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# Strategies and interactive beliefs in dynamic games

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## Abstract

Interactive epistemology in dynamic games studies forms of strategic reasoning like backward and forward induction by means of a formal representation of players' beliefs about each other, conditional on each history. Work on this topic typically relies on epistemic models where states of the world specify both strategies and beliefs. Strategies are conjunctions of behavioral conditionals of the form "if history  $h$  occurred, then player  $i$  would choose action  $a_i$ ." In this literature, strategies are literally interpreted as (objective) behavioral conditionals. But the intuitive interpretation of "strategy" is that of (subjective) "contingent plan of action." As players do not delegate their moves to devices that mechanically execute a strategy, plans cannot be anything but beliefs of players about their own behavior. In this paper we analyze strategic reasoning in dynamic games with perfect information by means of epistemic models where states of the world describe the actual play path (not behavioral conditionals) and the players' conditional probability systems about the path and about each other conditional beliefs. Therefore, the players' beliefs include their contingent plans. We define *rational planning* as a property of beliefs, whereas *material consistency* connects plans with choices on the actual play path. *Material rationality* is the conjunction of rational planning and material consistency. In perfect information games of depth two (the simplest dynamic games), correct belief in material rationality only implies a Nash outcome, not the backward induction one. We have to consider stronger assumptions of persistence of belief in material rationality in order to obtain backward and forward induction reasoning.

## 1 Introduction

Interactive epistemology in dynamic games studies forms of strategic reasoning like backward and forward induction by means of a formal representation of the players' beliefs about each other at each history.<sup>1</sup> Work on this topic typically relies on epistemic models where states of

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<sup>1</sup>For surveys on this topic see [Battigalli and Bonanno \(1999\)](#), [Perea \(2001\)](#) and [Brandenburger \(2007\)](#).

the world specify both strategies and beliefs. Strategies are conjunctions of behavioral conditionals of the form “if  $h$  then  $a_i$ ,” or more descriptively, “if  $h$  occurred, player  $i$  would choose  $a_i$ .” In this literature, strategies are literally interpreted as objective behavioral conditionals: if the strategy of player  $i$  at a state is  $s_i$ , this means that, at that state,  $s_i$  is necessarily executed and every conditional “if  $h$  occurred,  $i$  would choose  $a_i$ ” is true if and only if  $a_i = s_i(h)$ , independently of whether  $h$  occurs or is counterfactual at the state. But a more intuitive interpretation of “strategy” is that of a subjective “contingent plan of action.” If the extensive form describing the dynamic game is taken seriously, players cannot commit in advance (not even secretly) to play a strategy, as such commitment should appear as an explicit move in a larger, all-encompassing extensive form. Thus, “strategies as plans” cannot be anything but beliefs of players about their own behavior. Beliefs about own contingent behavior and beliefs about the contingent behavior of others provide a framework within which actual actions are rationally chosen at any given information set, as they allow to assign an expected value to each action. Some authors have explicitly addressed the issue of “strategies as behavioral conditionals vs. strategies as beliefs.” For example, [Battigalli and Siniscalchi \(1999\)](#) model first-order beliefs as *conditional probability systems* over strategy profiles (including own strategies) and assume that a rational player assigns probability one to her actual strategy whenever possible, i.e. conditional on each information set consistent with the given strategy.<sup>2</sup> Thus, a player  $j$  who believes in the rationality of  $i$ , equates the (rational) contingent plan of  $i$  to what  $i$  would actually do at each of her information sets; this allows  $j$  to make conditional predictions about  $i$ ’s behavior.<sup>3</sup>

Unlike strategies, that express behavioral conditionals, conditional probability systems express *epistemic* conditionals of the form “if  $i$  learned  $h$ , then  $i$  would believe that . . . .” Epistemic conditionals in the form of conditional probability systems have been axiomatized. Building on [Samet \(2000\)](#), [Di Tillio, Halpern, and Samet \(2010\)](#) show that an agent’s conditional beliefs  $B_i^p(E|C)$  ( $i$  believes  $E$  with probability at least  $p$  conditional on  $C$ ) satisfy a list of reasonable axioms if and only if there is a mapping assigning to each state of the world a conditional probability system on the set of states.

While we do not claim that including behavioral conditionals in the state of the world is conceptually incorrect, we submit that we have a better understanding of how epistemic conditionals can be embedded in states of the world. Therefore we propose to study interactive epistemology in dynamic games by means of epistemic models whereby states of the world describe the play path (*not* the behavioral conditionals) and the players’ conditional belief sys-

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<sup>2</sup>In their analysis, the specification of a strategy at information sets inconsistent with it is immaterial: only the classes of realization-equivalent strategies matter.

<sup>3</sup>See also the discussion in [Battigalli and Dufwenberg \(2009\)](#), who note that interpreting strategies as beliefs is crucial in the context of games with belief-dependent preferences (so called “psychological games”).

tems (that is, the epistemic conditionals). This methodological move, first advanced by Samet (1996) in the context of conditional *knowledge*,<sup>4</sup> forces the analyst to interpret strategies as systems of conditional beliefs about own behavior, which in our view is the correct interpretation in a context where players cannot delegate their choices to mechanical devices or trustworthy agents. On the other hand, this move presents interesting challenges in representing even the most elementary forms of strategic reasoning. We define a seemingly natural notion of rationality of a given player, say  $i$ , called *material rationality*,<sup>5</sup> made of two parts:

*Rational Planning*: this is a property of player  $i$ 's system of conditional beliefs, and in particular her beliefs about behavior (including her own) at the various nodes; these conditional beliefs define a profile of behavior strategies  $(b_i, b_{-i})$ ; the behavioral strategies of the opponents,  $b_{-i}$ , define a subjective decision tree, and rational planning of  $i$  requires that  $b_i$  is obtained by dynamic programming on this subjective decision tree.

*Material Consistency*: this property relates  $i$ 's conditional beliefs with her actual behavior: on the actual path,  $i$  never takes actions excluded by her beliefs, that is, by her plan of action.

Let us test the power of these concepts in the simplest dynamic games: generic perfect information games of depth two, e.g., leader-follower games. First, we rehearse the standard argument. The existing literature suggests that assuming rationality and *initial* (or, unconditional) *belief* in rationality is enough to obtain the standard backward induction solution: Rationality (in its sequential version) implies that the second mover best-responds to the first-mover choice; this pins down the strategy of the second mover. Initial belief in rationality implies that the first-mover *assigns probability one to this strategy*. Finally, rationality of the first mover implies that she best responds to it. Such intuitive argument can be given a formal representation when objective behavioral conditionals are part of the state of the world, therefore the conjunction of conditionals "for each action  $a_1$  of the leader, if  $a_1$  were chosen then the follower would take the best response to  $a_1$ " is an event to which the leader can assign probability one. But our methodological move of putting only the play path (not the behavioral conditionals) in the state of the world means that we cannot afford the luxury of having players who assign probabilities to behavioral conditionals. Instead, we work with material rationality and beliefs, including conditional beliefs.

As we have said, material rationality is the conjunction of rational planning and material consistency. What are the consequences of material rationality and belief in material rationality? Of course, this depends on how "belief" is defined. If we mean "initial belief," as above, then we cannot obtain the backward induction outcome even in leader-follower games. But we can recover the backward induction outcome if we assume stronger forms of belief. Suppose

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<sup>4</sup>We briefly discuss the relationship with Samet's paper in Section 5.

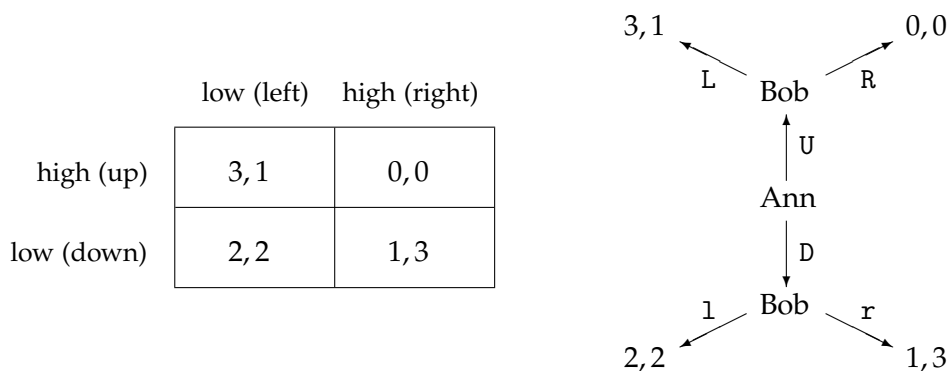
<sup>5</sup>See also Aumann (1998).

that material rationality holds and that the leader keeps believing in the material rationality of the follower whenever possible, in particular, after each of her own initial actions. Such post-action beliefs are crucial because they define the subjective decision tree of the leader: the subjective value to the leader of each action  $a_1$  is determined by what she believes about the follower *conditional on taking*  $a_1$ . If the leader believes in the material rationality of the follower, conditional on each initial action  $a_1$ , then each  $a_1$  has the backward induction value: thus, the materially rational leader's choice and the materially rational follower's response are their respective backward induction actions.

This brief discussion suggests that assuming material rationality and some degree of persistency in beliefs may allow an interesting formal analysis of strategic thinking in dynamic games. It is worth noting that the concept of "strong belief" in an event  $E$  ( $E$  is believed whenever possible) is at the heart of the epistemic analysis of forward induction reasoning, and that forward induction reasoning yields the backward induction path in generic games with perfect information (Battigalli and Siniscalchi, 2002). Such results are obtained in epistemic models that have behavioral conditionals in the states of the world. We show that similar results hold within the more parsimonious epistemic models considered here.

### 1.1 An illustrative example: the Stackelberg mini-game

Consider a quantity-setting duopoly where each of two firms, Ann and Bob, can choose either a low quantity or a high quantity, and Ann moves first. The extensive form of the game and the payoffs associated to the various combinations of outputs are given in the figure below; note that the backward induction path is the Stackelberg sequence (U,L).



Let us review the standard epistemic analysis of this game. A state of the world specifies a strategy and a type for each firm, where a type determines (conditional) beliefs about strategies and types of the other firm. In this context, rationality of Bob means that he *would* respond with L to U, and with r to D. If Ann believes in Bob's rationality, she assigns proba-

bility one to the strategy just described. Thus, if she is herself rational, she plays U, and the backward induction path obtains. Now consider the Nash equilibrium  $(D, R, r)$ . In the standard framework with strategies in the states, path  $(D, r)$  is inconsistent with rationality and belief in rationality. In the framework we propose, however, this imperfect Nash equilibrium is consistent with material rationality and initial belief in material rationality. In order to restore the backward induction solution, we have to add the assumption of Ann's *strong belief* in Bob's rationality.

In our framework, a state of the world (for this example) is a list  $(z, t_A, t_B)$ , where Ann's (Bob's) type  $t_A$  (resp.  $t_B$ ) specifies her beliefs on paths and Bob's (resp. Ann's) types, conditional on each decision node of the game. Thus, each type  $t_i$  of each player  $i$  has an unconditional belief  $\beta_i(t_i)(\cdot|\varnothing)$ , where  $\varnothing$  denotes the initial node, and two conditional beliefs  $\beta_i(t_i)(\cdot|U)$  and  $\beta_i(t_i)(\cdot|D)$ ,<sup>6</sup> related to the unconditional belief via the chain rule of conditional probabilities. Ann's plan is determined by her unconditional belief, while Bob's is determined by his conditional beliefs given U and D. In every state  $(z, t_A, t_B)$  consistent with Bob's material rationality,  $MR_B$ , we must have either  $z = (U, L)$  or  $z = (D, r)$ . However, if  $\beta_A(t_A)(U|\varnothing) = 0$ , then even assuming that  $\beta_A(t_A)(MR_B|\varnothing) = 1$ , we cannot conclude that  $\beta_A(t_A)(L|U) = 1$  and  $\beta_A(t_A)(r|D) = 1$ . Thus, it is indeed possible that  $z = (D, r)$ , while  $(z, t_A, t_B)$  exhibits material rationality for both players and also Ann's unconditional belief in Bob's material rationality.

In general, we show that in generic perfect information games of depth two, material rationality and (unconditional) belief in material rationality only imply a Nash equilibrium path, not the backward induction path (Proposition 1). Then we prove that in generic games of depth two, the backward induction path does obtain, if we assume material rationality and strong belief in material rationality within a sufficiently rich epistemic structure (Proposition 2, Corollary 1). For more complex games, such as centipede games of depth three or more, we show that (correct) common belief in material rationality does not even guarantee a Nash equilibrium path. On the other hand, building on Battigalli and Siniscalchi (2002), we show that *common strong belief in the other players' material rationality* (an assumption that captures forward induction reasoning) implies the backward induction path in generic games of perfect information (of any depth), provided that the epistemic structure is sufficiently rich (Proposition 5).

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<sup>6</sup>In our analysis we allow also beliefs conditional on terminal nodes, but we do so mainly for notational convenience.

## 2 Preliminaries

### 2.1 Games with perfect information

Throughout the paper we fix a finite game with perfect information, using the standard notation in [Osborne and Rubinstein \(1994\)](#). Thus we assume the following:

- a finite set  $I$  of *players* and a finite set  $A$  of *actions*;
- a finite set  $H$  of *histories*, that is, a finite set of finite sequences in  $A$ , containing the empty sequence  $\emptyset$ , which we call the *initial* history, and such that, for every  $(a^1, \dots, a^k) \in H$  and  $\ell < k$ , the corresponding subsequence is also a history, that is,  $(a^1, \dots, a^\ell) \in H$ ; for  $h \in H$ ,  $A(h) = \{a \in A : (h, a) \in H\}$  is the set of actions *available* at  $h$ ; if  $A(h) = \emptyset$ , then  $h$  is said to be *terminal*, and the set of terminal histories, or *paths*, is denoted by  $Z$ ; we let  $\delta$  denote the *depth* of the game, i.e. the number of elements in the longest sequence in  $H$ ;
- a function  $\iota : H \setminus Z \rightarrow I$ ; for each player  $i$ , we let  $H_i = \iota^{-1}(i)$ ;
- for each player  $i$ , a *payoff function*  $u_i : Z \rightarrow \mathbb{R}$ .

The induced weak and strict *precedence* relations on  $H$  will be denoted by  $\preceq$  and  $\prec$ , respectively. Thus, for  $h, h' \in H$  we write  $h \preceq h'$  whenever  $h$  is a subsequence of  $h'$ , and we write  $h \prec h'$  if in addition  $h \neq h'$ . The terminal successors of a history  $h$  are those in the set  $Z(h) = \{z \in Z : h \preceq z\}$ . For  $h \in H \setminus Z$  and  $h' \in H$  with  $h \prec h'$ , we write  $\alpha(h, h')$  for the unique  $a \in A$  such that  $(h, a) \preceq h'$ .

To avoid discussions of relatively minor issues, we focus our attention on the case where for every two distinct terminal histories  $z, z'$ , the player who moves at their last common predecessor is not indifferent between them:

**Assumption** (No relevant ties). For every  $i \in I$ ,  $h \in H_i$  and  $a, a' \in A(h)$ ,

$$a \neq a' \quad \Rightarrow \quad \{u_i(z) : z \in Z(h, a)\} \cap \{u_i(z) : z \in Z(h, a')\} = \emptyset.$$

### 2.2 Conditional probability systems

Given a compact metrizable space  $X$ , endowed with its Borel  $\sigma$ -algebra  $\mathcal{F}$ , and a finite family  $\mathcal{C} \subset \mathcal{F}$  of clopen (closed and open) events, containing  $X$  itself, called *conditions* or *hypotheses*, a *conditional probability system* is a collection of probability measures  $(\mu(\cdot|C))_{C \in \mathcal{C}}$  on  $X$  satisfying the following properties:

- $\mu(C|C) = 1$  for all  $C \in \mathcal{C}$ ;

- $\mu(E|C) = \mu(E|C')\mu(C'|C)$  for all  $E \in \mathcal{F}$  and  $C, C' \in \mathcal{C}$  with  $E \subseteq C' \subseteq C$ .

The set of all conditional probability systems is denoted  $\Delta^{\mathcal{C}}(X)$ . Under the stated assumptions,  $\Delta^{\mathcal{C}}(X)$  is a compact metrizable space—see [Battigalli and Siniscalchi \(1999\)](#). In our analysis, the family  $\mathcal{C}$  corresponds to the collection of events that the players can *observe* in the game, namely, the histories.

### 2.3 Strategies and conditional beliefs

Given the game described above, the standard definition of a *strategy* for player  $i$  is that of a mapping from  $i$ 's histories into available actions, that is,

$$s_i \in S_i = \prod_{h \in H_i} A(h).$$

In this paper we keep the latter formal definition of strategy, but we depart from the received literature on interactive epistemology for dynamic games, in that we do not assume that players can reason *directly* about strategies. Instead, we model a player's belief about behavior at a history (including her own), as her conditional belief on the set of paths, given the event that (she observes that) the path goes through that history.

Thus, in our framework a *state of the world* (or simply a *state*) specifies a path  $z \in Z$  and a *type*  $t_i \in T_i$  for each player  $i$ , where  $T_i$  is a compact metrizable space. Types encode the players' conditional beliefs, and conditional beliefs about each other's conditional beliefs, where for each player  $i$  the set of conditions  $\mathcal{C}$  is the family of events of the form  $Z(h) \times T_{-i}$ ,<sup>7</sup> with  $h \in H$ . This family of events is obviously isomorphic to  $H$  itself, and therefore in what follows, for every  $h \in H$ , we will often write  $h$  instead of the more cumbersome  $Z(h) \times T_{-i}$ . Moreover, we will write  $[h]$  instead of  $Z(h) \times T_i \times T_{-i}$  to denote the set of states where the path goes through  $h$ . Thus we assume: for each player  $i$ , there is a continuous function  $\beta_i : T_i \rightarrow \Delta^H(Z \times T_{-i})$  satisfying, for each  $t_i \in T_i$ ,

- $\beta_i(t_i)(h|h) = 1$  for all  $h \in H$ ;
- $\beta_i(t_i)(E|h) = \beta_i(t_i)(E|g)\beta_i(t_i)(g|h)$  for all  $g, h \in H$  with  $h \preceq g$  and  $E \subseteq Z(g) \times T_{-i}$ .

Note the obvious but important fact that that histories are *uninformative* (to player  $i$ ) about beliefs (of the other players), since our state space has a product structure and the conditioning events concern only one “side” of the product, namely the paths.

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<sup>7</sup>As usual, we define  $T_{-i} = \times_{j \in I \setminus \{i\}} T_j$ . Similar notations will be used, without notice, throughout the paper.



In what follows we call a tuple  $(T_i, \beta_i)_{i \in I}$  as described above, a *type structure*, and we say that it is *complete* if for every player  $i$ , the mapping  $\beta_i$  is onto.<sup>8</sup>

For each  $i \in I$  and  $t_i \in T_i$ , it is convenient to view the probability measures  $\beta_i(t_i)(\cdot|h)$  as probability measures on the whole state space  $Z \times T_i \times T_{-i}$ . Thus, given an event  $E \subseteq Z \times T_i \times T_{-i}$ , we say that player  $i$  *believes in E conditional on h* at a state  $(z, t_i, t_{-i})$ , provided that

$$\beta_i(t_i)(\{(z', t'_{-i}) \in Z \times T_{-i} : (z', t_i, t'_{-i}) \in E\} | h) = 1,$$

and we denote by  $B_i(E|h)$  the set of such states.<sup>9</sup> For  $h = \emptyset$  we write simply  $B_i(E)$ , and we say that  $i$  *believes in E*. Finally, there is (*correct*) *common belief in E* at every state in the event

$$CB(E) = E \cap B(E) \cap B(B(E)) \cap \dots,$$

where for every event  $E$  we write  $B(E)$  as an abbreviation for  $\bigcap_{i \in I} B_i(E)$ .

### 3 Rational planning and material consistency

For every player  $i$  and every type  $t_i$  of hers, the probabilities  $\beta_i(t_i)((h, a)|h)$  describe a *behavior strategy* for every player  $j$ , as  $h$  varies in  $H_j$  and  $a$  varies in  $A(h)$ . In particular, for  $j = i$ , they describe a behavior strategy for player  $i$  herself. However, we stress that despite this formal equivalence, such probabilities only represent  $i$ 's beliefs, and nothing in our basic framework requires that  $i$  will *act* accordingly; formally, a type structure can have a state  $(z, t_i, t_{-i})$  such that  $\beta_i(t_i)((h, \alpha(h, z))|h) = 0$  for some history  $h \in H_i$  with  $h \prec z$ .

Thus, we interpret those probabilities as the result of  $i$ 's *planning*: starting from conditional beliefs about the other players' behavior,  $i$  solves the corresponding subjective decision tree by dynamic programming, breaking ties arbitrarily; while this does not imply any commitment, it does deliver a *rational plan*, that is, a belief by player  $i$  that at each history of hers, she *would* follow the (optimal) recommendation of the dynamic programming solution, should that history indeed occur. Once the plan is in  $i$ 's mind, together with the beliefs about others that she started with, the entire profile of behavior strategies  $((\beta_i(t_i)((h, \cdot)|h))_{h \in H_i})_{j \in I}$  is determined.

The continuation value for type  $t_i$  of player  $i$ , corresponding to action  $a \in A(h)$  at history  $h \in H_i$ , is her expected payoff, conditional on history  $(h, a)$ , namely

$$\sum_{z \in Z(h, a)} \beta_i(t_i)(z|(h, a)) u_i(z).$$

<sup>8</sup>On completeness see [Brandenburger \(2003\)](#). [Battigalli and Siniscalchi \(1999\)](#) prove by construction that a complete type structure exists: it is the canonical structure where each  $T_i$  is the set of *hierarchies* of conditional probability systems satisfying collective coherency.

<sup>9</sup>This implies that  $i$  is always certain of her true type and, therefore, of her belief revision rule.

Thus, for all  $i \in I$ ,  $t_i \in T_i$  and  $h \in H_i$ , we define the set of locally optimal actions at  $h$  as

$$A^*(h, \beta_i(t_i)) = \arg \max_{a \in A(h)} \sum_{z \in Z(h,a)} \beta_i(t_i)(z|(h,a)) u_i(z). \quad (1)$$

If type  $t_i$  believes that she would indeed take an optimal action at  $h$ , should the latter be reached, then  $\beta_i(t_i)((h, \cdot)|h)$  must be supported in  $A^*(h, \beta_i(t_i))$ . This motivates the following:

**Definition 1.** Player  $i$  satisfies *rational planning* at state  $(z, t_i, t_{-i})$  if for all  $h \in H_i$  and  $a \in A(h)$ ,

$$a \notin A^*(h, \beta_i(t_i)) \quad \Rightarrow \quad \beta_i(t_i)((h, a)|h) = 0.^{10}$$

Let  $RP_i$  denote the set of such states, and let  $RP = \bigcap_{i \in I} RP_i$ .

As we have already argued, and as is indeed clear from its definition, rational planning is a property of beliefs, *per se* it has no implication about actual behavior. Hence, a player's belief in rational planning of other players need not have any implication on her belief about their behavior. In order to obtain such implications, we have to add the belief that the other players's behavior is consistent with their plan. This leads to the following:

**Definition 2.** Player  $i$  is *materially consistent* at state  $(z, t_i, t_{-i})$  if for every  $h \in H_i$ ,

$$h \prec z \quad \Rightarrow \quad \beta_i(t_i)((h, \alpha(h, z))|h) > 0.$$

Let  $MC_i$  denote the set of such states, and let  $MC = \bigcap_{i \in I} MC_i$ .

In other words,  $i$  is not materially consistent if at some history along the realized path, she takes an action that she planned to exclude, conditional on that history being reached.<sup>11</sup> The conjunction of rational planning and material consistency plays an important role in our analysis, and therefore it deserves its own name:<sup>12</sup>

**Definition 3.** Player  $i$  is *materially rational* at each state in  $MR_i = MC_i \cap RP_i$ .  $MR = \bigcap_{i \in I} MR_i$ .

### 3.1 Common belief in material rationality

A standard result of the literature on interactive epistemology for simultaneous moves games states that an outcome is consistent with rationality and common belief in rationality if and

<sup>10</sup>By the one-shot deviation principle, if  $i$  plans rationally at  $(z, t_i, t_{-i})$  then the behavioral strategy implied by  $t_i$  is dynamically optimal in the decision tree implied by  $t_i$ .

<sup>11</sup>In logic, a *material implication* is an if-then statement that holds false if and only if its antecedent is true while the consequent is false. We use the term "material" because here, analogously, lack of  $i$ 's material consistency occurs if the material implication "if  $h$  is reached, then  $i$  acts according to her plan" is false for some history  $h$  of  $i$ .

<sup>12</sup>Our definition of material rationality is similar to that of [Aumann \(1998\)](#).

only if it is rationalizable. It is therefore natural to consider similar epistemic assumptions in the present context. The assumptions of material rationality and common belief thereof are represented by the event

$$CB(MR) = MR \cap B(MR) \cap B(B(MR)) \cap \dots$$

The latter is not a vacuous assumption, since it holds in the *BI structure*, which is defined as follows. By no relevant ties, there is a unique backward induction (henceforth BI) strategy profile  $s^{BI}$  which induces the unique BI path  $z^{BI}$ . Then, the BI structure is the type structure where each player has only one type, and for each nonterminal history  $h \in H \setminus Z$ , this type assigns probability one to action  $s^{BI}(h)$  given  $h$ . In the BI structure,  $CB(MR)$  holds.

**Definition 4.** A type structure  $(T_i, \beta_i)_{i \in I}$  contains the BI structure if there is a profile of types  $(t_i^{BI})_{i \in I} \in \times_{i \in I} T_i$  such that for all  $i \in I$  and  $h \in H \setminus Z$ ,  $\beta_i(t_i^{BI})(Z(h, s^{BI}(h)) \times \{t_{-i}^{BI}\} | h) = 1$ .

We now present a preliminary result. Despite its simplicity, the proof requires some care and it illustrates the features of the adopted framework.

**Proposition 1.** Assume that  $\delta = 2$ . For every type structure  $(T_i, \beta_i)_{i \in I}$  and every state  $(z^*, t^*) \in MR \cap B(MR)$ ,  $z^*$  is a (mixed) Nash equilibrium path. Conversely, for every (mixed) Nash equilibrium path  $z$ , there exists a type structure  $(T_i, \beta_i)_{i \in I}$  and a profile of types  $t^*$  in it, such that  $(z^*, t^*) \in MR \cap B(MR)$ , and in fact,  $(z^*, t^*) \in CB(MR)$ .<sup>13</sup>

**Proof.** Since the game has depth two, we may assume without essential loss of generality that each state has the form  $((a_1, a_2), t)$ . Fix a type structure  $(T_i, \beta_i)_{i \in I}$  and a state  $((a_1^*, a_2^*), t^*) \in MR \cap B(MR)$ . We show that the behavioral strategy profile implied by the type of the first mover,  $t_{i(\emptyset)}^*$ , is a Nash equilibrium that gives positive probability (indeed, probability one) to  $(a_1^*, a_2^*)$ . Let  $i = i(\emptyset)$  be the first mover. By the assumption of no relevant ties,  $MR_{-i} \subseteq \{((a_1, a_2), t) : a_2 = s^{BI}(a_1)\}$ , where  $s^{BI}$  denotes the backward induction strategy profile. Since  $((a_1^*, a_2^*), t^*) \in B_i(MR_{-i})$ , we also have

$$\beta_i(t_i^*)(\{((a_1, a_2), t_{-i}) : a_2 = s^{BI}(a_1)\} | \emptyset) = 1.$$

Hence, for all  $a_1 \in A(\emptyset)$ ,  $\beta_i(t_i^*)(a_1 | \emptyset) = \beta_i(t_i^*)((a_1, s^{BI}(a_1)) | \emptyset)$ . But the right-hand side of the latter equals  $\beta_i(t_i^*)((a_1, s^{BI}(a_1)) | a_1) \beta_i(t_i^*)(a_1 | \emptyset)$  by the chain rule, so  $\beta_i(t_i^*)((a_1, s^{BI}(a_1)) | a_1) = 1$  for all  $a_1$  with  $\beta_i(t_i^*)(a_1 | \emptyset) > 0$ . Since  $((a_1^*, a_2^*), t^*) \in RP_i$ , for all  $a_1$  with  $\beta_i(t_i^*)(a_1 | \emptyset) > 0$ ,

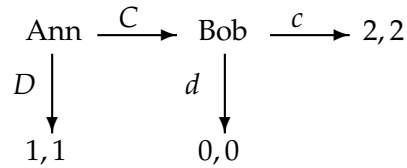
$$a_1 \in \arg \max_{a'_1 \in A(\emptyset)} \sum_{a'_2 \in A(a'_1)} \beta_i(t_i^*)((a'_1, a'_2) | a'_1) u_i(a'_1, a'_2).$$

<sup>13</sup>We can show that the converse holds for all perfect information games with no relevant ties.

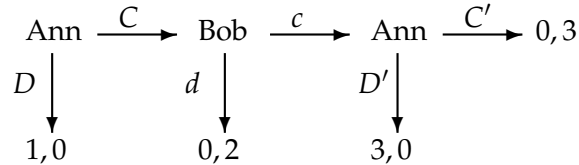
All this implies that the behavior strategy profile described by  $\beta_i(t_i^*)$  is a Nash equilibrium. Since  $((a_1^*, a_2^*), t^*) \in MC_i$ ,  $\beta_i(t_i^*)((a_1^*, a_2^*)|\emptyset) = \beta_i(t_i^*)(a_1^*|\emptyset) > 0$  and  $(a_1^*, a_2^*)$  is a Nash equilibrium path. For the second claim in the proposition, see Appendix A.1. ■

This simple result has obvious limitations, since it assumes depth two, and moreover, even games of depth two may have non-BI Nash equilibria. So we have not established that (correct, unconditional) common belief in material rationality yields the BI outcome. The following two examples address these limitations. The first example shows that assuming common belief in material rationality, even in games of depth two, we cannot go very far. The reason is that the material implication “if  $h$  is reached,  $i$ ’s action is consistent with his plan” is trivially satisfied at each state where  $h$  is not reached.

**Example 1.** In the game below, by Proposition 1, the imperfect Nash equilibrium outcome  $D$  is consistent with material rationality and common belief in it.



**Example 2.** In centipede games the BI outcome is also the unique Nash outcome. The following centipede example shows that in games with depth three or more, material rationality and common belief thereof, do not even give a Nash outcome.



The BI solution is to go always down. However, in Appendix A.2 we exhibit a type structure with a state in  $CB(MR)$  where the path is  $(C, c, D')$ . Intuitively, at this state Ann surprises Bob and tricks him into thinking that she is not materially rational; Bob’s beliefs are incorrect, but he is indeed unconditionally certain that path  $D$  occurs, that material rationality holds, and that Ann unconditionally believes in  $CB(MR)$ . Path  $(C, c, D')$  obtains from strategies that survive one round of weak dominance followed by iterated strict dominance, as in Dekel and Fudenberg (1990); we can show that all such paths are consistent with common belief in material rationality.

### 3.2 Common strong belief in material rationality

The examples in the previous section indicate that in order to obtain the BI path, some persistence of belief in material rationality is needed. In this section we introduce *strong belief* in material rationality, which captures precisely this idea.

**Definition 5.** Fix a type structure  $(T_i, \beta_i)_{i \in I}$  and an event  $E$  in it. Player  $i$  has *strong belief* in  $E$  at state  $(z, t_i, t_{-i})$  if for every  $h \in H$ ,

$$\{(z', t') \in E : h \prec z'\} \neq \emptyset \quad \Rightarrow \quad \beta_i(t_i)(E|h) = 1.$$

Let  $SB_i(E)$  denote the set of such states.

Strong belief in material rationality yields the BI outcome in simple games, provided that the type structure is sufficiently rich, as the following result shows.

**Proposition 2.** *Suppose that  $\delta = 2$  and fix a type structure  $(T_i, \beta_i)_{i \in I}$  that contains the BI structure. Then material rationality and strong belief of the first mover in the other players' material rationality imply the BI path: let  $i = \iota(\emptyset)$ , then for every  $(z, t) \in MR \cap SB_i(MR_{-i})$ ,  $z$  is the BI path.*

**Proof.** Let  $i = \iota(\emptyset)$  be the first mover and fix  $(z, t) \in MR \cap SB_i(MR_{-i})$ . Pick any  $a_1 \in A(\emptyset)$  and let  $j = \iota(a_1)$  be the player who moves after  $a_1$  (if it is not terminal). Let  $s_i^{BI}$  and  $s_j^{BI}$  be the strategies that select the BI move at every history. As a preliminary observation, note that for every  $(z', t') \in MR_j$ , if  $a_1 \prec z'$  then  $z' = (a_1, s_j^{BI}(a_1))$  (see the proof of Proposition 1). Now consider the conditional belief  $\beta_i(t_i)(\cdot|a_1)$ . If  $j = i$ , that is, if  $i$  plays again after  $a_1$ , then  $(z, t) \in RP_i$  implies  $\beta_i(t_i)((a_1, s_i^{BI}(a_1))|a_1) = 1$ . Otherwise, assume  $j \neq i$ , and let us show that  $(z, t) \in SB_i(MR_{-i})$  implies  $\beta_i(t_i)((a_1, s_j^{BI}(a_1))|a_1) = 1$ . Since the structure contains the BI structure, it contains the state  $((a_1, s_j^{BI}(a_1)), t_i, t_{-i}^{BI})$ . It is easily checked that  $((a_1, s_j^{BI}(a_1)), t_i, t_{-i}^{BI}) \in MR_{-i}$ . Thus,  $(z, t) \in SB_i(MR_{-i})$  implies  $\beta_i(t_i)(MR_{-i}|a_1) = 1$  and hence  $\beta_i(t_i)(MR_j|a_1)$ . By the preliminary observation above, we then obtain  $\beta_i(t_i)((a_1, s_j^{BI}(a_1))|a_1) = 1$ . This is true for every  $a_1 \in A(\emptyset)$ , so  $i$  assigns the BI value to every such action. Since  $(z, t) \in MR$ , this implies that  $z$  is the BI path. ■

**Corollary 1.** *Suppose that  $\delta = 2$  and fix a complete type structure  $(T_i, \beta_i)_{i \in I}$ . Then material rationality and strong belief of the first mover in the other players' material rationality imply the BI path in this structure: let  $i = \iota(\emptyset)$ , then for every  $(z, t) \in MR \cap SB_i(MR_{-i})$ ,  $z$  is the BI path.*

**Proof.** By Lemma 1 in Appendix A.3, each complete type structure contains the BI structure. The thesis then follows from Proposition 2. ■

Building on Battigalli and Siniscalchi (2002), the result can be extended to all perfect information games with no relevant ties, i.e. without restrictions on the depth of the game  $\delta$ ,

by looking at material rationality and *common* strong belief in the *other* players' material rationality. In order to establish this result, we define strong belief and common strong belief for *profiles* of events, as follows. For  $\underline{E} = (E_i)_{i \in I}$ , where for every player  $i$ ,  $E_i \subseteq Z \times T_i \times T_{-i}$  is measurable, define  $\cap \underline{E} = \cap_{i \in I} E_i$  and  $\cap_{-i} \underline{E} = \cap_{j \neq i} E_j$ , and let

$$\underline{CSB}(\underline{E}) = (E_i \cap SB_i(\cap_{-i} \underline{E}))_{i \in I}.$$

Note that with this notation,  $\cap \underline{CSB}(\underline{E})$  is the event that each  $E_i$  obtains, and each player  $i$  strongly believes in  $\cap_{j \neq i} E_j$ . Then, recursively for all  $m \geq 1$ , letting  $\underline{CSB}^0(\underline{E}) = \underline{E}$  by convention,

$$\underline{CSB}^m(\underline{E}) = \underline{CSB}(\underline{CSB}^{m-1}(\underline{E}))$$

**Definition 6.** There is (*correct*) *common strong belief in the other players' material rationality* at each state in

$$CSBMR = \bigcap_{m \geq 0} \cap \underline{CSB}^m((MR_i)_{i \in I}).$$

As a preliminary result, we show that in every type structure, there is a unique path consistent with common strong belief in material rationality; then we show that this path is, in fact, a Nash equilibrium path. Before we state the result, note that if player  $i$  strongly believes in the other players' material rationality, and the latter is compatible with a history  $h$  where  $i$  moves, then player  $i$  must be certain of the other players' material rationality, when conditioning on  $h$  being reached *and* any action  $a \in A(h)$ . Moreover, since histories are uninformative about beliefs, the same is true when considering the event that, in addition, the other players' beliefs lie in a certain set  $E_{-i}$ . Formally, for all  $i \in I$ ,  $h \in H_i$ ,  $a \in A(h)$  and measurable  $E_{-i} \subseteq T_{-i}$ , letting  $E = Z \times T_i \times E_{-i}$ ,

$$(Z(h) \times T_i \times T_{-i}) \cap E \cap MR_{-i} \neq \emptyset \quad \Rightarrow \quad SB_i(MR_{-i} \cap E) \subseteq B_i(MR_{-i} \cap E | (h, a)). \quad (2)$$

In the proof of the following proposition, this fact will be used inductively, to show that for every history  $h$  that is compatible with common strong belief in rationality, and for every action  $a$  available at  $h$ , the player moving at  $h$  expects a unique path following  $a$ , that is, her beliefs given  $(h, a)$  are concentrated on a single path.

**Proposition 3.** *Fix a type structure. Every state in CSBMR specifies the same path.*

**Proof.** Let  $\underline{MR} = (MR_i)_{i \in I}$ . For every path  $z$  and history  $h \prec z$ , let  $d(h, z)$  denote the length of the subpath from  $h$  to  $z$ .<sup>14</sup> We prove by induction in  $m \geq 0$ , that for every  $i \in I$ , every  $h \in H_i$  with  $\max_{z \in Z(h)} d(h, z) \leq m + 1$  and  $[h] \cap \cap \underline{CSB}^m(\underline{MR}) \neq \emptyset$ , and every  $a \in A(h)$  with  $[(h, a)] \cap MR_i \neq \emptyset$ , there exists  $z_{h,a} \in Z$  such that

$$MR_i \cap \bigcap_{k=0}^m SB_i(\cap_{-i} \underline{CSB}^k(\underline{MR})) \subseteq B_i([z_{h,a}] | (h, a)).$$

<sup>14</sup>If  $z = (h, \alpha(h, z))$  then  $d(h, z) = 1$ , if  $z = (h, \alpha(h, z), \alpha((h, \alpha(h, z)), z))$  then  $d(h, z) = 2$ , and so on.

Note that this implies that at every state  $(z, t_i, t_{-i})$  in the event on the left-hand side, if  $h \prec z$ , then  $\alpha(h, z)$  must be, by no relevant ties, the unique locally optimal action for  $t_i$  at  $h$ . In other words, there exists  $a_h^* \in A(h)$  such that

$$[h] \cap MR_i \cap \bigcap_{k=0}^m SB_i(\cap_{-i} \underline{CSB}^k(\underline{MR})) \subseteq [(h, a_h^*)].$$

For  $m = 0$ , our claim is trivially true. Let  $n \geq 1$ , assume the claim holds for all  $0 \leq m \leq n - 1$ , and fix  $i \in I$ ,  $h \in H_i$  with  $\max_{z \in Z(h)} d(h, z) \leq n + 1$  and  $[h] \cap \cap \underline{CSB}^n(\underline{MR}) \neq \emptyset$ , and  $a \in A(h)$  with  $[(h, a)] \cap MR_i \neq \emptyset$ . Note that the induction hypothesis implies

$$MR_i \cap B_i \left( \bigcap_{j \neq i} ([ (h, a) ] \cap MR_j \cap \bigcap_{k=0}^{n-1} SB_j(\cap_{-j} \underline{CSB}^k(\underline{MR}))) \middle| (h, a) \right) \subseteq MR_i \cap B_i([z_{h,a}] | (h, a)). \quad (3)$$

By our definitions,

$$\cap_{-i} \underline{CSB}^n(\underline{MR}) = \bigcap_{j \neq i} \left( MR_j \cap \bigcap_{k=0}^{n-1} SB_j(\cap_{-j} \underline{CSB}^k(\underline{MR})) \right). \quad (4)$$

From  $[h] \cap \cap \underline{CSB}^n(\underline{MR}) \neq \emptyset$  it follows that  $[h] \cap \cap_{-i} \underline{CSB}^k(\underline{MR}) \neq \emptyset$  for all  $0 \leq k \leq n - 1$  and hence, using (2) and the induction hypothesis,

$$\begin{aligned} MR_i \cap \bigcap_{k=0}^n SB_i(\cap_{-i} \underline{CSB}^k(\underline{MR})) &\subseteq MR_i \cap \bigcap_{k=0}^n B_i(\cap_{-i} \underline{CSB}^k(\underline{MR}) | (h, a)) \\ &= MR_i \cap B_i(\cap_{-i} \underline{CSB}^n(\underline{MR}) | (h, a)) \\ &= MR_i \cap B_i \left( \bigcap_{j \neq i} (MR_j \cap \bigcap_{k=0}^{n-1} SB_j(\cap_{-j} \underline{CSB}^k(\underline{MR}))) \middle| (h, a) \right) \quad (\text{by (4)}) \\ &= MR_i \cap B_i \left( \bigcap_{j \neq i} ([ (h, a) ] \cap MR_j \cap \bigcap_{k=0}^{n-1} SB_j(\cap_{-j} \underline{CSB}^k(\underline{MR}))) \middle| (h, a) \right) \\ &\subseteq MR_i \cap B_i([z_{h,a}] | (h, a)) \quad (\text{by (3)}). \end{aligned}$$

■

Using the latter proposition, we then obtain the following:<sup>15</sup>

**Proposition 4.** *Fix a type structure. Every state in CSBMR specifies a (mixed) Nash equilibrium path.*

**Proof.** See Appendix A.4. ■

In Appendix A.5 we prove that in a complete type structure, a path is consistent with common strong belief in rationality if and only if it is consistent with Pearce's (1984) extensive

<sup>15</sup>See Battigalli and Friedenberg (2010, Corollary 8.1, Proposition 8.2).

form rationalizability, a non-empty solution concept. The proof of the following proposition then follows from Battigalli's (1997) result that in perfect information games with no relevant ties, all extensive form rationalizable profiles induce the BI path.

**Proposition 5.** *Fix a complete type structure. Every state in CSBMR specifies the BI path.*

**Proof.** See Appendix A.5. ■

## 4 Discussion and Extensions

In this paper we used epistemic structures for games with perfect information where states include only the play path and interactive conditional beliefs to elucidate how assumptions about rationality and beliefs are related to standard solution concepts. Some of our results are similar to those obtained in the literature by means of epistemic structures that put “objective strategies” in the states of the world. Here we further discuss the relationships with that literature, and we hint about an extension to games with imperfect information.

### 4.1 Hypothetical knowledge

The epistemic structures of Samet (1996), like ours, constitute a major departure from the standard model in which strategies are specified at each state of the world. In such structures, each state specifies what each players hypothesizes about his epistemic state should any node  $h$  occur. Note that we can give a similar interpretation to the beliefs of a type  $t_i$  conditional on  $h$ ,  $\beta_i(t_i)(\cdot|h)$ . In Samet's epistemic structures, just like in ours, strategies are not primitive; instead, they are constructed from the *hypotheses* that the player makes about the actions he would take at each of his nodes, should it be reached. Unlike our structures, Samet's structures do not specify conditional probabilistic beliefs, but rather conditional knowledge. In the game-theoretic analysis of that paper, which also deals only with perfect information games, a hypothesis is a node: for any given state and node, the player at that node is *rational at the node* if the node is reached at the given state, and the player does not know that his action, at the given node, gives him strictly less than what he hypothesizes another action would give.

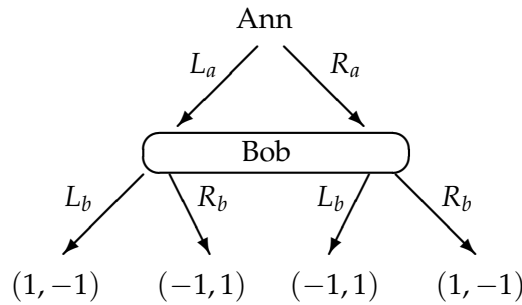
Similarly to what we illustrate in Proposition 1 and Example 1, Samet (1996) provides an example showing that common knowledge of rationality does not imply the backward induction path. And similarly to our Proposition 5, Samet proves that if there is a *common hypothesis* of node rationality, then the players play the backward induction path. Like strong belief captures persistence of belief, the notion of common hypothesis embodies persistence of knowledge: roughly, it requires that for every node  $h$ , starting from the root and iteratively



hypothesizing that the actions leading to  $h$  are taken, the players know that node rationality at  $h$  holds.

## 4.2 Imperfect information

Rational planning, an essential ingredient of our analysis, is based on the possibility to assign a continuation value to an action  $a$  taken at history  $h$  using beliefs conditional on  $(h, a)$ . In order to be consistent with the analysis of perfect information games given earlier, it becomes necessary to extend the set of hypotheses to include more than just information sets. Consider the following matching pennies example:



Write  $z^{LL}$  for the path  $(L_a, L_b)$  and define  $z^{LR}$ ,  $z^{RL}$  and  $z^{RR}$  analogously. What is Bob's expected utility, given that (his information set is reached and) he chooses  $L_b$ ? In order to give a formal answer that is consistent with our analysis of the perfect-information case, the set of paths  $\{z^{LL}, z^{RL}\}$  must be added as a hypothesis, and Bob's expected utility must be computed using his beliefs given this set. Similarly, in order to evaluate his other choice  $R_b$ , the set of paths  $\{z^{LR}, z^{RR}\}$  must be added as a hypothesis. This, however, opens the door to the following possibility: Bob believes, conditional on his information set being reached and on himself choosing  $L_b$ , that his expected utility is strictly negative, and he expects the same when considering  $R_b$ . This goes against the traditional expected utility argument that a rational Bob cannot expect to receive less than zero, whatever his beliefs about Ann may be.

For the said possibility to materialize, all we need is a type of Bob who attaches probability strictly smaller than  $1/2$  to  $R_a$ , conditional on  $\{z^{LL}, z^{RL}\}$ , and strictly larger than  $1/2$  to  $R_a$ , given  $\{z^{LR}, z^{RR}\}$ . Note that such beliefs violate a natural independence property: the sets of paths  $\{z^{LL}, z^{RL}\}$  and  $\{z^{LR}, z^{RR}\}$  only differ because of Bob's own action, and yet Bob holds different beliefs about Ann conditional on such events. Conversely, imposing independence would prevent Bob holding beliefs of this kind: if Bob's beliefs conditional on  $\{z^{LL}, z^{RL}\}$  and  $\{z^{LR}, z^{RR}\}$  are the same, it is clear that the expected values of  $L_b$  and  $R_b$  must sum to zero, whatever those beliefs are, as per the traditional expected utility argument.

# A Appendices

## A.1 Proof of the second claim in Proposition 1

Fix a Nash equilibrium in behavioral strategies  $b^* = (b_i^*)_{i \in I}$ . Since the game has perfect information and no relevant ties,  $b^*$  yields some path  $z^* = (a_1^*, a_2^*)$  with probability one. We construct a simple type structure with a state  $(z^*, t^*) \in CB(MR)$ .

For each player there is only one type with beliefs determined by  $b^*$  as follows: for all  $i \in I$ ,  $a_1 \in A(\emptyset)$ ,  $a_2 \in A(a_1)$ ,

$$\begin{aligned} T_i &= \{t_i^*\}, \\ \beta_i(t_i^*)(a_1^*, a_2^*, t_{-i}^* | \emptyset) &= 1, \\ \beta_i(t_i)(t_i^*)(a_1, a_2^*, t_{-i} | a_1) &= \begin{cases} b_{i(a_1)}^*(a_2 | a_1), & \text{if } i \neq \iota(a_1), \\ s^{BI}(a_1), & \text{if } i = \iota(a_1). \end{cases} \end{aligned}$$

First note that, just by construction, all players plan to use their optimal action in the second stage *off* the  $z^*$  path. Since  $b^*$  is an equilibrium,  $a_2^* = s^{BI}(a_1^*)$ . Thus, all players different from the first mover  $\iota(\emptyset)$  are materially rational at  $(z^*, t^*)$ . To see that also the first mover is materially rational at this state, consider that, since  $b^*$  is a Nash equilibrium, if  $\iota(a_1) = \iota(\emptyset)$  then  $u_{\iota(\emptyset)}(z^*) \geq u_{\iota(\emptyset)}(a_1, s^{BI}(a_1))$ , if  $\iota(\emptyset) \neq \iota(a_1)$  then  $u_{\iota(\emptyset)}(z^*) \geq \sum_{a_2 \in A(a_1)} u_i(a_1, a_2) b_{i(a_1)}^*(a_2 | a_1)$ . Thus  $(z^*, t^*) \in MR$ , furthermore, by construction, at  $(z^*, t^*)$  every player unconditionally believes  $(z^*, t^*)$ . This implies  $(z^*, t^*) \in CB(MR)$ .

## A.2 The type structure of Example 2

Ann's set of types is  $T_a = \{t_a^*, t_a^{BI}, \hat{t}_a\}$  and Bob's is  $T_b = \{t_b^*, t_b^{BI}\}$ . The beliefs of each type of Ann are described by a table where each row corresponds to a path  $z$ , each column corresponds to a conditioning event (a non-terminal history  $h$ ), and for each  $(z, h)$  the corresponding cell describes the type's beliefs over  $\{z\} \times \{t_b^*, t_b^{BI}\}$  conditional on  $h$  (for example, looking at row  $(C, d)$  and column  $C$  of matrix  $t_a^{BI}$ , we see that  $t_a^{BI}$  assigns probability one to path  $(C, d)$  and type  $t_b^{BI}$  conditional on  $C$ ). Analogous matrices represent, for each type of Bob, his beliefs

over  $\{z\} \times \{t_a^*, t_a^{BI}, \hat{t}_a\}$ .

$t_a^*$	$\emptyset$	$C$	$C, c$	$t_a^{BI}$	$\emptyset$	$C$	$C, c$	$\hat{t}_a$	$\emptyset$	$C$	$C, c$
$D$	0,0	0,0	0,0	$D$	0,1	0,0	0,0	$D$	0,0	0,0	0,0
$C, d$	0,0	0,0	0,0	$C, d$	0,0	0,1	0,0	$C, d$	0,0	0,0	0,0
$C, c, D'$	1,0	1,0	1,0	$C, c, D'$	0,0	0,0	0,1	$C, c, D'$	0,0	0,0	0,0
$C, c, C'$	0,0	0,0	0,0	$C, c, C'$	0,0	0,0	0,0	$C, c, C'$	1,0	1,0	1,0

$t_b^*$	$\emptyset$	$C$	$C, c$	$t_b^{BI}$	$\emptyset$	$C$	$C, c$
$D$	0,1,0	0,0,0	0,0,0	$D$	0,1,0	0,0,0	0,0,0
$C, d$	0,0,0	0,0,0	0,0,0	$C, d$	0,0,0	0,1,0	0,0,0
$C, c, D'$	0,0,0	0,0,0	0,0,0	$C, c, D'$	0,0,0	0,0,0	0,1,0
$C, c, C'$	0,0,0	0,0,1	0,0,1	$C, c, C'$	0,0,0	0,0,0	0,0,0

The beliefs of types  $t_a^{BI}$  and  $t_b^{BI}$  correspond to the backward induction plans and path. The backward induction state is  $((D), t_a^{BI}, t_b^{BI})$ , whereas  $((C, c, D'), t_a^*, t_b^*)$  is a state where Ann surprises Bob and tricks him into thinking that she is not materially rational. It is clear that  $t_a^{BI}$  and  $t_b^{BI}$  satisfy rational planning, and it can be checked that  $t_a^*$  and  $t_b^*$  also do. Indeed, at state  $((C, c, D'), t_a^*, t_b^*)$  there is (correct, unconditional) common belief in material rationality, and a non-Nash outcome occurs. To see this, first note that

$$((D), t_a^{BI}, t_b^{BI}) \in MR \cap B(MR) \cap B(B(MR)) \cap \dots \subseteq B_i(MR \cap B(MR) \cap B(B(MR)) \cap \dots).$$

It is easily checked that at  $((C, c, D'), t_a^*, t_b^*)$  both players are materially consistent and plan rationally and that Ann's beliefs (conditional and unconditional) are correct. Bob's beliefs are incorrect, but he unconditionally assigns probability one to state  $((D), t_a^{BI}, t_b^{BI})$ , hence Bob (unconditionally) believes  $MR$  and that Ann unconditionally believes  $MR \cap B(MR) \cap B(B(MR)) \cap \dots$ . This implies that  $((C, c, D'), t_a^*, t_b^*) \in MR \cap B(MR) \cap B(B(MR)) \cap \dots$ .

### A.3 Complete structures contain the BI structure

**Lemma 1.** *A complete type structure contains the BI structure.*

**Proof.** For every  $h$ , let  $z^{BI}(h)$  and  $s^{BI}(h)$  denote, respectively, the BI path in the subgame starting at  $h$ , and the BI action at  $h$  (if  $h$  is nonterminal). We construct by recursion a sequence of type profiles  $(t^n)_{n=0}^\infty$  that converges to some  $t^{BI}$  satisfying

$$\forall i \in I, \forall h \in H, \quad \beta_i(t_i^{BI})(\{(z^{BI}(h), t_{-i}^{BI})\} | h) = 1 \quad (5)$$

Fix a type profile  $t^0$  arbitrarily. For the inductive step of the recursive construction, suppose we have constructed a type profile  $t^{n-1}$ ,  $n = 1, 2, \dots$ ; then for each player  $i$  there is a unique

conditional probability system  $\mu_i^n \in \Delta^H(Z \times T_{-i})$  such that

$$\forall h \in H, \quad \mu_i^n(\{(z^{BI}(h), t_{-i}^{n-1})\} | h) = 1.$$

By completeness, there is some type  $t_i^n$  such that  $\beta_i(t_i^n) = \mu_i^n$ . Suppose we have constructed types  $t_i^n$ ,  $i \in I$ ,  $n = 1, 2, \dots$ . By compactness, the sequence  $(t_i^n)_{n=0}^\infty$  has a convergent subsequence  $(t_i^{n_k})_{k=1}^\infty$ . Call  $t_i^{BI}$  the limit of this subsequence. We must show that  $t_i^{BI}$  satisfies (5). By continuity of  $\beta_i$ ,  $\beta_i(t_i^{BI}) = \lim_{k \rightarrow \infty} \beta_i(t_i^{n_k})$ . All the conditional probability systems  $\beta_i(t_i^{n_k})$  are arrays of Dirac measures  $\delta_{i,h}^k$  ( $h \in H$ ) where each  $\delta_{i,h}^k$  is concentrated on point  $(z^{BI}(h), t_{-i}^{n_k-1})$ . Since  $(z^{BI}(h), t_{-i}^{n_k-1})$  converges to  $(z^{BI}(h), t_{-i}^{BI})$ ,  $\delta_{i,h}^k$  converges to  $\delta_i^{BI}$ , the Dirac measure concentrated on  $(z^{BI}(h), t_{-i}^{BI})$ . Therefore (5) is satisfied. ■

#### A.4 Proof of Proposition 4

Fix a state  $(z^*, t^*) \in CSBMR$  and an arbitrary player  $i \in I$ . At this state,  $i$  has beliefs about his own behavior represented by the behavioral strategy  $b_i^*$  with  $b_i^*(a|h) = \beta_i(t_i^*)((h, a)|h)$  for each  $h \in H_i$ , and he has beliefs about the opponents' behavior represented by the unique behavioral strategy profile  $b_{i,-i}^*$  with  $b_{i,-i}^*(a|h) = \beta_i(t_i^*)((h, a)|h)$  for each  $h \in H_{-i}$ . Since  $(z^*, t^*) \in RP_i$ ,  $b_i^*$  is a best response to  $b_{i,-i}^*$ , therefore every pure strategy  $s_i$  in the support of  $b_i^*$  is also a best response to  $b_{i,-i}^*$ . By inspection of the proof of Proposition 3, for each  $h \prec z^*$ ,  $\beta_i(t_i)((h, \alpha(h, z^*))|h) = 1$ , which implies that  $i$ 's beliefs are confirmed by path  $z^*$ . Therefore the behavior strategy profile  $(b_i^*)_{i \in I}$  is a self-confirming equilibrium with independent, unitary beliefs, as defined in Fudenberg and Levine (1993),<sup>16</sup> and  $z^*$  is the path resulting from  $(b_i^*)_{i \in I}$  with probability one. By the Corollary in Kamada (2010), in a perfect information game every self-confirming equilibrium with independent, unitary beliefs is realization-equivalent to a (mixed) Nash equilibrium. Hence  $z^*$  is a Nash equilibrium path.

#### A.5 Proof of Proposition 5

Fix a complete structure  $(T_i, \beta_i)_{i \in I}$ . The idea of the proof is that CSBMR in such a structure yields first-order beliefs consistent with extensive form rationalizability, henceforth abbreviated as EFR (see Pearce, 1984; Battigalli, 1997). By this we mean that for every  $(z, t) \in CSBMR$  and  $i \in I$  there is a corresponding conditional probability system  $\nu_{-i}(t_i) \in \Delta^H(S_{-i})$ , where the set of conditions is the family of sets of the form  $S_{-i}(h)$  with  $h \in H$ , and  $S_{-i}(h)$  is the set of strategy profiles of players  $-i$  that allow  $h$ , satisfying the following: for every  $h \in H_{-i}$ ,  $a \in A(h)$  and  $n \geq 0$ ,  $\beta_i(t_i)(a|h) = \nu_{-i}(t_i)(S_{-i}(h, a) | S_{-i}(h))$  and if  $S_{-i}(h) \cap S_{-i}^n \neq \emptyset$ , then  $\nu_{-i}(t_i)(S_{-i}^n | S_{-i}(h)) = 1$ , where  $S_{-i}^n$  is the nonempty set of strategies that survive  $n$  steps of the

<sup>16</sup>See Definitions 1, 4, 5 and note that every perfect information game has observed deviators.

EFR procedure, and  $S_{-i}^n = \times_{j \neq i} S_j^n$  (see (10), (11) and Theorem 1 below). Given such beliefs, the unique local best reply at each node on the BI path is the BI action, this follows from Battigalli (1997).

As a preliminary observation, note that for each  $v_{-i} \in \Delta^H(S_{-i})$  we obtain a corresponding decision tree  $\Gamma_i(v_{-i})$  for player  $i$  by assigning to each action  $a \in A(h)$  of the other players ( $h \in H_{-i}$ ) the conditional probability  $v_{-i}(S_{-i}(h, a)|S_{-i}(h))$ . Then we can determine by backward induction on  $\Gamma_i(v_{-i})$  the set of optimal actions  $A^*(h, v_{-i})$  for each  $h \in H_i$ . We will map types to conditional probability systems on  $S_{-i}$  and *vice versa*, so that both determine the same decision tree. The following lemma shows how to associate each  $t_i$  with a corresponding  $v_{-i}(t_i) \in \Delta^H(S_{-i})$ . Recall that for every  $i \in I$  and  $t_i \in T_i$ , the probabilities  $\beta_i(t_i)((h, a)|h)$  describe a behavior strategy profile as  $j$  varies in  $I$  and  $h$  in  $H_j$ .

**Lemma 2.** Fix  $i \in I$  and  $t_i \in T_i$ . For all  $h \in H$  and  $s_{-i} \in S_{-i}$ , define

$$v_{-i}(t_i)(s_{-i}|S_{-i}(h)) = \begin{cases} 0 & \text{if } s_{-i} \notin S_{-i}(h), \\ \prod_{h' \in H_{-i}: h' \neq h} \beta_i(t_i)((h', s'_{-i}(h'))|h') & \text{if } s_{-i} \in S_{-i}(h). \end{cases} \quad (6)$$

Then  $v_{-i}(t_i) \in \Delta^H(S_{-i})$ , and for all  $h \in H_{-i}$  and  $a \in A(h)$ ,

$$v_{-i}(t_i)(S_{-i}(h, a)|S_{-i}(h)) = \beta_i(t_i)((h, a)|h). \quad (7)$$

**Proof.** Regard  $-i$  as a coalition. By perfect information,  $-i$  has perfect recall.<sup>17</sup> Consider the following behavior strategy  $b_{-i}^h$  of  $-i$ : for all  $h' \in H_{-i}$  and  $a \in A(h')$ ,

$$b_{-i}^h(a|h') = \begin{cases} 1, & \text{if } h' \prec h \text{ and } a = \alpha(h', h), \\ 0, & \text{if } h' \prec h \text{ and } a \neq \alpha(h', h), \\ \beta_i(t_i)((h', a)|h') & \text{if } h' \not\prec h. \end{cases}$$

By Kuhn's theorem,  $b_{-i}^h$  induces a realization-equivalent mixed strategy  $v_{-i}^h \in \Delta(S_{-i})$  of  $-i$ , with

$$\begin{aligned} v_{-i}^h(s_{-i}) &= \prod_{h' \in H_{-i}} b_{-i}^h(s_{-i}(h')|h) \\ &= \begin{cases} \prod_{h' \in H_{-i}: h' \neq h} \beta_i(t_i)((h', s_{-i}(h'))|h') & \text{if } s_{-i}(h') = \alpha(h', h) \\ & \text{for all } h' \in H_{-i} \text{ with } h' \prec h, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

<sup>17</sup>In games with imperfect information  $-i$  does not have perfect recall and  $v_{-i}(t_i)(\cdot|S_{-i}(h))$  is a correlated strategy of  $-i$ . The proof still works.

where the second equality follows from the definition of  $b_{-i}^h$ . Thus,  $v_{-i}^h(s_{-i}) = 0$  for every  $s_{-i} \notin S_{-i}(h)$ , and therefore  $v_{-i}(t_i)(\cdot|S_{-i}(h)) = v_{-i}^h(\cdot) \in \Delta(S_{-i})$  and, moreover,  $v_{-i}(t_i)(S_{-i}(h)|S_{-i}(h)) = 1$ . Then (7) follows from the realization-equivalence of  $b_{-i}^h$  and  $v_{-i}(t_i)(\cdot|S_{-i}(h))$ . To show that  $v_{-i}(t_i) \in \Delta^H(S_{-i})$  we only have to verify the chain rule. Fix non terminal histories  $g \prec h$ , so that  $S_{-i}(h) \subseteq S_{-i}(g)$  and pick any  $s_{-i} \in S_{-i}(h)$ , so that  $s_{-i}$  selects every action of  $-i$  in  $h$  and hence also in  $g$ , then

$$\begin{aligned} v_{-i}(t_i)(s_{-i}|S_{-i}(g)) &= \prod_{h' \in H_{-i}: h' \not\prec g} \beta_i(t_i)((h', s_{-i}(h'))|h') \\ &= \prod_{h' \in H_{-i}: h' \not\prec h} \beta_i(t_i)((h', s_{-i}(h'))|h') \prod_{h' \in H_{-i}: g \preceq h' \prec h} \beta_i(t_i)((h', s_{-i}(h'))|h') \\ &= v_{-i}(t_i)(s_{-i}|S_{-i}(h))v_{-i}(t_i)(S_{-i}(h)|S_{-i}(g)), \end{aligned}$$

where the second equality follows from the fact that  $g \prec h$  implies that the two sets  $\{\bar{h} \in H_{-i} : \bar{h} \not\prec h\}$  and  $\{\bar{h} \in H_{-i} : g \preceq \bar{h} \prec h\}$  partition  $\{\bar{h} \in H_{-i} : \bar{h} \not\prec g\}$ , and the third equality follows from the construction of  $v_{-i}(t_i)(\cdot|S_{-i}(g))$  and its realization-equivalence with  $b_{-i}^g$ .  $v_{-i}(t_i)(S_{-i}(h)|S_{-i}(g))$  is just the probability of reaching  $h$  from  $g$  given that each action of  $i$  on this path has conditional probability one; since  $s_{-i}$  allows  $h$ , this transition probability is precisely

$$\prod_{h' \in H_{-i}: g \preceq h' \prec h} \beta_i(t_i)((h', s_{-i}(h'))|h') = \prod_{h' \in H_{-i}: g \preceq h' \prec h} \beta_i(t_i)((h', \alpha(h', h))|h').$$

■

The next lemma shows that, given some map  $\tau_{-i} : S_{-i} \rightarrow T_{-i}$ , we can associate to each pair  $(s_i, \nu_{-i}) \in S_i \times \Delta^H(S_{-i})$  a conditional probability system in  $\Delta^H(Z \times T_{-i})$  corresponding to  $(s_i, \nu_{-i})$  in the following sense. The marginal probabilities on  $Z$  (which are equivalent to a behavior strategy profile) agree with  $s_i$ , when the latter is viewed as  $i$ 's conditional belief about her own actions, whereas the conditional belief about the actions of others are derived from  $\nu_{-i}$  in the spirit of Kuhn's transformation from mixed to behavioral strategies. Finally, the marginal on  $T_{-i}$  is obtained from  $\tau_{-i}$ . An important feature of this conditional probability system is that it always assigns probability one to the material consistency of the other players. Let  $\pi(s|h)$  denote the path induced by strategy profile  $s$  starting from history  $h$ , writing just  $\pi(s)$  for  $h = \emptyset$ .

**Lemma 3.** Fix  $(s_i, \nu_{-i}) \in S_i \times \Delta^H(S_{-i})$  and a mapping  $\tau_{-i} : S_{-i} \rightarrow T_{-i}$ . There exists  $\psi_i \in \Delta^H(Z \times T_{-i})$  such that for all  $h \in H$  and  $s_{-i} \in S_{-i}(h)$ ,

$$\psi_i(\{(\pi(s_i, s_{-i}|h), \tau_{-i}(s_{-i}))\}|h) = \nu_{-i}(s_{-i}|S_{-i}(h)).$$

**Proof.** For every  $h \in H$ , let  $\psi_i(\cdot|h)$  be the finite-support probability measure defined as follows: for every  $h \in H$  and  $E \subseteq Z(h) \times T_{-i}$ ,

$$\psi_i(E|h) = \nu_{-i}(\{s_{-i} \in S_{-i}(h) : (\pi(s_i, s_{-i}|h), \tau_{-i}(s_{-i})) \in E\} | S_{-i}(h)).$$

We must show that  $\psi_i$  satisfies the chain rule: for all  $g \prec h$  and  $E \subseteq Z(h) \times T_{-i}$ ,

$$\psi_i(E|g) = \psi_i(E|h)\psi_i(h|g).$$

There are two cases: either  $s_i$  makes  $h$  unreachable from  $g$ , so that both  $\psi_i(E|g)$  and  $\psi_i(h|g)$  are zero, or  $s_i$  always selects the action in  $h$  at each  $h' \in H_i$  with  $g \preceq h' \prec h$ . In the first case the equality is trivially satisfied. In the second case, recalling  $E \subseteq Z(h) \times T_{-i}$  one can see that

$$\{s_{-i} \in S_{-i}(g) : (\pi(s_i, s_{-i}|g), \tau_{-i}(s_{-i})) \in E\} = \{s_{-i} \in S_{-i}(h) : (\pi(s_i, s_{-i}|h), \tau_{-i}(s_{-i})) \in E\},$$

and therefore

$$\begin{aligned} \psi_i(E|g) &= \sum_{s_{-i} \in S_{-i}(g) : (\pi(s_i, s_{-i}|g), \tau_{-i}(s_{-i})) \in E} \nu_{-i}(s_{-i} | S_{-i}(g)) \\ &= \sum_{s_{-i} \in S_{-i}(h) : (\pi(s_i, s_{-i}|h), \tau_{-i}(s_{-i})) \in E} \nu_{-i}(s_{-i} | S_{-i}(g)) \\ &= \nu_{-i}(S_{-i}(h) | S_{-i}(g)) \sum_{s_{-i} \in S_{-i}(h) : (\pi(s_i, s_{-i}|h), \tau_{-i}(s_{-i})) \in E} \nu_{-i}(s_{-i} | S_{-i}(h)) \\ &= \psi_i(E|h)\psi_i(h|g), \end{aligned}$$

where the first and last equalities follow from the definition of  $\psi_i$  and the third is implied by the chain rule for  $\nu_{-i}$ .  $\blacksquare$

Note that by construction  $\psi_i$  is such that  $\psi_i((h, s_i(h))|h) = 1$  for every  $h \in H_i$ . In words,  $s_i$  is the plan of  $i$  entailed by  $\psi_i$ . In what follows, write  $\psi_i(s_i, \nu_{-i}; \tau_{-i})$  instead of  $\psi_i$  to emphasize the dependence on  $s_i$ ,  $\nu_{-i}$  and  $\tau_{-i}$ .

To make the proof of the main result a bit shorter we adapt the recursive definition of EFR from the characterization given by [Battigalli \(1997\)](#). Let  $S_i^0 = S_i$ ; given  $S_i^n$  and  $S_{-i}^n = \prod_{j \neq i} S_j^n$ , the set of  $(n+1)$ -rationalizable strategies for  $i$  is defined as<sup>18</sup>

$$S_i^{n+1} = \left\{ \begin{array}{l} s_i : \exists \nu_{-i} \in \Delta^H(S_{-i}), \forall m \in \{1, \dots, n\}, \forall h \in H, \\ S_{-i}^m \cap S_{-i}(h) \neq \emptyset \Rightarrow \nu_{-i}(S_{-i}^m | S_{-i}(h)) = 1, \forall h \in H_i, s_i(h) \in A^*(h, \nu_{-i}) \end{array} \right\}. \quad (8)$$

<sup>18</sup>In this definition we are assuming that  $s_i$  prescribes a local best response to the justifying belief  $\nu_{-i}$  at each  $h \in H_i$ . This is realization-equivalent to assuming that  $s_i$  yields a ‘‘global’’ best response in each subtree  $\Gamma_i(h, \nu_{-i})$  with root  $h \in H_i$  allowed by  $s_i$ , which is the best response property required in papers about extensive form rationalizability.

(By convention, the condition  $\forall m \in \{1, \dots, n\}, \forall h \in H, S_{-i}^m \cap S_{-i}(h) \neq \emptyset \Rightarrow v_{-i}(S_{-i}^m | S_{-i}(h)) = 1$  is trivially satisfied for  $n = 0$ ). It can be shown that each set  $S_i^n$  is nonempty and that  $s_i \in S_i^n$  if and only if every strategy  $s_i'$  realization-equivalent to  $s_i$  survives  $n$  steps of the definition of EFR found in the literature.

We now use completeness to recursively define a profile of functions  $\tau^n = (\tau_i^n : S_i \rightarrow T_i)_{i \in I}$  for each  $n \geq 0$  so that, for every strategy profile  $s \in S^{n+1}$ , the profile of types  $\tau^{n+1}(s)$  satisfies  $(\pi(s), \tau^{n+1}(s)) \in \underline{CSB}^n((MC_i \cap RP_i)_{i \in I})$ . For each player  $i$ , fix  $\tau_i^0$  arbitrarily. Suppose  $(\tau_j^n)_{j \in I}$  has been defined and let  $\tau_{-i}^n(s_{-i}) = (\tau_j^n(s_j))_{j \neq i}$ . Now, for each  $s_i \in S_i \setminus S_i^{n+1}$ , let  $\tau_i^{n+1}(s_i) = \tau_i^n(s_i)$ . For each  $s_i \in S_i^{n+1}$  there is some  $v_{-i}(s_i)(\cdot | \cdot) \in \Delta^H(S_{-i})$  such that:

$$\begin{aligned} \forall h \in H, \forall m \in \{0, \dots, n\}, \quad S_{-i}^m \cap S_{-i}(h) \neq \emptyset \Rightarrow v_{-i}(s_i)(S_{-i}^m | S_{-i}(h)) = 1; \\ \forall h \in H_i, \quad s_i(h) \in A^*(h, v_{-i}(s_i)); \end{aligned} \quad (9)$$

then pick

$$\tau_i^{n+1}(s_i) \in \beta_i^{-1}(\psi_i(s_i, v_{-i}(s_i), \tau_{-i}^n))$$

arbitrarily, where  $\psi_i(\cdot)$  is given by Lemma 3 and  $\beta_i^{-1}(\psi_i(s_i, v_{-i}(s_i), \tau_{-i}^n))$  is non-empty because  $\beta_i$  is onto (completeness). Given  $s = (s_i)_{i \in I}$ , let  $\tau^{n+1}(s) = (\tau_i^{n+1}(s_i))_{i \in I}$ .

The following lemma decomposes event  $CSBMR$  into separate events about each player's material consistency, rational planning, and strong belief in different "degrees of strategic sophistication" of the other players.

**Lemma 4.** *Let  $RP_i^0 = RP_i$  and  $RP_i^{m+1} = RP_i^m \cap SB_i(\cap_{j \neq i} (MC_j \cap RP_j^m))$  for  $m \geq 0$ . Then*

$$RP_i^{m+1} = RP_i \cap \left( \cap_{k=0}^m SB_i(\cap_{j \neq i} MC_j \cap RP_j^k) \right) \quad \forall m \geq 0,$$

and

$$\underline{CSB}^{m+1}((MC_i \cap RP_i)_{i \in I}) = (MC_i \cap RP_i^{m+1})_{i \in I} \quad \forall m \geq 0.$$

Therefore,

$$CSBMR = \cap_{m=0}^{\infty} \cap_{i \in I} (MC_i \cap RP_i \cap (\cap_{k=0}^m SB_i(\cap_{j \neq i} MC_j \cap RP_j^k))).$$

**Proof.** The proof is by induction in  $m$ . The results are obvious for  $m = 0$ . Now let  $m \geq 0$  and suppose that

$$RP_i^m = RP_i \cap \left( \cap_{k=0}^{m-1} SB_i(\cap_{j \neq i} MC_j \cap RP_j^k) \right).$$

Then

$$\begin{aligned} RP_i^{m+1} &= RP_i^m \cap SB_i(\cap_{j \neq i} MC_j \cap RP_j^m) \\ &= RP_i \cap \left( \cap_{k=0}^{m-1} SB_i(\cap_{j \neq i} MC_j \cap RP_j^k) \right) \cap SB_i(\cap_{j \neq i} MC_j \cap RP_j^m) \\ &= RP_i \cap \left( \cap_{k=0}^m SB_i(\cap_{j \neq i} MC_j \cap RP_j^k) \right), \end{aligned}$$



where the first equality holds by definition of  $RP_i^{m+1}$  and the second follows from the inductive hypothesis. Now suppose that

$$\underline{CSB}^m((MC_i \cap RP_i)_{i \in I}) = (MC_i \cap RP_i^m)_{i \in I}.$$

Then

$$\begin{aligned} \underline{CSB}^{m+1}((MC_i \cap RP_i)_{i \in I}) &= \underline{CSB}(\underline{CSB}^m((MC_i \cap RP_i)_{i \in I})) \\ &= \underline{CSB}((MC_i \cap RP_i^m)_{i \in I}) \\ &= (MC_i \cap RP_i^m \cap SB_i(\cap_{j \neq i} MC_j \cap RP_j^m))_{i \in N} \\ &= ((MC_i \cap RP_i^{m+1})_{i \in I}), \end{aligned}$$

where the first equality holds by definition of  $\underline{CSB}^{m+1}$ , the second by the inductive hypothesis, the third by definition of  $\underline{CSB}$ , and the fourth by definition of  $RP_i^{m+1}$ .  $\blacksquare$

In what follows, for every  $i \in I$  and  $n \geq 0$  write  $\overline{RP}_i^n$  for the set of all  $t_i \in T_i$  such that  $(z, t_i, t_{-i}) \in RP_i^n$  for some (and hence for all)  $(z, t_{-i}) \in Z \times T_{-i}$ . Similarly, write  $\overline{MC}_{-i}$  for the set of all  $(z, t_{-i}) \in Z \times T_{-i}$  such that  $(z, t_i, t_{-i}) \in MC_{-i}$  for some (and hence for all)  $t_i \in T_i$ .

**Claim.** For every  $n \geq 0$ ,  $i \in I$  and  $s_i \in S_i$ ,

$$s_i \in S_i^{n+1} \Rightarrow \tau_i^{n+1}(s_i) \in \overline{RP}_i^n, \quad (10)$$

$$(t_i \in \overline{RP}_i^n, \forall h \in H_i, \beta_i(t_i)((h, s_i(h))|h) > 0) \Rightarrow s_i \in S_i^{n+1} \quad (11)$$

**Proof.** The proof is by induction in  $n$ . For the case  $n = 0$ , suppose that  $s_i \in S_i^1$ . Then  $\tau_i^1(s_i) = \psi_i(s_i, \nu_{-i}(s_i), \tau_{-i}^0) \in \overline{RP}_i^0$ . Conversely, pick  $t_i \in \overline{RP}_i^0$  with  $\beta_i(t_i)((h, s_i(h))|h) > 0$  for all  $h \in H_i$ . Derive  $\nu_{-i}(t_i)(\cdot|\cdot)$  from  $\beta_i(t_i)$  as in (6). By Lemma 2,  $\nu_{-i}(t_i)$  satisfies (7), therefore  $\beta_i(t_i)$  and  $\nu_{-i}(t_i)$  determine the same decision tree. Since  $t_i \in \overline{RP}_i^0$  and  $\beta_i(t_i)((h, s_i(h))|h) > 0$  for every  $h \in H_i$ ,  $s_i$  must be optimal in this decision tree. Hence  $s_i \in S_i^1$ .

Now suppose by way of induction that (10) and (11) hold for every natural number smaller than  $n > 0$ . First we show that this induction hypothesis implies that, for each  $m = 0, \dots, n-1$ ,

$$MC_{-i} \cap [h] \cap RP_{-i}^m \neq \emptyset \iff S_{-i}(h) \cap S_{-i}^{m+1} \neq \emptyset. \quad (12)$$

Indeed, fix any state  $(z, t_i, t_{-i})$  in the intersection on the left-hand side above, so that  $h \prec z$ . Then there is some  $s'_{-i}$  such that for every  $j \neq i$  and  $h' \in H_j$ ,  $\beta_j(t_j)((h', s_j(h'))|h') > 0$ , and  $s'_j(h') = \alpha(h', z)$  whenever  $h' \prec z$ . Since  $h \prec z$ ,  $s'_{-i} \in S_{-i}(h)$ . By the induction hypothesis,  $s'_{-i} \in S_{-i}^{m+1}$ . Thus,  $S_{-i}(h) \cap S_{-i}^{m+1} \neq \emptyset$ . Conversely, pick  $s'_{-i} \in S_{-i}(h) \cap S_{-i}^{m+1}$ . By the induction hypothesis,  $s'_{-i} \in S_{-i}^{m+1}$  implies  $\tau_j^{m+1}(s'_j) \in \overline{RP}_j^m$  for each  $j \neq i$ . Since  $s'_{-i} \in S_{-i}(h)$ , there is some  $z' \in Z(h)$  such that  $s'_{-i} = \alpha(h', z')$  for each  $h' \in H_{-i}$  with  $h' \prec z'$ . The construction of  $\tau_{-i}^{m+1}$

implies that  $\{(z', \tau_{-i}^{m+1}(s'_{-i}))\} \times T_i \subseteq MC_{-i}$ . Hence,  $\{(z', \tau_{-i}^{m+1}(s'_{-i}))\} \times T_i \subseteq MC_{-i} \cap [h] \cap RP_{-i}^m$ , and (12) follows.

To prove (10), let  $s_i \in S_i^{n+1}$ . We claim that  $\tau_i^{n+1}(s_i) \in \overline{RP}_i^n$ . By definition of  $\overline{RP}_i^n$  and Lemma 4, we have to show that  $\tau_i^{n+1}(s_i) \in \overline{RP}_i^0$  and that type  $\tau_i^{n+1}(s_i) = \psi_i(s_i, \nu_{-i}(s_i), \tau_{-i}^n)$  strongly believes each event  $MC_{-i} \cap RP_{-i}^m$  with  $m = 0, \dots, n-1$ . The former is true by construction. Now, fix  $m = 0, \dots, n-1$  and  $h$  with  $[h] \cap MC_{-i} \cap RP_{-i}^m \neq \emptyset$ . By (12),  $S_{-i}(h) \cap S_{-i}^{m+1} \neq \emptyset$ , and by (9),  $\nu_{-i}(s_{-i})(S_{-i}^{m+1} | S_{-i}(h)) = 1$ . This implies that

$$\beta_i(\tau_i^{n+1}(s_i))(\{(\pi(s_i, s'_{-i}|h), \tau_{-i}^n(s'_{-i})) : s'_{-i} \in S_{-i}^m \cap S_{-i}(h)\} | h) = 1.$$

By construction of  $\tau_{-i}^n$  and the induction hypothesis, this implies

$$\beta_i(\tau_i^{n+1}(s_i))(\overline{MC}_{-i} \cap (Z(h) \times \overline{RP}_{-i}^m) | h) = 1.$$

Thus,  $\tau_i^{n+1}(s_i) \in \overline{RP}_i^n$ .

Finally, to prove (11), suppose that  $t_i \in \overline{RP}_i^n$  and  $\beta_i(t_i)((h, s_i(h)) | h) > 0$  for all  $h \in H_i$ . Define  $\nu_{-i}(t_i) \in \Delta^H(S_{-i})$  as in (6) of Lemma 2. Since  $\beta_i(t_i)$  and  $\nu_{-i}(t_i)$  determine the same decision tree and  $t_i \in \overline{RP}_i^n$ ,  $s_i$  is optimal in this decision tree. We only have to check that  $\nu_{-i}(t_i)$  satisfies the conditions in (8). Fix  $h \in H$  and  $m < n$ . Suppose  $S_{-i}^{m+1} \cap S_{-i}(h) \neq \emptyset$ . Then  $MC_{-i} \cap [h] \cap RP_{-i}^m \neq \emptyset$  by (12), and by definition of  $RP_{-i}^m$ ,

$$\beta_i(t_i)(\overline{MC}_{-i} \cap (Z(h) \times \overline{RP}_{-i}^m) | h) = 1.$$

The definition of  $\nu_{-i}(t_i)$  implies  $\nu_{-i}(t_{-i})(S_{-i}^{m+1} | S_{-i}(h)) = 1$ . This proves that  $s_i \in S_i^{n+1}$ . ■

**Theorem 1.** Fix a complete type structure and a terminal history  $z \in Z$ . The following are equivalent:

- (i) There exists a profile of types  $t = (t_i)_{i \in N}$  such that CSBMR holds at state  $(z, t)$ .
- (ii) There exists a rationalizable strategy profile  $s$  such that  $z = \pi(s)$ .

**Proof.** (i)  $\Rightarrow$  (ii). Suppose that there is CSBMR at state  $(z, t)$ , so that  $(z, t) \in \cap_{i \in I} (MC_i \cap RP_i^n)$  for every  $n$ . For each player  $i$ , pick any strategy  $s_i$  such that  $\beta_i(t_i)((h, s_i(h)) | h) > 0$  for all  $h \in H_i$ . By (11),  $s_i$  is rationalizable, whereas material consistency implies that  $\pi(s) = z$ .

(ii)  $\Rightarrow$  (i). Suppose that  $s$  is rationalizable and  $\pi(s) = z$ . Let  $K$  be the first integer  $n$  such that  $S^{n+1} = S^n$ . By construction  $\tau^K = \tau^n$  for every  $n \geq K$ . By (10),  $Z \times \{\tau_i^{n+1}(s_i)\} \times T_{-i} \subseteq RP_i^n$  for every  $i$  and  $n$ . By construction, there is material consistency at state  $(z, t) = (\pi(s), (\tau^K(s)))$ . Therefore there is CSBMR at  $(z, t)$ . ■

**Corollary 2.** Fix a complete type structure; then CSBMR holds at some state  $(z, t)$  and  $z = z^{BI}$  for every such state.

**Proof.** The definition of rationalizability in (8) is realization-equivalent to the one analyzed by Battigalli (1997), who shows that in a perfect information game with no relevant ties, every rationalizable strategy profile induces the backward induction path. As the set of rationalizable profiles is nonempty,  $z = z^{BI}$  if and only if  $z = \pi(s)$  for some rationalizable  $s$ . Then, by Theorem 1,  $z = z^{BI}$  if and only if there is CSBMR at  $(z, t)$  for some profile of types  $t$ . ■

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