X$, since the best response graphs $Gr^u(BR_i^\star)$ and $Gr^{proj}(BR_i^{\star\star})$ become only valid in case of a decrease and an increase of firm i's unilateral price deviations respectively, the intersection $W_1 \cap W_2$ characterizes the undominated Bertrand-Nash equilibria of the game for possible c_{n^*+1} variations. Indeed, for each $c_{n^*+1} \in (\bar{c}, R_{n^*+1})$, $\triangle_{c_{n^*+1}}^{n^*-1}$ intersects with $W_1 \cap W_2 \setminus \{N_1, N_2\}$ at multiple price vectors proving the claim.

CASE 2: $R_{n^*+1} \geq c_{n^*+1} \geq c_{n^*} \geq \bar{c}$

Claim 2: Let $n^* \geq 2$. If $c_{n^*} < c_{n^*+1} < R_{n^*+1}$, then multiple equilibria exist.

Proof of Claim 2: The proof of this case is similar to Case 1. Assume that $R_{n^*+1} \geq c_{n^*+1} \geq c_{n^*} > \bar{c}$ and let $W_3 = \{p \in \mathbb{R}_+^{n^*} : \sum_{i \in X} p_i \geq c_{n^*}\}$. Next, we make the following subclaim.

Subclaim 7: For every $i \in X$, when $c_{n^*+1} \geq c_{n^*}$, $p_i^\star(c_{n^*+1}) \geq p_i^{\star\star}(c_{n^*+1}) > c_i$.

Proof of subclaim 7: For all $i \in X$, since $c_{n^*+1} \geq c_i$, the proof of Subclaim 2 of Case 1 also works here. Indeed when $c_{n^*+1} = c_{n^*}$, $p_{n^*}(c_{n^*}) = p_{n^*}^{\star\star}(c_{n^*})$ by eq.(41).

Next, we form N_1, N_2, Z_1, Z_2, W_1, and W_2 in an identical way to Case 1. Indeed, by subclaim 7, for each $i \in X$ and each $c_{n^*+1} > c_i$, we get $\sum_{i \in X} p_i(c_{n^*+1}) > \sum_{j \in X \setminus i} p_j^{\star\star}(c_{n^*+1}) + c_i$ proving that the price vectors in $W_1 \cap W_3$, so does in $W_1 \cap W_2 \cap W_3$, are defined in the firm rational region. Finally, we claim that for each $c_{n^*+1} \in (c_n^*, R_{n^*+1})$, $\triangle_{c_{n^*+1}} \cap W_1 \cap W_2 \cap W_3 \neq \emptyset$ at multiple price vectors and the firms in X have different pricing strategies along this equilibrium path. The proof is identical to last steps of the proof of Case 1. Moreover, by subclaim 7, when $n^* = 1$ and $c_1 = c_2$, firms in X have a unique pricing strategy (Please see Figure 10). However, as the dimensionality increases, i.e $n^* \geq 2$, we still find firms in X having different pricing strategies.

Last of all, we point out the undominated equilibrium strategies of firms in $N \setminus X$ and finish our proof.

Claim 2: Let p_i^m be the monopoly price charged by firm i. If $c_{n^*+1} > R_{n^*+1}$, then for each $i \in N \setminus X$, $p_i^* \in I = [c_i, p_i^m]$. Moreover, if $c_{n^*+1} \leq R_{n^*+1}$, then $p_{n^*+1} = c_{n^*+1}$ and $\forall j \in N \setminus \{X \cup \{n^*+1\}\}$, $p_j^* \in [c_j, p_j^m]$.

Proof: Let $c_{n^*+1} > R_{n^*+1}$. Hence, there is no constraint on firm n^*+1. Take a firm $i \in N \setminus X$. By iteration algorithm 0, $q_i = 0$. Every price $p < c_i$ is weakly dominated by charging c_i trivially. Similarly, when all firms in $N \setminus i$ charge a price of total demand A or higher, firm i optimally charges the monopoly price of $p_i^m = \frac{A+c_i}{2}$ found by eq.(10). Hence, setting the monopoly price of p_i^m weakly dominates charging a price above this level, which proves the claim. Now, consider the case $c_{n^*+1} < R_{n^*+1}$. Accordingly, firms in X lower their prices such that firm n^*+1 is indifferent between entering into or remaining as a potential entrant in the market. But, in such a case, $p_{n^*+1} = c_{n^*+1}$ in equilibrium. However, the inefficient firms in $N \setminus \{X \cup \{n^*+1\}\}$ still have the same pricing strategies as in the first case, as desired. ■

Proof of Theorem 2 (Characterization):

Consider a problem $\xi = (N, A, \theta, c)$ and let $X = \{1, 2..., n^*\}$ where n^* is found by iteration algorithm 2. We examine two cases:

CASE 1: Take any two firms i and j such that $c_i < c_j$. Suppose to the contrary there exist an equilibrium, V_2, where firm j is in it, but firm i is not. Let V_2 be formed by $\tilde{n}$ number of firms denoted by the set $\tilde{N}$. Let firm k be the least efficient firm among these firms. Next, we calculate the critical value $\bar{c}$ such that if we add one more firm, say firm l, with marginal cost level $\bar{c}$ into these $\tilde{n}$ number of firms, firm l would produce exactly zero. Use eq.(11) to get:

$$\bar{c} = \frac{a_{\tilde{n}+1}(2b_{\tilde{n}+1} + d_{\tilde{n}+1}) + d_{\tilde{n}+1}b_{\tilde{n}+1}c_T^*}{(b_{\tilde{n}+1} + d_{\tilde{n}+1})(2b_{\tilde{n}+1} - d_{\tilde{n}+1}\tilde{n}) - b_{\tilde{n}+1}d_{\tilde{n}+1}} \tag{46}$$

where $c_T^* = \sum_{i \in \tilde{N}} c_i$.
Also we have,

$$c_i < c_k \leq \bar{c} \tag{47}$$

Moreover, $q_k \geq 0$ in equilibrium V_2. Hence using eq.(11) and eq(12), we get

$$c_k \leq \frac{a_{\tilde{n}}(2b_{\tilde{n}} + d_{\tilde{n}}) + b_{\tilde{n}}d_{\tilde{n}}c_T^*}{(b_{\tilde{n}} + d_{\tilde{n}})(2b_{\tilde{n}} - d_{\tilde{n}}(\tilde{n}-1))} \tag{48}$$

But, the right hand side of this inequality is equal to the residual demand left to any firm outside the firms forming equilibrium V_2. Thus, in particular, we have $c_k \leq R_i$. Additionally, the relationship given in eq.(40) calculated at $n = \tilde{n}$ assures that $R_i > \bar{c}$. Altogether we would have,

$$c_i < c_k \leq \bar{c} < R_i \tag{49}$$

Thus, the residual demand left to firm i is greater than his marginal cost level. That is, he has an incentive to deviate. Accordingly, firms that are forming V_2 lower their prices. Let

$\triangle_{R_i}^{\tilde{n}-1}= \{p \in \mathbb{R}_+^{\tilde{n}} : \sum_{i\in\tilde{N}} p_i = \frac{b_{\tilde{n}+1}c_i - a_{\tilde{n}+1}}{d_{\tilde{n}+1}}\}$. We claim that at every price vector $p \in\triangle_{R_i}^{\tilde{n}-1}$, for every firm $j \in \tilde{N}$ such that $c_j > c_i$, firm j has an incentive to either decrease or increase his price level, p_i. W.O.L.G. let $j = k$ and consider a price vector $p^\circ \in\triangle_{R_i}^{\tilde{n}-1}$. Next, we take the derivative of the profit function that firm k faces that is calculated at $n = \tilde{n} + 1$ and is stated in eq.(9) and substitute $\sum_{i\in\tilde{N}} p_i^\circ = \frac{b_{\tilde{n}+1}c_i - a_{\tilde{n}+1}}{d_{\tilde{n}+1}}$ into the resulting equation as follows:

$$\frac{\partial \pi_k(A, \theta, \tilde{n} + 1, c_i, p)}{\partial p_k} \mid p^\circ = (b_{\tilde{n}+1} + d_{\tilde{n}+1})c_i + b_{\tilde{n}+1}c_k - (2b_{\tilde{n}+1} + d_{\tilde{n}+1})p_k^\circ \tag{50}$$

It is important to note that if $p_k^\circ < K_1 = \frac{(b_{\tilde{n}+1}+d_{\tilde{n}+1})c_i+b_{\tilde{n}+1}c_k}{2b_{\tilde{n}+1}+d_{\tilde{n}+1}}$, then $\frac{\partial \pi_k(A,\theta,\tilde{n}+1,c_i,p)}{\partial p_k} \mid p^\circ > 0$ showing that firm k has an incentive to increase his price level. Finally, the same derivative is calculated at $n = \tilde{n}$ and we substitute the same constraint into it thereafter as follows:

$$\frac{\partial \pi_k(A, \theta, \tilde{n}, c_i, p)}{\partial p_k} \mid p^\circ = \frac{b_{\tilde{n}+1}d_{\tilde{n}}c^i + b_{\tilde{n}}d_{\tilde{n}+1}c_k}{d_{\tilde{n}+1}} - (2b_{\tilde{n}} + d_{\tilde{n}})p_k^\circ \tag{51}$$

We stress that if $p_k^\circ > K_2 = \frac{b_{\tilde{n}}d_{\tilde{n}+1}c^i+b_{\tilde{n}+1}d_{\tilde{n}}c_k}{d_{\tilde{n}}(2b_{\tilde{n}+1}+d_{\tilde{n}+1})}$, then $\frac{\partial \pi(A,\theta,\tilde{n},c_i,p)}{\partial p_k} \mid p^\circ < 0$ proving that firm k has an incentive to decrease his price level. In the final step, we claim that $K_1 > K_2$. Consider the function $D : (0, 1) \times \mathbb{N}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_{++}$ such that $D(\theta, \tilde{n}, c_k, c_i) = K_2 - K_1$ that was defined in the proof of Subclaim 1 of Proposition 1 [14]. We have already noted that when $c_k = c_i = \bar{c}$, then $D(\theta, \tilde{n}, \bar{c}, \bar{c}) = 0$ and $\frac{\partial D(.)}{\partial c_i} = -\frac{\partial D(.)}{\partial c_k} > 0$. However, given eq.(47), we have $\bar{c} \geq c_k > c_i$, which proves that $K_2 < K_1$. Hence, for every feasible price vector on $p \in\triangle_{R_i}^{\tilde{n}-1}$, firm k has an incentive to deviate as desired.

CASE 2: Let $1 \leq \ddot{n} < n^*$. We claim that firms in $Y = \{1, 2..., \ddot{n}\}$ cannot form an equilibrium. Assume they can. By Iteration Algorithm 2, $q_{\ddot{n}+1} > 0$. Hence using eq.(11) and eq.(12), we get

$$c_{\ddot{n}+1} \leq \frac{a_{\ddot{n}+1}(2b_{\ddot{n}+1} + d_{\ddot{n}+1}) + b_{\ddot{n}+1}d_{\ddot{n}+1}c_T^*}{2b_{\ddot{n}+1}^2 - b_{\ddot{n}+1}d_{\ddot{n}+1}\ddot{n} + 2b_{\ddot{n}+1}d_{\ddot{n}+1} - d_{\ddot{n}+1}^2\ddot{n}} \tag{52}$$

where $c_T^* = \sum_{i\in Y} c_i$. To ensure that firm $\ddot{n}$ is out of the market, firms in Y lower their prices and we have $\sum_{i\in Y} p_i = \frac{bc_{\ddot{n}+1}-a}{d}$. We claim that there exist a firm $j \in Y$ such that $p_j < \frac{(b_{\ddot{n}+1}+d_{\ddot{n}+1})c_{\ddot{n}+1}+b_{\ddot{n}+1}c_i}{2b_{\ddot{n}+1}+d_{\ddot{n}+1}}$. Otherwise, we have $\frac{b_{\ddot{n}+1}c_{\ddot{n}+1}-a_{\ddot{n}+1}}{d_{\ddot{n}+1}} < \frac{(b_{\ddot{n}+1}+d_{\ddot{n}+1})\ddot{n}c_{\ddot{n}+1}+b_{\ddot{n}+1}c_T^*}{2b_{\ddot{n}+1}+d_{\ddot{n}+1}}$,which can be simplified as follows:

$$c_{\ddot{n}+1} > \frac{a_{\ddot{n}+1}(2b_{\ddot{n}+1} + d_{\ddot{n}+1}) + b_{\ddot{n}+1}d_{\ddot{n}+1}c_T^*}{2b_{\ddot{n}+1}^2 - b_{\ddot{n}+1}d_{\ddot{n}+1}\ddot{n} + b_{\ddot{n}+1}d_{\ddot{n}+1} - d_{\ddot{n}+1}^2\ddot{n}} \tag{53}$$

which is a direct contradiction to eq.(52). Using the proved claim and following the proof in Case 1, firm j has an incentive to increase its price. ■

[14] In this context, $c_{n^*+1} = c_i$ and $c_i = c_k$.

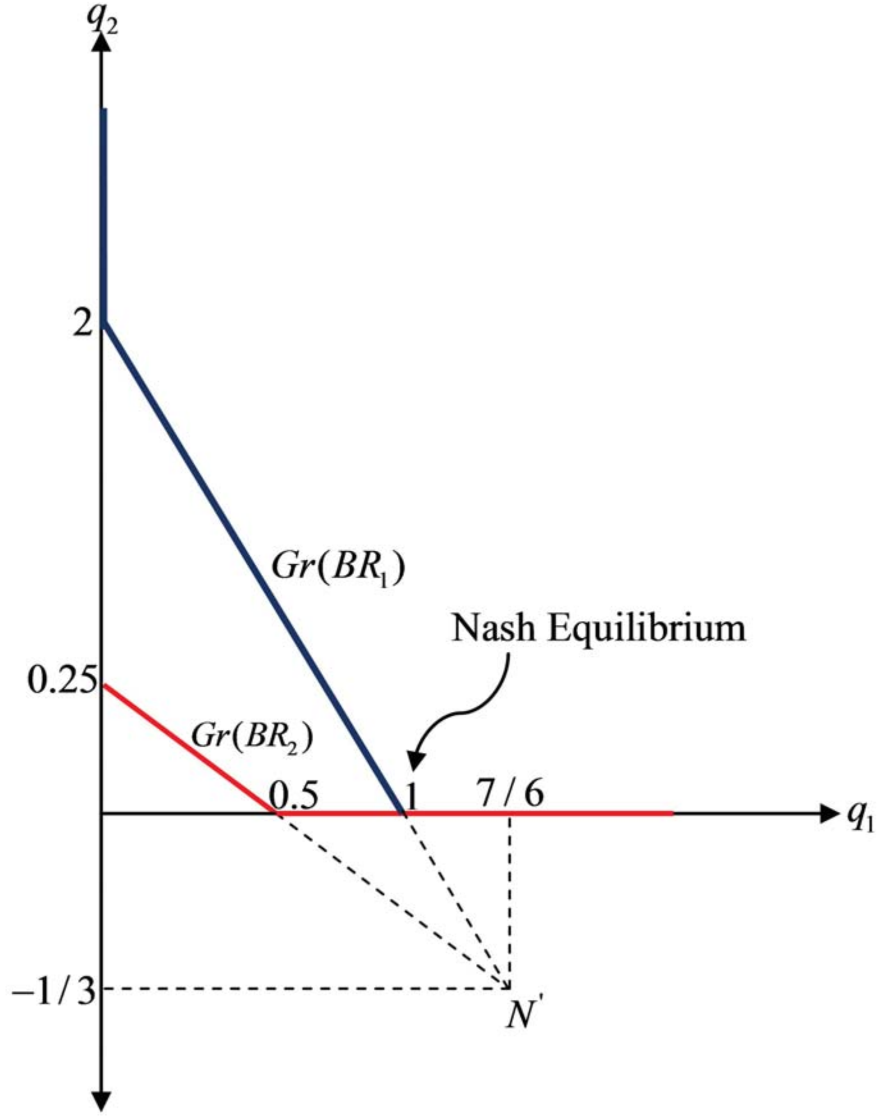

Figure 1: **Motivating Example:** Let $N = \{1, 2\}$, $c = (8, 9.5)$; and the market demand is $p = 10 - q$. Under Cournot competition, the best response functions of the firms are $BR_1(q_2) = \frac{2-q_2}{2}$ and $BR_2(q_1) = \frac{0.5-q_1}{2}$. The related graphs of these functions are for each $i \in \{1, 2\}$, $Gr(BR_i) = \{\mathbf{q} \in \mathbb{R}^2 : q_i = BR_i(q_{-i})\}$. However, these unrefined best response graphs intersect in the negative region and the coordinates of this intersection, i.e. point N', are $q_1^* = 7/6$ and $q_2^* = -1/3$. However, negative production is not feasible by definition. Therefore, if firm one produces more than 0.5, it is optimal for the inefficient firm to not produce. Similarly, if firm two produces more than 2, firm one does not produce. Under this refinement, it is easy to see that the best response graphs intersect at point N characterized by $q_1^* = 1$ and $q_2^* = 0$ as shown in Figure 2. Thus, in the unique Nash equilibrium of this game, firm one is a monopoly and firm two is the low threat potential entrant firm, which does not have any effect on the equilibrium strategy of the monopoly firm in our terminology.

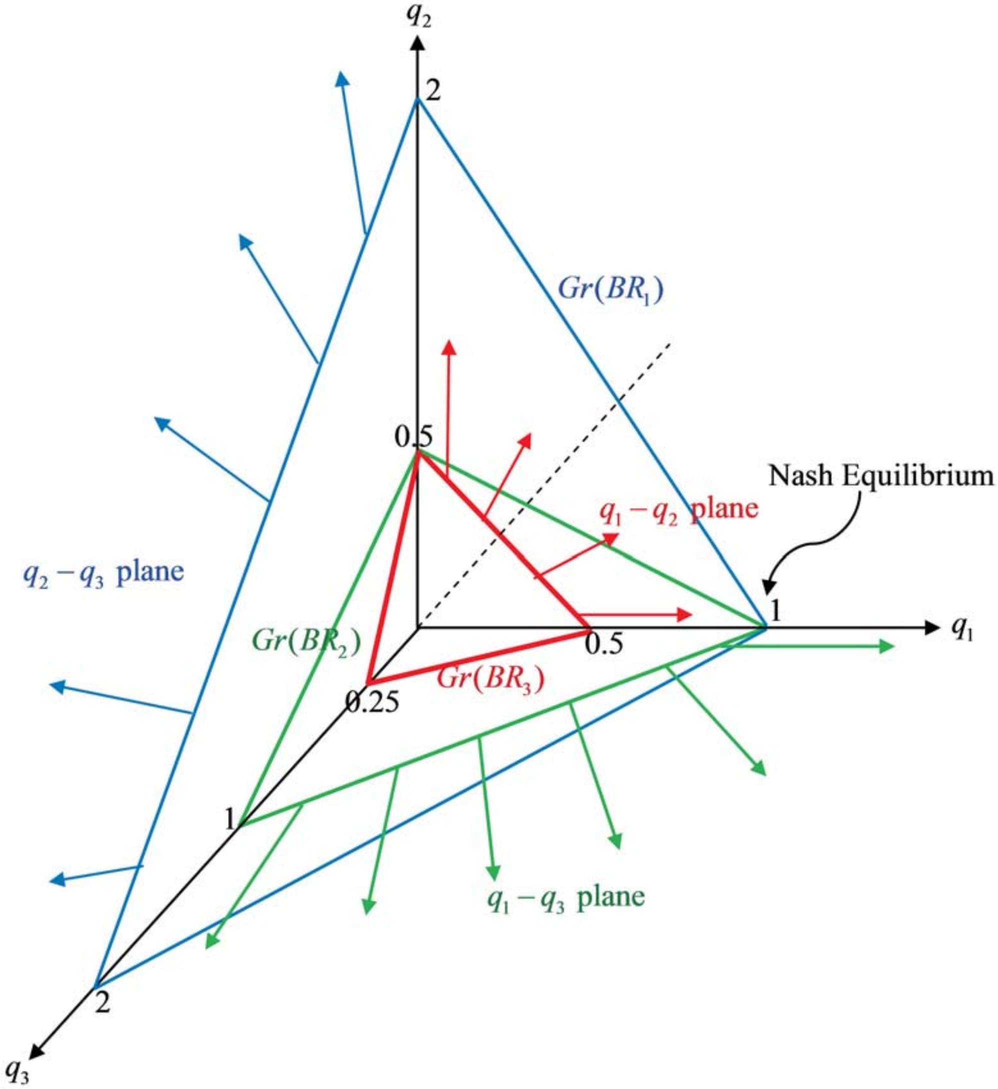

Figure 2: Let $N = \{1, 2, 3\}$, $(c_1, c_2, c_3) = (8, 9, 9.5)$, and $\theta = 1$. The market demand is given by $p_i = 10 - q_i - q_{-i}$. The best responses can be calculated as $BR_1(q_2, q_3) = \frac{2-q_2-q_3}{2}$, $BR_2(q_1, q_3) = \frac{1-q_1-q_3}{2}$, and $BR_3(q_1, q_2) = \frac{0.5-q_1-q_2}{2}$. For each $i \in \{1, 2, 3\}$, let the unrevised best response graphs be $GR(BR_i) = \{q \in \mathbb{R}^3 : q_i = BR_i(q_{-i})\}$. These unrefined best response graphs intersect at $q = (9/8, 1/8, -3/8)$. Iteration Algorithm 1 gives $N^* = \{1\}$. Note that monopoly output of firm 1 is one. Hence, Theorem 1 assures that $q^\star = (q_1^*, q_2^*, q_3^*) = (1, 0, 0)$ is the unique Nash equilibrium of this game. Indeed, we refine the best response graphs as follows: When firm 2 and firm 3 produce more than two in total, it is optimal for firm 1 to not produce. Therefore, firm 1's best response function becomes the $q_2 - q_3$ plane. Similarly, we refine the best response graphs of firms 2 and 3. Note that the refined best response functions graphs meet at a unique equilibrium point $N = (1, 0, 0)$, as desired.

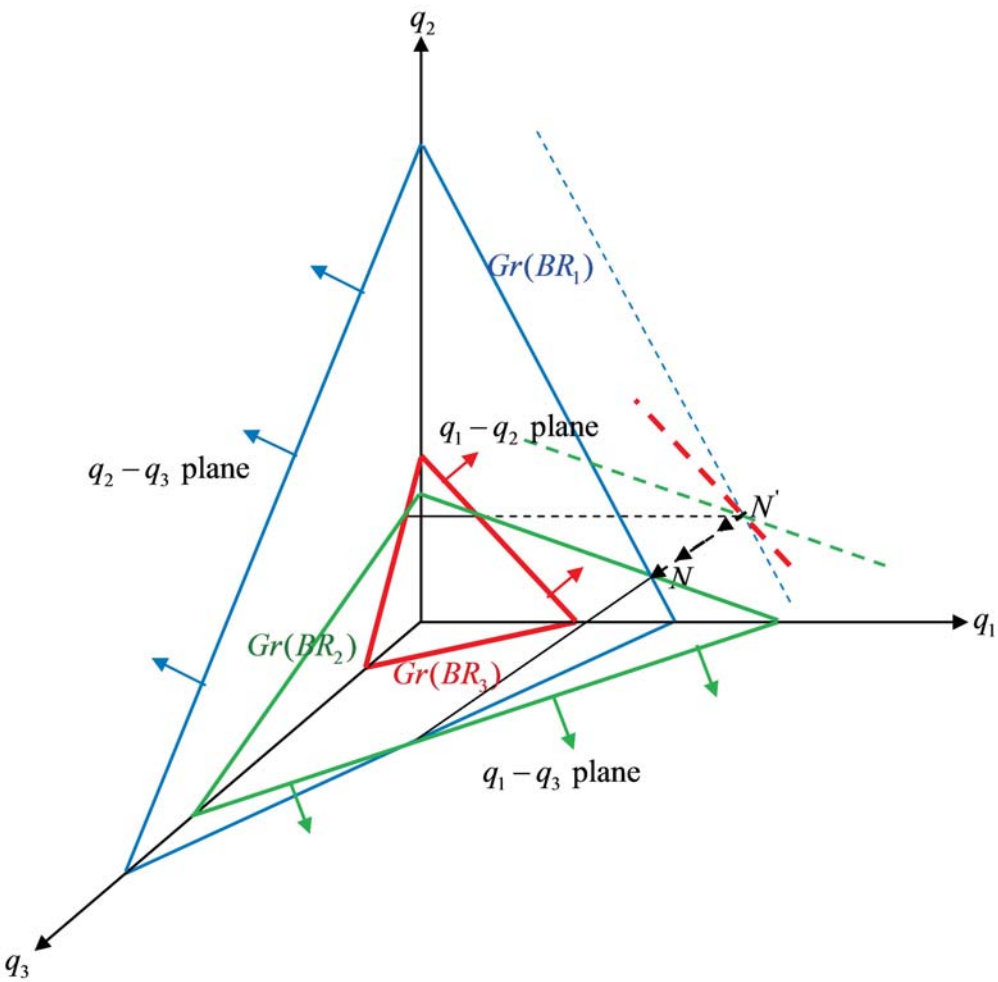

Figure 3: Let $N = \{1, 2, 3\}$, $(c_1, c_2, c_3) = (8, 8.5, 9.2)$, and $\theta = 1$. The unrefined best response graphs intersect at $N' = (\frac{3.7}{4}, \frac{1.7}{4}, \frac{-1.1}{4})$. Iteration Algorithm 1 gives $N^* = \{1, 2\}$. We refine best response graphs of the firms and these graphs become planes eventually in the related spaces. Hence, they intersect at a unique equilibrium point, say $N = (\frac{2.5}{3}, \frac{1}{3})$. At this point, the most efficient two firms play the Cournot game in the absence of the potential entrant firm, i.e. firm three.

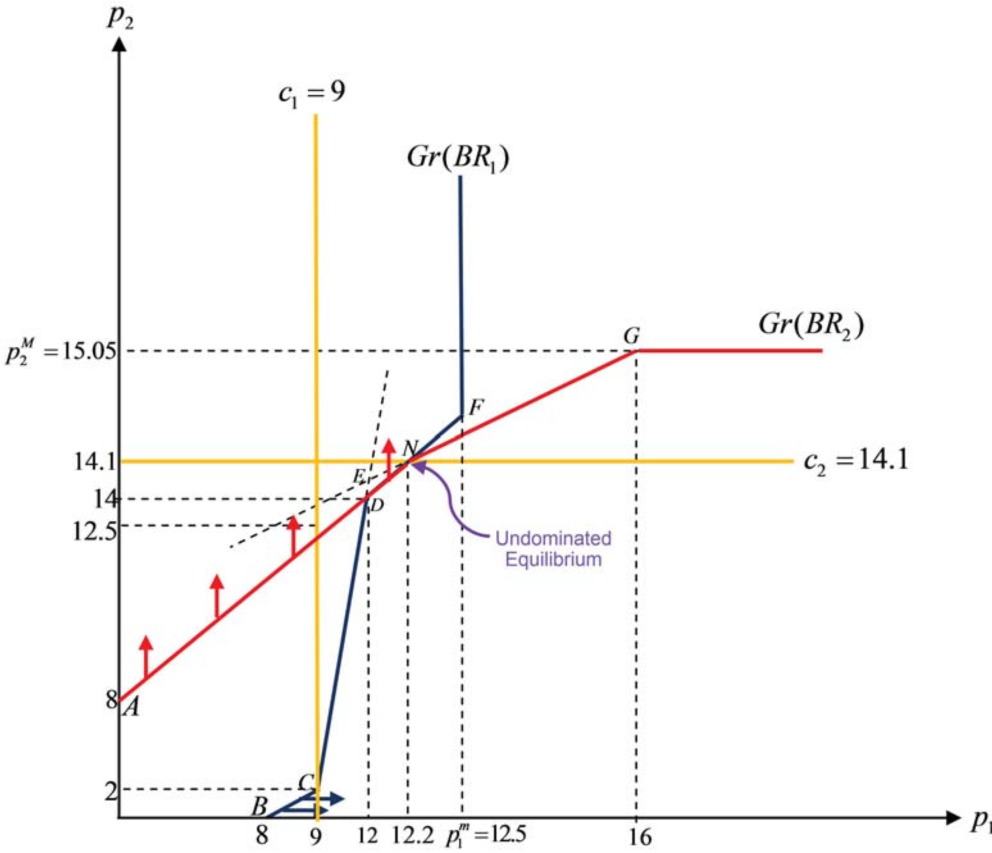

Figure 4: **Unique Bertrand-Nash Equilibrium in Two-Firm Case:** Let $N = \{1, 2\}$; $c = (9, 14.1)$; and $p_i = 16 - q_i - 0.5q_{-i}$. In this figure, the unrevised best response functions of firms 1 and 2 are $B_1^U(p_2) = 8.5 + p_2/4$ and $B_2^U(p_1) = 11.05 + p_1/4$ respectively. The related graphs of these functions are $Gr(B_1^U) = \{\mathbf{p} \in \mathbb{R}^2 : p_1 = B_1^U(p_2)\}$ and $Gr(B_2^U) = \{\mathbf{p} \in \mathbb{R}^2 : p_2 = B_2^U(p_1)\}$, which are shown by rays $]CE[$ and $]EG[$ respectively. They intersect at E, which is associated with the infeasible production vector $q = (4.02, -0.06)$. Accordingly, we draw the revised best response graphs of firms one and two, i.e. $Gr(BR_1)$ and $Gr(BR_2)$, which are shown by blue and red lines respectively. W.L.O.G., consider firm two. $Gr(BR_2^U)$ intersect with c_2 at $N = (12.2, 14, 1)$. Thus, if $p_1 \leq 12.2$, it is optimal for firm two to not produce. Note that charging any price weakly above $seg[AN]$ results in zero production for firm two by iteration algorithm 0. Moreover, if firm one charges a price higher or equal the total demand, e.g, A, firm two will be the monopoly firm and her best response is to charge the monopoly price. Similarly, we revise the best response function of firm one. In sum, both best responses meet at a unique undominated Nash equilibrium outcome given by $N = (12.2, 14, 1)$. $Seg[DN)$ constitutes the weakly dominated equilibria of the game.

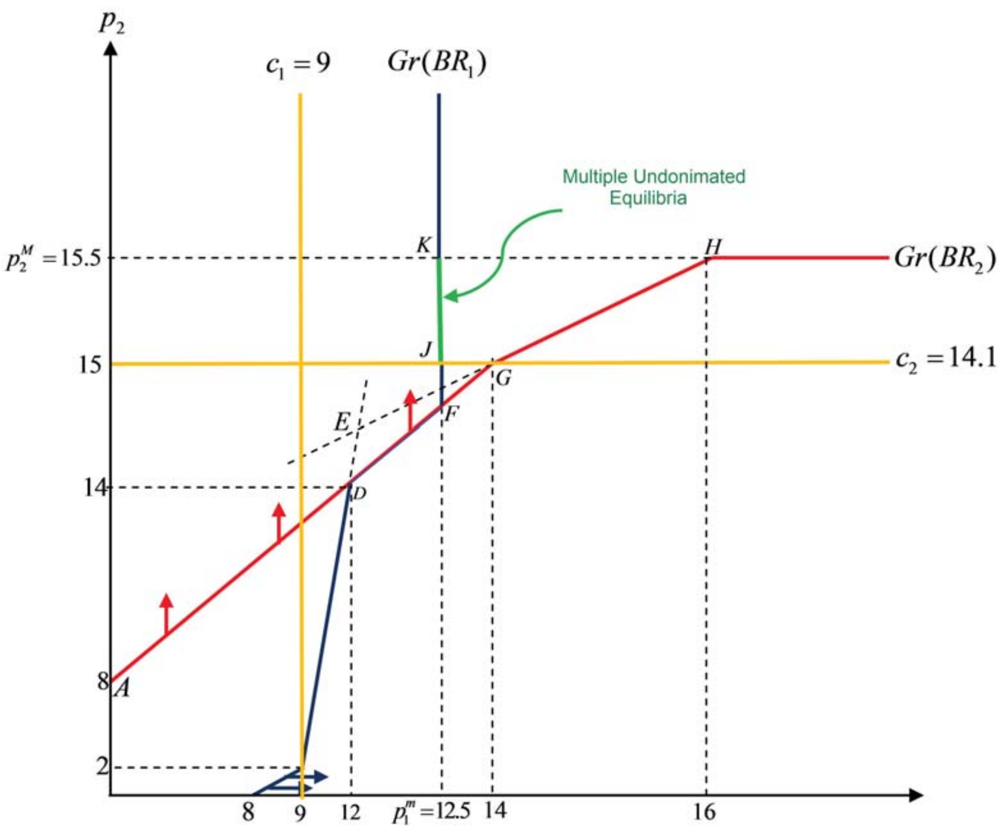

Figure 5: **Multiple Bertrand-Nash Equilibrium in Two Firms Case:** Let $N = \{1, 2\}$; $c = (9, 15)$; and $p_i = 16 - q_i - 0.5q_{-i}$. In this figure, the unrevised best response functions are given by $B_1(p_2) = 8.5 + p_2/4$ and $B_2(p_1) = 11.5 + p_1/4$. The related unrevised best response graphs intersect at E, which is associated with the infeasible production vector $q = (4.18, -0.62)$. Accordingly, we draw the revised best response graphs of firms one and two as above and see that they intersect at multiple points. Hence, we get multiple undominated equilibria in which firm one charges the monopoly price $p_1^M = 12.5$ and firm 2 charges any price between $c_2 = 15$ and $p_2^M = 15.5$.

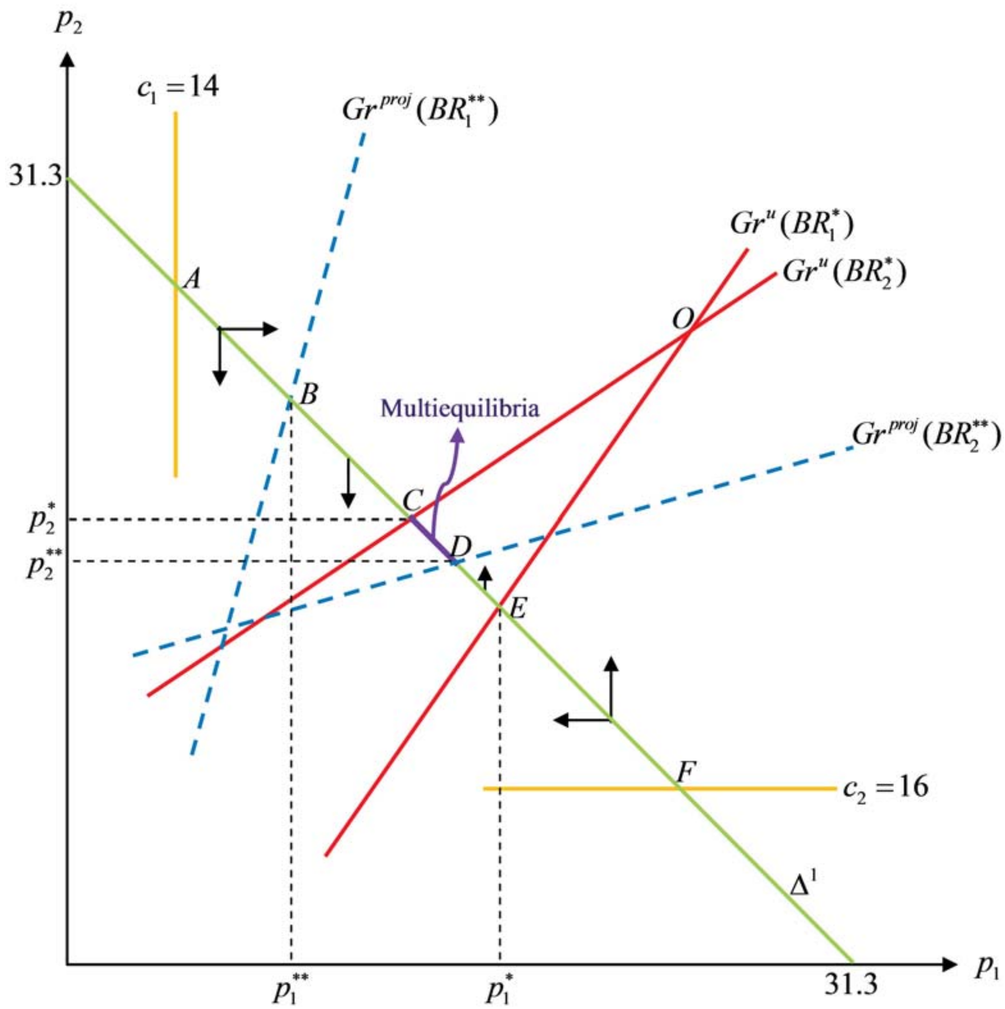

Figure 6: **Multiple Undominated Equilibria When Only Firms One and Two Actively Produce in the Bertrand Game:** Let $N = \{1, 2, 3\}$; $c = (14, 16, 16.1)$; and $p_i = 17 - q_i - 0.5q_{-i}$. When firms 1 and 2 play the Bertrand game in the absence of firm 3, the outcome is given point O. However, the potential entrant, i.e, firm 3, has an incentive to produce. Therefore, firms 1 and 2 decrease their total prices to 31.3 to eliminate firm 3's deviation. Each price combination in $seg(AF)$ is a candidate for undominated equilibrium. $seg(AF)$ is a border for firm 3's production and if either firm 1 or 2 charges a slightly higher price on this segment, firm 3 starts to produce and projected best response graphs become valid. Whereas the vertical arrows represent the directions of the possible deviations by firm 2, horizontal arrows show firm 1's possible deviations. For example, on $seg(EF]$, given other firms' prices, firm 2 has an incentive to deviate to a slightly higher price from p_2 to $p_2 + \varepsilon$ with $\varepsilon > 0$, because firm 3 starts to produce after this increase and the projected best response set of firm 2, i.e, $Gr^{proj}(BR_2^{\star\star})$, becomes valid. However, $p_2 + \varepsilon$ is more close to $Gr^{proj}(BR_2^{\star\star})$ than p_2 showing that firm 2 gains. Note that, there is no profitable deviation by firm 1 or firm 2 on line segment $seg[CD]$ where the feasibility (Green line) and firm rationality constraints (Yellow lines) are also satisfied. Hence all price combinations of firms 1 and 2 on the segment $seg[CD]$ constitute an undominated Bertrand-Nash equilibrium. As a check, for each $\hat{\mathbf{p}} \in seg[CD]$, for each $i \in \{1, 2\}$, $p_i^{\star\star} \leq \hat{p}_i \leq p_i^{\star}$ and $\sum_{i \in 1,2} \hat{p}_i = 31.3$ as claimed in part b of Proposition 1. Note that along this equilibrium path, firm 3 charges his marginal cost level, i.e. $p_3 = 16.1$, and produces nothing.

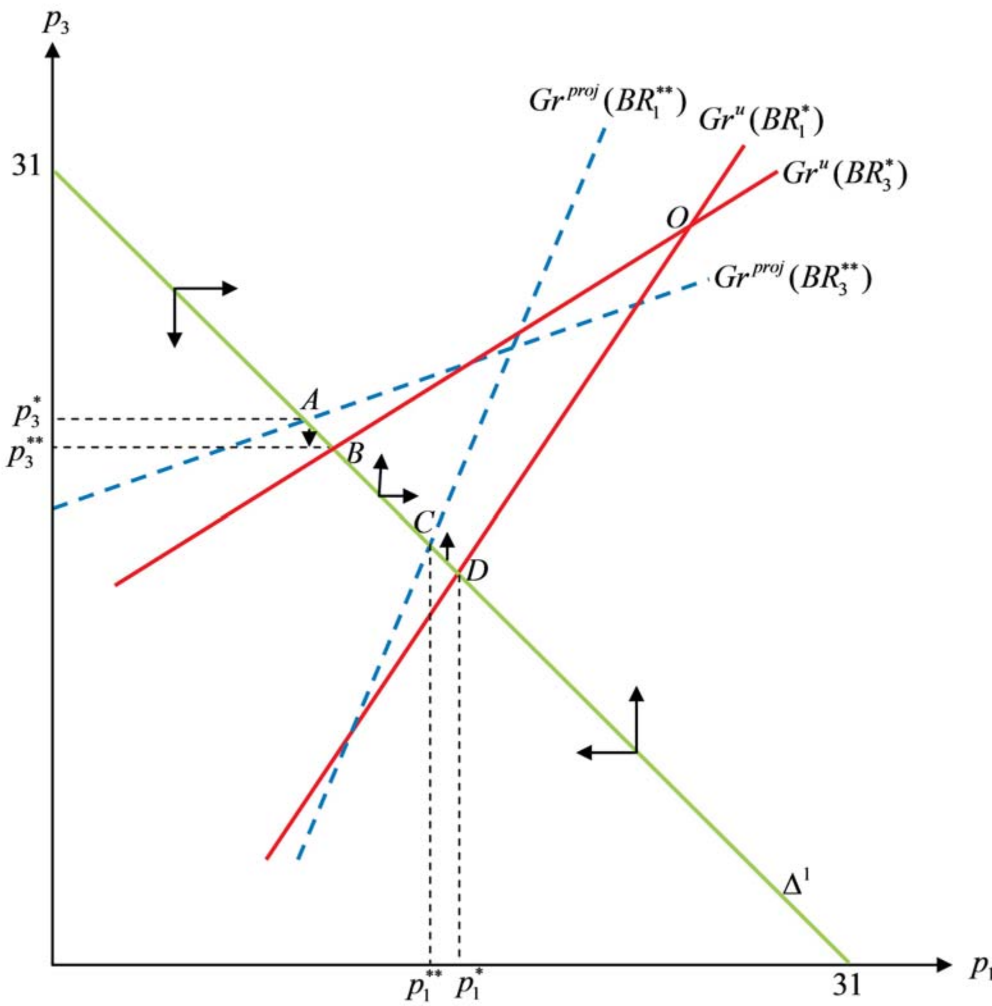

Figure 7: **Nonexistence of Equilibrium When Only Firms One and Three Actively Produce in the Bertrand Game:** Let $N = \{1, 2, 3\}$; $c = (14, 16, 16.1)$; and $p_i = 17 - q_i - 0.5q_{-i}$. In this figure, we consider a market candidate formed by firm one and three and search for the undominated Bertrand Nash equilibrium. The graphs of the best response functions are drawn in a similar way to Figure 6. The critical points are $A = (14.95, 16.05)$, $B = (14.96, 16.04)$, $C = (15.15, 15.85)$, and $D = (15.2, 15.8)$. When firms one and three play the game, the doupoly price vector is given by O but the demand left to firm 2 is greater than c_2. In order to eliminate entry incentives of firm 2, incumbent firms decrease their total prices to 31. Let $\triangle^1 = \{\mathbf{p} \in \mathbb{R}^2 : p_1 + p_3 = 31\}$. But, arrows show that given other firms' prices, for every price combination on Δ^1, either firm 1 or firm 3 deviates to decrease or increase their price levels. Accordingly, their is not any equilibrium in a game played among the most and least cost efficient firms. As a check, there is no price vector $\hat{\mathbf{p}}$ on $\triangle^1$ such that for each $i \in \{1, 3\}$, $p_i^{\star\star} \leq \hat{p}_i \leq p_i^{\star}$ and $\sum_{i \in 1,3} \hat{p}_i = 31$ as claimed in Theorem 2.

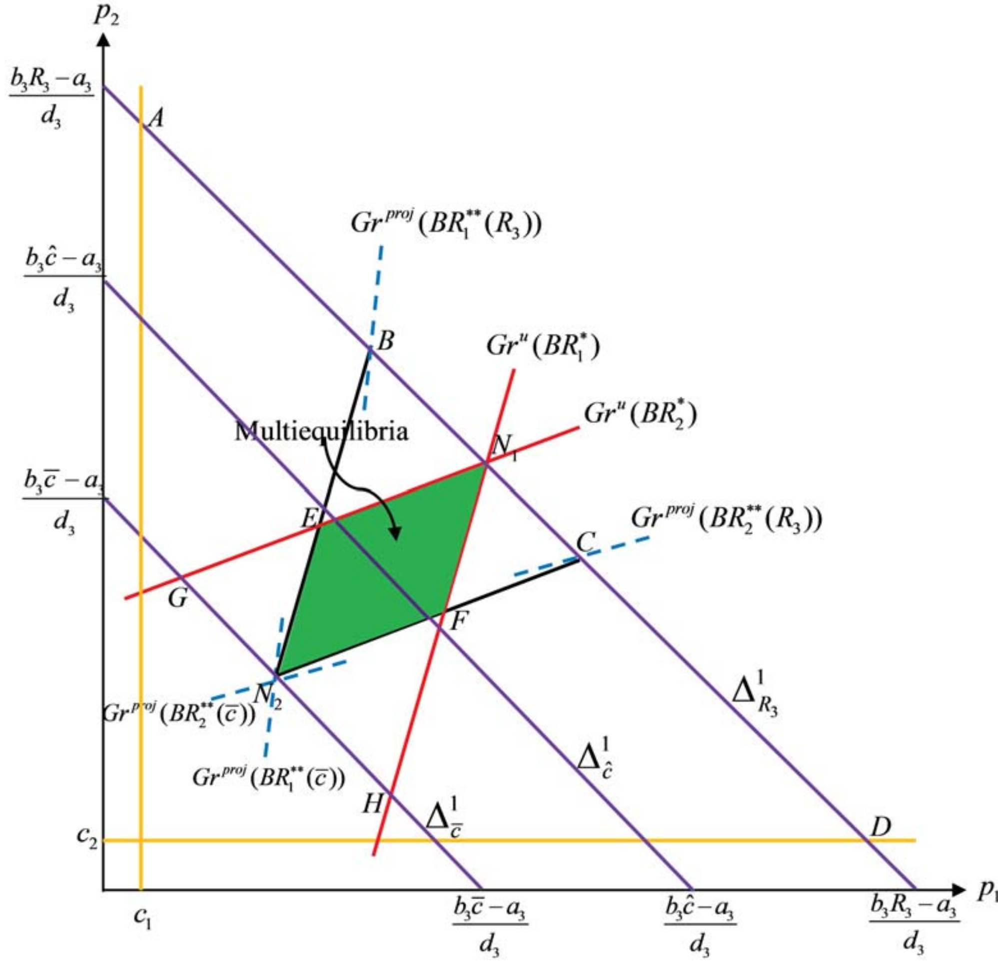

Figure 8: **The Proof of Proposition 1 is Sketched with Three Firms and** $n^* = 2$**:** Each problem is associated with a specific level of c_3 and this figure identifies multiple equilibria (or possibly unique) for possible c_3 variations in $[\bar{c}, R_3]$. For each $i \in \{1, 2\}$ and each $c_3 \in \{\bar{c}, R_3\}$, let $Gr^u(BR_i^\star)$ and $Gr^{proj}(BR_i^{\star\star}(c_3))$ denote the unprojected and projected best response graphs of firm i associated with a marginal cost level of c_3. Whereas the unprojected best response functions of firms is fix in c_3, the projected ones changes with c_3 monotonically. Moreover, the simplex, i.e, $\triangle^1_{c_3}$, is also a function of c_3. As c_3 changes from $\bar{c}$ to R_3, $seg[N_2B]$ and $seg[N_2C]$ are the locus of the critical intersections of $Gr^{proj}(BR_1^{\star\star}(c_3))$ and $Gr^{proj}(BR_2^{\star\star}(c_3))$ with $\triangle^1_{c_3}$ respectively. When $c_3 = \bar{c}$, point N_2 is the unique equilibrium of the game. Similarly, point N_1 is the unique equilibrium of the game whenever $c_3 = R_3$. However, for each $c_3 \in (\bar{c}, R_3)$, the multiple equilibria is given by the intersection of the green region with $\triangle^1_{c_3}$. We finally note that $\hat{c}$ cost level of firm 3 can cause a similar deviation arguments for the firms as in Example 6.

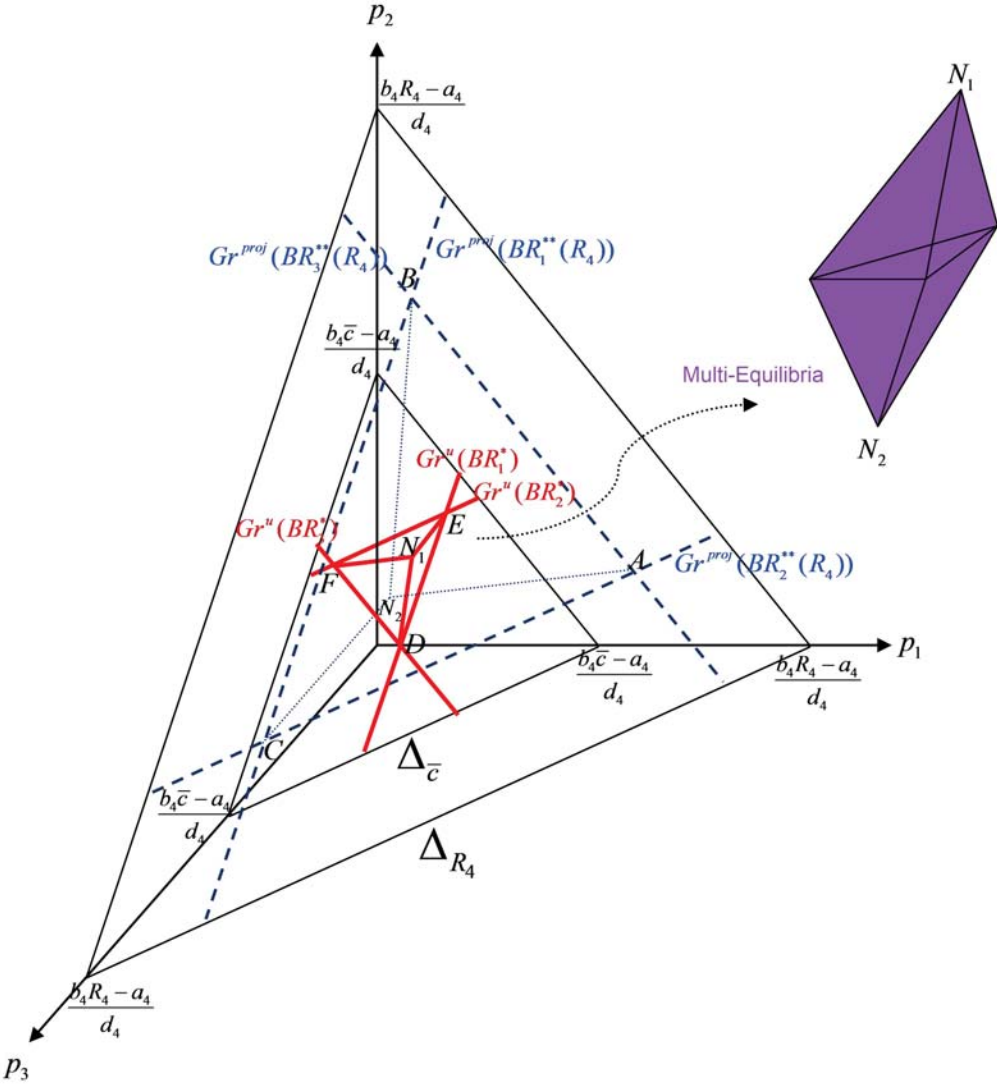

Figure 9: **The Proof of Case 1 of Proposition 1 is Sketched with Four Firms and $n^* = 3$:** In this figure, we show the multiple equilibrium price vectors by changing the marginal cost level of firm four along the interval $[\bar{c}, R_4]$. Note that $\{E, F, D, N_2\} \subseteq \triangle^2_{\bar{c}}$, $\{A, B, C, N_1\} \subseteq \triangle^2_{R_4}$, $Z_1 = conhull(ABC)$, $Z_2 = conhull(DEF)$, $N_1 \in Z_1$, and $N_2 \in Z_2$. Let $W_1 = conhull(Z_1, N_2)$ and $W_2 = conhull(Z_2, N_1)$. When $c_4 = \bar{c}$, the projected best response graphs of firms 1, 2, and 3, i.e, $Gr^{proj}BR_i^{\star\star}(\bar{c})$ intersect at N_2. Moreover, for each $i \in \{1, 2, 3\}$, its unprojected best response graph, i.e. $Gr^u(BR_i^\star)$ is drawn and we have $conhull\{N_1, E, D\} \subseteq Gr^u(BR_1^\star)$, $conhull\{N_1, E, F\} \subseteq Gr^u(BR_2^\star)$, and $conhull\{N_1, F, D\} \subseteq Gr^u(BR_3^\star)$. Accordingly, point N_1 lies at the intersection of these planes and is the unique equilibrium of the game whenever $c_4 = R_4$. The blue dotted lines represent the intersection of projected best response functions of firms 1, 2, and 3 with $\triangle^2_{R_4}$. For each $c_4 \in (\bar{c}, R_4)$, the multiple equilibria is given by the intersection of $W_1 \bigcap W_2$, which is shown by the purple pyramid on the right, with the simplex $\triangle^2_{c_4}$.

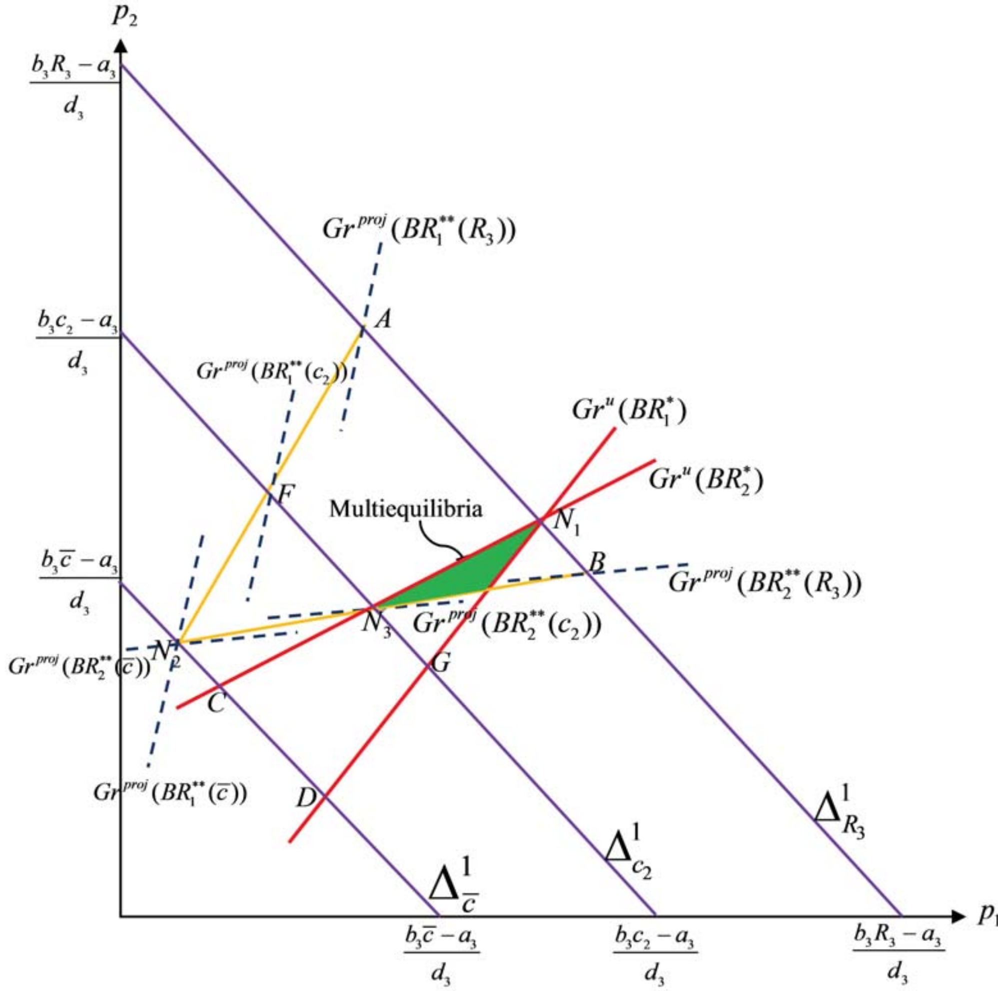

Figure 10: **The Proof of Case 2 of Proposition 1 is Sketched with Three Firms and $n^* = 2$:** Let $W_1 = conhull(A, B, N_2)$ and $W_2 = conhull(C, D, N_1)$. In this figure, we show the multiple equilibrium price vectors by changing the marginal cost level of firm three along the interval $[c_2, R_3]$. We draw the related best response functions and their critical intersections with the simplexes as in Figure 8 and see that for each $c_3 \in (c_2, R_3)$, the multiple equilibria is given by the intersection of $W_1 \bigcap W_2$ with the simplex $\triangle^1_{c_3}$.