

Perfect equilibrium in games with compact action spaces

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Abstract

We investigate the relations between different types of perfect equilibrium, introduced by Simon and Stinchcombe (1995) for games with compact action spaces and continuous payoffs. Simon and Stinchcombe distinguish two approaches to perfect equilibrium in this context, the classical "trembling hand" approach, and the so-called "finitistic" approach. We propose an improved definition of the finitistic approach, called global-limit-of-finite perfection, and prove its existence.

Despite the fact that the finitistic approach appeals to basic intuition, our results—specifically examples (1) and (2)—seem to imply a severe critique on this approach. In the first example any version of finitistic perfect equilibrium admits a Nash equilibrium strategy profile that is not limit admissible. The second example gives a completely mixed (and hence trembling hand perfect) Nash equilibrium that is not finitistically perfect. Further examples illustrate the relations between the two approaches to perfect equilibrium and the relation to admissibility and undominatedness of strategies.

JEL Codes. C72, D44.

Keywords. Strategic form games, perfect equilibrium, compact action spaces

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1 Introduction

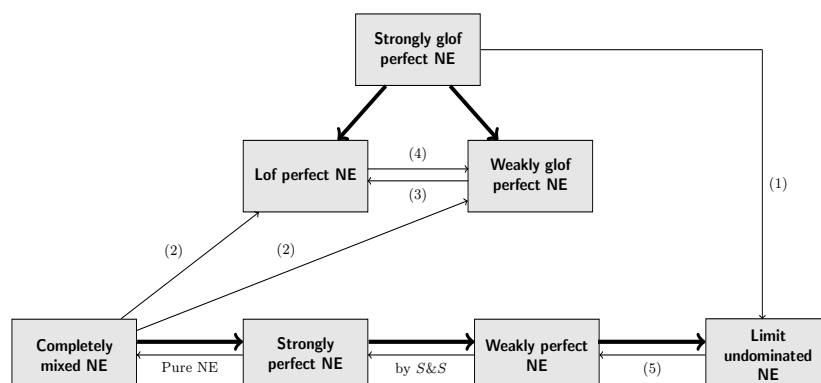
In 1995 Simon and Stinchcombe [21], for brevity referred to as S&S in this paper, defined perfect equilibrium for games with an infinite compact set of actions. They distinguished two main lines of definitions of perfection. The first line is based on the notion of a completely mixed strategy, giving rise to the notions of strongly perfect and weakly perfect equilibrium. This approach can be viewed as a direct generalization of the original trembling hand definition of perfect equilibrium introduced by Selten [20]. The second line of definitions, in S&S referred to as the finitistic approach, uses the notion of an ε -perfect equilibrium in finite approximations of the original game. The resulting notion of perfect equilibrium is called limit-of-finite perfect equilibrium¹. They showed that the first type of perfect equilibrium only admits limit admissible strategies. Furthermore, they conjectured that the two approaches are incomparable.

Perfect equilibrium in games with infinite action spaces is a useful tool to select Nash equilibria, for example in auction models with incomplete information, such as the ones studied in Compte and Jehiel [5] and Crémer et al. [6], where dominant strategies are not available as a selection criterion. Concrete applications of perfect equilibrium in infinite strategic form games can among others be found in Jackson et al. [10], S & S [21], and Anderson et al. [2].

In this paper we investigate the existing relations between the various types of perfect equilibrium within the framework of strategic form games with compact action spaces and continuous payoff functions.

The diagram below gives an overview of those relations between different types of perfect equilibrium that are currently known. Strongly perfect, weakly perfect, and lof (limit-of-finite) perfect equilibrium are introduced by S&S. Strongly glof (global limit-of-finite) and weakly glof perfect equilibrium are introduced in this paper. Limit undominated Nash equilibrium, a stronger version of limit admissibility from S&S, is also introduced in this paper.

¹Simon and Stinchcombe also introduce another variation on this type of perfection, called anchored perfection. In this paper we hardly ever consider this variation though.



Boldface arrows indicate valid implications. For example, every strongly glof perfect equilibrium is a lof perfect equilibrium. Roman face arrows indicate implications that do not hold in general. Counterexamples are provided by previous studies, and examples (1) to (5) in this paper.

Related literature Equilibrium selection and refinement theory already have a long and steady-going tradition, arguably starting with essential equilibrium (Wu Wen-Tsün and Jiang Jia-He [23]), perfect equilibrium (Selten [20]) and proper equilibrium (Myerson [17]) for finite strategic form games. Developments on refinements in the context of extensive form games are for example sequential equilibrium (Kreps and Wilson [13]) and theory on strategic stability (Hillas [9], Mertens [14] [15], and more recently Govindan and Wilson [8]).

The above line of literature mainly concerns what Andersson et al. [2] refer to as “strategic uncertainty” where a player is required to play strategies that guarantee robustness against possible mistakes by a player’s opponents. An alternative line of research on equilibrium refinement is on “structural uncertainty” where, due to informational uncertainty, iterated deletion of dominated strategies is used as a refinement technique. This approach, also known as the global games approach, was developed by Carlsson and van Damme [4], with applications in for example banking and finance (Morris and Shin [16]).

Results The paper makes two main contributions. The first contribution is the introduction of global limit-of-finite perfection, from now on referred to as glof perfection. S&S already introduced lof perfect equilibrium as the limit

of ε -perfect equilibria of large but finite approximations of the original game. Despite its relative straightforward intuition, lof perfect equilibrium does not always select dominant strategies.

Strongly glof perfect equilibrium is defined in the same way as lof perfect equilibrium, but with the added requirement that best responses in the approximating games are in fact best responses in the original game. Strongly glof perfect equilibrium is a stronger version of finitistic perfection than lof perfect equilibrium. Weakly glof perfect equilibrium adds to the definition of lof perfection the weak requirement that best responses in the approximating games are only close to best responses in the original game. We show existence of weakly glof perfect equilibrium. Moreover, we show that weakly (and therefore also strongly) glof perfect equilibrium does uniquely select dominant strategy Nash equilibria as soon as dominant strategies are available to a player.

Second, we provide five illustrative counterexamples. Example 1 presents a strongly glof perfect equilibrium that is not limit admissible. This result is in line with Example 2.4 of S&S of a lof perfect equilibrium that is not limit admissible. Example 2 verifies the conjecture of Simon and Stinchcombe that strong perfection does not imply finitistic perfection. More precisely, it is an example a completely mixed, and therefore strongly perfect, equilibrium that is neither lof perfect (and therefore also not anchored perfect) nor weakly (and therefore also not strongly) glof perfect. Example 3 presents a weakly glof perfect equilibrium that is not lof perfect, and therefore also not strongly glof perfect. Example 4 is in fact equivalent to Example 2.4 of S&S with the added feature that the action spaces are intervals instead of the union of an interval and an isolated point.

Example 5 is a comment on the theorem of van Damme that for bimatrix games a Nash equilibrium is perfect precisely when the equilibrium strategies are undominated. This equivalence no longer holds in the more general context of two-player games with compact action spaces and continuous payoff functions. Example 5 presents a Nash equilibrium of a $2 \times (\infty + 1)$ -bimatrix game in undominated strategies that is not weakly perfect. An added feature of the example is that there is another Nash equilibrium in this game that is payoff

equivalent to the first Nash equilibrium, while the second Nash equilibrium is weakly perfect. Thus, weakly perfect equilibrium is not invariant, in contrast to perfect equilibrium for finite strategic form games. The example highlights the fact that for infinite compact action spaces, the topology on the action spaces starts to matter for perfection.

Except for Example 4, all our examples use countably infinite pure action spaces. We specifically chose not to use compact intervals as action spaces, in order to highlight the underlying logic of the constructions. It is nevertheless evident from the constructions that, with a bit (well, for some examples perhaps a bit more than a bit) of extra work the examples can also be transformed to counterexamples on the unit interval.

2 Preliminaries

For a metric space (X, d) , a set $U \subseteq X$ is open if for every $x \in U$ there is an $\varepsilon > 0$ such that if $y \in X$ and $d(x, y) < \varepsilon$ then $y \in U$. The topology on X induced by metric d is the collection of all open sets we can thus construct. For a topological space X , the Borel σ -field on X is the smallest σ -field that contains all open sets. A probability measure on X is a function μ from a σ -field Σ on X to $[0, 1]$ such that $\mu(X) = 1$, and moreover

$$\mu\left(\bigcup_{i \in I} E_i\right) = \sum_{i \in I} \mu(E_i)$$

for every countable collection $(E_i)_{i \in I}$ of pairwise disjoint sets in Σ . For a compact set X , $\Delta(X)$ denotes the set of probability measures on the Borel σ -field on X . A probability measure $\mu \in \Delta(X)$ is completely mixed if $\mu(U) > 0$ for every non-empty open subset U of X .

In this paper we use two different metrics on the space $\Delta(X)$. For a non-empty set $B \subseteq X$, the distance $d(x, B)$ between a point $x \in X$ and the set B is

$$d(x, B) = \inf\{d(x, y) \mid y \in B\}.$$

The ε -neighborhood of B is $B^\varepsilon = \{x \in X \mid d(x, B) < \varepsilon\}$. The weak metric is defined for $\mu, \nu \in \Delta(X)$ by

$$\rho^w(\mu, \nu) = \inf\{\varepsilon \mid \forall B \in \Sigma : \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \text{ and } \nu(B) \leq \mu(B^\varepsilon) + \varepsilon\}$$

and the strong metric by

$$\rho^s(\mu, \nu) = \sup\{|\mu(B) - \nu(B)| \mid B \in \Sigma\}.$$

Note that for every μ and ν , $\rho^w(\mu, \nu) \leq \rho^s(\mu, \nu)$. This easily follows from the observation that for every $B \in \Sigma$, both $\mu(B) - \nu(B^\varepsilon)$ and $\nu(B) - \mu(B^\varepsilon)$ are smaller than or equal to $|\mu(B) - \nu(B)|$ and therefore smaller than or equal to $\rho^s(\mu, \nu)$ as well.

The weak (strong) topology on $\Delta(X)$ is the topology induced by the weak (strong) metric. The inequality $\rho^w(\mu, \nu) \leq \rho^s(\mu, \nu)$ implies that the weak topology is included in the strong topology. Since X is compact, $\Delta(X)$ is compact with respect to the weak metric. Further, compactness and sequential compactness are equivalent for metric spaces.

3 Strategic form games

In this section we introduce the game theoretic notation used in this paper. A strategic form game (with compact action spaces) is a triplet $\Gamma = (N, A, u)$ where $N = \{1, \dots, n\}$ is the set of the players, $A = \prod_{i \in N} A_i$ is the set of profiles of actions, and $u = (u_1, \dots, u_n)$ is the vector of payoff functions. We assume for every player $i \in N$ that the set A_i of actions is a non-empty compact subset of \mathbb{R} , and that the payoff functions $u_i: A \rightarrow \mathbb{R}$ are continuous ².

The set $\Delta(A_i)$ is the set of (mixed) strategies of player i . We identify every action a_i with the Dirac measure $\delta(a_i)$, the strategy that selects a_i with probability 1. A vector $\sigma = (\sigma_i)_{i \in N}$ with $\sigma_i \in \Delta(A_i)$ for all i is called a strategy profile. The set of strategy profiles is denoted ³ by $\Delta(A)$. For every strategy profile $\sigma \in \Delta(A)$ we define $u_i(\sigma) = \int u_i d\sigma$. Prokhorov's Theorem implies that $u_i(\sigma)$ is continuous in σ with respect to the weak metric ρ^w on $\Delta(A)$ ⁴.

²Everything we do in this paper immediately generalizes to general compact metric spaces. In this context note that a compact metric space is automatically separable, so that the finitistic approach also has a bite in the general case.

³With a slight abuse of notation we implicitly identify each strategy profile $\sigma = (\sigma_i)_{i \in N}$ with its induced probability distribution on $\Delta(A)$. Obviously not every element of $\Delta(A)$ can be obtained this way. We restrict attention to the independent elements of $\Delta(A)$ that are induced by strategy profiles.

⁴Throughout the paper we use several results from real analysis without further mention. All of the results we use are well-known. Good references for the relevant theory are for example Aliprantis and Border [1], and Billingsley [3].

We write $A_{-i} = \prod_{j \in N \setminus \{i\}} A_j$. A generic element of $\Delta(A_{-i})$ is denoted by $\sigma_{-i} = (\sigma_j)_{j \neq i}$. By (σ_{-i}, τ_i) or $(\sigma \mid \tau_i)$ we denote the strategy profile where every player $j \neq i$ adheres to strategy σ_j , while player i uses τ_i . A strategy $\tau_i \in \Delta(A_i)$ is called a best response of player i to the strategy profile $\sigma \in \Delta(A)$ if $u_i(\sigma \mid \tau_i) \geq u_i(\sigma \mid \rho_i)$ for all $\rho_i \in \Delta(A_i)$. A strategy profile $\sigma = (\sigma_i)_{i \in N}$ is a Nash equilibrium if

$$u_i(\sigma) \geq u_i(\sigma \mid \tau_i)$$

for all i and all $\tau_i \in \Delta(A_i)$. So, σ is a Nash equilibrium if each σ_i is a best response to σ .

For a strategy profile σ , $BR_i(\sigma)$ is the set of best responses of player i to σ . An action $a_i \in A_i$ is a pure best response of player i to σ if $\delta(a_i) \in BR_i(\sigma)$. By $PBR_i(\sigma)$ we denote the set of pure best responses of player i to σ . The set $PBR_i(\sigma)$ is non-empty by compactness of A_i and continuity of u_i . Note that σ_i is a best response to σ if and only if it puts weight 1 on pure best responses, i.e., $\sigma_i(PBR_i(\sigma)) = 1$. For a non-empty compact subset X_i of A_i ,

$$BR_i(\sigma \mid X_i) = \{\rho_i \in \Delta(X_i) \mid u_i(\sigma \mid \rho_i) \geq u_i(\sigma \mid \tau_i) \text{ for all } \tau_i \in \Delta(X_i)\}$$

is the set of player i 's best strategies against σ given that the pure choices of player i are restricted to actions in X_i . By $PBR_i(\sigma \mid X_i)$ we denote the set of those actions of player i that belong to $BR_i(\sigma \mid X_i)$.

4 Perfect equilibrium

Selten [20] introduced the notion of perfect equilibrium for non-cooperative extensive form games. The definition of perfect equilibrium is based on the idea that a satisfactory interpretation of equilibrium points should accommodate possible slight mistakes. Selten showed that perfect equilibrium exists for finite extensive form games with perfect recall.

Twenty years later Simon and Stinchcombe [21] provided a general treatment of perfect equilibria for infinite strategic form games with compact action spaces and continuous payoffs. S&S discuss three distinct approaches to perfect equilibrium for infinite normal form games. In the first two approaches, as in Selten's original definition, players may "tremble", modeled by having players choose

completely mixed strategies. Players are then required to play approximate best responses to opponents' trembling strategies. In the strong approach, a tremble assigns high probability to the set of pure best responses itself, while in the weak approach a tremble is only required to assign high probability to a neighborhood of this set. The third, limit-of-finite, approach applies traditional refinements to sequences of successively larger finite approximations of the original infinite game. S&S proved the existence of these three generalized notions of perfect equilibrium. Moreover, they showed that the set of strongly perfect equilibria is a closed, non-empty subset of the set of weakly perfect equilibria, which is a closed subset of the collection of limit admissible Nash equilibria.

Both the strong and weak approaches of S&S use a direct generalization of the notion of completely mixed strategies in the definition of perfect equilibrium. Let $\epsilon > 0$. A completely mixed strategy profile $\sigma = (\sigma_i)_{i \in N}$ in $\Delta(A)$ is a strongly ϵ -perfect equilibrium if for every player $i \in N$ it holds that $\rho^s(\sigma_i, BR_i(\sigma)) < \epsilon$. It is a weakly ϵ -perfect equilibrium if $\rho^w(\sigma_i, BR_i(\sigma)) < \epsilon$.

Definition 4.1 (Simon and Stinchcombe) A strategy profile σ is a strongly (respectively weakly) perfect equilibrium if there is a sequence $(\sigma^k)_{k=1}^\infty$ of strongly (weakly) $\frac{1}{k}$ -perfect equilibria with $\rho^w(\sigma^k, \sigma) \rightarrow 0$ as $k \rightarrow \infty$.

Finitistic perfection is based on a different approach where games with infinite strategy spaces are approximated by games with large, but still finite, strategy spaces. In order to guarantee existence of limit points, the above definition only considers weak limits of ϵ -perfect equilibria.

Definition 4.2 (Simon and Stinchcombe) A strategy profile $\sigma = (\sigma_i)_{i \in N}$ is a limit-of-finite (lof) perfect equilibrium if for every player $i \in N$ there is a sequence $(B_i^k)_{k=1}^\infty$ of finite subsets of A_i and a sequence $(\sigma_i^k)_{k=1}^\infty$ of completely mixed strategies in $\Delta(B_i^k)$ such that for every $i \in N$

[1] for every $\epsilon > 0$ there is a $K \in \mathbb{N}$ such that for each $k > K$ and for each

$$a_i \in A_i \text{ there is a } b_i \in B_i^k \text{ with } |a_i - b_i| < \epsilon$$

[2] $\rho^w(\sigma_i^k, \sigma_i) \rightarrow 0$ as $k \rightarrow \infty$

[3] $\sum_{x \in PBR_i(\sigma^k | B_i^k)} \sigma_i^k(x) \rightarrow 1$ as $k \rightarrow \infty$ (where $\sigma^k = (\sigma_i^k)_{i \in N}$).

This definition has quite some appeal given our basic intuition that infinite games often serve as a proxy for large finite games, and that therefore strategic behavior should carry over from the finite setting to infinite games. Despite this basic intuition, S&S already noted that even in games with dominant actions lof perfection may fail to eliminate Nash equilibria that do not select this dominant action. In order to circumvent this drawback, we propose the following modification of lof perfection, which exclusively selects the dominant strategies in case such strategies are available to a player.

Definition 4.3 A strategy profile $\sigma = (\sigma_i)_{i \in N}$ is a strongly global-limit-of-finite (strongly glof) perfect equilibrium if in Definition 4.2 condition 3) is replaced by

$$[4] \quad \rho^s(\sigma_i^k, BR_i(\sigma^k)) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

The strategy profile $\sigma = (\sigma_i)_{i \in N}$ is a weakly glof perfect equilibrium if in Definition 4.2 condition 3) is replaced by

$$[5] \quad \rho^w(\sigma_i^k, BR_i(\sigma^k)) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

S&S introduced the method of anchoring as a partial-remedy to the failure of lof perfection to select limit admissible strategies. Anchoring requires us to select a priori a collection of actions that are considered to be indispensable for a player. Our approach is more flexible in the sense that the indispensability of selected strategies is allowed to be contingent on the approximating sequence of strategy profiles we consider.

Simon and Stinchcombe established existence of lof perfect equilibrium. We show that weakly glof perfect equilibrium exists, and that every weakly glof perfect equilibrium is a Nash equilibrium. The rather lengthy proof is deferred to Appendix A.

Theorem 4.4 *Every strategic form game with compact action spaces has a weakly glof perfect equilibrium. Moreover, every weakly glof perfect equilibrium is a Nash equilibrium.*

Since clearly every strongly glof perfect equilibrium is also weakly glof, both strong and weak glof perfection refines the set of Nash equilibria. Existence of strongly glof perfect equilibrium remains an open question.

5 Admissibility

In this section we analyze the relation between the various notions of perfection and admissibility. We show that weak perfection exclusively selects limit undominated strategies. We also show that this statement is a genuine generalization of the classical result for finite games that perfection implies undominatedness.

For the finitistic approach the connection to admissibility is much weaker. As said before, in Example 2.4 S&S already present a lof perfect equilibrium that does not select the unique dominant strategy. We show that glof perfection performs better: for every game in which a player has dominant strategies, glof perfect equilibrium exclusively selects equilibria in which such a player plays one of his dominant strategies. However, the analogous result for undominated strategies again fails to hold. Example 1 presents a two-player game with a strongly glof perfect equilibrium in which both players play dominated strategies that are not limits of undominated strategies.

We say that $\tau_i \in \Delta(A_i)$ dominates $\sigma_i \in \Delta(A_i)$ if

$$u_i(\sigma \mid \tau_i) \geq u_i(\sigma \mid \sigma_i)$$

for all strategy profiles $\sigma \in \Delta(A)$, and the inequality is strict for at least one strategy profile $\sigma \in \Delta(A)$. A strategy τ_i is dominant if $u_i(\sigma \mid \tau_i) \geq u_i(\sigma \mid \sigma_i)$ for all $\sigma \in \Delta(A)$ and $\sigma_i \in \Delta(A_i)$. The set of all dominant strategies for player i is denoted by D_i . Obviously D_i may be empty.

If there is no τ_i that dominates σ_i , we say that σ_i is undominated. We denote the set of undominated strategies of player i by U_i . We write $\sigma_i \in LU_i$ if there exists a sequence $(\sigma_i^k)_{k \in \mathbb{N}}$ such that $\rho^w(\sigma_i^k, \sigma_i) \rightarrow 0$ as $k \rightarrow \infty$, and σ_i^k is undominated for all k ⁵. For $a_i \in A_i$ we write $a_i \in PU_i$, if a_i is not dominated by any $b_i \in A_i$. Following S&S (1995), a strategy σ_i is limit admissible if $\sigma_i(\overline{PU_i}) = 1$, where $\overline{PU_i}$ denotes the closure of the set PU_i .

Clearly every limit undominated strategy is also limit admissible. The converse does not hold though, not even for finite strategic form games. In the table

⁵In other words, the set LU_i is the closure of the set U_i under the weak metric. The letters LU stand for limit undominated.

below the payoff matrix of player 2 is given:

$$u_2 = \begin{bmatrix} 4 & 0 & 3 \\ 0 & 4 & 3 \end{bmatrix}.$$

The strategy $\sigma_2 = (\frac{1}{2}, \frac{1}{2}, 0)$ is limit admissible for player 2. Nevertheless, σ_2 is dominated by $(0, 0, 1)$.

5.1 Weak perfection and admissibility

S&S already showed that weakly perfect equilibrium is limit admissible. In this section we show the somewhat stronger claim that weakly perfect equilibrium strategies are limit undominated. We also show that this is the exact analogue of the classical result for finite games that perfect equilibrium strategies are undominated.

Theorem 5.1 *Let σ be a weakly perfect equilibrium. Then $\sigma_i \in LU_i$ for each i .*

Proof. By Definition 4.1 and Theorem 3.2 of Billingsley [3], there exists a sequence of completely mixed strategy profiles $\sigma^k = (\sigma_i^k)_{i \in N}$ such that for every $i \in N$ we have $\rho^w(\sigma_i^k, \sigma_i) \rightarrow 0$ and $\rho^w(\sigma_i^k, BR_i(\sigma^k)) \rightarrow 0$ as $k \rightarrow \infty$.

So, for every $i \in N$, there is a sequence $\tau_i^k \in BR_i(\sigma^k)$ with $\rho^w(\sigma_i^k, \tau_i^k) \rightarrow 0$ as $k \rightarrow \infty$. Therefore, by the triangle inequality, $\rho^w(\tau_i^k, \sigma_i) \rightarrow 0$ as $k \rightarrow \infty$. Since τ_i^k is a best response against the completely mixed profile σ^k , by Lemma B.2, τ_i^k is undominated. Therefore σ_i is limit undominated. ■

At first glance Theorem 5.1 seems to be a compromise between the well-known result for finite normal form games that the strategies in a perfect equilibrium are undominated, and the transition to infinite compact action spaces. The following Proposition shows that this compromise is only seemingly, since for finite games limit undominated strategies are in fact undominated.

Proposition 5.2 *For strategic form games with finite action spaces, the set U_i is closed, so that $U_i = LU_i$ for such games.*

Proof. To prove that the set of undominated strategies is closed, we show that the set of dominated strategies is open. Suppose σ_i is dominated by τ_i . We prove that there is an open set $V \ni \sigma_i$ such that every $\nu_i \in V$ is dominated as

well. Define $\varepsilon = \frac{1}{2} \min\{\sigma_i(a) \mid a \in A_i \text{ and } \sigma_i(a) > 0\}$. Since A_i is a finite set, $\varepsilon > 0$. Define

$$V = \{\nu_i \in \Delta(A_i) \mid \|\nu_i - \sigma_i\|_\infty < \varepsilon\}.$$

Here, $\|\cdot\|_\infty$ denotes the maximum norm. Take $\nu_i \in V$. Write $\mu_i = 2\nu_i - \sigma_i$. We show that $\mu_i \in \Delta(A_i)$. Note that if $\sigma_i(a) = 0$ then $\mu_i(a) \geq 0$. Now assume that $\sigma_i(a) > 0$. Since $\|\nu_i - \sigma_i\|_\infty < \varepsilon$, we have

$$|\nu_i(a) - \sigma_i(a)| < \varepsilon \leq \frac{1}{2}\sigma_i(a).$$

Consequently, $\mu_i(a) = 2\nu_i(a) - \sigma_i(a) = 2\nu_i(a) - 2\sigma_i(a) + \sigma_i(a) > 0$. Because $\sum_{a \in A_i} \mu_i(a) = 1$ clearly holds, we obtain $\mu_i \in \Delta(A_i)$.

We show that $\nu_i = \frac{1}{2}\sigma_i + \frac{1}{2}\mu_i$ is dominated by $\frac{1}{2}\tau_i + \frac{1}{2}\mu_i$. As σ_i is dominated by τ_i , there is a strategy profile $\rho_{-i} \in \Delta(A_{-i})$ such that $u_i(\rho_{-i}, \tau_i) > u_i(\rho_{-i}, \sigma_i)$. Since u_i is linear in each player's strategy we have

$$\begin{aligned} u_i(\rho_{-i}, \frac{1}{2}\tau_i + \frac{1}{2}\mu_i) &= \frac{1}{2}u_i(\rho_{-i}, \tau_i) + \frac{1}{2}u_i(\rho_{-i}, \mu_i) \\ &> \frac{1}{2}u_i(\rho_{-i}, \sigma_i) + \frac{1}{2}u_i(\rho_{-i}, \mu_i) \\ &= u_i(\rho_{-i}, \frac{1}{2}\sigma_i + \frac{1}{2}\mu_i). \end{aligned}$$

Similarly we conclude for every $\sigma_{-i} \in \Delta(A_{-i})$ that

$$u_i(\sigma_{-i}, \frac{1}{2}\tau_i + \frac{1}{2}\mu_i) \geq u_i(\sigma_{-i}, \frac{1}{2}\sigma_i + \frac{1}{2}\mu_i).$$

Hence, $\nu_i = \frac{1}{2}\sigma_i + \frac{1}{2}\mu_i$ is dominated by $\frac{1}{2}\tau_i + \frac{1}{2}\mu_i$. ■

5.2 Finitistic perfection and admissibility

The relation between finitistic perfection and admissibility is much weaker than it is for weakly perfect equilibrium. S&S already have an example of a game in which one of the players has a unique dominant action (see example 2.4 of Simon and Stinchcombe [21]), while nevertheless there is a lof perfect equilibrium in this game in which the player in question does play a dominated action. Hence, even in games with dominant actions, lof perfection does not exclusively select limit admissible strategies.

We show that our alternative definition of finitistic perfection, glof perfect equilibrium, does select only dominant strategies in case such strategies are available

to a player. In an example we show that for games without dominant strategies also glof perfect equilibrium may use strategies that are not limit admissible.

Write $PD_i := \{a_i \in A_i \mid \delta(a_i) \in D_i\}$. In Appendix B we show that D_i is closed with respect to the weak metric in $\Delta(A_i)$, that PD_i is closed in A_i , and that $D_i = \Delta(PD_i)$.

Theorem 5.3 *Let $\sigma \in \Delta(A)$ be a weakly glof equilibrium. Suppose that D_i is not empty. Then $\sigma_i \in D_i$.*

Proof. Because σ is a weakly glof equilibrium, for every player i there is a sequence $(\sigma_i^k)_{k=1}^\infty$ such that $\rho^w(\sigma_i^k, \sigma_i) \rightarrow 0$ and $\rho^w(\sigma_i^k, BR_i(\sigma^k)) \rightarrow 0$ as $k \rightarrow \infty$. So, for every k we can take $\tau_i^k \in BR_i(\sigma^k)$ such that $\rho^w(\sigma_i^k, \tau_i^k) \rightarrow 0$ as $k \rightarrow \infty$. Then, by the triangle inequality for ρ^w , also $\rho^w(\sigma_i, \tau_i^k) \rightarrow 0$ as $k \rightarrow \infty$.

Suppose that D_i is not empty. Then, by Proposition B.4, PD_i is not empty either. Let $U \supset PD_i$ be open. First we show that $\tau_i^k(U) = 1$ for sufficiently large k . Take $a_i \in A_i \setminus U$ and $b_i \in PD_i$. By Lemma B.1, there is an $a_{-i} \in A_{-i}$ such that

$$u_i(b_i, a_{-i}) > u_i(a_i, a_{-i}).$$

Since u_i is continuous, there are open sets $V(a_i) \ni a_i$ and $W(a_{-i}) \ni a_{-i}$ such that for every $x_i \in V(a_i)$ and $x_{-i} \in W(a_{-i})$ we have

$$u_i(b_i, x_{-i}) > u_i(x_i, x_{-i}).$$

Take $x_i \in V(a_i)$. Since b_i is a dominant strategy, we have

$$u_i(b_i, y_{-i}) \geq u_i(x_i, y_{-i})$$

for every $y_{-i} \in A_{-i}$. Further, define $B_{-i}^k = \prod_{j \neq i} B_j^k$. Then $B_{-i}^k \cap W(a_{-i}) \neq \emptyset$ for large k by Definition 4.3. So, since σ_{-i}^k is a completely mixed strategy in $\Delta(B_{-i}^k)$, we know that

$$u_i(b_i, \sigma_{-i}^k) > u_i(x_i, \sigma_{-i}^k)$$

for large k . Hence, for large k , x_i is not a best response to σ^k . We conclude for every $a_i \in A_i \setminus U$ that

$$V(a_i) \cap PBR_i(\sigma^k) = \emptyset$$

for large k . Now note that $A_i \setminus U$ is compact, since it is a closed subset of a compact set. Therefore, there are actions a_i^1, \dots, a_i^p in $A_i \setminus U$ such that $A_i \setminus U$ is a subset of $\bigcup_{r=1}^p V(a_i^r)$. Moreover,

$$PBR_i(\sigma^k) \cap \bigcup_{j=1}^p V(a_i^j) = \emptyset$$

for large k . Consequently $PBR_i(\sigma^k) \subseteq U$ for large k . But $\tau_i^k \in BR_i(\sigma^k)$, so that $\tau_i^k(PBR_i(\sigma^k)) = 1$ for all k . Hence, $\tau_i^k(U) = 1$ for large k , as claimed.

We show that $\sigma_i \in D_i$. Take $\varepsilon > 0$. Since $\rho^w(\sigma_i, \tau_i^k) \rightarrow 0$ as $k \rightarrow \infty$,

$$\tau_i^k(U) - \varepsilon \leq \sigma_i(U^\varepsilon)$$

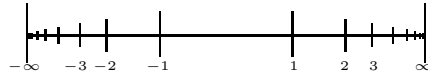
for large k . Because $\tau_i^k(U) = 1$ for large k , we get $1 - \varepsilon \leq \sigma_i(U^\varepsilon)$. Since $\varepsilon > 0$ and $U \supseteq PD_i$ were chosen arbitrarily, and since PD_i is a closed set, we can conclude that $\sigma_i(PD_i) = 1$. Hence, $\sigma_i \in \Delta(PD_i) = D_i$. ■

Theorem 5.3 shows that in games where players have dominant strategies, weak glof perfection can only select those equilibria in which dominant strategies are played. However, the next example shows that, in games where players do not have dominant strategies, even strong glof perfection may fail to uniquely select limit admissible strategies.

EXAMPLE 1. The action spaces are

$$A_1 = A_2 = \{-\infty, \dots, -3, -2, -1, 1, 2, 3, \dots, \infty\},$$

in which each integer is an isolated point, whereas $-\infty$ is the limit of the sequence $-1, -2, \dots$ and ∞ is the limit of the sequence $1, 2, \dots$. The topological structure is shown in the picture below.



The payoff functions u_1 and u_2 are symmetric and u_1 is given in the table below. Player 1 is the row player, and player 2 is the column player.

u_1	$-\infty$	\dots	-3	-2	-1	1	2	3	\dots	∞
$-\infty$	0	\dots	0	0	0	0	0	0	\dots	0
\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots
-3	0	\dots	$\frac{1}{3}$	0	0	0	0	$\frac{1}{3}$	\dots	0
-2	0	\dots	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	\dots	0
-1	0	\dots	0	0	1	1	0	0	\dots	0
1	0	\dots	0	0	0	1	0	0	\dots	0
2	0	\dots	0	0	0	0	$\frac{1}{2}$	0	\dots	0
3	0	\dots	0	0	0	0	0	$\frac{1}{3}$	\dots	0
\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots
∞	0	\dots	0	0	0	0	0	0	\dots	0

Claim. The action pair (∞, ∞) is a strongly glof perfect equilibrium. However, ∞ is not limit admissible for either player.

Proof. For either player, ∞ is not limit admissible because ∞ is dominated by every action $k \in \mathbb{Z}$, and every action $k \in \mathbb{N}$ is dominated by action $-k$.

Now, we prove that $\sigma = (\infty, \infty)$ is a strongly glof perfect equilibrium. For every player $i = 1, 2$ and $k \in \mathbb{N}$, let $B_i^k = \{-k, \dots, -1, 1, \dots, k+1\}$ and let σ_i^k be the completely mixed strategy on B_i^k which assigns probability $\frac{1}{(k+2)^2}$ to every action in B_i^k except action $k+1$ and assigns probability $(1 - \frac{2k}{(k+2)^2})$ to action $k+1$. It is clear that $(B_i^k)_{k=1}^\infty$ satisfies the first condition of Definition 4.3. Since $\rho^w(\sigma_i^k, \sigma_i) \rightarrow 0$ for $i = 1, 2$ as $k \rightarrow \infty$, the second condition also holds. Using the payoff table, one can verify that strategies $(-k-1)$ and $(k+1)$ are the best responses for player 1 against σ_2^k in A_1 . Due to symmetry, the same holds for player 2. Thus,

$$\rho^s(\sigma_i^k, BR_i(\sigma^k)) = 1 - (1 - \frac{2k}{(k+2)^2}) = \frac{2k}{(k+2)^2},$$

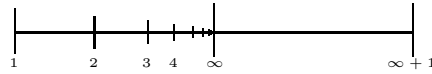
which tends to zero as $k \rightarrow \infty$. Hence, condition 4 of Definition 4.3 is valid as

well, and σ is a strongly glof perfect equilibrium. ■

6 Relations between definitions of perfect equilibrium

Simon and Stinchcombe [21] conjectured (on page 1433) that there are strongly perfect equilibria that are not anchored perfect. We provide a concrete example of a completely mixed, and therefore surely strongly perfect, equilibrium that is neither lof perfect nor glof perfect (and therefore also not anchored perfect). Thus, the finitistic approach to perfection does not have a straightforward logical relation to the trembling hand approach.

EXAMPLE 2. The action spaces are $A_1 = A_2 = \{1, 2, 3, \dots, \infty, \infty + 1\}$, where all natural numbers and $\infty + 1$ are isolated points, whereas ∞ is the limit of the sequence $1, 2, \dots$. The topology is shown in the picture below:



The payoff functions u_1 and u_2 are symmetric and u_1 is given in the table below.

u_1	1	2	3	4	...	∞	$\infty + 1$
1	$\frac{4}{8^2}$	$\frac{-1}{8}$	0	0	...	0	0
2	$\frac{4}{16^2}$	0	$\frac{-1}{16}$	0	...	0	0
3	$\frac{4}{32^2}$	0	0	$\frac{-1}{32}$...	0	0
4	$\frac{4}{64^2}$	0	0	0	...	0	0
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
∞	0	0	0	0	...	0	0
$\infty + 1$	0	0	0	0	...	0	0

Define the strategies σ_1 and σ_2 by

$$\sigma_1 = \sigma_2 = \frac{1}{4} \cdot \delta(1) + \frac{1}{8} \cdot \delta(2) + \frac{1}{16} \cdot \delta(3) + \dots + \frac{1}{4} \cdot \delta(\infty) + \frac{1}{4} \cdot \delta(\infty + 1).$$

Claim. The strategy profile (σ_1, σ_2) is a completely mixed, and hence strongly

perfect equilibrium. However, it is neither lof perfect nor weakly glof perfect. This verifies the conjecture of S&S [21] ⁶.

Proof. It is clear that (σ_1, σ_2) is a completely mixed Nash equilibrium. Hence, it is a strongly perfect equilibrium.

We show that the completely mixed equilibrium is not lof perfect. Suppose by way of contradiction that (σ_1, σ_2) is lof perfect. For every player i , take a sequence $(B_i^k)_{k=1}^\infty$ of finite subsets of A_i and a sequence of strategies $(\sigma_i^k)_{k=1}^\infty$ that satisfy the conditions in Definition 4.2. Let m_k and n_k be the largest positive integers in the sets B_1^k and B_2^k , respectively. Since $\infty + 1$ is an isolated point of the action sets, for large k each B_i^k includes $\infty + 1$. Moreover, for large k , $\infty + 1$ must be a best response for player 1 against σ_2^k among B_1^k , because σ_1 places positive probability on $\infty + 1$. As strategy $\infty + 1$ has payoff zero for player 1, for large k player 1 cannot have an action in B_1^k which gives him a positive payoff against σ_2^k . Since player 1 would get a positive payoff for every action $m \geq n_k$, we have $m_k < n_k$. Then also $n_k < m_k$ by symmetry, which is a contradiction. By similar arguments the completely mixed equilibrium is not weakly glof either. ■

We continue with a discussion of the relations between the different approaches to finitistic perfect equilibrium.

Proposition 6.1 *Every strongly glof perfect equilibrium is lof perfect.*

Proof. Suppose that $\sigma = (\sigma_i)_{i \in N}$ is a strongly glof perfect equilibrium. Then, for every $i \in N$, there is a sequence $(B_i^k)_{k=1}^\infty$ of finite subsets of A_i and a sequence of strategies $(\sigma_i^k)_{k=1}^\infty$ that satisfy the conditions in Definition 4.3. We show that they also satisfy condition 3 of Definition 4.2. Since for every i , we have $\rho^s(\sigma_i^k, BR_i(\sigma^k)) \rightarrow 0$ as $k \rightarrow \infty$, there exists a sequence $(\mu_i^k)_{k=1}^\infty$ such that $\mu_i^k \in BR_i(\sigma^k)$ for every $k \in \mathbb{N}$ and $\rho^s(\sigma_i^k, \mu_i^k) \rightarrow 0$ as $k \rightarrow \infty$. As u_i is a continuous function, $PBR_i(\sigma^k)$ is a closed, hence Borel, subset of the action space A_i . Moreover, since best responses put weight 1 on pure best responses,

$$\rho^s(\sigma_i^k, \mu_i^k) \geq \left| \sigma_i^k(PBR_i(\sigma^k)) - \mu_i^k(PBR_i(\sigma^k)) \right| = \left| \sigma_i^k(PBR_i(\sigma^k)) - 1 \right|$$

⁶Their conjecture only concerns *anchored* perfect equilibrium. We show the *stronger* claim that the completely mixed equilibrium is even not a lof perfect equilibrium.

for every i and k . So, $\sigma_i^k(PBR_i(\sigma^k)) \rightarrow 1$ as $k \rightarrow \infty$. This implies that for large k , player i has a pure best response to σ^k which lies in B_i^k . Consequently, $\sigma_i^k(PBR_i(\sigma^k | B_i^k)) \rightarrow 1$ as $k \rightarrow \infty$, and σ is a lof perfect equilibrium. ■

Theorem 5.3 shows that the lof perfect equilibrium in Example 2.4 in S&S is not weakly glof perfect (and hence not strongly glof perfect either). Next we present an example of a weakly glof perfect equilibrium that is not lof perfect. Consequently, lof perfection and weak glof perfection are not comparable.

EXAMPLE 3. In this game the action spaces are

$$A_1 = A_2 = \{-1, -2, -3, \dots, 0, \dots, 3, 2, 1\},$$

where all integers except 0 are isolated points, whereas 0 is the limit point of the sequences $-1, -2, \dots$ and $1, 2, \dots$. The topological structure is shown in the picture below.



The topology can be metrized by the following metric. For every $k, l \in A_i$,

$$d(k, l) = \begin{cases} 0 & \text{if } k = l = 0 \\ \frac{1}{7} & \text{if } k = 0 \text{ and } l \neq 0 \\ \frac{1}{k} & \text{if } k \neq 0 \text{ and } l = 0 \\ \left| \frac{1}{k} - \frac{1}{l} \right| & \text{if } k \neq 0 \text{ and } l \neq 0. \end{cases}$$

The payoff functions u_1 and u_2 are symmetric and u_1 is given in the table below.

u_1	-1	-2	-3	\dots	0	\dots	+3	+2	+1
-1	1	$\frac{1}{2}$	$\frac{1}{3}$	\dots	0	\dots	0	0	1
-2	0	0	0	\dots	0	\dots	0	$\frac{1}{2}$	0
-3	0	0	0	\dots	0	\dots	$\frac{1}{3}$	0	0
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots	\vdots
0	0	0	0	\dots	0	\dots	0	0	0
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\ddots	\vdots	\vdots	\vdots
+3	0	0	0	\dots	0	\dots	0	0	0
+2	0	0	0	\dots	0	\dots	0	0	0
+1	0	0	0	\dots	0	\dots	0	0	0

Claim. The Nash equilibrium $(\sigma_1, \sigma_2) = (0, 0)$ is weakly glof perfect. However, $(\sigma_1, \sigma_2) = (0, 0)$ is not lof perfect, and therefore by Proposition 6.1 not strongly glof perfect either.

Proof. We show that (σ_1, σ_2) is a weakly glof perfect. For every $k \in \mathbb{N}$, define $\varepsilon_k = \frac{1}{k(k+3)}$. Moreover, for $k \in \mathbb{N}$ and $i = 1, 2$, write

$$B_i^k = \{-1, -2, \dots, -k\} \cup \{k, \dots, 2, 1\}.$$

Let σ_i^k be the completely mixed strategy on B_i^k that assigns probability ε_k to every action in B_i^k except action k , to which it assigns probability $1 - (2k - 1)\varepsilon_k$. Note that due to the choice of ε_k , we have $1 - (2k - 1)\varepsilon_k > \varepsilon_k > 0$. It is clear that conditions 1 and 2 of Definition 4.3 are satisfied. It remains to verify that condition 5 holds as well. According to the payoff table, player 1's pure best response to σ_2^k is either -1 or $-k$. Since for every $k \geq 2$ we have

$$\begin{aligned} u_1(\delta(-1), \sigma_2^k) &= \varepsilon_k \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} + 1\right) \\ &\leq \varepsilon_k(k + 1) \\ &< (1 - (2k - 1)\varepsilon_k) \frac{1}{k} \\ &= u_1(\delta(-k), \sigma_2^k), \end{aligned}$$

player 1's unique best response against σ_2^k is $-k$. We prove that

$$\rho^w(\sigma_1^k, BR_1(\sigma_2^k)) = \rho^w(\sigma_1^k, \delta(-k)) \leq \frac{2}{k}.$$

Let $\varepsilon > \frac{2}{k} = d(-k, k)$. It is easy to see that $(2k-1)\varepsilon_k \leq \varepsilon$. Take an arbitrary set $B \subseteq A_1$.

A. We show that $\delta(-k)(B) \leq \sigma_1^k(B^\varepsilon) + \varepsilon$. If $-k \notin B$. Then

$$\delta(-k)(B) = 0 \leq \sigma_1^k(B^\varepsilon) + \varepsilon.$$

If $-k \in B$. Then $k \in B^\varepsilon$. So

$$\delta(-k)(B) = 1 \leq (1 - (2k-1)\varepsilon_k) + \varepsilon = \sigma_1^k(k) + \varepsilon \leq \sigma_1^k(B^\varepsilon) + \varepsilon.$$

B. We show that $\sigma_1^k(B) \leq \delta(-k)(B^\varepsilon) + \varepsilon$. If $-k \in B$. Then

$$\sigma_1^k(B) \leq 1 < 1 + \varepsilon = \delta(-k)(B^\varepsilon) + \varepsilon$$

If $-k \notin B$ and $k \notin B$. Then

$$\sigma_1^k(B) \leq (2k-1)\varepsilon_k \leq \varepsilon \leq \delta(-k)(B^\varepsilon) + \varepsilon.$$

If $-k \notin B$ and $k \in B$. Then $-k \in B^\varepsilon$. So

$$\sigma_1^k(B) \leq 1 + \varepsilon = \delta(-k)(B^\varepsilon) + \varepsilon.$$

So, by **A**, **B**, and the choice of ε , $\rho^w(\sigma_1^k, \delta(-k)) \leq \frac{2}{k}$. Since the payoff functions are symmetric this argument is also valid for the second player. Therefore, condition 5 of Definition 4.3 holds and σ is a weakly glof perfect equilibrium as claimed.

We show that $(0, 0)$ is not lof perfect. Suppose the opposite. Then by Definition 4.2 for each $i = 1, 2$ there is non-empty finite subset B_i of A_i and a completely mixed strategy σ_i in $\Delta(B_i)$ such that

$$\sigma_i(PBR_i(\sigma | B_i)) \geq \frac{2}{3},$$

where $\sigma = (\sigma_1, \sigma_2)$. Since -1 is an isolated point, w.l.o.g. $-1 \in B_i$ for each $i = 1, 2$. Furthermore, $PBR_i(\sigma | B_i)$ is a subset of $\{-1, -2, \dots\}$. Define

$$M_i = -\min \{x \mid x \in PBR_i(\sigma | B_i)\} \quad \text{for } i = 1, 2.$$

Assume w.l.o.g. that $M_1 \geq M_2$. Because $-M_1 \in PBR_1(\sigma \mid B_1)$ and $-1 \in B_1$ we have

$$u_1(\delta(-1), \sigma_2) \leq u_1(\delta(-M_1), \sigma_2).$$

Thus, according to the payoff table,

$$\frac{2}{3} \cdot \frac{1}{M_2} \leq u_1(\delta(-1), \sigma_2) \leq u_1(\delta(-M_1), \sigma_2) \leq \frac{1}{3} \cdot \frac{1}{M_1}.$$

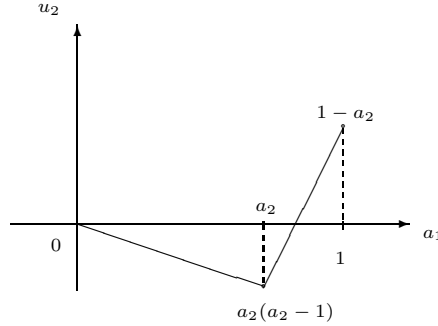
Rewriting yields $\frac{M_2}{2} \geq M_1$. Consequently $\frac{M_2}{2} \geq M_2$. Contradiction. ■

Finally, in this section we present a modification of the example in S&S of a pure equilibrium that is limit of finite, while one of the players plays an action that is not limit admissible. Theorem 5.3 shows that the equilibrium is not weakly glof.

EXAMPLE 4 The game has two players with action spaces $A_1 = A_2 = [0, 1]$ and utility functions $u_1 = 0$ and

$$u_2(a_1, a_2) = \begin{cases} (a_2 - 1) \cdot a_1 & \text{if } a_1 \leq a_2 \\ (a_2 + 1) \cdot a_1 - 2a_2 & \text{if } a_1 \geq a_2. \end{cases}$$

In the following picture the function $a_1 \mapsto u_2(a_1, a_2)$ is depicted for a fixed action a_2 of player 2.



Claim. The action pair $(0, 1)$ is a lof perfect equilibrium. However, $a_2 = 0$ is the unique dominant strategy, so that $a_2 = 1$ is not limit admissible. Consequently, the equilibrium is not weakly glof perfect.

Proof. The strategy $a_2 = 0$ is the unique dominant strategy for player 2, so that $a_2 = 1$ is not limit admissible. Hence, by Theorem 5.3, the pair $(0, 1)$ is not weakly glof perfect.

In order to show that $(0, 1)$ is lof perfect, let $\varepsilon > 0$. For $n \in \mathbb{N}$, consider the finite subset

$$B_1^n = B_2^n = \left\{ \frac{i}{n} \mid i = 1, \dots, n \right\}$$

of $A_1 = A_2$. For $s < \frac{1}{n+1}\varepsilon$, define the strategy

$$\tau_1(\varepsilon, s) = (1 - \varepsilon)\delta(0) + (\varepsilon - s)\delta\left(\frac{1}{n}\right) + \frac{s}{n-1} \sum_{i=2}^n \delta\left(\frac{i}{n}\right).$$

Then for every $a_2^j = \frac{j}{n}$, $j = 1, \dots, n-1$ we have

$$\begin{aligned} u_2\left(\tau_1(\varepsilon, s), \delta(a_2^j)\right) &\leq (\varepsilon - s) \cdot \left(\frac{1}{n}\right) \cdot \left(\frac{j}{n} - 1\right) + \frac{s}{n-1} \cdot \left[\sum_{i=2}^n \left(\frac{j}{n} + 1\right) \frac{i}{n} - \frac{2j}{n}\right] \\ &\leq (\varepsilon - s) \left(\frac{1}{n}\right) \left(\frac{j}{n} - 1\right) + s\left(1 - \frac{j}{n}\right) \\ &= (\varepsilon - s) \left(\frac{1}{n}\right) \left(\frac{j-n}{n}\right) + s\left(\frac{n-j}{n}\right) \\ &= \frac{n-j}{n} \left(\frac{s(n+1)-\varepsilon}{n}\right). \end{aligned}$$

This outcome is less than zero, but player 2 can get zero by choosing 1. Hence, action pair $(0, 1)$ is a lof perfect equilibrium. \blacksquare

7 Two players

For finite strategic form games it is well-known that perfection implies undominatedness. A similarly well-known theorem of van Damme (see for example van Damme [7], Theorem 3.2.2) states that for bimatrix games the converse implication also holds.

Theorem 7.1 *Let (p, q) be a strategy pair of the bimatrix game (A, B) . Equivalent are*

- [1] *The strategy pair (p, q) is a perfect equilibrium.*
- [2] *The strategy pair (p, q) is a Nash equilibrium and both p and q are undominated strategies.*

Theorem 5.1 and Proposition 5.2 show that, in the appropriately adjusted sense, the implication from (1) to (2) is also valid in the setting with infinite compact action spaces. A natural guess would be that the appropriately adjusted converse would also be true for two-player infinite games. However, the following example shows that the converse implication from (2) to (1) no longer holds for games with infinite action spaces.

EXAMPLE 5. The action spaces are $A_1 = \{T, B\}$ and $A_2 = \{1, 2, 3, 4, \dots, \infty\}$, where all natural numbers are isolated points, whereas ∞ is the limit point of the sequence $1, 2, 3, \dots$. The payoffs for the players are given by

$$u_1 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \dots & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{9} & -\frac{1}{16} & \dots & 0 \end{bmatrix}$$

Claim. The action pair $(B, 1)$ is a Nash equilibrium in undominated strategies, yet $(B, 1)$ is not a weakly perfect equilibrium.

Proof. Clearly $(B, 1)$ is a Nash equilibrium and actions B and 1 are undominated in A_1 and A_2 respectively.

We prove that the action pair $\sigma = (B, 1)$ is not a weakly perfect equilibrium. Take any completely mixed strategy $\tau_1 = (p, 1 - p)$ of player 1. Take a natural number $k > \frac{1-p}{p}$. Then $\frac{1-p}{k} < p$, so that

$$\begin{aligned} u_2(\tau_1, \delta(k)) &= p \cdot \frac{1}{k} - (1-p) \cdot \frac{1}{k^2} \\ &= \frac{1}{k} \cdot \left[p - \frac{1-p}{k} \right] \\ &> 0. \end{aligned}$$

Since $u_2(\tau_1, \delta(1)) = 0$, this implies that $\delta(1)$ is not a best response for player 1 against any completely mixed strategy of player 2. Hence, $\sigma = (B, 1)$ is not a weakly perfect equilibrium. ■

Note that the above example is in a very strong sense minimal. The action space A_2 is in a very natural way the “smallest” example of a compact set that is not finite. And also A_1 is minimal in the sense that it is the smallest example of an action space in which a player actually has a genuine choice to make.

8 Invariance

Another remarkable feature of the game in Example 5 is that (B, ∞) is a strongly (and hence also weakly) perfect equilibrium. This is surprising in the sense that ∞ is payoff equivalent to 1 for player 2. Thus, both strong and weak perfection violate invariance in this setting, while perfect equilibrium satisfies invariance in

the setting of finite strategic form games (see for example Kohlberg and Mertens [12] and Vermeulen and Jansen [22]).

The underlying cause of this seeming breakdown of invariance is that in the finite case the space of actions is, from a topological perspective, a very simple object. All actions are isolated points, and hence there is a homeomorphism that transforms one into the other while the induced transformation on payoffs equals the identity in the case of payoff equivalence.

However, in the infinite case two actions giving the same payoff might be totally different from a topological perspective. Therefore, only conditional on having a homeomorphism that preserves payoffs, we can conclude that strongly and weakly perfection are invariant too. This result is stated in the following theorem.

Theorem 8.1 *Both strongly and weakly perfect equilibrium are invariant under payoff preserving homeomorphic transformations of the action spaces.*

Note that, for finite games, invariance with respect to payoff preserving homeomorphic transformations is in fact equivalent to invariance. Hence, the above Theorem implies that for finite strategic form games perfect equilibrium is invariant.

9 Discussion

We studied the relations between various generalizations of the basic notion of perfect equilibrium to the context of strategic form games with compact action spaces and continuous payoff functions. We showed the existence of weakly glof perfect equilibrium, and presented several examples of games with compact action spaces that clarify the relationships between admissibility, trembling hand perfect equilibrium, and finitistic perfect equilibrium.

Specifically, we showed in Example 5 that the equivalence between perfection and undominatedness for bimatrix games no longer holds for general two-person strategic form games with compact action spaces and continuous payoffs.

We also verified a conjecture of Simon and Stichcombe [21] with an example of a completely mixed Nash equilibrium that is not perfect in any of the finitistic

definitions. Despite its appeal to basic intuition of the finitistic approach, our results (especially Examples 1 and 2) seem to imply a severe critique on the finitistic approach. Example 1 shows that finitistic perfection may easily fail to be limit admissible, a fact that was already observed by Simon and Stinchcombe for lof perfection. Example 2 shows that a completely mixed equilibrium need not be perfect in any of the finitistic approaches.

In Example 2 the completely mixed equilibrium is part of a component of Nash equilibria, one of which is still finitistically perfect. It remains an open question whether an isolated completely mixed equilibrium is automatically perfect in the finitistic approach. Another possible question for future research is the existence of strongly glof perfect equilibrium.

A Proof of Theorem 4.4

The proof of Theorem 4.4 is based on the fixed point Theorem of Kakutani. We state the Theorem of Kakutani first.

Let X and Y be two metric spaces. A correspondence between X and Y is a map from the elements of X to the collection of all subsets of Y . So, for every $x \in X$, we have $F(x) \subseteq Y$. A correspondence F between X and Y is upper-hemicontinuous if for every $x \in X$ and every sequence $(x_n)_{n=1}^\infty$ in X converging to x and every sequence $(y_n)_{n=1}^\infty$ in Y converging to y with $y_n \in F(x_n)$ it holds that $y \in F(x)$. A point $x \in X$ is called a fixed point for a correspondence $F : X \rightarrow X$ when $x \in F(x)$.

Theorem A.1 (*Kakutani's fixed point theorem*) *Let X be a non-empty, compact and convex subset of an Euclidean space. Let $F : X \rightarrow X$ be an upper-hemicontinuous correspondence such that $F(x)$ is non-empty, closed and convex for all $x \in X$. Then F has a fixed point.*

We show Theorem 4.4. W.l.o.g. suppose that $A_i \subseteq [0, 1]$ for every $i \in N$. Fix $k \in \mathbb{N}$. For every i , construct a finite subset B_i^k of A_i with $|B_i^k| \leq 3k$ and for every $a \in A_i$ there is a $b \in B_i^k$ such that $|a - b| \leq \frac{1}{k}$. Write $n_{ik} = |B_i^k|$ and $\varepsilon = \frac{1}{3k^2}$. For every i , define

$$\Delta_i^k(\varepsilon) = \{\nu_i \in \Delta(B_i^k) \mid \nu_i(b) \geq \varepsilon \text{ for every } b \in B_i^k\}$$

and $\Delta^k(\varepsilon) = \prod_{i \in N} \Delta_i^k(\varepsilon)$. Define the correspondence $ABR_i^k: \Delta^k(\varepsilon) \rightarrow \Delta_i^k(\varepsilon)$ by

$$ABR_i^k(\sigma) = \{\mu_i \in \Delta_i^k(\varepsilon) \mid \rho^w(\mu_i, BR_i(\sigma)) \leq \frac{2}{k}\}$$

for every $\sigma = (\sigma_i)_{i \in N} \in \Delta^k(\varepsilon)$ ⁷. Furthermore, define the correspondence $ABR^k: \Delta^k(\varepsilon) \rightarrow \Delta^k(\varepsilon)$ by, for every $\sigma \in \Delta(A)$,

$$ABR^k(\sigma) = \prod_{i \in N} ABR_i^k(\sigma).$$

We verify the conditions of Theorem A.1.

A. Clearly $\Delta^k(\varepsilon)$ is compact and convex. Further, since $\varepsilon = \frac{1}{3k^2}$ and $|B_i^k| \leq 3k$, the set $\Delta^k(\varepsilon)$ is not empty.

B. We verify that $ABR^k(\sigma)$ satisfies the conditions of the Theorem of Kakutani.

B1. We show for every $i \in N$ that $ABR_i^k(\sigma)$ is non-empty. Take $a \in PBR_i(\sigma)$. Then, there is a $b \in B_i^k$ such that $|a - b| \leq \frac{1}{k}$. Hence, $\rho^w(\delta(a), \delta(b)) \leq \frac{1}{k}$. Now, for $x \in B_i^k$ define

$$\mu_i(x) = \begin{cases} \varepsilon & \text{if } x \neq b \\ 1 - (n_{ik} - 1)\varepsilon & \text{if } x = b. \end{cases}$$

Then, $\mu_i \in \Delta_i^k(\varepsilon)$. Moreover,

$$\rho^w(\mu_i, \delta(b)) \leq \|\mu_i - \delta(b)\|_\infty = (n_{ik} - 1)\varepsilon \leq 3k \cdot \frac{1}{3k^2} = \frac{1}{k}.$$

Then, by the triangle inequality we have

$$\rho^w(\mu_i, \delta(a)) \leq \frac{1}{k} + \frac{1}{k} = \frac{2}{k}.$$

Thus, $\rho^w(\mu_i, BR_i(\sigma)) \leq \frac{2}{k}$, which implies $\mu_i \in ABR_i^k(\sigma)$. Hence, $ABR^k(\sigma)$ is non-empty.

B2. Clearly, for every i , $ABR_i^k(\sigma)$ is closed and therefore $ABR^k(\sigma)$ is closed as well.

B3. We prove that $ABR_i^k(\sigma)$ is convex for every i . Take strategies $\mu, \nu \in ABR_i^k(\sigma)$. We prove that $c\mu + (1 - c)\nu \in ABR_i^k(\sigma)$ for every $c \in [0, 1]$. Since

⁷ ABR stands for approximate best response.

$\mu, \nu \in ABR_i^k(\sigma)$, we have $\rho^w(\mu, BR_i(\sigma)) \leq \frac{2}{k}$ and $\rho^w(\nu, BR_i(\sigma)) \leq \frac{2}{k}$. So there are strategies $\mu', \nu' \in BR_i(\sigma)$ such that

$$\rho^w(\mu, \mu') \leq \frac{2}{k} \quad \text{and} \quad \rho^w(\nu, \nu') \leq \frac{2}{k}.$$

Take an arbitrary $\delta > \frac{2}{k}$. Then, for every non-empty measurable set $A \subseteq A_i$, we have

$$\mu(A) \leq \mu'(A^\delta) + \delta$$

$$\mu'(A) \leq \mu(A^\delta) + \delta$$

and

$$\nu(A) \leq \nu'(A^\delta) + \delta$$

$$\nu'(A) \leq \nu(A^\delta) + \delta.$$

Then,

$$c\mu(A) + (1-c)\nu(A) \leq c\mu'(A^\delta) + (1-c)\nu'(A^\delta) + \delta$$

$$c\mu'(A) + (1-c)\nu'(A) \leq c\mu(A^\delta) + (1-c)\nu(A^\delta) + \delta.$$

Hence,

$$\rho^w(c\mu + (1-c)\nu, c\mu' + (1-c)\nu') \leq \delta,$$

which by the choice of δ implies

$$\rho^w(c\mu + (1-c)\nu, c\mu' + (1-c)\nu') \leq \frac{2}{k}.$$

Since $BR_i(\sigma)$ is a convex set, $c\mu' + (1-c)\nu' \in BR_i(\sigma)$. Thus,

$$\rho^w(c\mu + (1-c)\nu, BR_i(\sigma)) \leq \frac{2}{k},$$

which implies $c\mu + (1-c)\nu \in ABR_i^k(\sigma)$.

C. We show that ABR^k is upper-hemicontinuous. Take sequences $(\sigma^m)_{m=1}^\infty$ and $(\tau^m)_{m=1}^\infty$ in $\Delta^k(\epsilon)$ such that $\lim_{m \rightarrow \infty} \sigma^m = \sigma$ and $\lim_{m \rightarrow \infty} \tau^m = \tau$ in Euclidean distance, and such that $\tau^m \in ABR^k(\sigma^m)$ for every $m \in \mathbb{N}$. We show $\tau_i \in ABR_i^k(\sigma)$ for every $i \in N$, so that $\tau \in ABR^k(\sigma)$.

Take $m \in \mathbb{N}$. Since $\tau_i^m \in ABR_i^k(\sigma^m)$, we have $\rho^w(\tau_i^m, BR_i(\sigma^m)) \leq \frac{2}{k}$. As $BR_i(\sigma^m)$ is a closed set with respect to the weak metric, there is a strategy $\mu_i^m \in BR_i(\sigma^m)$ such that $\rho^w(\tau_i^m, \mu_i^m) \leq \frac{2}{k}$. Because $\Delta(A_i)$ is sequentially compact with respect to the weak metric, the sequence $(\mu_i^m)_{m=1}^\infty$ has a subsequence

$(\mu_i^{m_l})_{l=1}^\infty$ with $\rho^w(\mu_i^{m_l}, \mu_i) \rightarrow 0$ as $l \rightarrow \infty$ for some $\mu_i \in \Delta(A_i)$. Then, by continuity of u_i with respect to ρ^w , we have $\mu_i \in BR_i(\sigma)$. Moreover, $\rho^w(\tau_i, \mu_i) \leq \frac{2}{k}$. Hence, $\rho^w(\tau_i, BR_i(\sigma)) \leq \frac{2}{k}$. This shows that the correspondence ABR^k is upper-hemicontinuous. This concludes **C**.

By **A**, **B**, and **C** we know that each ABR^k satisfies the conditions of Theorem A.1. Hence, for every k , there is a $\sigma^k \in \Delta^k(\varepsilon)$ such that $\sigma^k \in ABR^k(\sigma^k)$, which means that $\sigma_i^k \in ABR_i^k(\sigma^k)$ for every i . Hence, $\rho^w(\sigma_i^k, BR_i(\sigma^k)) \leq \frac{2}{k}$ for every i . Since $\Delta(A)$ is sequentially compact with respect to the weak metric, we can assume w.l.o.g. that the sequence $\sigma^k = (\sigma_i^k)_{i \in N}$ converges to a strategy profile $\sigma = (\sigma_i)_{i \in N}$ when $k \rightarrow \infty$. Note that $\rho^w(\sigma_i^k, BR_i(\sigma^k)) \rightarrow 0$ as $k \rightarrow \infty$. Hence, $\sigma = (\sigma_i)_{i \in N}$ is a weakly glof perfect equilibrium.

It remains to show that every weakly glof perfect equilibrium is a Nash equilibrium. Suppose that $\sigma = (\sigma_i)_{i \in N}$ is a weakly glof perfect equilibrium. Then, by definition, for every i there is a sequence $(\sigma_i^k)_{k=1}^\infty$ such that $\rho^w(\sigma_i^k, \sigma_i) \rightarrow 0$ and $\rho^w(\sigma_i^k, BR_i(\sigma^k)) \rightarrow 0$ as $k \rightarrow \infty$. Therefore, for every i there is also a sequence $(\mu_i^k)_{k=1}^\infty$ such that $\mu_i^k \in BR_i(\sigma^k)$ for every $k \in \mathbb{N}$ and $\rho^w(\sigma_i^k, \mu_i^k) \rightarrow 0$ as $k \rightarrow \infty$. By the triangle inequality for ρ^w , this implies that for every i , $\rho^w(\mu_i^k, \sigma_i) \rightarrow 0$ as $k \rightarrow \infty$. Hence, again by continuity of u_i , we obtain $\sigma_i \in BR_i(\sigma)$ for every $i \in N$, as desired. ■

B Proofs for Section 5

Lemma B.1 *Suppose that τ_i dominates σ_i . Then there is an $a_{-i} \in A_{-i}$ such that $u_i(a_{-i}, \tau_i) > u_i(a_{-i}, \sigma_i)$.*

Proof. Suppose that $u_i(a_{-i}, \tau_i) = u_i(a_{-i}, \sigma_i)$ for every $a_{-i} \in A_{-i}$. Then, for every $\sigma_{-i} \in \Delta(A_{-i})$ we have $u_i(\sigma_{-i}, \tau_i) = u_i(\sigma_{-i}, \sigma_i)$. Contradiction. ■

Lemma B.2 *Suppose that strategy $\tau_i \in \Delta(A_i)$ dominates $\sigma_i \in \Delta(A_i)$. Then*

$$u_i(\sigma \mid \tau_i) > u_i(\sigma \mid \sigma_i)$$

for any completely mixed strategy profile $\sigma \in \Delta(A)$. Consequently, any best response to a completely mixed strategy profile is undominated.

Proof. This is a direct consequence of Lemma B.1 and the continuity of the payoff function u_i on $\Delta(A)$. ■

The next Theorem is a direct consequence of the monotonicity of measure.

Lemma B.3 *Suppose that f is a real function on X which is strictly positive and μ is a measure on X such that $\mu(X) > 0$. Then, $\int_X f d\mu > 0$.*

Proof. Let $F_n = \{x \in X \mid f(x) \geq \frac{1}{n}\}$. It is clear that $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ and $\bigcup_{n=1}^{\infty} F_n = X$. Consequently, $\mu(F_n) \uparrow \mu(X)$ when $n \rightarrow \infty$. Since $\mu(X) > 0$, there exists an $n \in \mathbb{N}$ such that $\mu(F_n) > 0$. But for every $x \in X$, $f(x) \geq \frac{1}{n} \mathbf{1}_{F_n}(x)$.

Therefore,

$$\int_X f d\mu \geq \frac{1}{n} \int_X \mathbf{1}_{F_n} d\mu = \frac{1}{n} \mu(F_n).$$

Thus, $\int_X f d\mu > 0$, as claimed. \blacksquare

Proposition B.4 *The set D_i is closed with respect to the weak metric in $\Delta(A_i)$ and PD_i is closed in A_i . Moreover, $D_i = \Delta(PD_i)$.*

Proof. Assume that D_i is non-empty, otherwise the statements are obvious.

Take a $\rho_i \in D_i$. For every $\sigma_{-i} \in \Delta(A_{-i})$ define

$$F_{\sigma_{-i}} = \{\sigma_i \in \Delta(A_i) \mid u_i(\sigma_{-i}, \sigma_i) = u_i(\sigma_{-i}, \rho_i)\}.$$

Since u_i is a continuous function on $\Delta(A_i)$ with respect to the weak metric, $F_{\sigma_{-i}}$ is a closed set. Moreover, $D_i = \bigcap_{\sigma_{-i} \in \Delta(A_{-i})} F_{\sigma_{-i}}$. Therefore, D_i is also a closed set with respect to the weak metric. With a similar argument we can conclude that PD_i is a closed subset of A_i .

Now we show that $D_i = \Delta(PD_i)$. Since $\Delta(PD_i) \subseteq D_i$, we only need to prove that $\rho_i \in D_i$ implies $\rho_i \in \Delta(PD_i)$, or equivalently $\rho_i(PD_i) = 1$. Suppose by way of contradiction that $\rho_i(PD_i) < 1$, implying $\rho_i(A_i \setminus PD_i) > 0$. Take a completely mixed strategy profile $\tau \in \Delta(A)$. Let $c = u_i(\tau \mid \rho_i)$. By Lemma B.2 we have

$$u_i(\tau \mid x_i) < u_i(\tau \mid \rho_i) = c$$

for every $x_i \in A_i \setminus PD_i$, while $u_i(\tau \mid x_i) = u_i(\tau \mid \rho_i) = c$ for every $x_i \in PD_i$.

Then, using Lemma B.3 to get the strict inequality, we have

$$\begin{aligned} c = u_i(\tau \mid \rho_i) &= \int_{A_i} u_i(\tau \mid x_i) d\rho_i \\ &= \int_{PD_i} u_i(\tau \mid x_i) d\rho_i + \int_{A_i \setminus PD_i} u_i(\tau \mid x_i) d\rho_i \\ &< c \rho_i(PD_i) + c \rho_i(A_i \setminus PD_i) = c \end{aligned}$$

which is a contradiction. ■

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